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Let $\phi = \{\phi(n)\}_{n\geq 0}$ be a filtration on a ring R. Then the concept of a reduction of ϕ is introduced, several basic properties of such reductions are established, and then these results are used to characterize analytically unramified semi-local rings and locally quasi-unmixed Noetherian rings.

1. Introduction. Reductions of ideals were introduced in [**6**] and they have proved to be very useful in many research problems. Recently, reductions of modules were introduced and developed in [**16**], and their "dual" concept was investigated in [**20**]. Also, there have been several recent papers in which a number of important theorems for ideals in Noetherian rings have been extended to Noetherian filtrations (for example, see [**1**, **8**, **9**, **14**, **15**, **18**]). (Filtrations are generalizations of the sequence of powers of a given ideal, and there are many important filtrations (such as the sequence $\{q^{(n)}\}_{n\geq 0}$ of symbolic powers of a primary ideal q and the sequence $\{(I^n)_a\}_{n\geq 0}$ of integral closures of the powers of an ideal I) which are generally not powers of an ideal, but which are quite often Noetherian filtrations. So extension of these results to filtrations is of some interest and importance.)

In §2 we introduce reductions of filtrations and show that many of the basic properties of reductions of ideals have a natural extension to reductions of filtrations. (Actually, basic reductions of Noetherian filtrations were first considered and briefly used in [**15**], but general reductions and their properties were not considered in [**15**].) Among these properties is the very useful result that if ϕ and γ are filtrations on a Noetherian ring R, then ϕ is a reduction of γ if and only if the Rees ring of R with respect to γ is a finite integral extension ring of the Rees ring of R with respect to ϕ. It readily follows from this that if ϕ is a reduction of γ, then ϕ is Noetherian if and only if γ is Noetherian. Also, if ϕ is Noetherian, then ϕ is a reduction of γ if and only if $\phi \leq \gamma$ and ϕ and γ determine linearly equivalent ideal topologies on R if and only if γ is a Noetherian filtration between ϕ and ϕ_w, the weak integral closure of ϕ, and then there exists a positive integer e such that $\gamma(n+e) = \phi(e)\gamma(n)$ for all large n. Then §2 is closed by showing that most Noetherian filtrations on a local ring do not have minimal reductions (in contradistinction to the ideal case), but they do have what are called minimal e-reductions (see (2.12)).

Finally, §3 contains several applications of these results. Among these are characterizations of analytically unramified semi-local rings and of locally quasi-unmixed Noetherian rings, and filtration analogs of the theorems of Sakuma-Okuyama and Briancon-Skoda.

2. Reductions for filtrations. In this section we define reductions of a filtration and prove several of their basic properties. We begin with several definitions.

(2.1) DEFINITION. If R is a ring, then:

(2.1.1) A *filtration* $\phi = \{\phi(n)\}_{n\geq 0}$ on R is a decreasing sequence of ideals $\phi(n)$ of R such that $\phi(0) = R$ and $\phi(m)\phi(n) \subseteq \phi(m+n)$ for all nonnegative integers m and n.

(2.1.2) If ϕ and γ are filtrations on R, then $\phi = \gamma$ in case $\phi(n) = \gamma(n)$ for all $n \geq 0$, and $\phi \leq \gamma$ in case $\phi(n) \subseteq \gamma(n)$ for all $n \geq 0$.

(2.1.3) If ϕ and γ are filtrations on R, then ϕ is a *reduction* of γ in case $\phi \leq \gamma$ and there exists a positive integer d such that $\gamma(n) = \sum_{i=0}^{d} \phi(n-i)\gamma(i)$ for all $n \geq 1$. (Here, and throughout this paper, $\phi(i) = R$ if $i \leq 0$.)

(2.1.4) If R is Noetherian, then a filtration ϕ on R is *Noetherian* in case there exists a positive integer d such that

$$\phi(n) = \sum_{i=1}^{d} \phi(n-i)\phi(i) \quad \text{for all } n \geq 1.$$

(2.1.5) The *integral closure* ϕ_a of a filtration ϕ on R is the sequence of ideals $\phi_a = \{(\phi(n))_a\}_{n\geq 0}$, where $(\phi(n))_a$ is the *integral closure of* $\phi(n)$; therefore $(\phi(n))_a = \{x \in R; x$ satisfies an equation of the form $x^k + b_1x^{k-1} + \cdots + b_k = 0$, where $b_i \in (\phi(n))^i$ for $i = 1, \ldots, k\}$ (see (2.2.2)).

(2.1.6) The *weak integral closure* ϕ_w of a filtration ϕ on R is the sequence of ideals $\phi_w = \{(\phi(n))_w\}_{n\geq 0}$, where $(\phi(n))_w$ is the *weak integral closure of* $\phi(n)$; therefore $(\phi(n))_w = \{x \in R; x$ satisfies an equation of the form $x^k + b_1x^{k-1} + \cdots + b_k = 0$, where $b_i \in \phi(ni)$ for $i = 1, \ldots, k\}$ (see (2.2.2)).

(2.1.7) If ϕ is a filtration on R, then the *Rees ring of* R *with respect to* ϕ is the graded subring $\mathbf{R}(R, \phi) = R[u, t\phi(1), t^2\phi(2), \ldots]$ of $R[u, t]$, where t is an indeterminate and $u = 1/t$.

Concerning (2.1.3), since $\phi(i) = R$ when $i \leq 0$, it is clear that every filtration ϕ is a reduction of itself. Also, note that if d is such that $\gamma(n) = \sum_{i=0}^{d} \phi(n-i)\gamma(i)$, then a similar equation holds for all integers greater than or equal to d. Further, if $I \subseteq J$ are ideals in R, then I is a reduction of J if and only if there exists a positive integer d such that $J^n = \sum_{i=0}^{d} I^{n-i}J^i$ $(= I^{n-d}J^d)$ for all $n \geq 1$. Therefore, if $\phi(n) = I^n$ and $\gamma(n) = J^n$ for all $n \geq 0$, then ϕ is a reduction of γ if and only if I is a reduction of J, so (2.1.3) extends the definition of a reduction from ideals to filtrations.

The other definitions in (2.1) have previously appeared in the literature, but a Noetherian filtration, as defined in (2.1.4), is sometimes called an essentially powers filtration in the literature. Also, the weak integral closure ϕ_w of ϕ (see (2.1.6)) is called the integral closure of ϕ in **[15]**. This seems like appropriate terminology to us, but there is already quite a lot of literature where ϕ_w is called the weak integral closure of ϕ and the filtration ϕ_a (see (2.1.5)) is called the integral closure of ϕ, so we decided to stay with the older terminology.

(2.2) lists two facts concerning these definitions that will be needed below.

(2.2) REMARK. (2.2.1) It is shown in **[1**, (2.2) and (3.6)] together with **[12**, (2.7)] that if R is Noetherian, then the following are equivalent: (a) ϕ is Noetherian; (b) there exists a positive integer e such that $\phi(n+e) = \phi(e)\phi(n)$ for all $n \geq e$; (c) $\mathbf{R}(R, \phi)$ is Noetherian; and (d) $\mathbf{R}(R, \phi)$ is finitely generated over R.

(2.2.2) If ϕ is a filtration on R, then it is shown in **[8**, (4.2.1) and (2.2)] that ϕ_a and ϕ_w are filtrations on R such that $\phi \leq \phi_a \leq \phi_w$. Also, if e is an integer such that $\phi(ne) = (\phi(e))^n$ for all $n \geq 1$ (this holds if ϕ is Noetherian, by (2.2.1)), then it follows from (2.1.5) and (2.1.6) that $(\phi(ne))_a = (\phi(ne))_w$ for all $n \geq 1$.

(2.3) gives a useful characterization of when ϕ is a reduction of γ.

(2.3) THEOREM. *Let* ϕ *and* γ *be filtrations on a ring* R *such that all the ideals* $\gamma(n)$ *are finitely generated. Then* ϕ *is a reduction of* γ *if and only if* $\mathbf{S} = \mathbf{R}(R, \gamma)$ *is a finite module over* $\mathbf{R} = \mathbf{R}(R, \phi)$.

Proof. Assume first that ϕ is a reduction of γ. Then since $\mathbf{R}$ and $\mathbf{S}$ are graded subrings of $R[u, t]$, it readily follows that $\mathbf{S}$ is generated as a module over $\mathbf{R}$ by $t\gamma(1), \ldots, t^d\gamma(d)$, where d is as in (2.1.3). Therefore the hypothesis on the ideals $\gamma(n)$ implies that $\mathbf{S}$ is a finite $\mathbf{R}$-module.

For the converse, assume that $\mathbf{S}$ is a finite module over $\mathbf{R}$. Now $\mathbf{R}$ and $\mathbf{S}$ are graded subrings of $R[u, t]$, so let $\Theta_1, \ldots, \Theta_m$ be homogeneous elements in $\mathbf{S}$ that are a linear basis of $\mathbf{S}$ considered as an $\mathbf{R}$-module, let d_i be the degree of Θ_i, and let $d = \max\{d_i; i = 1, \ldots, m\}$. Now, if $n > d$, then using the basis it follows that

$$\begin{aligned}\gamma(n) &\subseteq \sum_{i=1}^{m} \phi(n-d_i)\gamma(d_i) \subseteq \sum_{i=0}^{d} \phi(n-i)\gamma(i)\\ &\subseteq \sum_{i=0}^{d} \gamma(n-i)\gamma(i) \subseteq \gamma(n),\end{aligned}$$

so $\gamma(n) = \sum_{i=0}^{d} \phi(n-i)\gamma(i)$ for all $n \geq 1$, so ϕ is a reduction of γ. □

(2.4) COROLLARY. *Let* ϕ *be a filtration on a Noetherian ring* R. *Then*:

(2.4.1) *There exists a one-to-one correspondence between the filtrations* γ *on* R *such that* ϕ *is a reduction of* γ *and the graded subrings* $\mathbf{S}$ *of* $R[u, t]$ *that are finite integral extensions of* $\mathbf{R}(R, \phi)$.

(2.4.2) *There exists a one-to-one correspondence between the filtrations* δ *on* R *that are reductions of* ϕ *and the graded subrings* $\mathbf{U}$ *of* $\mathbf{R}(R, \phi)$ *such that* $R[u] \subseteq \mathbf{U}$ *and* $\mathbf{R}(R, \phi)$ *is a finite module over* $\mathbf{U}$.

Proof. For (2.4.1), assume first that $\mathbf{S}$ is a graded subring of $R[u, t]$ that is a finite module over $\mathbf{R}(R, \phi)$ and let $I_n = u^n\mathbf{S} \cap R$. Then since $\mathbf{S}$ is a graded subring of $R[u, t]$ it is readily checked that $\gamma = \{I_n\}_{n \geq 0}$ is a filtration on R and that $\mathbf{S} = \mathbf{R}(R, \gamma)$, so ϕ is a reduction of γ, by (2.3) (the ideals $\gamma(n) = I_n$ are finitely generated, since R is Noetherian). And distinct rings $\mathbf{S}$ give distinct filtrations γ, since $\mathbf{S} = \mathbf{R}(R, \gamma)$.

For the converse of (2.4.1), if γ is a filtration on R that has ϕ as a reduction, then since R is Noetherian it follows from (2.3) that $\mathbf{S} = \mathbf{R}(R, \gamma)$ is a finite integral extension of $\mathbf{R}(R, \phi)$. Also, $\mathbf{S}$ is a graded subring of $R[u, t]$, and it is clear that $u^n\mathbf{S} \cap R = \gamma(n)$, so distinct filtrations γ yields distinct rings $\mathbf{S}$.

The proof of (2.4.2) is similar, so it will be omitted. □

A result related to (2.4) for Noetherian filtrations is given in (2.8).

In (2.5.1) we consider the sum $\delta + \phi = \{\sum_{i=0}^{n} \delta(n-i)\phi(i)\}_{n\geq 0}$ of two filtrations δ and ϕ; it is readily checked that $\delta + \phi$ is a filtration on R and that it is the smallest filtration γ such that $\gamma \geq \delta$ and $\gamma \geq \phi$.

(2.5) COROLLARY. *If δ, ϕ, γ and θ are filtrations on a Noetherian ring R, then:*

(2.5.1) *If δ is a reduction of γ and ϕ is a reduction of θ, then $\delta + \phi$ is a reduction of $\gamma + \theta$.*

(2.5.2) *If ϕ is a reduction of γ and γ is a reduction of θ, then ϕ is a reduction of θ.*

(2.5.3) *If ϕ is a reduction of θ and if $\phi \leq \gamma \leq \theta$, then γ is a reduction of θ. And if ϕ is Noetherian, then ϕ is a reduction of γ.*

Proof. For (2.5.1), it is readily checked that

$$\mathbf{R}(R, \delta + \phi) = R[u, t\delta(1), t\phi(1), t^2\delta(2), t^2\phi(2), \ldots] \quad \text{and}$$
$$\mathbf{R}(R, \gamma + \theta) = R[u, t\gamma(1), t\theta(1), t^2\gamma(2), t^2\theta(2), \ldots].$$

Therefore, since the hypothesis and (2.3) imply that $\mathbf{R}(R, \gamma)$ (resp., $\mathbf{R}(R, \theta)$) is a finite module over $\mathbf{R}(R, \delta)$ (resp., $\mathbf{R}(R, \phi)$), it follows that $\mathbf{R}(R, \gamma + \theta)$ is a finite module over $\mathbf{R}(R, \delta + \phi)$, so $\delta + \phi$ is a reduction of $\gamma + \theta$ by (2.3).

(2.5.2) and the first part of (2.5.3) follow readily from (2.3).

Finally, if R and ϕ are Noetherian, then $\mathbf{R}(R, \phi)$ is Noetherian, by (2.2.1). Also, if ϕ is a reduction of θ, then $\mathbf{R}(R, \theta)$ is a finite module over $\mathbf{R}(R, \phi)$, by (2.3), and if $\phi \leq \gamma \leq \theta$, then $\mathbf{R}(R, \gamma)$ is an intermediate ring. Therefore $\mathbf{R}(R, \gamma)$ is a finite module over $\mathbf{R}(R, \phi)$, and hence ϕ is a reduction of γ, by (2.3). □

In (2.6.2) and (2.6.3), for a filtration ϕ and a given positive integer k we consider the sequence $\phi^{(k)} = \{\phi(nk)\}_{n\geq 0}$. It is readily checked that $\phi^{(k)}$ is a filtration on R.

(2.6) COROLLARY. *Assume that R is Noetherian and let $\phi \leq \gamma$ be filtrations on R. Then:*

(2.6.1) *If ϕ is a reduction of γ, then ϕ is Noetherian if and only if γ is Noetherian.*

(2.6.2) *If ϕ is Noetherian and a reduction of γ, then $\phi^{(n)}$ is Noetherian and a reduction of $\gamma^{(n)}$ for all $n \geq 1$.*

(2.6.3) *If R is a local ring and $\phi^{(k)}$ is Noetherian and a reduction of $\gamma^{(k)}$ for some $k \geq 1$, then $\phi^{(n)}$ is Noetherian and a reduction of $\gamma^{(n)}$ for all $n \geq 1$.*

Proof. For (2.6.1) let $\mathbf{R} = \mathbf{R}(R, \phi)$ and $\mathbf{S} = \mathbf{R}(R, \gamma)$, so $\mathbf{S}$ is a finite module over $\mathbf{R}$ by (2.3) (since ϕ is a reduction of γ). Therefore $\mathbf{S}$ is Noetherian if and only if $\mathbf{R}$ is, by Eakin's Theorem [**2**], so γ is Noetherian if and only if ϕ is by (2.2.1).

For (2.6.2) assume that ϕ is Noetherian and a reduction of γ and for $n \geq 1$ let

$$\mathbf{R}_n = R[u^n, t^n\phi(n), t^{2n}\phi(2n), \ldots] \quad \text{and}$$
$$\mathbf{S}_n = R[u^n, t^n\gamma(n), t^{2n}\gamma(2n), \ldots].$$

Then $\mathbf{R}_n \cong \mathbf{R}(R, \phi^{(n)})$ and $\mathbf{S}_n \cong \mathbf{R}(R, \gamma^{(n)})$, so since ϕ is a reduction of γ, it follows from (2.3) that $\mathbf{S}_1$ is a finite module over $\mathbf{R}_1$. Also, since $(t^i\phi(i))^n = t^{in}(\phi(i))^n \subseteq t^{in}\phi(in) \subseteq \mathbf{R}_n$, it follows that $\mathbf{R}_1$ is integral over $\mathbf{R}_n$, and similarly $\mathbf{S}_1$ is integral over $\mathbf{S}_n$ for all $n \geq 1$. Further, $\mathbf{R}_1$ is finitely generated over R, since R and ϕ are Noetherian (see (2.2.1)), so $\mathbf{R}$ is finitely generated and integral over $\mathbf{R}_n$ for all $n \geq 1$. Hence $\mathbf{R}_n$ is Noetherian by Eakin's Theorem, and so $\phi^{(n)}$ is Noetherian by (2.2.1). And, as already noted, $\mathbf{S}_1$ is a finite module over $\mathbf{R}_1$, so $\mathbf{S}_1$ is a finite module over $\mathbf{R}_n$ for all $n \geq 1$. Therefore, since $\mathbf{R}_n \subseteq \mathbf{S}_n \subseteq \mathbf{S}_1$ and $\mathbf{R}_n$ is Noetherian, it follows that $\mathbf{S}_n$ is a finite module over $\mathbf{R}_n$. Hence (2.3) and the isomorphisms noted above show that $\phi^{(n)}$ is a reduction of $\gamma^{(n)}$ for all $n \geq 1$.

For (2.6.3), using (2.6.2) it suffices to show that ϕ is Noetherian and a reduction of γ. For this, $\phi^{(k)}$ is Noetherian and a reduction of $\gamma^{(k)}$, by hypothesis, so with the notation of the preceding paragraph, (2.6.1) and (2.2.1) show that $\mathbf{S}_k$ is Noetherian (since $\mathbf{S}_k \cong \mathbf{R}(R, \gamma^{(k)})$). Therefore, since R is local, [**19**, (1.3)] shows that $\mathbf{S}_1$ is Noetherian, and hence γ is Noetherian by (2.2.1). Also, $\mathbf{S}_k$ is a finite module over $\mathbf{R}_k$, by (2.3), and $\mathbf{S}_1$ is integral over $\mathbf{S}_k$, by the preceding paragraph, so $\mathbf{S}_1$ is integral over $\mathbf{R}_k$. Hence $\mathbf{S}_1$ is integral over $\mathbf{R}_1$, since $\mathbf{R}_k \subseteq \mathbf{R}_1 \subseteq \mathbf{S}_1$. Further, $\mathbf{S}_1$ is finitely generated over R (since it is Noetherian), so it follows that $\mathbf{S}_1$ is a finite module over $\mathbf{R}_1$. Therefore ϕ is a reduction of γ, by (2.3), so ϕ is Noetherian by (2.6.1). □

(2.7) COROLLARY. *Assume that R is Noetherian and let ϕ and γ be filtrations on R. If both ϕ and γ are reductions of $\phi + \gamma$, then*

$\mathbf{R}(R, \phi)$ *and* $\mathbf{R}(R, \gamma)$ *have the same integral closure. The converse holds if both* ϕ *and* γ *are Noetherian.*

Proof. Assume first that both ϕ and γ are reductions of $\phi + \gamma$. Then $\mathbf{R}(R, \phi+\gamma)$ is a finite module over both $\mathbf{R}(R, \phi)$ and $\mathbf{R}(R, \gamma)$ by (2.3). Therefore it readily follows that all three rings have the same integral closure, so $\mathbf{R}(R, \phi)$ and $\mathbf{R}(R, \gamma)$ do.

For the converse, if ϕ and γ are Noetherian, then $\phi+\gamma$ is Noetherian, by [**12**, (3.1.1)], so $\mathbf{R}(R, \phi+\gamma)$ is finitely generated over R, by (2.2.1). Also, the hypothesis implies that $\mathbf{R}(R, \phi+\gamma)$ is integral over both $\mathbf{R}(R, \phi)$ and $\mathbf{R}(R, \gamma)$, so it follows from (2.3) that both ϕ and γ are reductions of $\phi+\gamma$. □

(2.8) gives a useful characterization of the filtrations that are reduced by a given Noetherian filtration.

(2.8) Theorem. *If* ϕ *is a Noetherian filtration on a Noetherian ring* R, *then the filtrations that* ϕ *reduces are the Noetherian filtrations between* ϕ *and* ϕ_w (*see* (2.1.6)). *Therefore, the filtrations that reduce* ϕ *are the Noetherian filtrations* γ *such that* $\gamma \le \phi$ *and* $\gamma_w = \phi_w$.

Proof. Let $\mathbf{S} = \mathbf{R}' \cap R[u, t]$, where $\mathbf{R}'$ is the integral closure of $\mathbf{R} = \mathbf{R}(R, \phi)$. Then $\mathbf{S}$ is a graded subring of $R[u, t]$, and by considering an equation of integral (resp., weak integral) dependence it follows that if $b \in R$, then $bt^m \in \mathbf{S}$ if and only if $b \in (\phi(m))_w$. Therefore $u^n\mathbf{S} \cap R = (\phi(n))_w$ for all $n \ge 1$, so $\mathbf{S} = \mathbf{R}(R, \phi_w)$.

Now assume that ϕ is a reduction of a filtration γ on R. Then γ is Noetherian, by (2.6.1), and $\mathbf{R}(R, \gamma)$ is a finite $\mathbf{R}$-module, by (2.3). Therefore $\mathbf{R} \subseteq \mathbf{R}(R, \gamma) \subseteq \mathbf{S}$, so $u^n\mathbf{R} \cap R \subseteq u^n\mathbf{R}(R, \gamma) \cap R \subseteq u^n\mathbf{S} \cap R$ for all $n \ge 1$; that is, $\phi(n) \subseteq \gamma(n) \subseteq (\phi(n))_w$, so $\phi \le \gamma \le \phi_w$.

Next, if γ is a Noetherian filtration on R such that $\phi \le \gamma \le \phi_w$, then $\mathbf{R} \subseteq \mathbf{R}(R, \gamma) \subseteq \mathbf{S}$, and $\mathbf{R}(R, \gamma)$ is finitely generated over R. Therefore $\mathbf{R}(R, \gamma)$ is finitely generated and integral over $\mathbf{R}$, so (2.3) shows that ϕ is a reduction of γ.

For the final statement, γ is Noetherian by (2.6.1) and it is clear that $\gamma \le \phi$, so $\phi \le \gamma_w$, by what has already been shown. Finally, it is shown in [**8**, (2.4)] that if δ and θ are filtrations such that $\delta \le \theta$, then $\delta_w \le \theta_w$ and $(\theta_w)_w = \theta_w$, so it follows that $\phi_w = \gamma_w$. □

(2.9) gives three useful characterizations of reductions of Noetherian filtrations. The first of these shows that ϕ is a reduction of γ if and only if $\phi \le \gamma$ and ϕ and γ determine *linearly equivalent ideal* *topologies* (that is, $\phi(n) \subseteq \gamma(n)$ for all $n \geq 1$ and there exists a positive integer d such that $\gamma(n) \subseteq \phi(n-d)$ for all $n \geq d$), and the third characterization resembles the ideal equality $J^{n+e} = I^e J^n$.

(2.9) THEOREM. *Assume that R is Noetherian and let ϕ and γ be Noetherian filtrations on R. Then the following are equivalent:*

(2.9.1) *ϕ is a reduction of γ.*

(2.9.2) *$\phi \leq \gamma$ and there exists a positive integer d such that $\gamma(n) \subseteq \phi(n-d)$ for all $n \geq d$.*

(2.9.3) *$\phi \leq \gamma$ and there exists a positive integer d such that $\gamma(n) = \sum_{i=0}^{d} \phi(n-i)\gamma(i)$ for all $n \geq d$.*

(2.9.4) *$\phi \leq \gamma$ and there exists a positive integer e such that $\gamma(n+e) = \phi(e)\gamma(n)$ for all large n.*

Proof. Assume that (2.9.1) holds, so $\phi \leq \gamma$ and there exists a positive integer d such that $\gamma(n) = \sum_{i=0}^{d} \phi(n-i)\gamma(i)$ for all $n \geq 1$. Since ϕ is Noetherian, by (2.2.1) let e be a positive integer such that $\phi(n+e) = \phi(e)\phi(n)$ for all $n \geq e$, and let $n \geq d+e$. Then

$$\begin{aligned}\gamma(n+e) &= \sum_{i=0}^{d} \phi(n+e-i)\gamma(i) \\ &= \sum_{i=0}^{d} \phi(e)\phi(n-i)\gamma(i), \quad \text{since } n-i \geq n-d \geq e, \\ &\subseteq \sum_{i=0}^{d} \phi(e)\gamma(n-i)\gamma(i) \subseteq \phi(e)\gamma(n) \subseteq \gamma(n+e).\end{aligned}$$

Therefore $\gamma(n+e) = \phi(e)\gamma(n)$ for all large n, so (2.9.1) $\Rightarrow$ (2.9.4).

Now assume that (2.9.4) holds and let k be such that $\gamma(n+e) = \phi(e)\gamma(n)$ for all $n \geq k$; it may clearly be assumed that $k \geq e$. Let $d = k+e-1$, let $n \geq d$, and write $n = qe + r$ with $k \leq r < k+e$. Then $\gamma(n) = \gamma(qe+r) = (\phi(e))^q\gamma(r)$ (by hypothesis and the choice of r) $\subseteq \sum_{i=0}^{k+e-1} \phi(n-i)\gamma(i) \subseteq \gamma(n)$. Therefore $\gamma(n) = \sum_{i=0}^{d} \phi(n-i)\gamma(i)$ for all $n \geq d$, so (2.9.4) $\Rightarrow$ (2.9.3).

If (2.9.3) holds, then it follows that $\phi \leq \gamma$ and

$$\gamma(n) = \sum_{i=0}^{d} \phi(n-i)\gamma(i) \subseteq \phi(n-d) \sum_{i=0}^{d} \gamma(i) \subseteq \phi(n-d)$$

for all $n \geq d$, so (2.9.3) $\Rightarrow$ (2.9.2).

Finally, assume that (2.9.2) holds. Then it readily follows that $\mathbf{R}(R, \phi) \subseteq \mathbf{R}(R, \gamma)$ and that $u^n\mathbf{R}(R, \gamma) \subseteq \mathbf{R}(R, \phi)$ for all $n \geq d$. Also, u is a regular element in $\mathbf{R}(R, \phi)$, and $\mathbf{R}(R, \phi)$ is Noetherian, by (2.2.1). Therefore $\mathbf{R}(R, \gamma)$ is a finite module over $\mathbf{R}(R, \phi)$, so (2.9.2) $\Rightarrow$ (2.9.1) by (2.3). □

(2.10) gives two related results. The proof of the first of these is an application of (2.9.1) $\Rightarrow$ (2.9.3). And the second shows that a nice characterization of when a regular ideal $I \subseteq J$ is a reduction of J (namely that there exists a regular ideal K such that $IK = JK$) does not extend to filtrations. For this second result, a filtration δ is said to be *regular* in case $\delta(n)$ contains a regular element for some $n \geq 1$ (equivalently (since $\mathrm{Rad}(\delta(n)) = \mathrm{Rad}(\delta(1))$), if $\delta(1)$ contains a regular element). (The restriction that δ be regular in (2.10.2) is needed, since $\phi 0 = 0 = \gamma 0$, where 0 is the filtration such that $0(n) = (0)$ for all $n \geq 1$.)

(2.10) REMARK. Let ϕ and γ be Noetherian filtrations on a Noetherian ring R such that ϕ is a reduction of γ. Then the following hold:

(2.10.1) $\phi\delta$ is a reduction of $\gamma\delta$ for all Noetherian filtrations δ on R.

(2.10.2) There need not exist a regular filtration δ on R such that $\phi\delta = \gamma\delta$.

Proof. For (2.10.1), since ϕ (resp., δ) is Noetherian, by (2.2.1) let e_1 (resp., e_2) be a positive integer such that $\phi(n + e_1) = \phi(e_1)\phi(n)$ for all $n \geq e_1$ (resp., $\delta(n + e_2) = \delta(e_2)\delta(n)$ for all $n \geq e_2$). Then it is readily checked that both of these hold with $e = e_1e_2$ in place of e_1 (resp., e_2). Also, since ϕ is a reduction of γ, by the proof that (2.9.1) $\Rightarrow$ (2.9.3) it follows that $\gamma(n+e) = \phi(e)\gamma(n)$ for all large n. Therefore $(\gamma\delta)(n + e) = \gamma(n + e)\delta(n + e) = \phi(e)\gamma(n)\delta(e)\delta(n) = [(\phi\delta)(e)][(\gamma\delta)(n)]$ for all large n, so $\phi\delta$ is a reduction of $\gamma\delta$ by (2.9.3) $\Rightarrow$ (2.9.1).

For (2.10.2) let I be a regular ideal in R and define filtrations γ and ϕ on R by $\gamma(n) = I^n$ for all $n \geq 0$ and $\phi(2n-1) = \phi(2n) = I^{2n}$ for all $n \geq 1$ (and $\phi(0) = R$). Then it follows as in the proof of (2.11) below that ϕ is a reduction of γ, and it is clear that γ is regular and Noetherian, so ϕ is, by (2.6.1). And it is also clear that $\phi(1) = I^2$ is not a reduction of $\gamma(1) = I$. But if there exists a regular filtration δ such that $\phi\delta = \gamma\delta$, then $(\phi\delta)(n) = (\gamma\delta)(n)$ for all $n \geq 1$. Therefore

$\phi(n)\delta(n) = \gamma(n)\delta(n)$, so $\phi(n)$ is a reduction of $\gamma(n)$ by [**6**, Theorem 2, p. 156], so there does not exist such a filtration δ. □

(2.10.2) showed that one important result concerning reductions of ideals does not extend to Noetherian filtrations. (2.11) shows that another such result (namely that there exist minimal reductions when R is local) also fails to extend to Noetherian filtrations.

(2.11) REMARK. If ϕ is a Noetherian filtration on a local ring R such that $\phi(n) \neq (0)$ for all large n, then ϕ does not have a minimal reduction.

Proof. Fix a positive integer $e = e_1$ and define ϕ_1 by $\phi_1(0) = R$, $\phi_1(1) = \cdots = \phi_1(e) = \phi(e)$, $\phi_1(e+1) = \cdots = \phi_1(2e) = (\phi(2e))$, etc. Then it is readily checked that ϕ_1 is a filtration on R. Let $\mathbf{S} = \mathbf{R}(R, \phi)$ and $\mathbf{R} = \mathbf{R}(R, \phi_1)$. Then for $i \geq 1$ it holds that $(t^i\phi(i))^e \subseteq t^{ie}\phi(ie) \subseteq \mathbf{R}$, so $\mathbf{S}$ is an integral over $\mathbf{R}$. Also, $\mathbf{S}$ is finitely generated over $\mathbf{R}$ since it is finitely generated over R, so ϕ_1 is a reduction of ϕ, by (2.3), and hence ϕ_1 is Noetherian by (2.6.1). Now repeat this with $e_2 > e_1$ and with ϕ_1 in place of ϕ, etc. to obtain a descending sequence $\phi_0 = \phi \geq \phi_1 \geq \phi_2 \geq \cdots$ of Noetherian filtrations on R such that ϕ_i is a reduction of ϕ_{i-1} for $i = 1, 2, \ldots$. Now if $\phi(n) \neq (0)$ for all large n, then since $\mathbf{S}$ is Noetherian it follows that for each n there exists a positive integer $k(n)$ such that $\phi(n) \supset \phi(n+k(n))$, and it readily follows from this that this sequence of filtrations can be chosen to be strictly decreasing. □

If γ is a Noetherian filtration on a Noetherian ring R and if e is a positive integer such that $\gamma(n+e) = \gamma(e)\gamma(n)$ for all $n \geq e$ (see (2.2.1)), then a reduction ϕ of γ such that $\gamma(n+e) = \phi(e)\gamma(n)$ for all large n will be called an *e-reduction* of γ. In view of (2.11) this definition is useful, since it will now be shown that γ has minimal e-reductions when R is local.

(2.12) THEOREM. *If θ is a Noetherian filtration on a local ring, then θ has a minimal e-reduction.*

Proof. By (2.2.1) let e be a positive integer such that $\theta(n+e) = \theta(e)\theta(n)$ for all $n \geq e$, let I be a reduction of $\theta(e)$ and define a filtration ϕ on R by $\phi(0) = R$, $\phi(1) = \cdots = \phi(e) = I$, $\phi(e+1) = \cdots = \phi(2e) = I^2$, etc. Also, define γ similarly, but use $\theta(e)$ in place of I. Then the proof of (2.11) shows that $\mathbf{R}(R, \theta)$ is a finite integral extension of $\mathbf{R}(R, \gamma)$ (since $(\theta(e))^i = \theta(ie)$ for all $i \geq 1$). Also,

$\mathbf{R}(R, \gamma)$ is an integral extension of $\mathbf{R}(R, \phi)$, since I is a reduction of $\phi(e)$, so $\mathbf{R}(R, \theta)$ is integral over $\mathbf{R}(R, \phi)$. Further, $\mathbf{R}(R, \theta)$ is finitely generated over R, so it follows that $\mathbf{R}(R, \theta)$ is finitely generated and integral over $\mathbf{R}(R, \phi)$. Hence ϕ is a reduction of θ by (2.3). Moreover, since I is a reduction of $\theta(e)$ there exists a positive integer m such that $I(\theta(e))^n = (\theta(e))^{n+1}$ for all $n \geq m$; that is, $I\theta(ne) = \theta(ne + e)$. Therefore, if $n \geq me + e$, then $\phi(e)\theta(n) = I\theta(n) = I\theta(me)\theta(n - me) = \theta(me + e)\theta(n - me) = \theta(n + e)$, so ϕ is an e-reduction of θ.

Finally, assume that I (in the preceding paragraph) is a minimal reduction of $\theta(e)$ and let δ be an e-reduction of θ such that $\delta \leq \phi$, so $\delta(n) \subseteq \phi(n)$ for all $n \geq 1$. Also $\delta(e)\theta(n) = \theta(n + e)$ for all large n, since δ is an e-reduction of θ, so in particular $\delta(e)(\theta(e))^n = \delta(e)\theta(ne) = \theta(ne + e) = (\theta(e))^{n+1}$ for all large n. Therefore $\delta(e)$ is a reduction of $\theta(e)$, and $\delta(e) \subseteq \phi(e) = I$, so $\delta(e) = I$ since I is a minimal reduction of $\theta(e)$. Then $\delta(i) \supseteq \delta(e) = I = \phi(i)$ for $i = 1, \dots, e$. Also, for $i = 1, \dots, e$, $\delta(e + i) \supseteq \delta(2e) \supseteq (\delta(e))^2 = I^2 = \phi(e + i)$, and it is similarly seen that $\delta(n) \supseteq \phi(n)$ for all $n \geq 1$, so $\delta = \phi$. Hence ϕ is a minimal e-reduction of θ. □

(2.13) REMARK. The proof of (2.12) showed that if θ is a Noetherian filtration on a Noetherian ring R and e is a positive integer such that $\theta(n + e) = \theta(e)\theta(n)$ for all $n \geq e$, then the filtration γ such that $\gamma(ke + i) = \theta((k + 1)e) = (\theta(e))^{k+1}$ (for $i = 1, \dots, e$ and for all $k \geq 0$) is a reduction of θ, and it is readily checked that γ is an e-reduction of θ. The proof of (2.12) also showed that if I is a reduction of $\theta(e)$, then the filtration ϕ such that $\phi(ke + i) = I^{k+1}$ (for $i = 1, \dots, e$ and for all $k \geq 0$) is an e-reduction of θ, and if I is a minimal reduction of $\theta(e)$, then ϕ is a minimal e-reduction of θ. In §3, a reduction ϕ of θ of this form will be called *the e-repeated ideal reduction of θ generated by I* (since the ideals I^{k+1} are repeated e times in the filtration), and if I is a minimal reduction of $\theta(e)$, then ϕ will be called *the basic e-repeated ideal reduction of θ generated by I* (since it is a minimal e-reduction of θ and also an e-repeated ideal reduction of θ). Note that by (2.6.1), such reductions of θ are Noetherian filtrations.

In closing this section we note that reductions of a filtration γ are closely related to γ-good filtrations on R (see **[14]**). Because of this, a couple of additional necessary and sufficient conditions (in terms of restricted Rees rings) for ϕ to be a reduction of γ are given in **[14**, (3.3)**]**.

3. Applications. This section contains several applications of the results of §2. The first of these gives four characterizations of when a semi-local ring R is analytically unramified. (In [**18**, Lemma 3], Sakuma and Okuyama showed that the following are equivalent for a semi-local ring R: R is analytically unramified; there exists an open ideal J in R and a positive integer d such that $(J^n)_a \subseteq J^{n-d}$ for all $n \geq d$; and, for each ideal J in R there exists a positive integer d such that $(J^n)_a \subseteq J^{n-d}$ for all $n \geq d$. Now (2.9.1) $\Leftrightarrow$ (2.9.2) (applied to $\phi = \{J^n\}_{n\geq 0}$ and $\gamma = \{(J^n)_a\}_{n\geq 0}$) shows that $(J^n)_a \subseteq J^{n-d}$ for all $n \geq d$ is equivalent to ϕ is a reduction of γ, so it follows that the equivalence of (3.1.1)–(3.1.3) extends the theorem of Sakuma-Okuyama from ideals to Noetherian filtrations. Also, it should be noted that the equivalence of (3.1.1), (3.1.4), and (3.1.5) was first shown in [**15**, (4.1) and (4.3)].)

(3.1) THEOREM. *If R is a semi-local ring, then the following are equivalent*:

(3.1.1) *R is analytically unramified.*

(3.1.2) *ϕ is a reduction of ϕ_w for all Noetherian filtrations ϕ on R.*

(3.1.3) *There exists a Noetherian filtration ϕ on R such that $\phi(1)$ is open and ϕ is a reduction of ϕ_w.*

(3.1.4) *ϕ_w is Noetherian for all Noetherian filtrations ϕ on R.*

(3.1.5) *There exists a Noetherian filtration ϕ on R such that $\phi(1)$ is open and ϕ_w is Noetherian.*

Proof. (3.1.2) $\Leftrightarrow$ (3.1.4) and (3.1.3) $\Leftrightarrow$ (3.1.5) by (2.8), and it is clear that (3.1.2) $\Rightarrow$ (3.1.3) and (3.1.4) $\Rightarrow$ (3.1.5).

Now assume that (3.1.1) holds and let ϕ be a Noetherian filtration on R. Let $\mathbf{R} = \mathbf{R}(R, \phi)$ and let $\mathbf{S} = \mathbf{R}' \cap R[u, t]$ be as in the first paragraph of the proof of (2.8), so $\mathbf{S} = \mathbf{R}(R, \phi_w)$. Since ϕ is Noetherian, $\mathbf{R}$ is finitely generated over R, so since R is analytically unramified it is shown in [**10**, Lemma 2.4] that $\mathbf{R}'$ is a finite $\mathbf{R}$-module. Therefore $\mathbf{S}$ is a finite $\mathbf{R}$-module, so ϕ is a reduction of ϕ_w by (2.3), and hence (3.1.1) $\Rightarrow$ (3.1.2).

Finally, assume that (3.1.3) holds and let ϕ be a Noetherian filtration on R such that $\phi(1)$ is open and ϕ is a reduction of ϕ_w, so $\mathbf{S} = \mathbf{R}(R, \phi_w) = \mathbf{R}' \cap R[u, t]$ is a finite module over $\mathbf{R} = \mathbf{R}(R, \phi)$, by (2.3). By (2.2.1) let e be a positive integer such that $\phi(n+e) = \phi(e)\phi(n)$ for all $n \geq e$, let $J = \phi(e)$, and let $\mathbf{A} = R[u^e, t^e J]$. Then $(t^i\phi(i))^e \subseteq t^{ie}\phi(ie) = (t^e\phi(e))^i \subseteq \mathbf{A}$ for all $i \geq 1$, so $\mathbf{R}$ is integral over $\mathbf{A}$. Also, $\mathbf{R}$ is finitely generated over R, so $\mathbf{R}$ is a finite module over $\mathbf{A}$. Therefore $\mathbf{S}$ is a finite module over $\mathbf{A}$, so $\mathbf{A}' \cap R[u^e, t^e]$ is a finite module over $\mathbf{A}$, since $\mathbf{A} \subseteq \mathbf{A}' \cap R[u^e, t^e] \subseteq \mathbf{S}$ and $\mathbf{A}$ is Noetherian (here, $\mathbf{A}'$ is the integral closure of $\mathbf{A}$). Also, $\mathbf{A} \cong \mathbf{B}$, where $\mathbf{B} = \mathbf{R}(R, J)$, so it follows that $\mathbf{B}' \cap R[u, t]$ is a finite module over $\mathbf{B}$. Further, $\mathbf{B} \subseteq \mathbf{R}(R, \gamma_w) \subseteq \mathbf{B}' \cap R[u, t]$, where $\gamma = \{J^n\}_{n\geq 0}$, so $\mathbf{R}(R, \gamma_w)$ is a finite module over $\mathbf{B}$. Hence γ is a reduction of γ_w by (2.3). Also, $\gamma_w = \gamma_a$, by (2.2.2) (where the integer e of (2.2.2) for the present γ is 1), so the proof of (2.9.1) $\Rightarrow$ (2.9.3) shows that $(J^{n+1})_a = J(J^n)_a$ for all large n. It follows that there exists a positive integer h such that $(J^{n+h})_a = J^n(J^h)_a \subseteq J^n$ for all $n \geq 1$, and J is open (since $\phi(1)$ and $\phi(e) = J$ have the same radical). Therefore it is shown in **[18**, Lemma 3] that this implies that R is analytically unramified, so (3.1.3) $\Rightarrow$ (3.1.1). □

If ϕ is a filtration on a ring R and S is a multiplicatively closed set in R such that $0 \notin S$, then it is readily checked that $\phi R_s = \{\phi(n)R_S\}_{n\geq 0}$ is a filtration on R_S and that ϕR_S is Noetherian if ϕ is. This will be used several times in the remainder of this section.

(3.2) Corollary. *Let P be a prime ideal in a Noetherian ring R. Then the following are equivalent:*

(3.2.1) *R_P is analytically unramified.*

(3.2.2) *For all Noetherian filtrations ϕ on R such that $\phi(1) \subseteq P$, ϕR_P is a reduction of $(\phi R_P)_w$.*

(3.2.3) *There exists a Noetherian filtration ϕ on R such that P is a minimal prime divisor of ϕ and ϕR_P is a reduction of $(\phi R_P)_w$.*

(3.2.4) *For all Noetherian filtrations ϕ on R such that $\phi(1) \subseteq P$, $(\phi R_P)_w$ is Noetherian.*

(3.2.5) *There exists a Noetherian filtration ϕ on R such that P is a minimal prime divisor of ϕ and $(\phi R_P)_w$ is Noetherian.*

Proof. This follows immediately from (3.1). □

By virtue of (2.9.1) $\Leftrightarrow$ (2.9.2), (3.1.2) could be restated: there exists a positive integer d such that $(\phi(n))_w \subseteq \phi(n-d)$ for all $n \geq d$. Here, the integer d depends on ϕ. However, the beautiful theorem of Briancon-Skoda, as developed by Lipman, Sathaye, and Teissier in [3] and [4], shows that often d can be chosen independent of ϕ; specifically, if I is an ideal in a regular local ring R, then $(I^n)_a \subseteq I^{n-d}$ for all $n \geq d$, where $d+1 = \text{altitude}(R)$. In (3.3) we extend this theorem to Noetherian filtrations by showing that the integer d depends only on altitude(R) and the integer e such that $\phi(n+e) = \phi(e)\phi(n)$ for all $n \geq e$. and not on the particular ϕ. To do this we need to use the following result, [4, Corollary, p. 200]: if R is a regular local ring and $b_1, c_1, \ldots, b_n, c_n$ are nonunits in R, and if A' is the integral closure of $A = R[c_1/b_1, \ldots, c_n/b_n]$, then $(b_1 \cdots b_n)A' \subseteq A$.

(3.3) THEOREM. *Let ϕ be a Noetherian filtration on a regular local ring (R, M), let $d = \text{altitude}(R)$, and let e be a positive integer such that $\phi(n+e) = \phi(e)\phi(n)$ for all $n \geq e$. Then $(\phi(n+de))_w \subseteq \phi(n)$ for all $n \geq 0$.*

Proof. It is readily seen that if X is an indeterminate, if $A = R[X]_{MR[X]}$, and if $\gamma(n) = \phi(n)A$ for $n = 0, 1, \ldots,$ then $(\gamma(n+de))_w \subseteq \gamma(n)$ if and only if $(\phi(n+de))_w \subseteq \phi(n)$, so it may be assumed that R/M is infinite. Therefore each ideal in R has a reduction that is generated by $d = \text{altitude}(R)$ elements, so let $X = (b_1, \ldots, b_d)R$ be a reduction of $\phi(e)$, let δ be the e-repeated ideal reduction of ϕ generated by X (see (2.1.3)), and let $\mathbf{A} = \mathbf{R}(R, \delta)$. Then the proof of (2.12) shows that $\mathbf{R} = \mathbf{R}(R, \phi)$ is a finite integral extension ring of $\mathbf{A}$, so it follows that $\mathbf{A} \subseteq \mathbf{R} \subseteq \mathbf{R}' \subseteq \mathbf{A}'$. Also, it is readily seen that $\mathbf{A} = R[u, t^e X]$. Further, $L = R[u]_{(M,u)R[u]}$ is a regular local ring, and if $S = R[u] - (M, u)R[u]$, then $\mathbf{A}_S = L[X/u^e]$ and $\mathbf{R}'_S$ is the integral closure of $\mathbf{A}_S$. Therefore it follows from the comment preceding this proof that if $g = de$ then $u^g \mathbf{R}'_S \subseteq \mathbf{A}_S \subseteq \mathbf{R}_S$. Hence $u^{n+g}\mathbf{R}'_S \subseteq u^n \mathbf{R}_S$ for all $n \geq 0$.

Now if p is a prime divisor of $u\mathbf{R}$, then

$$p \subseteq (u, M, t\phi(1), t^2\phi(2), \ldots)\mathbf{R},$$

so $p \cap R[u] \subseteq (M, u)R[u]$, so $p \cap S = \varnothing$, and so it follows that $u^k \mathbf{R}_S \cap \mathbf{R} = u^k \mathbf{R}$ for all $k \geq 1$. Also, it is readily seen that $u^n \mathbf{R}' \cap R = (\phi(n))_w$ for all $n \geq 1$ (since $R = R'$, it follows that $\mathbf{B} = \mathbf{R}(R, \phi_w) = \mathbf{R}'$), so it follows from the end of the preceding paragraph that $(\phi(n+g))_w = u^{n+g}\mathbf{R}' \cap R \subseteq u^{n+g}\mathbf{R}'_S \cap R \subseteq u^n \mathbf{R}_S \cap R = u^n \mathbf{R} \cap R = \phi(n)$ for all $n \geq 0$. □

For the remaining applications we need two additional definitions. The first is of the analytic spread of a filtration, and (3.4.1) agrees with the definition of spread in **[15**, p. 28], and it extends the definition of $a(I)$ $(= \text{altitude}(\mathbf{R}(R, I)/(u, M)\mathbf{R}(R, I))$, see **[6**, Theorem 2, p. 149]) from ideals I in a local ring (R, M) to filtrations on R.

(3.4) Definition. Let ϕ be a filtration on a Noetherian ring R. Then:

(3.4.1) If R is local with maximal ideal M, then the *analytic spread* $a(\phi)$ of ϕ is defined by $a(\phi) = \text{altitude}(\mathbf{R}(R, \phi)/(u, M)\mathbf{R}(R, \phi))$.

(3.4.2) $\mathbf{A}_w(\phi) = \{P; P \in \text{Ass}(R/(\phi(n))_w) \text{ for some } n \geq 1\}$. Members of $\mathbf{A}_w(\phi)$ are called the *asymptotic prime divisors of* ϕ.

(3.4.3) If I is an ideal in R, then $\widehat{A}^*(I) = \{P; P \in \text{Ass}(R/(I^n)_a)$ for some (equivalently, by **[5**, (3.4)], for all large) $n \geq 1\}$. Members of $\widehat{A}^*(I)$ are called the *asymptotic prime divisors of* I.

(3.5) Lemma. *Let ϕ be a filtration on a local ring (R, M) and let δ be the e-repeated ideal reduction of ϕ generated by $\phi(e)$ (see* (2.13)). *Then $a(\phi) = a(\delta) = a(\phi(e))$.*

Proof. Let $\mathbf{R} = \mathbf{R}(R, \phi)$, $\mathbf{B} = \mathbf{R}(R, \delta)$, and $\mathbf{A} = R[u^e, t^e\phi(e)]$, so $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{R}$. Also, $(t^i\phi(i))^e = t^{ie}(\phi(i))^e \subseteq t^{ie}\phi(ie) = t^{ie}(\phi(e))^i = (t^e\phi(e))^i \subseteq \mathbf{A}$ for all $i \geq 1$, so $\mathbf{R}$ and $\mathbf{B}$ are integral over $\mathbf{A}$. Therefore it follows from integral dependence that $\text{altitude}(\mathbf{R}/(u, M)\mathbf{R}) = \text{altitude}(\mathbf{B}/(u, M)\mathbf{R} \cap \mathbf{B}) = \text{altitude}(\mathbf{A}/(u, M)\mathbf{R} \cap \mathbf{A})$. Also, $(u, M)\mathbf{B} \subseteq (u, M)\mathbf{R} \cap \mathbf{B} \subseteq ((u, M)\mathbf{B})_a$, by integral dependence, so

$$\begin{aligned} a(\delta) &= \text{altitude}(\mathbf{B}/(u, M)\mathbf{B}) = \text{altitude}(\mathbf{B}/(u, M)\mathbf{R} \cap \mathbf{B}) \\ &= \text{altitude}(\mathbf{R}/(u, M)\mathbf{R}) = a(\phi). \end{aligned}$$

Further, $\mathbf{A} \cong \mathbf{R}(R, \phi(e))$, so $a(\phi(e)) = \text{altitude}(\mathbf{A}/(u, M)\mathbf{A})$, so it follows similarly that $a(\phi(e)) = \text{altitude}(\mathbf{A}/(u, M)\mathbf{R} \cap \mathbf{A}) = \text{altitude}(\mathbf{R}/(u, M)\mathbf{R}) = a(\phi)$. Hence $a(\phi) = a(\delta) = a(\phi(e))$. □

A proof similar to that of (3.5) shows that if ϕ and γ are filtrations on a Noetherian ring R such that $\phi_w = \gamma_w$, then $a(\phi) = a(\gamma)$. And, similarly, $a(\phi) = a(\phi^{(n)})$ for all $n \geq 1$.

(3.6) extends McAdam's Theorem **[5**, (4.1)] from ideals to Noetherian filtrations.

(3.6) Theorem. *Let R be a locally quasi-unmixed Noetherian ring, let ϕ be a Noetherian filtration on R, and let $P \in$* Spec(R)

such that $\phi(1) \subseteq P$ (*see* (2.13)). *Then* $P \in \mathbf{A}_w(\phi)$ *if and only if* $\text{height}(P) = a(\phi R_P)$.

Proof. Let $P \in \text{Spec}(R)$ such that $\phi(1) \subseteq P$. Then ϕR_P is Noetherian, since ϕ is, so $P \in \mathbf{A}_w(\phi)$ if and only if $P \in \widehat{A}^*(\phi(e))$ for e such that $\phi(n+e) = \phi(e)\phi(n)$ for all $n \geq e$, by [**8**, (3.4.3)], if and only if $PR_P \in \widehat{A}^*(\phi(e)R_P)$, by [**13**, (2.9.2)], if and only if $PR_P \in \mathbf{A}_w(\phi R_P)$, by [**8**, (3.4.3)]. Also, R_P is quasi-unmixed, so it may be assumed to begin with that R is local with maximal ideal P, and it remains to show that $P \in \mathbf{A}_w(\phi)$ if and only if $\text{height}(P) = a(\phi)$.

For this, by (2.13) let δ be the e-repeated ideal reduction of ϕ generated by $\phi(e)$, so $a(\phi) = a(\delta) = a(\phi(e))$, by (3.5). Also, it is shown in [**8**, (3.4.3)] that $\mathbf{A}_w(\phi) = \widehat{A}^*(\phi(e))$. Further, since R is quasi-unmixed, $P \in \widehat{A}^*(\phi(e))$ if and only if $\text{height}(P) = a(\phi(e))$, by [**5**, (4.1)]. Therefore, $P \in \mathbf{A}_w(\phi)$ if and only if $P \in \widehat{A}^*(\phi(e))$ if and only if $\text{height}(P) = a(\phi(e)) = a(\phi)$. □

(3.7) COROLLARY. *A Noetherian ring* R *is locally quasi-unmixed if and only if whenever* $P \in \text{Spec}(R)$ *is in* $\mathbf{A}_w(\phi)$ *for some Noetherian filtration* ϕ *on* R *it holds that* $\text{height}(P) = a(\phi R_P)$.

Proof. It is shown in [7, (2.8)] that R is locally quasi-unmixed if and only if whenever $P \in \text{Spec}(R)$ is in $\widehat{A}^*(I)$ for some ideal I in R it holds that $\text{height}(P) = a(IR_P)$. Also, it is clear that for each ideal I in R, $\phi = \{I^n\}_{n \geq 0}$ is a Noetherian filtration on R such that $\mathbf{A}_w(\phi) = \widehat{A}^*(\phi)$, so this follows immediately from (3.6). □

(3.8) extends the main result in [**11**] to Noetherian filtrations. For this result we use $\nu(I)$ to denote the number of elements in a minimal basis of the ideal I.

(3.8) THEOREM. *Let* R *be a locally quasi-unmixed Noetherian ring, let* ϕ *be a Noetherian filtration on* R, *and assume that* ϕ *has a* (*necessarily basic*) *e-repeated ideal reduction* δ *generated by an ideal* I *with* $\nu(I) = \text{height}(\phi(1))$ (*see* (2.13)). *Then* $\text{height}(P) = \text{height}(\phi(1))$ *for every* $P \in \mathbf{A}_w(\phi)$.

Proof. By (2.13), note that I is a reduction of $\phi(e)$ (where e is such that $\phi(n+e) = \phi(e)\phi(n)$ for all $n \geq e$), and it is readily checked that $\text{height}(\phi(1)) = \text{height}(\phi(n))$ for all $n \geq 1$. Therefore $\nu(I) = \text{height}(\phi(1))$ shows that I must be a minimal reduction of

$\phi(e)$, and hence (2.13) shows that δ is a basic e-repeated ideal reduction of ϕ. Also, if $P \in \mathbf{A}_w(\phi)$, then $P \in \widehat{A}^*(\phi(e)) = \widehat{A}^*(I)$, by [**8**, (3.4.3)] (and since $(I^n)_a = ((\phi(e))^n)_a$ for all $n \geq 1$), so since $\nu(I) = \text{height}(\phi(e))$ and R is locally quasi-unmixed, [**11**, Corollary 2.14] shows that $\text{height}(P) = \text{height}(\phi(1))$. □

Our final result characterizes locally quasi-unmixed Noetherian rings in terms of reductions of Noetherian filtrations.

(3.9) Corollary. *A Noetherian ring R is locally quasi-unmixed if and only if whenever $P \in \text{Spec}(R)$ and ϕ is a Noetherian filtration on R that has a basic e-repeated ideal reduction generated by an ideal I with $\nu(I) = \text{height}(\phi(1))$ it holds that $P \in \mathbf{A}_w(\phi)$ if and only if $\phi(1) \subseteq P$ and $\text{height}(P) = \text{height}(\phi(1))$.*

Proof. It was shown in [**11**, (2.29)] that a Noetherian ring R is locally quasi-unmixed if and only if whenever $P \in \text{Spec}(R)$ and I is an ideal in R with $\nu(I) = \text{height}(I)$ it holds that $P \in \widehat{A}^*(I)$ if and only if $I \subseteq P$ and $\text{height}(P) = \text{height}(I)$. Therefore this follows from (3.8), since for each ideal I in R it holds that $\{I^n\}_{n\geq 0}$ is a Noetherian filtration on R such that $\mathbf{A}_w(\phi) = \widehat{A}^*(I)$. □

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