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The only known absolute invariant for a general system of pde is its maximal character. The purpose of this paper is to prove that conservation laws are also absolute invariants, that is, they are preserved under partial prolongations in a natural way. We also show that the property, closely related to soliton behavior, of having an infinite number of conservation laws is an absolute invariant.

I. Introduction and background. Partial prolongations of systems of pde were introduced by E. Cartan to define when two continuous infinite dimensional transformation pseudogroups are equivalent as abstract groups [**1**, p. 625ff]. With this relation he was able to classify the simple infinite pseudogroups [**1**, p. 857ff].

In [**1**, p. 1133] Cartan studied partial prolongations of general systems of pde and used them to make precise the notion due to D. Hilbert of when two systems possess a one-to-one correspondence between their solutions [**6**]. Two systems are "absolutely equivalent" when they can be joined by a finite sequence of systems where for each adjacent pair in the sequence, one is a partial prolongation of the other.

Thus, the main use of partial prolongations is geometric: a property associated with systems of pde is an "absolute invariant" property when it is shared by both systems in any partial prolongation.

In a series of papers the author showed that several classical concepts (hyperbolicity, characteristics) are not, in fact, absoluted invariants [**7**, **8**, **9**, **10**, **11**]. This means that these concepts are not necessarily intrinsically related to the systems they are defined on: if one changes the form of the system by a partial prolongation, such concepts may change or no longer exist. Any theory based on these non-absolute invariants may predict different kinds of behavior for systems which are in fact completely and naturally equivalent.

In physics the problem is even more crucial. Concepts which are not absolute invariants of the systems of pde that describe the physical

process are not "physical" at all, since they could change or disappear if one alters the describing system of pde in some inessential way: they depend on how you happen to write down the pde. Just as modern physics commonly seeks quantities which are shared by systems equivalent under geometric Lie groups, they ought to also check for invariance under partial prolongations.

A rare concept which does turn out to be an absolute invariant would thus be particularly important and the focus of special interest. Until the present paper, the *only* known absolute invariant is the maximal character, which determines, roughly, the number of arbitrary functions in a maximal number of independent variables which can be assigned arbitrarily in the initial value problem [**10**, p. 237]. In this paper we prove conservations laws to be absolutely invariant.

In some works, pde which are derived from a conservation law are studied [**12**]. Our viewpoint is the opposite: to study the conservation laws associated with a given system of pde. Studies of the KdV equation and others reveal that they have conservation laws related to soliton solutions.

Solitons are special solutions associated with certain pde [**2**]. Since we shall show that there is a one-to-one correspondence between the solutions of absolutely equivalent systems, the property of having soliton solutions is an absolute invariant. The property of having infinitely many different conservation laws seems closely connected with existence of solitons [**2**, p. 36]. We show that this property is also an absolute invariant, thus strengthening this conjectured connection.

To make this paper as readable and self-contained as possible, we illustrate the ideas with numerous examples and often use local coordinates. Experts can easily translate the proofs into the language of abstract fiber spaces and bundles.

We do take for granted a familiarity with the calculus of exterior differential forms, however [**4**].

II. Examples of conservation laws.

EXAMPLE 2.1 [**2**, p. 32]. Let a compressible fluid of density $r(x, t)$ be flowing with velocity $u(x, t)$ in the x-direction at time t. Then

$$\frac{\partial r}{\partial t} + \frac{\partial (ru)}{\partial x} = 0. \tag{2.1}$$

Consider the 1-form $C = r\,dx - (ru)\,dt$. If we compute dC, the exterior derivative of C, using exterior algebra and calculus, we get

$$dC = \left(\frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial t}dt\right) \wedge dx - \left(\frac{\partial (ru)}{\partial x}dx + \frac{\partial (ru)}{\partial t}dt\right) \wedge dt$$
$$= \left(\frac{\partial r}{\partial t} + \frac{\partial (ru)}{\partial x}\right) dt \wedge dx = 0,$$

when one "takes into account" equation (2.1).

When E is a plane region in (x, t)-space with oriented boundary ∂E,

$$\oint_{\partial E} C = \iint_{E} dC = 0.$$

If $E = \{(x, t) | -\infty < x < \infty, t_1 \le t \le t_2\}$ then the last equation says the total mass at time t_1 equals the total mass at time t_2.

On the other hand, when $E = \{(x, t) | x_1 \le x \le x_2, -\infty < t < \infty\}$, it expresses conservation of total flux across two planes at x_1 and x_2.

The 3-dimensional analogue works the same way, only now C is a 3-form.

EXAMPLE 2.2. The plane motion of N particles of masses m_i at (x^i, y^i) under gravity is governed by

$$m_i \frac{d^2 x^i}{dt^2} = k \sum_{j \ne i} \frac{x^j - x^i}{d_{ij}} m_i m_j, \tag{2.2}$$
$$m_i \frac{d^2 y^i}{dt^2} = k \sum_{j \ne i} \frac{y^j - y^i}{d_{ij}} m_i m_j,$$

where

$$d_{ij} = \left[(x^i - x^j)^2 + (y^i - y^j)^2\right]^{3/2}, \qquad i, j = 1, \ldots, N.$$

Let

$$L = \sum_{i=1}^{N} m_i \left(x^i \frac{dy^i}{dt} - y^i \frac{dx^i}{dt}\right).$$

If we replace

$$dx^i \text{ by } \frac{dx^i}{dt}dt, \qquad d\left(\frac{dx^i}{dt}\right) \text{ by } \frac{d^2 x^i}{dt^2}dt, \quad \text{etc.}$$

then $dL = 0$ when we "take into account" equations (2.2).

Notice that we must treat dx^i/dt and dy^i/dt as variables in some extended space. Ehresmann's "jet" manifolds are just such a space [3] where, indeed, they are coordinates.

EXAMPLE 2.3 (KdV Equation).

$$\frac{\partial y}{\partial t} - 6y\frac{\partial y}{\partial x} + \frac{\partial^3 y}{\partial x^3} = 0. \tag{2.3}$$

This is known to have soliton solutions [**2**, pp. 3–8]. Then, if

$$C = y\,dx - \left(\frac{\partial^2 y}{\partial x^2} - 3y^2\right) dt,$$

$$\begin{aligned} dC &= dy \wedge dx - \left(d\left(\frac{\partial^2 y}{\partial x^2}\right) - 6y\,dy\right) \wedge dt \\ &= \left(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial t}dt\right) \wedge dx \\ &\quad - \left[\frac{\partial^3 y}{\partial x^3}dx + \frac{\partial^3 y}{\partial x^2 \partial t}dt - 6y\left(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial t}dt\right)\right] \wedge dt \\ &= \left(\frac{\partial y}{\partial t} + \frac{\partial^3 y}{\partial x^3} - 6y\frac{\partial y}{\partial x}\right) dt \wedge dx = 0, \end{aligned}$$

if one "takes into account" (2.3).

EXAMPLE 2.4. Consider next the "formal derivative" of C in the last example:

$$\partial_x C = \frac{\partial y}{\partial x}dx - \left(\frac{\partial^3 y}{\partial x^3} - 6y\frac{\partial y}{\partial x}\right) dt. \tag{2.4}$$

By using not only equation (2.3) but the derivative of equation (2.3) with respect to x, one finds that $d(\partial_x C) = 0$ if one replaces

$$d\left(\frac{\partial y}{\partial x}\right) \quad \text{by} \quad \frac{\partial^2 y}{\partial x^2}dx + \frac{\partial^2 y}{\partial x \partial t}dt, \text{ etc.}$$

We thus find that the formal derivative of the conservation law C is also a conservation law. This is true in general. The KdV equation is unusual in that it has an infinite number of conservation laws which are *not* formal derivatives of some lower order laws [**2**, pp. 34–36].

EXAMPLE 2.5.

$$\frac{\partial^2 y}{\partial x^1 \partial x^2} = 0, \qquad C = \left(\frac{\partial y}{\partial x^1}\right)^2 dx^1 + \frac{\partial y}{\partial x^2}dx^2. \tag{2.5}$$

Then

$$dC = \left[2\left(\frac{\partial y}{\partial x^1}\right)\left(\frac{\partial^2 y}{\partial x^1 \partial x^2}\right) - \frac{\partial^2 y}{\partial x^1 \partial x^2}\right] dx^2 \wedge dx^1 = 0$$

when one uses equation (2.5).

We observe that in each case an exterior form C over the differentials of the independent variables is given, whose coefficients are functions of the dependent and independent variables and certain partial derivatives. Moreover, in computing the exterior derivative of C one must use certain 1-forms to obtain dC in terms of the differentials of the independent variables. Then, when the original equation or some of its formal derivatives are "taken into account," $dC = 0$.

We want first to make this process precise and reduced to ordinary computations on the geometry of manifolds. This can be done using Ehresmann's jet spaces.

II. Jet spaces. C. Ehresmann's theory of jets [3] enables us to solve problems about the structure of pde by differential geometric methods [13]. We introduce notations and state elementary results that will be needed.

If $X = \mathbb{R}^n$ has coordinates $(x^1, \ldots, x^n)$ and $Y = \mathbb{R}^m$ has coordinates $(y^1, \ldots, y^m)$ then the space J_p of p-jets of maps from X to Y has coordinates that correspond to the variables in a pde of order p with dependent variables y^j and independent variables x^i, namely,

$$x^i, y^j, y^j_k, y^j_{hj}, \ldots \quad \text{up through order } p$$

(where y^j_{hj} represents $\partial y^j/\partial x^h \partial x^k$, etc.)

Ehresmann gave the elegant definition: a p-jet at $x \in X$ is the equivalence class $j_p(f)(x)$ of a local C^∞ function $f: X \to Y$ whose domain includes x, the equivalence relation being:

$f \approx g$ if $f(x) = g(x)$ and all partials of f and g are the same at x up to and including order p. Then in coordinates the jet $j_p(f)(x)$ is

$$\left(x^i, f^j(x), \frac{\partial f^j}{\partial x^k}(x), \frac{\partial^2 f}{\partial x^h \partial x^k}(x), \ldots\right).$$

The fact that $y^j_{ik} = y^j_{ki}$ complicates things. A very convenient notation from pde theory is

$$y^j_{i_1 i_2 \cdots i_n} = \frac{\partial^r f^j}{\partial (x^1)^{i_1} \partial (x^2)^{i_2} \cdots \partial (x^n)^{i_n}} = \partial_{i_1 \cdots i_n} f^j$$

rather than $y_{11\cdots 22\cdots n}$. For this "n-tuple" notation we designate $\nu = (i_1, \ldots, i_n)$ where i_k are non-negative integers, so

$$y^j_{i_1, \ldots, i_n} = y^j_\nu = \partial_\nu f^j.$$

Let $|\nu| = i_1 + \cdots + i_n$. If we let $y^j_{0\cdots 0} = y^j$, then the coordinates of J_p are (x^i, y^j_ν), for $|\nu| \leq p$.

It is understood that the i and j depend on dimensions of X and Y. *We often omit these upper indices* i, j *which greatly simplifies the notation, and merely write* (x, y_ν), $|\nu| \leq p$ *for coordinates in* J_p [**14, 15**].

We also simplify things by using the summation convention.

The reason why it is necessary to involve the geometry of J_p is that we want results independent of choice of coordinates on X or Y. If we allow changes of variables of the form $y' = F(x, y)$, $x' = G(x)$, we are really studying jets $X \to E$, where E is a fibered manifold $\pi: E \to X$. That is the viewpoint in [**14, 15**].

(The physicist may need even more general coordinate changes $y' = F(x, y)$, $x' = G(x, y)$. The author knows of no extension of jet formalism to this situation. Cartan's theory of exterior systems of equations is, however, of this genre, since it makes no implicit designation of independent variables.)

Notation. If $\nu = (i_1, \ldots, i_n)$, then

$$\nu + 1_j = (i_1, \ldots, i_j + 1, \ldots, i_n). \tag{3.1}$$

If $f: X \to Y$ is a local C^∞ function then $x \to j_p(f)(x)$ defines a map from X to J_p, given by

$$j_\nu(f) = \begin{cases} x = x, \\ y_\nu = \partial_\nu f, \quad \nu = (i_1, \ldots, i_n). \end{cases}$$

For such an f,

$$dy_\nu = \sum_{i=1}^{n} \partial_{\nu+1_i} f \, dx^i = \sum y_{\nu+1_i} \, dx^i, \qquad |\nu| < p.$$

Hence, the "pull-backs" by $j_p(f)$ of the 1-forms

$$w_\nu = dy_\nu - \sum y_{\nu+1_i} \, dx^i, \qquad |\nu| < p,$$

are all zero: $j_p(f)^*(w_\nu) = 0$.

DEFINITION 3.1. The 1-forms $w_\nu^j = dy_\nu^j - y_{\nu+1_i}^j \, dx^i$, are the *contact* 1-forms of J_p. They generate the *contact ideal* Ω_p on J_p. It is known that $g: X \to J_p$ equals $j_p(f)$ for some $f: X \to Y$ if and only if $g^*(\Omega_p) = 0$ [**13**, p. 20ff].

Now we consider real-valued functions $\Phi: J_p \to \mathbb{R}$.

DEFINITION 3.2. The *formal* (*partial*) *derivative* of Φ with respect to x^i is $\partial_i \Phi(j_{p+1}(f(x))) = \left[\frac{\partial[\Phi \circ j_p(f)]}{\partial x^i}\right](x)$. This is well-defined, and

in coordinates looks like

$$\partial_i \Phi = \frac{\partial \Phi}{\partial x^i} + \sum_{|\nu| \leq p} \frac{\partial \Phi}{\partial y_\nu} y_{\nu+1_i}.$$

EXAMPLE 3.1. Let $p = 2$, $\dim X = 2$, $\dim Y = 1$. Let

$$\Phi(x^1, x^2, y, y_{10}, y_{01}, y_{20}, y_{11}, y_{02}) = x^1 y_{01} + y_{20} y_{11}.$$

Then $\partial_1 \Phi = y_{01} + x^1 y_{11} + y_{30} y_{11} + y_{30} y_{11} + y_{20} y_{21}$.

We could also use the n-tuple notation $\partial_{10} \Phi$ instead of $\partial_1 \Phi$. More generally, for any n-tuple μ, $\partial_\mu \Phi \colon J_{p+|\mu|} \to \mathbb{R}$. is defined, and

PROPOSITION 3.1. $\partial_\nu(\partial_\mu \Phi) = \partial_\mu(\partial_\nu \Phi) = \partial_{\nu+\mu} \Phi$.

Instead of studying systems of equations on J_p we study their zero sets, which we assume to be submanifolds of J_p.

DEFINITION 3.3. A *system of pde of order* p is a submanifold R_p of J_p. A local map $f \colon X \to Y$ is a *solution* of R_p when, for all $x \in \text{domain}(f)$, $j_p(f)(x) \in R_p$.

The "prolongation" R_{p+1} of R_p is obtained by formally differentiating the equations that define R_p. For example, (2.4) would be one of the equations defining the prolongation of the KdV equation.

DEFINITION 3.4. Let $R_p = \{(x, y_\lambda) | g(x, y_\lambda) = 0\}$. Then

$$R_{p+1} = \{(x, y_\mu) \in J_{p+1} | g(x, y_\lambda) = 0, \partial_j g(x, y_\mu) = 0, 1 \leq j \leq n\}$$

is the *first prolongation* of R_p. More generally

$$R_{p+q} = \{(x, y_\mu) \in J_{p+q} | \partial_\nu g(x, y_\mu) = 0, 0 \leq |\nu| \leq q\}.$$

Let $\Pi_p^{p+q} \colon J_{p+q} \to J_p$ be the projection. Then obviously

PROPOSITION 3.2. $\Pi_p^{p+q}(R_{p+q}) \subseteq R_p$.

Moreover, it can be shown that

PROPOSITION 3.3. *If* $F \colon J_p \to \mathbb{R}$ *is zero on* R_p, *then* $\partial_\nu F \colon J_{p+q} \to \mathbb{R}$ *is zero on* R_{p+q}, *for all* $|\nu| \leq q$. (*Note that though* R_p *is assumed to be a manifold,* R_{p+q} *need not be.*)

More generally, suppose Y' is another manifold and J_r' is the manifold of r-jets of maps $f' \colon X \to Y'$. We need to consider maps

$F: J_p \to X \times Y'$ which preserve X, i.e., $F(j_p(f)(x)) = (x, y')$ (same point x). In coordinates:

$$F: \begin{cases} x = x, \\ y' = \varphi(x, y_\lambda), \quad |\lambda| \leq p. \end{cases}$$

DEFINITION 3.5. Let $F: J_p \to X \times Y'$ preserve X. In this case the *prolongation* of F, $\rho_q(F): J_{p+q} \to J'_q$, is defined by

$$\rho_q(F): \begin{cases} x = x, \\ y'_\mu = \partial_\mu \Phi(x, y_\theta), \quad |\theta| \leq p+q, |\mu| \leq q. \end{cases}$$

(For invariant definitions of 3.4 and 3.5 see [**14**, pp. 40, 49].)

PROPOSITION 3.4. *Let $F: J_p \to X \times Y'$ preserve X. Suppose $R_{p+q} \subseteq J_{p+q}$ and $R'_q \subseteq J'_q$ are defined as the zero sets of some functions. If $\rho_q(F)(R_{p+q}) \subseteq R'_q$, then, for all $r \geq 0$,*

$$\rho_{q+r}(F)(R_{p+q+r}) \subseteq R'_{q+r}.$$

IV. Partial prolongations. To explain the idea of a partial prolongation consider the following examples.

EXAMPLE 4.1. Let $\dim X = 2$, $\dim Y = 1$, $R_2 \subset J_2$ defined by

$$\frac{\partial^2 y}{\partial x^1 \partial x^2} = y_{11} = 0.$$

Let R'_1 be the system on $X \times Y \times Z$, $\dim Z = 2$, defined by

$$R'_1: z^1 = y_{10}, \qquad z^2 = y_{01}, \qquad z^1_{01} = z^2_{10} = 0.$$

Obviously, R_2 and R'_1 are related. The new variables z^1, z^2 are the first derivatives of the variable y in R_2. The last equations of R'_1 say the same thing as the equation $y_{11} = 0$. R'_1 is called a partial prolongation of R_2.

(This device of introducing new variables for lower order derivatives of variables in a pde can be used to reduce any pde to a first order system, called its "lowered system" [**14**, p. 109ff].)

EXAMPLE 4.2.

$$R_2: y_{11} = 0,$$
$$R'_1: y_{01} = y - Az, \quad y_{10} - Az_{10} = 0 \qquad (A = \text{ constant}).$$

Suppose $y(x^1, x^2)$, $z(x^1, x^2)$ solves R'_1. Differentiating the first equation that defines R'_1 gives $y_{11} = y_{10} - Az_{10} = 0$, so y solves R_2.

On the other hand, when y solves R_2, if $z = (y - y_{01})/A$ then $y_{10} - Az_{10} = 0$, so (y, z) solves R_1'. There is thus a natural one-to-one correspondence between the solutions of the two systems. R_1' is a "partial prolongation" of R_2.

EXAMPLE 4.3.

$$R_1\colon y^1_{01} = 0, \quad y^2_{01} = y^2_{10}, \quad y^3_{01} = y^1_{10},$$
$$R_1'\colon y^1_{01} = 0, \quad y^2_{01} = y^2_{10}, \quad y^3_{01} = y^1_{10}, \quad z = y^1_{10}, \quad z_{01} = 0.$$

Again, there is a one-to-one natural correspondence between the solutions of R_1 and R_1'.

DEFINITION 4.1. Let $R_p \subset J_p$ be a system with independent variable manifold X and dependent variable manifold Y. Let Z be a manifold and

$$F\colon J_{p-r} \to X \times Y \times Z \quad \text{for some } 0 < r \leq p-1.$$

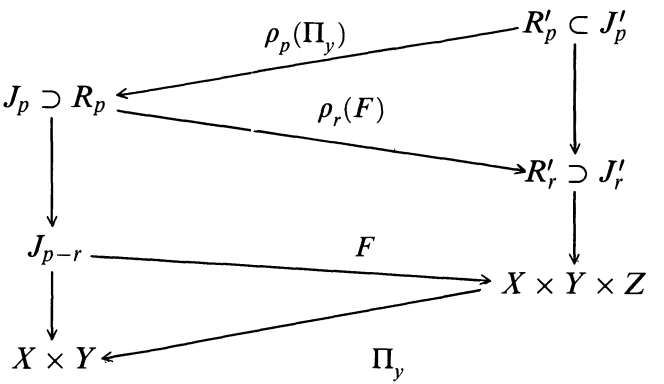
Let J_r' be the r-jets of maps $X \to Y \times Z$. Denote by $\Pi_y\colon X \times Y \times Z \to X \times Y$ the projection.

Let $R_r' \subset J_r'$ be a system of pde, R_p' its $(p-r)$ th projection. Assume

(1) $\Pi_y \circ F = \Pi_0^{p-r}\colon J_{p-r} \to X \times Y$,
(2) $\rho_r(F)\colon J_p \to J_r'$ satisfies $\rho_r(F)(R_p) \subset R_r'$,
(3) $\rho_p(\Pi_y)\colon J_p' \to J_p$ satisfies $\rho_p(\Pi_y)(R_p') \subset R_p$, and
(4) $F \circ \Pi_{p-r}^p \circ \rho_p(\Pi_y)\colon J_p' \to X \times Y \times Z$ satisfies $F \circ \Pi_{p-r}^p \circ \rho_p(\Pi_y) = \Pi_0^{\prime p}$ on R_p'.

Then we call R_r' a *partial prolongation* of R_p

COMMUTATIVE DIAGRAM



COORDINATES

Choose (x) on X, (y) on Y and (z) on Z. Then condition (1) implies

$$(4.1)\qquad F:\begin{cases} x = x, \\ y = y, \\ z = \varphi(x, y_\theta), \ |\theta| \le p - r \end{cases} \qquad \Pi_y:\begin{cases} x = x \\ y = y \end{cases}$$

$$(4.2)\qquad \rho_r(F):\begin{cases} x = x, \\ y_\nu = y_\nu, & |\nu| \le r, \\ z_\nu = (\partial_\nu \varphi)(x, y_\lambda), & |\lambda| \le r + p - r = p. \end{cases}$$

$$(4.3)\qquad \rho_p(\Pi_y):\begin{cases} x = x, \\ y_\lambda = y_\lambda & |\lambda| \le p. \end{cases}$$

Condition (2) says

$$(4.4)\qquad \text{if } (x, y_\lambda) \in R_p \quad \text{then } (x, y_\nu, (\partial_\nu \varphi)(x, y_\lambda)) \in R'_r.$$

Condition (3) says

$$(4.5)\qquad \text{if } (x, y_\lambda, z_\lambda) \in R'_p \quad \text{then } (x, y_\lambda) \in R_p.$$

Condition (4) says

$$(4.6)\qquad \text{if } (x, y_\lambda, z_\lambda) \in R'_p \quad \text{then } z = \varphi(x, y_\theta) \text{ for } |\theta| \le p - r.$$

That is, $z - \varphi(x, y_\theta) = 0$ on R'_p.

EXAMPLE 4.1. (See above.)

$$\rho_1(F):\begin{cases} F:\begin{cases} x = x, \\ y = y, \\ z^1 = y_{10} = \varphi^1(x, y_\nu), \\ z^2 = y_{01} = \varphi^2(x, y_\nu), \end{cases} \\ y_\nu = y_\nu, \qquad |\nu| = 1, \\ z^1_{10} = y_{20}, \\ z^1_{01} = y_{11}, \\ z^2_{10} = y_{11}, \\ z^2_{01} = y_{02}. \end{cases}$$

Condition (2): if $(x, y_\lambda) \in R_2$, $y_{11} = 0$, so $z^1_{01} = z^2_{10} = 0$, and hence $(x, y_\nu, z_\nu) = \rho_1(F)(x, y_\lambda) \in R'_1$.

Condition (3): R_2' is defined by the equations defining R_1' together with $z_{10}^1 = y_{20}$, $0 = z_{01}^1 = y_{11} = z_{10}^2$, $y_{02} = z_{01}^2$, $z_{11}^1 = z_{20}^2 = z_{02}^1 = z_{11}^2 = 0$. If $(x, y_\lambda, z_\lambda) \in R_2'$, then y_{11} must be zero; hence $(x, y_\lambda) \in R_2$.

Condition (4): if $(x, y_\lambda, z_\lambda) \in R_2'$ then $z^1 = y_{10} = \varphi^1(x, y_\theta)$ and $z^2 = y_{01} = \varphi^2(x, y_\theta)$, since these equations are among the defining equations of R_2'.

EXAMPLE 4.2. This is equally easy to check, using

$$\rho_1(F)\colon \begin{cases} F\colon x = x\,,\ y = y\,,\ z = \frac{y - y_{01}}{A} = \varphi(x, y_0)\,, \\ y_\nu = y_\nu\,,\ z_{10} = \frac{y_{10} - y_{11}}{A}\,,\ z_{01} = \frac{y_{01} - y_{02}}{A}. \end{cases}$$

EXAMPLE 4.3. Here, $r = 1$, $p - r = 1$, so p should be 2. To fit the definition we must first prolong R_1 to R_2. Then R_1' can be shown to be a partial prolongation of R_2.

We next show how the solutions of a system and a partial prolongation correspond in a natural one-to-one way.

Let R_p and R_r' be as in Definition 4.1.

THEOREM 4.1. *For every local solution* $f\colon X \to Y$ *of* R_p,

$$P(f) = \Pi_{yz} \circ F \circ j_{p-r}(f)\colon X \to J_{p-r} \to X \times Y \times Z \to Y \times Z$$

is a solution of R'.

For every local solution $f'\colon X \to Y \times Z$ *of* R_r',

$$Q(f') = \tilde{\Pi}_y \circ f'\colon X \to Y \times Z \to Y$$

is a solution of R_p.

Furthermore, $P \circ Q(f') = f'$ *and* $Q \circ P(f) = f$.

Proof. Most of the complexity in this proof is notational, arising out of the usual confusion between functions and their graphs. In the local coordinates that follow Definition 4.1 it is quite transparent.

$$F\colon x = x\,, \quad y = y\,, \quad z = \varphi(x, y_\theta)\,, \quad |\theta| \le p - r\,,$$
$$f\colon y = f(x) \quad \text{hence}\,,$$

$$P(f)\colon \begin{cases} y = f(x)\,, \\ z = \varphi(x, (j_\theta f)(x)) \end{cases} \tag{4.7}$$

so by Definition 3.5,

$$j_r(P(f))\colon \begin{cases} x = x\,, \\ y_\nu = (j_\nu f)(x)\,, \\ z_\nu = (\partial_\nu \varphi)(x\,, (j_\lambda f)(x)). \end{cases}$$

Since f solves R_p, $(x\,, (j_\lambda f)(x)) \in R_p$ for all $x \in \text{domain}(f)$, and by (4.4) $(x\,, \partial_\nu f(x)\,, (\partial_\nu \varphi)(x\,, (j_\lambda f)(x))) \in R'_r$, so $P(f)$ solves R'_r. Let

$$f'\colon \begin{cases} y = h(x)\,, \\ z = k(x)\,, \end{cases} \qquad \text{so } Q(f')\colon y = h(x). \tag{4.8}$$

Since f' solves R'_r, it also solves R'_p, and hence

$$(x\,, (j_\lambda h)(x)\,, (j_\lambda k)(x)) \in R'_p \quad \text{for } |\lambda| \le p$$

for all $x \in \text{domain}(f')$. By (4.5), $(x\,, (j_\lambda h)(x)) \in R_p$ and hence $Q(f')$ solves R_p.

By (4.6), when $(x\,, y_\lambda\,, z_\lambda) = (x\,, (j_\lambda h)(x)\,, (j_\lambda k)(x)) \in R'_p$, $z = k(x) = \varphi(x\,, (j_\theta h)(x))$. Hence, $P \circ Q(f') = f'$.

By (4.7) and (4.8), $Q \circ P(f) = f$. □

V. Conservation laws.

DEFINITION 5.1. Let $R_p \subset J_p$ be a system of pde. Let C be an exterior differential k-form on J_s which depends only on the differentials of the independent variables $(dx^1\,, \ldots\,, dx^n)$. Let $q > s$, and consider $d[(\Pi^q_s)^*(C)]$. Modulo the contact forms on J_q, this exterior derivative is congruent to a unique $(k+1)$-form in the differentials of the independent variables. We denote it by dC/Ω_q.

Then C is defined to be a *conservation law* for R_p when, for some q where $q \ge p$ and $q > s$,

$$\text{if } (x\,, y_\lambda) \in R_q \quad \text{then } dC/\Omega_q(x\,, y_\lambda) = 0.$$

(This is a slightly more general definition than given in [**5**, p. 253].)

LOCAL COORDINATES

Let

$$C = \sum_I{}' A_I\, dX^I\,, \qquad A_I(x\,, y_\mu)\colon J_s \to \mathbb{R}\,, \qquad |\mu| \le s.$$

Here, $I = (i_1\,, i_2\,, \ldots\,, i_k)$, $1 \le i_1 \le i_2 \le \cdots \le n = \dim X$, and the $'$ means summation over all such ordered k-tuples, and dX^I means

$dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then

$$(\Pi_s^q)^*(C) = \sum_I{}' A_I\, dX^I \qquad \text{(same expression)},$$

$$d\left[(\Pi_s^q)^*(C)\right] = \sum_I{}' dA_I \wedge dX^I$$

$$= \sum_I{}' \left(\sum_j \frac{\partial A_I}{\partial x^j} dx^j + \sum_{|\nu| \le s} \frac{\partial A_I}{\partial y_\nu} dy_\nu \right) \wedge dX^I$$

$$\equiv \sum_I{}' \sum_j \left(\frac{\partial A_I}{\partial x^j} + \sum_\nu \frac{\partial A_I}{\partial y_\nu} y_{\nu+1_j} \right) dx^j \wedge dX^I \qquad \mod \Omega_q$$

$$dC/\Omega_q = \sum_I{}' \sum_j (\partial_j A_I)\, dx^j \wedge dX^I. \tag{5.1}$$

Hence, C is a conservation law for R_q iff

$$(x, y_\lambda) \in R_q \Rightarrow \sum_I{}' \sum_j (\partial_j A_I)(x, y_\mu)\, dx^j \wedge dX^I = 0, \tag{5.2}$$

for all $|\lambda| \le q$, $|\mu| \le s+1$. By exterior algebra, this in fact means

$$(x, y_\lambda) \in R_q \Rightarrow \left[\sum_{j=1}^{k+1} \partial_{i_j} A_{i_1, \ldots, \hat{i}_j, \ldots, i_{k+1}} (-1)^{j-1} \right] (x, y_\mu) = 0 \tag{5.3}$$

for every sequence $i_1 < i_2 < \cdots < i_{k+1}$ and where $\hat{i}_j$ means that this item is missing in the list.

LEMMA 5.1. *If C is a conservation law for R_p, then it is a conservation law for all R_t when $t > p$.*

Proof. When $p < t < q$, this follows from Definition 5.1. For $t > q$, recall that $(\Pi_q^t)(R_t) \subset R_q$ by Proposition 3.2, so (5.2) must hold when $(x, y_\lambda) \in R_t$. □

EXAMPLE 2.1. Replace (t, x, r, u) by (x^1, x^2, y^1, y^2).

$$R_1 : y^1_{10} + y^1 y^2_{01} + y^2 y^1_{01} = 0, \qquad C = y^1\, dx^2 - y^1 y^2\, dx^1$$

on $J_0 = X \times Y$, so $s = 0$.

$$dC = dy^1 \wedge dx^2 - (y^1\,dy^2 + y^2\,dy^1) \wedge dx^1,$$

$$\begin{aligned}(\Pi_0^1)^*(dC) &\equiv (y_{10}^1\,dx^1 + y_{01}^1\,dx^2) \wedge dx^1 \\ &\quad - \left[y^1(y_{10}^2\,dx^1 + y_{01}^2\,dx^2)\right. \\ &\quad \left. + y^2(y_{10}^1\,dx^1 + y_{01}^1\,dx^2)\right] \wedge dx^1 \mod \Omega_1, \\ dC/\Omega_1 &= (y_{10}^1 + y^1 y_{01}^2 + y^2 y_{01}^1)\,dx^1 \wedge dx^2 = 0\end{aligned}$$

when $(x, y_\lambda) \in R_1$.

EXAMPLE 2.2. Replace $(t, x^1, y^1, \ldots, x^N, y^N)$ by $(x, y^1, y^2, \ldots, y^{2N-1}, y^{2N})$. Then

$$C = \sum_{i=1}^{N} m_i(y^{2i-1}y_1^{2i} - y^{2i}y_1^{2i-1})$$

is a 0-form (function) on J_1, so $s = 1$. One finds that

$$dC/\Omega_2 = \sum m_i(y^{2i-1}y_2^{2i} - y^{2i}y_2^{2i-1})\,dx^i$$

and, with some algebra, this is zero when $(x, y_\lambda) \in R_2$.

To each conservation law C belongs an infinite set of its formal derivatives, each of which is also a conservation law, as we now show.

DEFINITION 5.2. Let C on J_s be a conservation law for R_p:

$$C = \sum_I{}' A_I\,dX^I, \qquad A_I(x, y_\alpha)\colon J_s \to \mathbb{R}.$$

Define $\partial_h C = \sum_I'(\partial_h A_I)\,dX^I$, $1 \le h \le n = \dim X$. This is called the *formal derivative of C with respect to x^h*. More generally, $\partial_\nu C = \sum_I'(\partial_\nu A_I)\,dX^I$, $\nu = (i_1, \ldots, i_n)$.

Note: since $A_I\colon J_s \to \mathbb{R}$, $\partial_\nu A_I\colon J_{s+|\nu|} \to \mathbb{R}$, so $\partial_\nu C$ is a k-form on $J_{s+|\nu|}$.

EXAMPLES 2.3, 2.4. Change (t, x, y) to (x^1, x^2, y), so

$$R_3\colon y_{10} - 6y\,y_{01} + y_{03} = 0.$$

$$C = y\,dx^2 - (y_{02} - 3y^2)\,dx^1 \quad \text{on } J_2.$$

Instead of $\partial_x C$ we have

$$\partial_2 C = \partial_{01} C = y_{01}\,dx^2 - (y_{03} - 6y\,y_{01})\,dx^1 \quad \text{on } J_3.$$

$$d(\partial_2 C)/\Omega_4 = \left[y_{11} + y_{04} - 6(y_{01})^2 - 6y\,y_{02}\right] dx^1 \wedge dx^2.$$

But on R_4 $\partial_{01}(y_{10} - 6y\,y_{01} + y_{03}) = y_{11} - 6(y_{01})^2 - 6y\,y_{02} + y_{04} = 0$; we conclude that $(x, y_\lambda) \in R_4$ implies $d(\partial_2 C)/\Omega_4(x, y_\lambda) = 0$.

PROPOSITION 5.1. *If the k-form C is a conservation law for R_p, then $\partial_h C$ is also a conservation law for R_p.*

Proof. Using local coordinates with $C = \sum_I' A_I\, dX^I$, where $A_I\colon J_s \to \mathbb{R}$, we know that for some $q > s$, when $(x, y_\lambda) \in R_q$, by (5.3),

$$\left[\sum_{j=1}^{k+1} \partial_{i_j} A_{i_1,\dots,\hat{i}_j,\dots,i_{k+1}} (-1)^{j-1}\right](x, y_\lambda) = 0.$$

Hence by Proposition 3.3, if $(x, y_\lambda) \in R_{q+1}$ then

$$\partial_h \left[\sum_{j=1}^{k+1} \partial_{i_j} A_{i_1,\dots,\hat{i}_j,\dots,i_{k+1}} (-1)^{j-1}\right](x, y_\lambda) = 0$$

$$= \left[\sum_{j=1}^{k+1} \partial_{i_j} (\partial_h A_{i_1,\dots,\hat{i}_j,\dots,i_{k+1}} (-1)^{j-1}\right](x, y_\lambda) = 0$$

by Proposition 3.1. So, by (5.1) and (5.2) when $(x, y_\lambda) \in R_{q+1}$, $d(\partial_h C)/\Omega_{q+1} = 0$. □

DEFINITION 5.3. R_p has an *infinite number of conservation* laws if its conservation laws cannot be generated algebraically from laws at some level J_p using formal derivatives.

We now prove our main goal, that when R_r' is a partial prolongation of R_p, then a conservation law for one of these systems corresponds in a natural way to a conservation law for the other.

Let R_p and R_r' be as in Definition 4.1. Let C on J_s be a conservation law for R_p so that, for some $q > s$, $q \geq p$,

$$C = \sum_I' A_I\, dX \quad \text{and} \quad dC/\Omega_q = \sum_I' \sum_i (\partial_i A_i)\, dx_i \wedge dX$$

and $(x, y_\lambda) \in R_q$ implies $dC/\Omega_q(x, y_\lambda) = 0$.

In fact, of course, $\partial_i A_I$ depends only on $J_{s+1} = \Pi_{s+1}^q(J_q)$. We also have $\Pi_y\colon J_0' = X \times Y \times Z \to J_0 = X \times Y$ and

$$\rho_s(\Pi_y)\colon \left\{\begin{matrix} x = x \\ y_\mu = y_\mu \end{matrix}\right\}\colon J_s' \to J_s \qquad |\mu| \leq s.$$

Let $P(C) = \rho_s(\Pi_y)^*(C) = \sum_I' A_I\, dX^I$. In these coordinates, Ω_q', the contact form on J_q', contains all the forms in Ω_q, so

$$d\left[(\Pi_s'^q)^* \circ \rho_s(\Pi_y)^*(C)\right] \equiv \sum_I' \sum_i (\partial_i A_I)\, dx^i \wedge dX^I \qquad \text{mod } \Omega_q',$$

hence $d(P(C))/\Omega_q' = \rho_q(\Pi_y)^*(dC/\Omega_q)$.

By condition (3) of Definition 4.1 and Proposition 3.4, $\rho_q(\Pi_y)(R'_q) \subset R_q$, so if $(x, y_\lambda, z_\lambda) \in R'_q$, then $(x, y_\lambda) \in R_q$ and since $dP(C)/\Omega'_q$ and dC/Ω_q have exactly the same expressions which depend only on (x, y_λ), $(x, y_\lambda, z_\lambda) \in R'_q$ implies that $dP(C)/\Omega'_q(x, y_\lambda, z_\lambda) = 0$. We have proved

PROPOSITION 5.2. *If C is a conservation law for R_p then C corresponds in a natural way to a conservation law $P(C)$ for R'_r.*

Now, let C' on J'_t be a conservation law for R'_r. Suppose that $dC'/\Omega'_q = 0$ at points $(x, y_\lambda, z_\lambda) \in R'_q$ for some $q > t$, $q \geq r$. There is no reason why we cannot suppose $q \geq p$. Let

$$C' = \sum_I{}' B_I(x, y_\alpha, z_\alpha)\, dX^I, \qquad |\alpha| \leq t,$$

$$dC'/\Omega'_q = \sum_I{}' \sum_i (\partial_i B_I)(x, y_\beta, z_\beta)\, dx^i \wedge dX^I, \qquad |\beta| \leq t+1,$$

and this is zero whenever $(x, y_\gamma, z_\gamma) \in R'_q$. Using the notations of Definition 4.1,

$$F: \begin{cases} x = x, \\ y = y, \\ z = \varphi(x, y_\theta), \end{cases} \qquad \rho_t(F): \begin{cases} x = x, \\ y_\alpha = y_\alpha, \\ z_\alpha = (\partial_\alpha \varphi)(x, y_\beta), \end{cases}$$

where $|\theta| \leq p - r$, $|\alpha| \leq t$, and $|\beta| \leq p - r + t$, so the "pull-back,"

$$Q(C') = \rho_t(F)^*(C') = \sum_I{}' B_I(x, y_\alpha, (\partial_\alpha \varphi)(x, y_\beta))\, dX^I$$

is a k-form on J_{p-r+t}.

$$\begin{aligned} d[Q(C')] &\equiv \sum_I{}' \sum_i \partial_i \left[B_I(x, y_\alpha, (\partial_\alpha \varphi)(x, y_\beta))\right] dx^i \wedge dX^I \mod \Omega_{p-r+t+1} \\ &= d[Q(C')]/\Omega_q \end{aligned}$$

by definition. But

$$\begin{aligned} &\partial_i B_I(x, y_\alpha, (\partial_\alpha \varphi)(x, y_\beta)) \\ &\quad = \frac{\partial B_I}{\partial x^i} + \frac{\partial B_I}{\partial z_\alpha}\frac{\partial(\partial_\alpha \varphi)}{\partial x^i} + \frac{\partial B_I}{\partial y_\alpha} y_{\alpha+1_i} + \frac{\partial B_I}{\partial z_\alpha}\frac{\partial(\partial_\alpha \varphi)}{\partial y_\beta} y_{\beta+1_i} \\ &\quad = \frac{\partial B_I}{\partial x^i} + \frac{\partial B_I}{\partial y_\alpha} y_{\alpha+1_i} + \frac{\partial B_I}{\partial z_\alpha}[\partial_i(\partial_\alpha \varphi)] \\ &\quad = (\partial_i B_I)(x, y_\beta, \partial_\beta \varphi), \qquad |\beta| \leq p - r + t. \end{aligned}$$

Thus, $d[Q(C')]/\Omega_q = (\rho_{t+1}(F))^*[dC'/\Omega'_q]$. If, now, $(x, y_\lambda) \in R_{p-r+q}$, then $(x, y_\lambda, (\partial_\lambda\varphi)(x, y_\beta)) \in R'_q$ by Condition (2) of Definition 4.1 and Proposition 3.4. Since C' is a conservation law, when

$$(x, y_\lambda, (\partial_\lambda\varphi)(x, y_\beta)) \in R'_q, \qquad (dC'/\Omega'_q)(x, y_\lambda, (\partial_\lambda\varphi)(x, y_\beta)) = 0.$$

But, since dC'/Ω'_q involves only $dx^1, \ldots, dx^n$,

$$\begin{aligned}(dC'/\Omega'_q)&(x, y_\lambda, (\partial_\lambda\varphi)(x, y_\beta)) \\ &= \rho_{t+1}(F)^*[dC'/\Omega'_q](x, y_\lambda) = d[Q(C')]/\Omega_q.\end{aligned}$$

Hence, the last expression is zero when $(x, y_\lambda) \in R_{p-r+q}$, so $Q(C')$ is a conservation law for R_p. We have proved the

PROPOSITION 5.3. *If C' is a conservation law for R'_r, then C' corresponds in a natural way to a conservation law for R_p.*

In order to prove that having an infinite number of conservation laws is shared by systems and their partial prolongations, we must show the correspondences P and Q commute with formal differentiation.

PROPOSITION 5.4. *If C' on J'_t is a conservation law for R'_r then $\partial_j Q(C') = Q(\partial_j C')$.*

Proof. Let $C' = \sum'_I B_I(x, y_\alpha, z_\alpha)\, dX^I$, $|\alpha| \le t$, so

$$Q(C') = \sum_I{}' B_I(x, y_\alpha, (\partial_\alpha\varphi)(x, y_\beta))\, dX^I, \qquad |\beta| \le p - r + t,$$

$$\partial_j Q(C') = \sum_I{}' \partial_j \left[B_I(x, y_\alpha, (\partial_\alpha\varphi)(x, y_\beta))\right] dX^I,$$

$$= \sum_I{}' \left(\frac{\partial B_I}{\partial x^j} + \frac{\partial B_I}{\partial z_\alpha}\frac{\partial(\partial_\alpha\varphi)}{\partial x^j} + \frac{\partial B_I}{\partial y_\alpha} y_{\alpha+1_j} + \frac{\partial B_I}{\partial z_\alpha}\frac{\partial(\partial_\alpha\varphi)}{\partial y_\beta} y_{\beta+1_j}\right) dX^I$$

$$= \sum_I{}' \left(\frac{\partial B_I}{\partial x^j} + \frac{\partial B_I}{\partial y_\alpha} y_{\alpha+1_j} + \frac{\partial B_I}{\partial z_\alpha}(\partial_{\alpha+1_j}\varphi)\right) dX^I$$

$$= Q(\partial_j C').$$

PROPOSITION 5.5. *If C on J_s is a conservation law for R_p then $\partial_j P(C) = P(\partial_j C)$.*

The proof is similar to that of Proposition 5.4. Combining these results we have the

THEOREM 5.1. *There is a natural correspondence between the conservation laws of a system of pde and one of its partial prolongations. This correspondence commutes with formal differentiation.*

In particular, if one system has an infinite number of conservation laws, so does the other.

We now present E. Cartan's response to D. Hilbert's question, when can one say that two systems of pde have a one-to-one correspondence between their solutions?

DEFINITION 5.4 (Recursive). Two systems R_p and R'_r of pde are *absolutely equivalent* if

(1) R'_r is a partial prolongation or prolongation of R_p, or

(2) R_p is a partial prolongation or prolongation of R'_r, or

(3) R_p is absolutely equivalent to some R''_s and R''_s is absolutely equivalent to R'_r.

A property of R_p is an *absolute invariant* when it is shared in a natural way by all systems absolutely equivalent to that system of pde.

We have proved

THEOREM 5.2. *The property of having a conservation law is an absolute invariant. The property of having infinitely many conservation laws is an absolute invariant.*

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