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# ON SOME TOTALLY ERGODIC FUNCTIONS

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*Dedicated to Dagmara Klim and Nina Tomaszewska*

**We study some classes of totally ergodic functions on locally compact Abelian groups. Among other things, we establish the following result: If $R$ is a locally compact commutative ring, $\mathscr{R}$ is the additive group of $R$, $\chi$ is a continuous character of $\mathscr{R}$, and $p$ is the function from $\mathscr{R}^n$ ($n \in \mathbb{N}$) into $\mathscr{R}$ induced by a polynomial of $n$ variables with coefficients in $R$, then the function $\chi \circ p$ either is a trigonometric polynomial on $\mathscr{R}^n$ or all of its Fourier-Bohr coefficients with respect to any Banach mean on $L^\infty(\mathscr{R}^n)$ vanish.**

**1. Introduction.** Let $G$ be a locally compact Abelian group, $\lambda_G$ be the Haar measure in $G$, and $L^\infty(G)$ be the space of all classes of complex-valued $\lambda_G$-measurable $\lambda_G$-essentially bounded functions on $G$ endowed with the $\lambda_G$-essential supremum norm.

A linear continuous functional $m$ on $L^\infty(G)$ is called a Banach mean on $L^\infty(G)$ if it satisfies the following conditions:

(i) $m(1) = 1 = \|m\|$,
(ii) $m(T_a f) = m(f)$ for each $a \in G$ and each $f \in L^\infty(G)$, where $T_a f(b) = f(a+b)$ for any $b \in G$.

When $G$ is finite, there is precisely one Banach mean on $L^\infty(G)$. When $G$ is infinite, then the set of all Banach means on $L^\infty(G)$ has at least the cardinality of the continuum (cf. [**6**, Propositions 22.26 and 22.41]).

Let $\widehat{G}$ be the dual group of $G$. Given $f \in L^\infty(G)$, $\chi \in \widehat{G}$, and a Banach mean $m$ on $L^\infty(G)$, let $\mathscr{F}_m f(\chi)$ stand for the Fourier-Bohr coefficient of $f$ at $\chi$ with respect to $m$, defined to be $m(f\bar{\chi})$.

A function $f$ in $L^\infty(G)$ is said to be ergodic if its mean value $m(f)$ is independent of the choice of the Banach mean $m$ on $L^\infty(G)$. A function $f$ in $L^\infty(G)$ is said to be totally ergodic if, for every $\chi \in \widehat{G}$, the function $f\chi$ is ergodic (cf. [**7**, **8**]). Let $E(G)$ be the space of all ergodic functions in $L^\infty(G)$, $TE(G)$ be the space of all totally ergodic functions in $L^\infty(G)$, and $TE_0(G)$ be the subspace of $TE(G)$ consisting of those $f \in L^\infty(G)$ for which $\mathscr{F}_m f(\chi) = 0$ for any $\chi \in \widehat{G}$ and any Banach mean $m$ on $L^\infty(G)$. Let $P(G)$ be the space of all functions in $L^\infty(G)$ which, to within modification on a $\lambda_G$-null set, are trigonometric polynomials on $G$. It is readily verified that

$$P(G) \subset TE(G)$$

and that

$$P(G) \cap TE_0(G) = \{0\}.$$

The chief aim of the present paper is to show that certain subsets of $L^\infty(G)$, determined by conditions formulated with use of some coboundary operator, are contained in $P(G) \cup TE_0(G)$. One consequence of the main result about those subsets reads as follows: If $R$ is a locally compact commutative ring, $\mathscr{R}$ is the additive group of $R$, $\chi$ is an element of $\widehat{\mathscr{R}}$, and $p$ is the function from $\mathscr{R}^n$ $(n \in \mathbb{N})$ into $\mathscr{R}$ induced by a polynomial of $n$ variables with coefficients in $R$, then the function $\chi \circ p$ is an element either of $P(\mathscr{R}^n)$ or of $TE_0(\mathscr{R}^n)$.

**2. Preliminaries.** Given a set $A$, $\#A$ denotes the cardinality of $A$. If $A$ is subset of a larger set, then $1_A$ stands for the characteristic function of $A$.

Given $a \in G$ and a subset $A$ of $G$, let

$$a + A = \{b \in G \colon b - a \in A\}.$$

A complex-valued function $f$ on $G$ with values of unit modulus will be called unitary. A function in $L^\infty(G)$ which, to within modification on a $\lambda_G$-null set, is unitary will be called almost unitary. We denote by $U(G)$ the set of all almost unitary functions in $L^\infty(G)$, and write $U_0(G)$ for $U(G) \cap P(G)$.

Let $f$ be function in $U(G)$. For each $a \in G$, put

$$\delta_a f = \bar{f} \cdot T_a f$$

and, for any $a_1, \dots, a_n \in G$, set inductively

$$\delta_{a_1 \cdots a_n} f = \delta_{a_n}(\delta_{a_1 \cdots a_{n-1}} f).$$

For each $1 \le p < +\infty$, let $L^p(G)$ be the $p$th Lebesgue space based on $\lambda_G$.

Given $f \in L^1(G)$, let $\mathscr{F}f$ denote the Fourier transform of $f$, defined by

$$\mathscr{F}f(\chi) = \int_G f(a)(a, -\chi)\, d\lambda_G(a) \qquad (\chi \in \widehat{G});$$

here $(a, -\chi)$ stands for the value of the character $-\chi$ at $a$. Let $\sigma(f)$ denote the spectrum of $f$, that is, the support of $\mathscr{F}f$.

If $f \in P(G)$ is $\lambda_G$-essentially equal to a trigonometric polynomial $\sum_{\chi\in\hat{G}} a_\chi \chi$, then the set $\{\chi \in \widehat{G}: a_\chi \neq 0\}$ will also be denoted as $\sigma(f)$ and referred to as the spectrum of $f$.

For each $n \in \mathbb{N}$, let

$$P_n(G) = \{f \in P(G): \#\sigma(f) \leq n\}$$

and

$$U_n(G) = \{f \in U(G): \delta_{a_1\cdots a_n} f \in P(G) \text{ for } a_1, \dots, a_n \in G\}.$$

For each $m \in \mathbb{N}$, let

$$U_{0,m}(G) = U(G) \cap P_m(G)$$

and, for any $n, m \in \mathbb{N}$, let

$$U_{n,m}(G) = \{f \in U(G): \delta_{a_1\cdots a_n} f \in P_m(G) \text{ for } a_1, \dots, a_n \in G\}.$$

Given a probability triple $(\Omega, \mathscr{B}, \mathbb{P})$ and a $\sigma$-subalgebra $\mathscr{A}$ of $\mathscr{B}$, we write $\mathbb{E}^{\mathscr{A}}$ for the conditional expectation operator relative to $\mathscr{A}$.

For a subset $A$ of a vector space, the linear span of $A$ is denoted by span $A$.

For a subset $A$ of a set $B$ with a topology, we denote by $\overline{A}$ the closure of $A$ in $B$.

**3. A characterization of $U_0(G)$.** In this section, we give a characterization of the set $U_0(G)$ for an arbitrary locally compact Abelian group $G$. We start with the following.

PROPOSITION 3.1. *Let $G$ be a locally compact Abelian group such that $\widehat{G}$ is torsion-free. Then*

$$U_0(G) = U_{0,1}(G).$$

*Proof.* Clearly, it suffices to show that $U_0(G) \subset U_{0,1}(G)$.

Let $f$ be a function in $U_0(G)$ and let $\sum_{i=1}^n a_i\chi_i$ be the trigonometric polynomial on $G$ $\lambda_G$-essentially equal to $f$, with $\sigma(f) = \{\chi_i: 1 \leq i \leq n\}$. Suppose that $n \geq 2$. Let $\Gamma$ be the subgroup of $\widehat{G}$ generated by $\sigma(f)$. Of course, $\Gamma$ is countable and torsion-free. Hence there exists a monomorphism $h$ from $\Gamma$ into the group of reals (cf. [**9**, Theorem 8.1.2]). Changing, if necessary, the enumeration of the elements of $\sigma(f)$, we may assume that $h(\chi_i) < h(\chi_j)$ whenever $1 \leq i < j \leq n$. Since

$$h(\chi_n\bar{\chi}_1) = h(\chi_n) - h(\chi_1) > h(\chi_i) - h(\chi_j) = h(\chi_i\bar{\chi}_j)$$

whenever $(i, j) \neq (n, 1)\,(1 \leq i \leq n, 1 \leq j \leq n)$, it follows that the Fourier coefficient of $\sum_{i,j=1}^{n} a_i\bar{a}_j\chi_i\bar{\chi}_j$ at $\chi_n\bar{\chi}_1$ is equal to $a_n\bar{a}_1$. Moreover, since

$$h(\chi_n\bar{\chi}_1) > h(\chi_n\bar{\chi}_n) = 0,$$

we see that $\chi_n\bar{\chi}_1$ is a non-trivial character of $G$. But

$$\sum_{i,j=1}^{n} a_i\bar{a}_j\chi_i\bar{\chi}_j = \left|\sum_{i=1}^{n} a_i\chi_i\right|^2 = 1,$$

so uniqueness of the Fourier expansion implies that $a_n\bar{a}_1 = 0$. This contradiction shows that $\sigma(f)$ is a singleton.

The proof is complete.

Passing to the characterization of $U_0(G)$ in the general case, we first show that the problem reduces to characterizing $U_0(G)$ for a compact Abelian group $G$ such that the component of 0 in $G$ (which is a closed subgroup of $G$) has finite index.

With $G$ an arbitrary locally compact Abelian group, let $f$ be an element of $U_0(G)$. Denote by $(\widehat{G})_d$ the group $\widehat{G}$ furnished with the discrete topology. Let $\Gamma$ be the subgroup of $(\widehat{G})_d$ generated by $\sigma(f)$, $\mathrm{Per}(\Gamma)$ be the subgroup of $\Gamma$ consisting of all elements of finite order, and $H$ be the component of 0 in $\widehat{\Gamma}$. Then the dual of $\widehat{\Gamma}/H$ coincides with $\mathrm{Per}(\Gamma)$ (cf. [5, Corollary 24.20]). Since $\Gamma$ is finitely generated, it follows that $\mathrm{Per}(\Gamma)$ is finite and hence $H$ has finite index. Let $\alpha$ be the canonical homomorphism from $\Gamma$ into $\widehat{G}$. Then the dual homomorphism $\hat{\alpha}$, defined by

$$(\hat{\alpha}(g), \chi) = (g, \alpha(\chi)) \qquad (g \in G, \chi \in \Gamma),$$

maps $G$ onto a dense subgroup of $\widehat{\Gamma}$. Moreover, there exists a unique $p$ in $U_0(\widehat{\Gamma})$ such that $f = p \circ \hat{\alpha}$. Thus it is clear that the passage from $f$ to $p$ yields the desired reduction.

Now we may and do assume that $G$ is a compact Abelian group such that the component $H$ of 0 in $G$ has finite index. Let $\{a_i \colon 1 \leq i \leq n\}$ be a subset of $G$ such that the sets $a_i + H$ $(1 \leq i \leq n)$ form the collection of all cosets of $H$ in $G$. We claim that

$$U_0(G) = \{f \in \mathbb{T}^G \colon f(a_i + g) = c_i\chi_i(g) \text{ for } g \in H, \\ c_i \in \mathbb{T}, \chi_i \in \widehat{H}\ (1 \leq i \leq n)\},$$

where $\mathbb{T}$ denotes the circle group.

Indeed, if we let $A$ denote the right-hand side set, the containment of $U_0(G)$ in $A$ follows from Proposition 3.1 and the fact that the dual of a connected locally compact Abelian group is torsion-free (cf. [**5**, Corollary 24.19]). Conversely, if $f \in A$, then

$$\operatorname{span}\{T_a f\colon a \in G\} \subset \operatorname{span}\{\chi_i 1_{a_j+H}\colon 1 \le i \le n,\ 1 \le j \le n\},$$

so $\operatorname{span}\{T_a\colon a \in G\}$ is finite dimensional, and hence $f$ is a trigonometric polynomial on $G$ (cf. [**9**, Theorem 7.8.3]). Thus $A \subset U_0(G)$ and the claim follows.

**4. The main results.** The starting point of our main considerations is the following.

THEOREM 4.1. *Let $G$ be a compact Abelian group. Then*

$$U_n(G) = U_0(G)$$

*for each $n \in \mathbb{N}$.*

*Proof.* Clearly, it suffices to prove that $U_n(G) \subset U_0(G)$ for each $n \in \mathbb{N}$. A simple induction argument shows that in fact it suffices to establish the containment of $U_1(G)$ in $U_0(G)$.

Given $f \in U_1(G)$, let $\Sigma$ be the subgroup of $\widehat{G}$ generated by $\sigma(f)$. Clearly, $\Sigma$ is countable. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a family of finite subsets of $\Sigma$ such that $\sigma_n \subset \sigma_{n+1}$ for each $n \in \mathbb{N}$, and

$$\Sigma = \bigcup_{n=1}^{\infty} \sigma_n.$$

Given $n \in \mathbb{N}$, let

$$F_n = \{a \in G\colon \sigma(\delta_a f) \subset \sigma_n\}.$$

Each $F_n$ is clearly closed. Since, for each $a \in G$, $\sigma(\delta_a f)$ is a finite subset of $\Sigma$, it follows that

$$G = \bigcup_{n=1}^{\infty} F_n.$$

By Baire's theorem, there exist an open subset $V$ of $G$ and a positive integer $m$ such that $V \subset F_m$. By the compactness of $G$, there exists a finite subset $\{a_i\colon 1 \le i \le k\}$ of $G$ such that

$$G = \bigcup_{i=1}^{\infty} (a_i + V).$$

For each $a \in G$, if $1 \le i \le k$ and $v \in V$ are such that $a = a_i + v$, then

$$T_a f = T_{a_i}(\delta_v f) \cdot T_{a_i} f.$$

Thus

$$\operatorname{span}\{T_a f\colon a \in G\} \subset \operatorname{span}\{\chi T_{a_i} f\colon \chi \in \sigma_m,\ 1 \le i \le k\}.$$

Consequently, $\operatorname{span}\{T_a f\colon a \in G\}$ is finite dimensional, and hence $f$ is in $U_0(G)$.

The proof is complete.

LEMMA 4.2. *Let $G$ be a compact Abelian group. Let $f$ be an almost unitary function in $G$, $S$ be a dense subset of $G$, and $n$ and $m$ be positive integers such that $\#\sigma(\delta_{s_1 \cdots s_n} f) \le m$ for any $s_1, \ldots, s_n \in S$. Then $f \in U_0(G)$.*

*Proof*. Suppose that for some $a_1, \ldots, a_n \in G$, the spectrum of $\delta_{a_1 \cdots a_n} f$ contains $m+1$ distinct elements $\chi_1, \ldots, \chi_{m+1}$. Then, in view of the continuity of the functions

$$G^n \ni (b_1, \ldots, b_n) \to \mathscr{F}\delta_{b_1 \cdots b_n} f(\chi_i) \qquad (1 \le i \le m+1)$$

and the denseness of $S$ in $G$, there exist $s_1, \ldots, s_n \in S$ such that

$$\{\chi_1, \ldots, \chi_{m+1}\} \subset \sigma(\delta_{a_1 \cdots s_n} f),$$

a contradiction. Thus $f \in U_{m,n}(G)$, and hence, by the preceding theorem, $f \in U_0(G)$.

The proof is complete.

The next theorem is the main result of this section.

THEOREM 4.3. *Let $G$ be a locally compact Abelian group. Then*

$$U_{n,m}(G) \subset U_0(G) \cup TE_0(G)$$

*for each $n \in \mathbb{N} \cup \{0\}$ and each $m \in \mathbb{N}$.*

*Proof*. We shall proceed by induction on $n$ with $m$ arbitrarily fixed.

The case $n = 0$ is obvious.

Assume the assertion for $n-1$. Suppose that $f \in U_{n,m}(G) \setminus TE_0(G)$. Then there exist $\chi \in \widehat{G}$ and a Banach mean $m$ on $L^\infty(G)$ such that $\mathscr{F}_m f(\chi) \neq 0$. Let $h = f\bar{\chi}$. Then, clearly, $m(h) \neq 0$. Moreover, for each $a \in G$, $\delta_a h \in U_{n-1,m}(G)$, and hence, by the inductive hypothesis, either $\delta_a h \in U_0(G)$ or $\delta_a h \in TE_0(G)$. Since, for each $a \in G$,

$$\delta_{-a} h = T_{-a} \delta_a \bar{h}$$

and, for any $a, b \in G$,

$$\delta_{a+b} h = \delta_a h \cdot T_a \delta_b h,$$

it follows that

$$G_0 = \{a \in G\colon \delta_a h \in U_0(G)\}$$

is a subgroup of $G$. We claim that the index of $G_0$ is finite.

Suppose, on the contrary, that there exists an infinite subset $\{a_n\colon n \in \mathbb{N}\}$ of $G$ such that $a_n - a_m \notin G_0$ whenever $n \neq m$. Then, if $n \neq m$, then $\delta_{a_n - a_m} h$ is in $TE_0(G)$, and hence

$$m(\delta_{a_n}\bar{h} \cdot \delta_{a_m} h) = m(T_{a_m}\delta_{a_n - a_m}\bar{h}) = m(\delta_{a_n - a_m}\bar{h}) = 0.$$

We see that the image of $\{\delta_{a_n}\bar{h}\colon n \in \mathbb{N}\}$ by the canonical mapping from $L^\infty(G)$ onto the pre-Hilbert space

$$H_m(G) = L^\infty(G)/\{f \in L^\infty(G)\colon m(|f|^2) = 0\}$$

is an orthonormal set. For each $n \in \mathbb{N}$, we have

$$m(h) = m(T_{a_n}h) = m(h \cdot \delta_{a_n} h).$$

Thus the Fourier coefficients of the image of $h$ in $H_m(G)$ relative to the image of $\{\delta_{a_n}\bar{h}\colon n \in \mathbb{N}\}$ in $H_m(G)$ are equal to $m(h)$, and hence, by Bessel's inequality, $m(h) = 0$. This contradiction establishes the claim.

Let $bG$ be the Bohr compactification of $G$ and $\alpha\colon G \to bG$ be the canonical monomorphism from $G$ into $bG$. For each $\chi \in \widehat{G}$, let $\tilde{\chi}$ be the continuous character of $bG$ such that $\tilde{\chi} \circ \alpha = \chi$. As is known, the Fourier transformation sets up a one-to-one correspondence between $L^2(bG)$ and $l^2(\widehat{G})$ $(= L^2((\widehat{G})_d))$. Since by Bessel's inequality, the function

$$\widehat{G} \ni \chi \longrightarrow \mathscr{F}_m f(\chi) \in \mathbb{C}$$

is in $l^2(\widehat{G})$, there exists a unique element $X$ in $L^2(bG)$ such that

$$\text{(4.1)} \qquad \mathscr{F}X(\tilde{\chi}) = \mathscr{F}_m f(\chi) \qquad (\chi \in \widehat{G}).$$

Since

$$G_0 = \{a \in G\colon \delta_a f \in U_0(G)\},$$

it follows that for each $a \in G_0$, there exists a unique unitary trigonometric polynomial $P_a$ on $bG$ such that

$$\text{(4.2)} \qquad \delta_a f = P_a \circ \alpha \qquad \lambda_G\text{-a.e.}$$

If, for each $a \in G_0$, we let

$$P_a = \sum_{\gamma \in \hat{G}} b_{\alpha,\gamma}\tilde{\gamma},$$

then, in view of (4.1), for each $\chi \in \widehat{G}$,

$$\begin{aligned}\mathscr{F}T_{\alpha(a)}X(\tilde{\chi}) &= (\alpha(a), \tilde{\chi})\mathscr{F}X(\tilde{\chi}) = (a, \chi)\mathscr{F}_m f(\chi) \\ &= \mathscr{F}_m T_a f(\chi) = \mathscr{F}_m(f\delta_a f)(\chi) \\ &= \sum_{\gamma\in\hat{G}} b_{a,\gamma}\mathscr{F}_m f(\tilde{\chi} - \tilde{\gamma}) \\ &= \sum_{\gamma\in\hat{G}} b_{a,\gamma}\mathscr{F}X(\tilde{\chi} - \tilde{\gamma}) = \mathscr{F}(XP_a)(\tilde{\chi})\end{aligned}$$

whence

$$T_{\alpha(a)}X = XP_a \qquad \lambda_{bG}\text{-a.e.} \tag{4.3}$$

Let $\{a_i: 1 \leq k\}$ be a subset of $G$ such that the sets $a_i + G_0$ $(1 \leq i \leq k)$ form the collection of all cosets of $G_0$ in $G$. Since

$$bG \supset \bigcup_{i=1}^{k}(\alpha(a_i) + \overline{\alpha(G_0)}) \supset \overline{\bigcup_{i=1}^{k}\alpha(a_i + G_0)} = bG,$$

the closures being taken in $bG$, it follows that the index of $\overline{\alpha(G_0)}$ in $bG$ is no greater than $k$. Thus $\overline{\alpha(G_0)}$ is an open subgroup of $bG$ and, in particular, the Haar measure in $\overline{\alpha(G_0)}$ is, to within normalization, the restriction to $\overline{\alpha(G_0)}$ of the Haar measure in $bG$.

Let $\{b_j: 1 \leq j \leq l\}$ be a subset of $\{a_i: 1 \leq i \leq k\}$ such that the sets $\alpha(b_j) + \overline{\alpha(G_0)}$ $(1 \leq j \leq l)$ form the collection of all cosets of $\overline{\alpha(G_0)}$ in $bG$. For each $1 \leq j \leq l$, let $X_j$ denote the restriction of $T_{\alpha(b_j)}X$ to $\overline{\alpha(G_0)}$. In view of (4.3), for each $1 \leq j \leq l$ and each $a \in G_0$,

$$T_{\alpha(a)}|X_j| = |X_j| \qquad \lambda_{\overline{\alpha(G_0)}}\text{-a.e.}$$

Applying the Fourier transformation to both sides of the latter equality, we readily find that for $1 \leq j \leq l$, $|X_j|$ is $\lambda_{\overline{\alpha(G_0)}}$-essentially constant. Choose $j_0$ so that

$$X_{j_0} \neq 0 \quad \lambda_{\overline{\alpha(G_0)}}\text{-a.e.}$$

and set

$$Y = |X_{j_0}|^{-1}X_{j_0}.$$

For each $a \in G_0$, let $R_a$ denote the restriction of $T_{\alpha(b_{j_0})}P_a$ to $\overline{\alpha(G_0)}$. Since, by (4.3), for each $a \in G_0$,

$$\delta_{\alpha(a)}Y = R_a \qquad \lambda_{\overline{\alpha(G_0)}}\text{-a.e.}, \tag{4.4}$$

it follows from Lemma 4.2 that $Y \in U_0(\overline{\alpha(G_0)})$. Hence in particular the set

$$\Gamma = \{\gamma \in (\overline{\alpha(G_0)})^\frown : \gamma = \gamma_1\bar{\gamma}_2 \text{ for } \gamma_1, \gamma_2 \in \sigma(Y)\}$$

is finite.

Let $(\alpha(G_0))^\perp$ be the annihilator of $\alpha(G_0)$ in $(bG)^\frown$, that is, the set

$$\{\gamma \in (bG)^\frown : (\alpha(a), \gamma) = 1 \text{ for } a \in G_0\}.$$

Being the dual of the quotient group $bG/\overline{\alpha(G_0)}$, the group $(\alpha(G_0))^\perp$ is finite. Let $\pi$ be the canonical homomorphism from $(bG)^\frown$ onto $(\alpha(G_0))^\frown$. Since the kernel of $\pi$ coincides with $(\alpha(G_0))^\perp$, we see that the set $\pi^{-1}(\Gamma)$ is finite, and hence the set

$$\Xi = \{\chi \in \widehat{G} : \chi = \gamma \circ \alpha \text{ for } \gamma \in \pi^{-1}(\Gamma)\}$$

is also finite.

In view of (4.4), $\Gamma$ contains the spectra of all the $R_a$ $(a \in G_0)$. Consequently, $\pi^{-1}(\Gamma)$ contains the spectra of all the $T_{\alpha(b_{j_0})}P_a$ $(a \in G_0)$, and hence the spectra of all the $P_a$ $(a \in G_0)$. Now Eq. (4.2) implies that

$$\{T_a f : a \in G_0\} \subset \operatorname{span}\{\chi f : \chi \in \Xi\}$$

whence

$$\{T_a f : a \in G\} \subset \operatorname{span}\{\chi T_{a_i} f : \chi \in \Xi,\ 1 \le i \le k\}.$$

We see that $\operatorname{span}\{T_a f : a \in G\}$ is finite dimensional, and so $f$ is in $U_0(G)$.

The proof is complete.

**5. Applications.** Let $R$ be a locally compact commutative ring, $\mathscr{R}$ be the additive group of $R$, $\chi$ be an element of $\widehat{\mathscr{R}}$, and $p$ be the function from $\mathscr{R}^n$ $(n \in \mathbb{N})$ into $\mathscr{R}$ induced by a polynomial

$$\sum_{|\alpha| \le k} a_\alpha x^\alpha \qquad (\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n,\ |\alpha| = \alpha_1 + \cdots + \alpha_n)$$

of $n$ variables with coefficients in $R$, of degree $k$. Then, of course, $\chi \circ p$ is in $U_{k,1}(\mathscr{R}^n)$. Applying Theorem 4.3 to $\chi \circ p$, we obtain the following.

THEOREM 5.1. *The function* $\chi \circ p$ *is an element either of* $U_0(\mathscr{R}^n)$ *or of* $TE_0(\mathscr{R}^n)$.

Let

$$R^{(2)} = \{r \in R : r = st \text{ for } s, t \in \mathbb{R}\}.$$

THEOREM 5.2. *Suppose that* $R = R^{(2)}$, *that* $\widehat{\mathscr{R}}$ *is torsion-free, that* $k$ *is not less than* $2$, *and that for some* $\alpha$ *with* $|\alpha| = k$, *the character* $r \to \chi(a_\alpha r)$ *of* $\mathscr{R}$ *is non-trivial. Then* $\chi \circ p$ *is in* $TE_0(\mathscr{R}^n)$.

*Proof*. Suppose, on the contrary, that $\chi \circ p$ is not in $TE_0(\mathscr{R}^n)$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index with $|\alpha| = k$ such that the character $r \to \chi(a_\alpha r)$ of $\mathscr{R}$ is non-trivial. Given $r_1, \dots, r_k \in R$, put

$$\begin{aligned} a_1 &= (r_1, 0, \dots, 0), \\ a_{\alpha_1} &= (r_{\alpha_1}, 0, \dots, 0), \\ a_{\alpha_1+1} &= (0, r_{\alpha_1+1}, \dots, 0), \\ a_{\alpha_1+\alpha_2} &= (0, r_{\alpha_1+\alpha_2}, \dots, 0), \\ a_k &= (0, \dots, 0, r_k). \end{aligned}$$

A straightforward calculation shows that

$$\delta_{a_1 \cdots a_k}(\chi \circ p) = \chi(\alpha! a_\alpha r_1 \cdots r_k) \qquad (\alpha! = \alpha_r! \cdot \cdots \cdot \alpha_n!).$$

Now Proposition 3.1, Theorem 4.3, and the fact that $\widehat{\mathscr{R}}$ is torsion-free imply that $\chi \circ p$ is in $U_{0,1}(\mathscr{R}^n)$. Since $k \geq 2$, it follows that $\delta_{a_1 \cdots a_k}(\chi \circ p) = 1$ and, consequently, that $\chi(\alpha! a_\alpha r_1 \cdots r_k) = 1$ for any $r_1, \dots, r_k \in R$. Taking into account that $R = R^{(2)}$ and that $\widehat{\mathscr{R}}$ is torsion-free, we infer that $\chi(a_\alpha r) = 1$ for each $r \in R$, a contradiction. The proof is complete.

As an immediate consequence of Theorem 5.2, we get the following generalization of a result of **[1]**:

THEOREM 5.3. *Let* $K$ *be a locally compact commutative field,* $\mathscr{K}$ *be the additive group of* $K$, $\chi$ *be an element of* $\widehat{\mathscr{K}}$, *and* $p$ *be the function from* $\mathscr{K}^n$ $(n \in \mathbb{N})$ *into* $\mathscr{K}$ *induced by a polynomial of* $n$ *variables with coefficients in* $K$, *of degree not less than* $2$. *Then* $\chi \circ p$ *is in* $TE_0(\mathscr{K}^n)$.

**6. A counter-example.** In this section we show that Theorem 4.3 fails in general if in the statement the set $U_{n,m}(G)$ is replaced by the set $U_n(G)$.

For each $n \in \mathbb{N}$, let $G_n$ be a non-zero finite Abelian group with a pair number of elements. Let $G$ be the direct sum of the $G_n$ $(n \in \mathbb{N})$, and $\Sigma$ be the direct product of the $G_n$ $(n \in \mathbb{N})$. Endow $G$ with the discrete topology, and $\Sigma$ with the product topology (of course, each $G_n$ is given the discrete topology). For each $n \in \mathbb{N}$, let $\pi_n$ be the canonical projection from $G$ onto $G_n$, and $\rho_n$ be the canonical projection from $\Sigma$ onto $G_n$. Let $\alpha$ be the canonical monomorphism from $G$ into $\Sigma$. Given $n \in \mathbb{N}$, let $e_n$ be a function from $G_n$ onto $\{-1, 1\}$ such that

$$\sum_{g \in G_n} e_n(g) = 0,$$

and put

$$f_n = e_n \circ \rho_n.$$

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} |a_n|^2 < +\infty, \qquad \sum_{n=1}^{\infty} |a_n| = +\infty,$$

and $|a_n| < \pi/4$ for each $n \in \mathbb{N}$. Given $\sigma \in \Sigma$ and $a \in G$, set

$$A(\sigma, a) = \exp\left[i \sum_{n=1}^{\infty} a_n(f_n(\sigma) - f_n(\sigma + \alpha(a)))\right].$$

To see that the above definition makes sense, note that given $a \in G$, there exists $m \in \mathbb{N}$ such that $\pi_n(a) = 0$ whenever $n > m$, and so, for each $\sigma \in \Sigma$,

$$A(\sigma, a) = \exp\left[i \sum_{n=1}^{m} (f_n(\sigma) - f_n(\sigma + \alpha(a)))\right]. \tag{6.1}$$

One verifies at once that the mapping $A$: $(\sigma, a) \to A(\sigma, a)$ is a Borel unitary function on $\Sigma \times G$ satisfying

$$A(\sigma, a + b) = A(\sigma, a)A(\sigma + \alpha(a), b)$$

for all $\sigma \in \Sigma$ and all $a, b \in G$. $A$ is an example of what is called a cocycle on $\Sigma$ (cf. [**2, 3, 4**]).

Since, clearly, $(f_n)_{n \in \mathbb{N}}$ is a Bernoulli sequence on the probability triple $(\Sigma, \mathscr{B}(\Sigma), \lambda_\Sigma)$, where $\mathscr{B}(\Sigma)$ stands for the Borel $\sigma$-algebra of $\Sigma$ and $\lambda_\Sigma$ is the normalized Haar measure in $\Sigma$, it follows that the series $\sum_{n=1}^{\infty} a_n f_n(\sigma)$ converges for $\lambda_\Sigma$-almost all $\sigma$ in $\Sigma$. Let $Z$ be a real Borel function on $\Sigma$ $\lambda_\Sigma$-almost everywhere equal to the sum of the above series. On putting

$$Y = \exp(iZ), \tag{6.2}$$

we see that given $a \in G$, the identity

$$A(\sigma, a) = Y(\sigma)\overline{Y(\sigma + \alpha(a))} \tag{6.3}$$

holds for $\lambda_\Sigma$-almost all $\sigma$ in $\Sigma$. The existence of a representation of $A$ as above is usually expressed as saying that $A$ is a coboundary.

Each function of the form $a \to A(\sigma, a)$ $(\sigma \in \Sigma)$ is called a trajectory of $A$. In view of (5.1), for each $a \in G$, the function $\sigma \to A(\sigma, a)$ is a unitary trigonometric polynomial on $\Sigma$. Hence, for each $\sigma \in \Sigma$ and each $b \in G$, the function $a \to A(\sigma+\alpha(a), b)$ is a unitary trigonometric polynomial on $G$. Taking into account the identity

$$\overline{A(\sigma, a)}A(\sigma, a+b) = A(\sigma+\alpha(a), b) \qquad (\sigma \in \Sigma, \quad a, b \in G),$$

we thus see that each trajectory of $A$ is in $U_1(G)$. On the other hand, a modification of an argument used in the proof to [2, Theorem 2.4] shows that if some trajectory of $A$ is totally ergodic, then $A$ is a so-called $c$-coboundary, that is, there exists a unitary continuous function $X$ on $\Sigma$ such that

$$A(\sigma, a) = X(\sigma)\overline{X(\sigma+\alpha(a))} \tag{6.4}$$

for each $\sigma \in \Sigma$ and each $a \in G$. Below we shall show that $A$ is not a $c$-coboundary. Consequently, each trajectory of $A$ will provide an example of an element of $U_1(G)$ that is not in $U_0(G) \cup TE_0(G)$.

To show that $A$ is not a $c$-coboundary, suppose, contrariwise, that there exists a unitary continuous function $X$ on $\Sigma$ satisfying (6.4). Then in view of (6.3), given $a \in G$, the identity

$$Y(\sigma+\alpha(a))\overline{X(\sigma+\alpha(a))} = Y(\sigma)\overline{X(\sigma)}$$

holds for $\lambda_\Sigma$-almost all $\sigma$ in $\Sigma$. Applying the Fourier transformation to both sides of the latter equality, we see that there exist $c \in \mathbb{T}$ such that

$$Y(\sigma) = cX(\sigma) \tag{6.5}$$

for $\lambda_\Sigma$-almost all $\sigma$ in $\Sigma$.

Let $M$ be a positive number such that

$$|z| \le M|e^z - 1| \tag{6.6}$$

for each complex number $z$ with $|z| < \pi$. Since $\Sigma$ is compact, it follows that $X$ is uniformly continuous, and so there exists $k \in \mathbb{N}$ such that if $\sigma$ is in

$$U_k = \{\theta \in \Sigma : \pi_n(\theta) = 0 \text{ for } n < k\},$$

then

$$\|T_\sigma X - X\|_\infty < \pi/2M, \tag{6.7}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. For each $n \geq k$, let $\mathscr{A}_n$ be the $\sigma$-subalgebra of $\mathscr{B}(\Sigma)$ generated by the $f_j$ with $k \leq j \leq n$. Then, by (6.7), for each $n \geq k$ and each $\sigma \in U_k$,

$$\|\mathbb{E}^{\mathscr{A}_n} T_\sigma X - \mathbb{E}^{\mathscr{A}_n} X\|_\infty < \pi/2M$$

whence, in view of (6.2), (6.3), and (6.5),

$$\left\| \exp\left[ i \sum_{j=k}^{m} a_j (T_\sigma f_j - f_j) \right] - 1 \right\|_\infty < \pi/2M. \tag{6.8}$$

Proceeding by induction on $n$, we show now that for each $n \geq k$ and each $\sigma \in U_k$,

$$\left\| \sum_{j=k}^{n} a_j (T_\sigma f_j - f_j) \right\|_\infty < \pi/2. \tag{6.9}$$

For $n = k$, the inequality follows from the estimates

$$\|a_k (T_\sigma f_k - f_k)\|_\infty \leq 2|a_k| < \pi/2.$$

Assume the validity of the inequality for $n - 1 \geq k$. Then

$$\left\| \sum_{j=k}^{n} a_j (T_\sigma f_j - f_j) \right\|_\infty < \pi/2 + 2|a_n| < \pi$$

and now (6.9) results from (6.6) and (6.8).

Choose $\theta$ in $\Sigma$ so that the series $\sum_{j=1}^{\infty} a_j f_j(\theta)$ converges. Then, in view of (6.9), for each $n \geq k$ and each $\sigma \in U_k$,

$$\left| \sum_{j=k}^{n} a_j f_j(\sigma + \theta) \right| < \pi/2 + \sup \left\{ \left| \sum_{j=k}^{m} a_j f_j(\theta) \right| : m \geq k \right\}.$$

On the other hand, it is easily seen that for each $n \geq k$,

$$\sup \left\{ \left| \sum_{j=k}^{n} a_j f_j(\sigma + \theta) \right| : \sigma \in U_k \right\} = \sum_{j=k}^{n} |a_j|.$$

The last two relations show that $(a_n)_{n \in \mathbb{N}}$ is summable, a contradiction. Thus $A$ is not a $c$-coboundary, as was to be shown.

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