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A theory of vector singular integral operators in the context of the local fields, is established. Applications to maximal functions, a diagonal multiplier theorem of Mihlin-Hörmander type and applications to Besov and Hardy-Sobolev spaces are given.

Introduction. The theory of the vector singular operators with operator valued kernels on Euclidean space was treated systematically by Rubio de Francia, Ruiz and Torrea [**6**] (see also Garcia-Cuerva and Rubio de Francia [**3**]). On the other hand, the classical singular integral operators of the Calderón-Zygmund type on finite product of local fields were considered by Phillips and Taibleson [**5**].

The goal of the present paper is to give a version for local fields of some results of Francia-Ruiz-Torrea [**6**] that generalize from several perspectives the quoted paper by Phillips-Taibleson.

The contents of the paper is as follows. We begin in §1 some basic notations, definitions and results that we can find in [**9**]. In §2 we state an inequality of Fefferman-Stein type and, we apply it to obtain an interpolation theorem of Marcinkiewicz-Riviere type. The main results are in §3 where we state the version of the integral singular operator theorem given in [**6**], for local fields, giving also sequential extensions. Next in §4 we obtain maximal inequalities of F. Zó and Fefferman-Stein type. A diagonal multiplier theorem of Mihlin-Hörmander type (for the Euclidean case see Triebel [**11**]) that generalize the scalar multiplier theorem of Taibleson [**8**] is given in §5. Finally, in §6 we give applications of some results obtained in the foregoing sections to Besov and Hardy-Sobolev spaces in local fields.

The extension of all results in this paper for a finite product of local fields will be an immediate consequence of a M. H. Taibleson's theorem (see [**10**], pp. 548–549) which states that, if $\mathbb{K}$ is a local field and d is an integer greater than 1, then $\mathbb{K}^d e$, the d-dimensional vector space over $\mathbb{K}$, has a field structure, as a local field, which is compatible with the usual vector space norm of $\mathbb{K}^d$.

1. Preliminaries. A local field is any locally compact, non-discrete and totally disconnected field. Let $\mathbb{K}$ be a fixed local field and dx a

Haar measure of the additive group $\mathbb{K}^+$ of $\mathbb{K}$. The measure of the measurable set A of $\mathbb{K}$ with respect to dx we denote for $|A|$. Let m be the modular function for $\mathbb{K}^+$, that is, $M(\lambda)|A| = |\lambda A|$ for $\lambda \in \mathbb{K}$ and A measurable. We also let $|x| = m(x)$. The sets

$$\mathbb{D} = \{x \in \mathbb{K}: |x| \leq 1\} \quad \text{and} \quad \mathbb{B} = \{x \in \mathbb{K}: |x| < 1\}$$

are the ring of integers of $\mathbb{K}$ and the unique maximal ideal of $\mathbb{D}$, respectively. Let $q = p^c$ (p prime) be the order of the finite field $\mathbb{D}/\mathbb{B}$ and π a fixed element of maximum absolute value of $\mathbb{B}$. The Haar measure dx is normalized such that $|\mathbb{D}| = 1$ and thus $|\pi| = |\mathbb{B}| = q^{-1}$. We observe that $dx/|x|$ is a Haar measure on the multiplicative group $\mathbb{K}^*$ of $\mathbb{K}$. We let

$$\mathbb{B}^k = \{x \in \mathbb{K}: |x| \leq q^{-k}\}, \qquad k \in \mathbb{Z}.$$

If B and R are two balls of $\mathbb{K}$ such that $B \cap R \neq \emptyset$, then $B \subset R$ or $R \subset B$, For each $k \in \mathbb{Z}$, there is only one sequence $(B_j)_{j \in \mathbb{N}}$ of balls with radius q^k that is a partition of $\mathbb{K}$. We fix a character χ on $\mathbb{K}^+$ that is trivial on $\mathbb{D}$ but is non-trivial on $\mathbb{B}^{-1} = \{x \in \mathbb{K}: |x| \leq q\}$. If we take $\chi_y(x) = \chi(x \cdot y)$, then the mapping $y \mapsto \chi_y$ is a topological isomorphism of $\mathbb{K}$ onto the group of characters of $\mathbb{K}^+$. The Fourier transform of a function $f \in L^1(\mathbb{K})$ is defined by

$$\hat{f}(x) = \int_{\mathbb{K}} f(y) \overline{\chi}_x(y)\, dy, \tag{1}$$

and the inverse Fourier transform of a function $f \in L^\infty_c(\mathbb{K})$ is defined by

$$f^\vee(x) = \int_{\mathbb{K}} f(y) \chi_x(y)\, dy. \tag{2}$$

We denote by $S(\mathbb{K})$ the space of all finite linear combinations of characteristic functions of balls of $\mathbb{K}$. The space $S(\mathbb{K})$ is an algebra of continuous functions with compact support that is dense in $L^p(\mathbb{K})$, $1 \leq p < \infty$. We observe that the Fourier transform is a homeomorphism of $S(\mathbb{K})$ onto $S(\mathbb{K})$. The space $S'(\mathbb{K})$ of continuous linear functionals on $S(\mathbb{K})$ is called the space of distributions. We will consider $S'(\mathbb{K})$ with the weak topology.

Let E be a Banach space. The space $l^s(E)$ is the set of all sequences $(c_j)_{j \in \mathbb{Z}}$ of elements of E, such that the sequence of its norms is in l^s. The space of the quasi-null sequences of elements of E, i.e. of the sequences (c_j) such that $c_j = 0$ for $|j| \geq N$, for some $N \geq 0$, will be denoted by $l_0^\infty(E)$. We denote by $S(\mathbb{K}, l_0^\infty)$ the space of the quasi-null sequences of functions of $S(\mathbb{K})$. The space $S(\mathbb{K}, l_0^\infty)$ is dense in the space $L^p(\mathbb{K}, l^s)$ for $1 \le p, s < \infty$.

The space $l_s^r(E)$, for $1 \le r \le \infty$ and $s \in \mathbb{R}$, will be the set of all sequences $(x_j)_{j\ge 0}$ of elements of E, such that

$$\|(x_j)_{j\ge 0}\|_{l_s^r(E)} = \|(q^{sj}\|x_j\|)_{j\ge 0}\|l^r < \infty.$$

The Hardy-Littlewood maximal function of $f \in L^1_{\mathrm{loc}}(\mathbb{K}, E)$ is defined by

$$Mf(x) = \sup_{k\in\mathbb{Z}} q^k \int_{|y-x|\le q^{-k}} \|f(y)\|_E \, dy. \tag{3}$$

The function $Mf(x)$ is measurable,

$$\|f(x)\|_E = \lim_{k\to\infty} q^k \int_{|y-x|\le q^{-k}} \|f(y)\|_E \, dy, \tag{4}$$

and

$$\|f(x)\|_E \le Mf(x), \tag{5}$$

for almost all $x \in \mathbb{K}$. Moreover, Mf is of the weak type (1, 1) and of the strong type (p, p), $1 < p \le \infty$.

For the details see **[9]**.

2. The BMO(E) space.

2.1. Definition. Let $f \in L^1_{\mathrm{loc}}(\mathbb{K}, E)$. The sharp maximal function $M^\# f$ is defined by

$$M^\# f(x) = \sup_{k\in\mathbb{Z}} q^k \int_{|y-x|\le q^{-k}} \|f(y) - f_k(x)\|_E \, dy,$$

where

$$f_k(x) = q^k \int_{|y-x|\le q^{-k}} f(y)\, dy.$$

2.2. Definition. The space BMO(E) of the functions of bounded mean oscillation is the set of the functions $f \in L^1_{\mathrm{loc}}(\mathbb{K}, E)$ such that

$$\|f\|_* = \|M^\# f\|_\infty < \infty. \tag{1}$$

2.3. Remarks. (a) The application $f \mapsto \|f\|_*$ is a seminorm on BMO(E) and $\|f\|_* = 0$ if and only if f is constant. We consider the space BMO(E) like a quotient space with respect to constant functions. (b) We can prove that BMO(E) is a Banach space analogously to the real case (see **[4]**). (c) We have $L^\infty(\mathbb{K}, E) \subset \mathrm{BMO}(E)$,

$L^\infty(\mathbb{K}, E) \neq \mathrm{BMO}(E)$ because the function $f(x) = \log|x|$ if $x \in \mathbb{K}^*$ and $f(0) = 0$ is in $\mathrm{BMO}(E)$ but is not in $L^\infty(\mathbb{K}, E)$.

A classical inequality of Fefferman-Stein also holds in the local field setting.

2.4. THEOREM. *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{K}, E)$ *such that* $Mf \in L^r(\mathbb{K})$ *for some* r *with* $0 < r < \infty$. *Then for every* p *with* $r \leq p < \infty$, *there is a constant* C_p *depending only on* p, *such that*

$$(1) \qquad \|Mf\|_p \leq C_p \|M^\# f\|_p.$$

The proof of this theorem is an adaptation of the Euclidean case (see [3], Chapter 2, Theorem 3.6). To obtain this adaptation we must remember that the balls of $\mathbb{K}$ have the same properties of the dyadic cubes. We do not need to take dilations of balls, the number 2 that appears in the proof of [3] is the prime number q here, and the functions $\alpha(t)$ and $\beta(t)$ that are considered in [3] are equal in this case.

2.5. REMARK. The inequality 2.4(1) is not true when $p = \infty$ (see 2.3(c)).

As a consequence of the Fefferman-Stein inequality we obtain an interpolation theorem of Marcinkiewicz-Riviere type, which will be fundamental in the study of the singular integrals.

2.6. THEOREM. *Let* E *and* F *be Banach spaces and let* T *be a linear operator from* $L^\infty(\mathbb{K}, E)$ *into* $L^0(\mathbb{K}, F)$ *such that,* T *has a bounded extension from* $L^r(\mathbb{K}, E)$ *into* $L^r(\mathbb{K}, F)$, *for some* r *with* $1 < r < \infty$, *and*

$$(1) \qquad \|Tf\|_* \leq C\|f\|_{L^\infty(E)}, \qquad f \in L^\infty_c(\mathbb{K}, E).$$

Then T *has a bounded extension from* $L^p(\mathbb{K}, E)$ *into* $L^p(\mathbb{K}, F)$, *for all* p *with* $r \leq p < \infty$.

3. Singular integral operators.

3.1. DEFINITION. Let E and F be Banach spaces. A linear operator T defined on $L^\infty_c(\mathbb{K}, E)$, the space of the E-valued L^∞-functions with compact support, with values in $L^0(\mathbb{K}, F)$, the space of all F-valued strongly measurable functions, is a singular integral operator with an operator valued kernel, if the following two conditions are fulfilled:

SIO 1. T has a bounded extension from $L^r(\mathbb{K}, E)$ into $L^r(\mathbb{K}, F)$, for some r with $1 < r \leq \infty$.

SIO 2. There is an operator valued kernel K, locally integrable from $\mathbb{K} \times \mathbb{K} \backslash \Delta$ into $L(E, F)$, such that

$$(1) \qquad Tf(x) = \int_{\mathbb{K}} K(x, y) f(y)\, dy,$$

for all $f \in L_c^\infty(\mathbb{K}, E)$ and for a.e. $x \notin \operatorname{supp} f$.

3.2. Definition. Let T be a singular integral operator with a kernel K. We say that K satisfies (H_1) if

$$(1) \qquad \int_{|x-y'|>|y-y'|} \|K(x, y) - K(x, y')\|_{L(E,F)}\, dx \le C$$

for all $y \neq y'$, and we say that K satisfies (H_∞) if

$$(2) \qquad \|K(x, y) - K(x, y')\|_{L(E,F)} \le C \frac{|y-y'|}{|x-y'|^2}$$

for $|x - y'| > |y - y'|$. Moreover, we say that K satisfies (H_r'), for $r = 1$ or $r = \infty$, if $K'(x, y) = K(y, x)$ satisfies (H_r).

3.3. Remark. The condition (H_∞) implies the condition (H_1). In fact, if $|y - y'| = q^l$ and $|x - y'| > |y - y'|$, then

$$\begin{aligned} \int_{|x-y'|>|y-y'|} \|K(x, y) - K(x, y')\|_{L(E,F)}\, dx &= Cq^l \int_{|z| \ge q^{l+1}} \frac{dz}{|z|^2} \\ &= Cq^l \sum_{k=l+1}^{\infty} \int_{|z|=q^k} \frac{dz}{|z|^2} = Cq^{-1}(1 - q^{-1})(1 - q^{-1})^{-1}. \end{aligned}$$

Analogously, (H_∞') implies (H_1').

Now we are ready to state the main theorem.

3.4. Theorem. *Let T be a singular integral operator with kernel K, which has a bounded extension from $L^r(\mathbb{K}, F)$, for some r with $q < r \le \infty$. The following hold:*

(i) *if K satisfies (H_1), then T is of weak type* (1, 1) *and of strong type (p, p) for p with $q < p \le r$;*

(ii) *if K satisfies (H_1'), then T is of strong type* (L^∞, BMO) *and of strong type (p, p), for p with $r \le p < \infty$.*

The proof of the above theorem is obtained like the Euclidean case (see **[3]** or **[6]**). The crucial part uses a decomposition of the Calderón-Zygmund type (see **[9]**, Chapter 3, results 7.6 and 7.9). Thanks to the decomposition it follows that T is of weak type (1, 1). The Marcinkiewicz interpolation theorem then shows that T is of strong type (p, p), $1 < p \leq r$. The proof that T is of strong type (L^∞, BMO) is similar to the Euclidean case. Finally, to conclude that T is of strong type (p, p) for $r \leq p < \infty$, we need the Marcinkiewicz-Riviere interpolation Theorem 2.6.

3.5. THEOREM. *Let $(T_j)_{j\in\mathbb{Z}}$ be a sequence of singular integral operators uniformly bounded from $L^r(\mathbb{K}, E)$ into $L^r(\mathbb{K}, F)$, for some r with $1 < r \leq \infty$. Suppose further that the sequence of associated kernels $(K_j)_{j\in\mathbb{Z}}$ satisfies*

$$(1) \qquad \int_{|x-y'|>|y-y'|} \sup_j \|K_j(x, y) - K_j(x, y')\|_{L(E,F)}\, dx \leq C, \qquad y \neq y',$$

and

$$(2) \qquad \int_{|y-x'|<|x-x'|} \sup_j \|K_j(x, y) - K_j(x', y)\|_{L(E,F)}\, dy \leq C, \qquad x \neq x'.$$

Then, given p and s with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p, s, C and r, such that

$$(3) \qquad \left|\left\{x \colon \sum_j \|T_j f_j(x)\|_F^s > \lambda^s\right\}\right| \leq A_{1,s}\lambda^{-1}\|(f_j)_j\|_{L^1(l^s(E))}$$

and

$$(4) \qquad \|(T_j f_j)_j\|_{L^p(l^s(F))} \leq A_{p,s}\|(f_j)_j\|_{L^p(l^s(E))}, \qquad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$. Moreover, the inequality (4) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s(E))$.

Proof. For each positive integer m, let $\widetilde{T}_m$ be the operator from $L_c^\infty(\mathbb{K}, l^s(E))$ into $L^0(\mathbb{K}, l^s(F))$ defined by

$$(5) \qquad \widetilde{T}_m(f_j)_j = (T_j f_j)_{m\leq j\leq m}, \qquad (f_j)_j \in L^\infty(\mathbb{K}, l^s(E)),$$

and let $\widetilde{K}_m$ be the kernel from $\mathbb{K}\times\mathbb{K}\backslash\Delta$ into $L(l^s(E), l^s(F))$ defined by

$$(6) \qquad \widetilde{K}_m(x, y)(\alpha_j)_j = (K_j(x, y)\alpha_j)_{-m\leq j\leq m}, \qquad (\alpha_j)_j \in l^s(E).$$

We observe that the operators T_j are uniformly bounded from $L^p(\mathbb{K}, E)$ into $L^p(\mathbb{K}, F)$ for all p, $1 < p < \infty$. Now, we fix

s, $1 < s < \infty$. The operators $\widetilde{T}_m$ are uniformly bounded from $L^s(\mathbb{K}, l^s(E))$ into $L^s(\mathbb{K}, l^s(F))$ and it is clear that

$$\widetilde{T}_m(f_j)_j(x) = \int_{\mathbb{K}} \widetilde{K}_m(x, y)(f_j(y))_j \, dy$$

for all $(f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$ and a.a. $x \notin \operatorname{supp}(f_j)_j$. Since

$$\|\widetilde{K}_m(x, y)\|_{L(l^s(E), l^s(F))} \le \sup_{|j|\le m} \|K_j(x, y)\|_{L(E, F)},$$

then it follows by (1) and (2) that the kernel $\widetilde{K}_m$ verifies (H_1) and (H_1'). Therefore, by Theorem 3.4, for each p with $1 \le p < \infty$, there is a constant $A_{p,s}$ depending only on p, s, C and r, such that

$$(7) \qquad \left|\left\{x \colon \sum_{|j|\le m} \|T_j f_j(x)\|_F^s > \lambda^s\right\}\right| \le A_{1,s}\lambda^{-1}\|(f_j)_j\|_{L^1(l^s(E))}$$

and

$$(8) \qquad \|\widetilde{T}_m(f_j)_j\|_{L^p(l^s(F))} \le A_{p,s}\|(F_j)_j\|_{L^p(l^s(E))}, \qquad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$. Moreover, the inequality (8) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s(E))$. Then, letting $m \to \infty$ on both sides of the inequalities (7) and (8) we obtain (3) and (4).

3.6. Corollary. *Let T be a singular integral operator with kernel K satisfying (H_1) and (H_1'). Then, given p and s with $1 \le p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p, s, C and r, such that*

$$(1) \qquad \left|\left\{x \colon \sum_{j} \|Tf_j(x)\|_F^s > \lambda^s\right\}\right| \le A_{1,s}\lambda^{-1}\|(f_j)_j\|_{L^1(l^s(E))}$$

and

$$(2) \qquad \|(Tf_j)_j\|_{L^p(l^s(F))} \le A_{p,s}\|(f_j)_j\|_{L^p(l^s(E))}, \qquad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s(E))$. Moreover, the inequality (2) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}^s(E))$.

3.7. Remark. In our applications we shall consider singular integral operators of convolution type, that is, with kernels of the type $K(x, y) = K'(x - y)$ where K' is locally integrable from $\mathbb{K}\backslash\{0\}$ into $L(E, F)$.

4. Applications to maximal functions.

4.1. Definition. Let $\varphi \in L^1(\mathbb{K})$ and for each $t \in \mathbb{K}^*$, let $\varphi_t(x) = |t|^{-1}\varphi(t^{-1}x)$. The maximal operator M^φ is defined by

$$M^\varphi f(x) = \sup_{t\neq 0} |(\varphi_t * f)(x)|, \qquad f \in L_c^\infty(\mathbb{K}).$$

The Euclidean version of the following theorem is due to F. Zó (see **[6]** or **[12]**).

4.2. Theorem. *Let $\varphi \in C_c(\mathbb{K})$ such that*

$$(1) \qquad \int_{|x|>|y|} \sup_{t\neq 0} |\varphi_t(x-y) - \varphi_t(x)|\, dx \leq C, \qquad y \neq 0.$$

Then, given p and s with $1 \leq p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p, s, C and $\|\varphi\|_1$, such that

$$(2) \qquad \left|\left\{x : \sum_j |M^\varphi f_j(x)|^s > \lambda^s\right\}\right| \leq A_{1,s}\lambda^{-1}\|(f_j)_j\|_{L^1(l^s)}$$

and

$$(3) \qquad \|(M^\varphi f_j)_j\|_{L^p(l^s)} \leq A_{p,s}\|(f_j)_j\|_{L^p(l^s)}, \qquad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s)$. Moreover, the inequality (3) *can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$.*

Proof. Step 1. Owing to continuity of the function $t \mapsto (\varphi_t * f)(x)$, it is enough to calculate the supremum, in the definition of M^φ, on a countable dense subset $\{t_j\}_{j\in N}$ of $\mathbb{K}^*$, that is,

$$M^\varphi f(x) = \sup_j |(\varphi_{t_j} * f)(x)|.$$

Consider the operators M_m^φ defined by

$$M_m^\varphi f(x) = \sup_{1\leq j\leq m} |(\varphi_{t_j} * f)(x)|.$$

We have that $M_m^\varphi f(x) \uparrow M^\varphi f(x)$ for all $x \in \mathbb{K}$. Therefore, obtaining estimates for $M_m^\varphi f$ that do not depend on m, we shall be obtaining also estimates for $M^\varphi f$.

Step 2. For each positive integer m, let T_m be the linear operator from $L_c^\infty(\mathbb{K})$ into $L^0(\mathbb{K}, l^\infty)$ defined by

$$(4) \qquad T_m f = (\varphi_{t_j} * f)_{1\leq j\leq m}, \qquad f \in L_c^\infty(\mathbb{K}),$$

and let K_m be the kernel (of convolution type) from $\mathbb{K}$ into $L(\mathbb{C}, l^\infty)$ defined by

$$K_m(x)\lambda = (\varphi_{t_j}(x)\lambda)_{1\le j\le m}, \qquad \lambda \in \mathbb{C}. \tag{5}$$

Since $\|\varphi_t\|_1 = \|\varphi\|_1$ for all $t \neq 0$, we have

$$\begin{aligned}\|T_m f\|_{L^\infty(l^\infty)} &= \operatorname*{ess\,sup}_{x\in\mathbb{K}} \sup_{1\le j\le m} |(\varphi_{t_j} * f)(x)| \\ &\le \operatorname*{ess\,sup}_{x\in\mathbb{K}} \sup_{1\le j\le m} \|f\|_\infty \|\varphi_{t_j}\|_1 = \|\varphi\|_1 \|f\|_\infty,\end{aligned} \tag{6}$$

i.e., the operator T_m is bounded from $L^\infty(\mathbb{K})$ into $L^\infty(l^\infty)$. On the other hand, we have

$$\begin{aligned}\int_{\mathbb{K}} \|K_m(x)\|_{L(\mathbb{C}, l^\infty)}\, dx &= \int_{\mathbb{K}} \sup_{1\le j\le m} |\varphi_{t_j}(x)|\, dx \\ &\le \sum_{1\le j\le m} \int_{\mathbb{K}} |\varphi_{t_j}(x)|\, dx = m\|\varphi\|_1 < \infty,\end{aligned}$$

and

$$\begin{aligned}T_m f(x) &= \left(\int_{\mathbb{K}} \varphi_{t_j}(x-y) f(y)\, dy\right)_{1\le j\le m} \\ &= \int_{\mathbb{K}} (\varphi_{t_j}(x-y) f(y))_{1\le j\le m}\, dy = \int_{\mathbb{K}} K_m(x-y) f(y)\, dy,\end{aligned}$$

for all $f \in L_c^\infty(\mathbb{K})$ and for a.e. $x \notin \operatorname{supp} f$. Consequently T_m is a singular integral operator of convolution type with kernel K_m. Moreover, the kernel K_m satisfies, for all $y \neq 0$,

$$\begin{aligned}&\int_{|x|>|y|} \|K_m(x-y) - K_m(x)\|_{L(\mathbb{C}, l^\infty)}\, dx \\ &\quad = \int_{|x|<|y|} \sup_{1\le j\le m} |\varphi_{t_j}(x-y) - \varphi_{t_j}(x)|\, dx \\ &\quad \le \int_{|x|>|y|} \sup_{t\neq 0} |\varphi_t(x-y) - \varphi_t(x)|\, dx \le C.\end{aligned} \tag{7}$$

Step 3. The inequalities (6) and (7) show that the operators T_m and its kernels K_m satisfy uniformly the hypothesis of the Corollary 3.6. Therefore, given p and s with $1 \le p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$, depending only on p, s, C and $\|\varphi\|_1$, such that

$$\left|\left\{x \colon \sum_j \|T_m f_j(x)\|_{l^\infty}^s > \lambda^s\right\}\right| \le A_{1,s}\lambda^{-1} \|(f_j)_j\|_{L^1(l^s)} \tag{8}$$

and

$$(9)\qquad \|(T_m f_j)_j\|_{L^p(l^s(l^\infty))} \le A_{p,s}\|(f_j)_j\|_{L^p(l^s)}, \qquad 1 < p < \infty,$$

for all $\lambda > 0$, $m \in \mathbb{N}$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s)$. Moreover, the inequality (9) can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$. Since

$$\|T_m f_j(x)\|_{l^\infty} = M_m^\varphi f_j(x),$$

then, letting $m \to \infty$ on both sides of (8) and (9), we obtain (2) and (3).

From 4.2 we obtain the maximal theorem of Fefferman-Stein (see [2] or [6]) in the context of the local fields.

4.3. THEOREM. *Given p and s with $1 \le p < \infty$ and $1 < s < \infty$, there is a constant $A_{p,s}$ depending only on p and s, such that*

$$(1)\qquad \left|\left\{x \colon \sum_j |Mf_j(x)|^s > \lambda^s\right\}\right| \le A_{1,s}\lambda^{-1}\|(f_j)_j\|_{L^1(l^s)}$$

and

$$(2)\qquad \|(Mf_j)_j\|_{L^p(l^s)} \le A_{p,s}\|(f_j)_j\|_{L^p(l^s)}, \qquad 1 < p < \infty,$$

for all $\lambda > 0$ and $f = (f_j)_j \in L_c^\infty(\mathbb{K}, l^s)$. Moreover, the inequality (2) *can be extended for all $f = (f_j)_j \in L^p(\mathbb{K}, l^s)$.*

Proof. Let φ be the characteristic function of the ball $\mathbb{B}^0$. If $|x| > |y|$, then $|t^{-1}(x-y)| = |t^{-1}x|$ and hence $\varphi(t^{-1}(x-y)) = \varphi(t^{-1}x)$. Therefore

$$|\varphi_t(x-y) - \varphi_t(x)| = |t|^{-1}|\varphi(t^{-1}(x-y)) - \varphi(t^{-1}x)| = 0$$

and consequently

$$(3)\qquad \int_{|x|>|y|} \sup_{t\neq 0} |\varphi_t(x-y) - \varphi_t(x)|\,dx = 0.$$

On the other hand, we have

$$\begin{aligned}(|f| * \varphi_t)(x) &= \int_{\mathbb{K}} |f(x-y)|\varphi_t(y)\,dy\\ &= |t|^{-1}\int_{\mathbb{K}} |f(x-y)|\varphi(t^{-1}y)\,dy\\ &= |t|^{-1}\int_{|y|\le|t|} |f(x-y)|\,dy\\ &= |t|^{-1}\int_{|y-x|\le|t|} |f(y)|\,dy\end{aligned}$$

and hence

$$(4)\qquad \begin{aligned} M^{\varphi}|f|(x) &= \sup_{t\neq 0}(|f| * \varphi_t)(x) \\ &= \sup_{t\neq 0}|t|^{-1}\int_{|y-x|\leq|t|}|f(y)|\,dy \\ &= \sup_{k\in\mathbb{Z}} q^k \int_{|y-x|\leq q^{-k}}|f(y)|\,dy = Mf(x). \end{aligned}$$

From (3) it follows that the maximal operator M^{φ} satisfies the inequalities 4.2(2) and 4.2(3). Then, by (4) we obtain the inequalities (1) and (2) for the Hardy-Littlewood maximal operator.

5. A multiplier theorem on $L^p(\mathbb{K}, l^s)$-spaces.

5.1. LEMMA. *Let $g \in L^2(\mathbb{K})$ and $\alpha > 0$. Then, there is a constant A_α depending only on α, such that*

$$(1)\qquad \begin{aligned} &q^{-\alpha}\int_{\mathbb{K}}|x|^{\alpha}|\hat{g}(x)|^2\,dx \\ &\qquad \leq A_\alpha \iint_{\mathbb{K}\times\mathbb{K}}|g(x+y)-g(x)|^2|y|^{-(1+\alpha)}\,dx\,dy. \end{aligned}$$

Proof. See **[9]**, page 220.

5.2. LEMMA. *Let $(g_j)_{j\in\mathbb{Z}}$ be a sequence of elements of $L^2(\mathbb{K})$ and suppose that there are $B > 0$ and $\varepsilon > 0$, such that*

$$(1)\qquad \iint_{\mathbb{K}\times\mathbb{K}}\sum_{j=-\infty}^{+\infty}|g_j(x+y)-g_j(x)|^2|y|^{-(2+\varepsilon)}\,dx\,dy \leq B^2.$$

Then, there is a constant A_ε depending only on ε, such that, for all $k \in \mathbb{Z}$,

$$(2)\qquad \int_{|x|\geq q^k}\sup|\hat{g}_j(x)|\,dx \leq A_\varepsilon B q^{-k\varepsilon/2}.$$

Proof. It follows from Hölder's Inequality that

$$\begin{aligned} &\int_{|x|\geq q^k}\sup_j|\hat{g}_j(x)|\,dx \\ &\quad\leq \left(\int_{\mathbb{K}}|x|^{(1+\varepsilon)}\sup_j|\hat{g}_j(x)|^2\,dx\right)^{1/2}\left(\int_{|x|\geq q^k}|x|^{-(1+\varepsilon)}\,dx\right)^{1/2} \\ &\quad= \left(\int_{\mathbb{K}}|x|^{(1+\varepsilon)}\sup_j|\hat{g}_j(x)|^2\,dx\right)^{1/2}\left(\frac{1-q^{-1}}{1-q^{-\varepsilon}}\right)^{1/2} q^{-k\varepsilon/2}. \end{aligned}$$

Now, setting $\alpha = 1+\varepsilon$ and applying Lemma 5.1, we obtain

$$q^{-\alpha j}\int_{\mathbb{K}} |x|^\alpha \sup_j |\hat{g}_j(x)|^2\,dx$$
$$\leq A_\alpha \iint_{\mathbb{K}\times\mathbb{K}} \sum_{j=-\infty}^{+\infty} |g_j(x+y)-g_j(x)|^2|y|^{-(2+\varepsilon)}\,dx\,dy \leq A_\alpha B^2$$

and consequently

$$\int_{|x|\geq q^k} \sup_j |\hat{g}_j(x)|\,dx \leq (A_\alpha B^2 q^\alpha)^{1/2}\left(\frac{1-q^{-1}}{1-q^{-\varepsilon}}\right)^{1/2} q^{-k\varepsilon/2}$$
$$= A_\varepsilon B q^{-k\varepsilon/2}.$$

5.3. Theorem. *Let $(m_j)_{j\in\mathbb{Z}} \in L^\infty(\mathbb{K}, l^2)$ and suppose that there are $B>0$ and $\varepsilon>0$, such that, for all $j\in\mathbb{Z}$,*

$$(1)\quad \int_{|y|<q^j}\int_{|x|=q^j} \sum_{i=-\infty}^{+\infty} |m_i(x+y)-m_i(x)|^2|y|^{-(2+\varepsilon)}\,dx\,dy \leq B^2 q^{-\varepsilon j}.$$

Then, for all $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$ and $1<p, s<\infty$, we have

$$(2)\qquad \|((m_j\hat{\varphi}_j)^\vee)_j\|_{L^p(l^s)} \leq C\|(\varphi_j)_j\|_{L^p(l^s)},$$

where C is independent of $(\varphi_j)_j$.

Proof. Step 1. Let ϕ_k be the characteristic function of the ball $\mathbb{B}^k$ and $m_j^k = m_j\phi_k$, $k\in\mathbb{Z}$. Since $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$ has compact support we see that $((m_j^k\hat{\varphi}_j)^\vee)_j = ((m_j\hat{\varphi}_j)^\vee)_j$ for k small enough. Hence, if we wish to show (2), we only need to show that, for all $(\varphi_j)_j \in S(\mathbb{K}, l_0^\infty)$, $k\in\mathbb{Z}$ and $1<p, s<\infty$, we have

$$(3)\qquad \|((m_j^k\hat{\varphi}_j)^\vee)_j\|_{L^p(l^s)} \leq C\|(\varphi_j)_j\|_{L^p(l^s)},$$

where the constant C is independent of k and $(\varphi_j)_j$.

Step 2. For each $k, j\in\mathbb{Z}$, let T_j^k be the linear operator defined by

$$(4)\qquad T_j^k\varphi = (m_j^k\hat{\varphi})^\vee = (m_j^k)^\vee * \varphi, \qquad \varphi\in S(\mathbb{K}).$$

For all $k, j\in\mathbb{Z}$ and $\varphi\in S(\mathbb{K})$ we have

$$(5)\qquad \|T_j^k\varphi\|_2 = \|(m_j^k\hat{\varphi})^\vee\|_2 = \|m_j^k\hat{\varphi}\|_2$$
$$\leq \|m_j^k\|_\infty\|\hat{\varphi}\|_2 \leq \|(m_j)_j\|_{L^\infty(l^2)}\|\varphi\|_2.$$

Therefore $(T_j^k)_{j\in\mathbb{Z}}$ is a sequence of singular integral operators of convolution type uniformly bounded from $L^2(\mathbb{K})$ into $L^2(\mathbb{K})$, with sequence of associated kernels $((m_j^k)^\vee)_{j\in\mathbb{Z}}$.

Step 3. Let $m_{jl} = m_j^{-l} - m_j^{1-l}$ for $j, l \in \mathbb{Z}$. It follows from (1) that

$$\text{(6)} \quad \int_{|y|<q^l}\int_{|x|=q^l}\sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)}\,dx\,dy$$
$$= \int_{|y|<q^l}\int_{|x|=q^l}\sum_{j=-\infty}^{+\infty} |m_j(x+y) - m_j(x)|^2 |y|^{-(2+\varepsilon)}\,dx\,dy$$
$$\le B^2 q^{-\varepsilon l}.$$

We have also

$$\text{(7)} \quad \int_{|y|\ge q^l}\int_{|x|=q^l}\sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)}\,dx\,dy$$
$$\le \int_{|y|\ge q^l}\int_{|x|=q^l} 2\sum_{j=-\infty}^{+\infty} (|m_{jl}(x+y)|^2 + |m_{jl}(x)|^2)|y|^{-(2+\varepsilon)}\,dx\,dy$$
$$\le 4\|(m_j)_j\|^2_{L^\infty(l^2)}(1-q^{-1})^2 q^l \left(\frac{q^{-(1+\varepsilon)l}}{1-q^{-(1+\varepsilon)}}\right) = C_1 q^{-l\varepsilon};$$

$$\text{(8)} \quad \int_{|y|=q^l}\int_{|x|<q^l}\sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)}\,dx\,dy$$
$$= q^{-(2+\varepsilon)l}\int_{|y|=q^l}\int_{|x|<q^l}\sum_{j=-\infty}^{+\infty} |m_j(x+y)|^2\,dx\,dy$$
$$\le \|(m_j)_j\|^2_{L^\infty(l^2)}(1-q^{-1})q^{-1}q^{-\varepsilon l} = C_2 q^{-\varepsilon l};$$

$$\text{(9)} \quad \iint_{|x|=|y|>q^l}\sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-2(+\varepsilon)}\,dx\,dy$$
$$\le \iint_{|xj|=|y|>q^l}\sum_{j=-\infty}^{+\infty} |m_{jl}(x+y)|^2 |y|^{-(2+\varepsilon)}\,dx\,dy$$
$$\le \|(m_j)_j\|_{L^\infty(l^2)}(1-q^{-1})^2 q^{-\varepsilon l}(q^{-\varepsilon}/1-q^{-\varepsilon}) = C_3 q^{-\varepsilon l}.$$

Therefore from (6), (7), (8) and (9) we obtain

$$(10)\qquad \iint_{\mathbb{K}\times\mathbb{K}} \sum_{j=-\infty}^{+\infty} |m_{jl}(x+y) - m_{jl}(x)|^2 |y|^{-(2+\varepsilon)}\, dx\, dy \le C^2 q^{-\varepsilon l},$$

for all $l \in \mathbb{Z}$, where the constant C depends only on $\|(m_j)_j\|_{L^\infty(l^2)}$, B and ε. Then, it follows by Lemma 5.2 that, for all $k \in \mathbb{Z}$,

$$(11)\qquad \int_{|x|\ge q^k} \sup_j |(m_{jl})^\vee(x)|\, dx = \int_{|x|\ge q^k} \sup_j |(m_{jl})^\wedge(x)|\, dx \le A_\varepsilon C q^{-(l+k)/2}.$$

Since $m_{jl}\phi_{-1} = m_{jl}$, the $(m_{jl})^\vee(x+y) = (m_{jl})^\vee(x)$ for all $x, y \in \mathbb{K}$ with $|y| \le q^{-l}$ (see **[9]**, page 126). Therefore, for all $t, j, k \in \mathbb{Z}$ and $x, y \in \mathbb{K}$ with $|y| \le q^t$, we have

$$|(m_j^k)^\vee(x+y) - (m_j^k)^\vee(x) \le \sum_{l=-t+1}^{\infty} |(m_{jl})^\vee(x+y) - (m_{jl})^\vee(x)|.$$

Hence we obtain by (11) that, for all $t, k \in \mathbb{Z}$,

$$(12)\qquad \begin{aligned} &\int_{|x|>q^t} \sup_j |(m_j^k)^\vee(x+y) - (m_j^k)^\vee(x)|\, dx \\ &\le 2 \sum_{l=-t+1}^{\infty} \int_{|x|>q^t} \sup_j |(m_{jl})^\vee(x)|\, dx \\ &\le 2A_\varepsilon C(q^{-\varepsilon/2}/1 - q^{-\varepsilon/2}) = C', \end{aligned}$$

and consequently for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} &\sup_{y\neq 0} \int_{|x|>|y|} \sup_j |(m_j^k)^\vee(x-y) - (m_j^k)^\vee(x)|\, dx \\ &= \sup_{t\in\mathbb{Z}} \sup_{|y|\le q^t} \int_{|x|>q^t} \sup_j |(m_j^k)^\vee(x+y) - (m_j^k)^\vee(x)|\, dx \le C'. \end{aligned}$$

Therefore, the sequences of kernels of convolution type $((m_j^k)^\vee)_{j\in\mathbb{Z}}$ satisfy uniformly 3.5(1) and 3.5(2). Consequently we obtain (3), which proves the theorem.

6. Applications to Besov and Hardy-Sobolev spaces. In this section we will give some applications of some foregoing results to Besov and Hardy-Sobolev spaces and to spaces of Bessel potentials.

6.1 Let $A^j = \mathbb{B}^j - \mathbb{B}^{j+1} = \{x \in \mathbb{K} : |x| = q^{-j}\}$ for $j \in \mathbb{Z}$. We will consider the sequence $(\Phi_j)_{j\ge 0}$ of elements of $S(\mathbb{K})$, where $\widehat{\Phi}_j$ is the characteristic function of A^{-j} for $j \geq 1$, and $\widehat{\Phi}_0$ is the characteristic function of $\mathbb{D}$.

For each distribution $f \in S'(\mathbb{K})$ and $j \geq 0$ we have that $\Phi_j * f$ is a function (see [9], p. 126). We can easily see that the function Φ_j satisfies:

$$\Phi_j * \Phi_j = \Phi_j \quad \text{and} \quad \Phi_j * \Phi_i = 0 \quad \text{for } i \neq j; \tag{1}$$

$$\widehat{\Phi}_j(x+y) = \widehat{\Phi}_j(x) \quad \text{for } |x| > |y|; \tag{2}$$

$$\sum_{j=0}^{\infty} \widehat{\Phi}_j = 1. \tag{3}$$

6.2. Definitions. Let $s \in \mathbb{R}$ and $1 < p < \infty$. For $1 \leq r \leq \infty$, the distribution $f \in S'(\mathbb{K})$ is in $B^s_{pr}(\mathbb{K})$ if

$$\|f\|_{B^s_{pr}} = \|(\Phi_j * f)_{j \geq 0}\|_{l^r_s(L^p)} < \infty.$$

For $1 < r < \infty$, the distribution $f \in S'(\mathbb{K})$ is in $F^s_{pr}(\mathbb{K})$ if

$$\|f\|_{F^s_{pr}} = \|(\Phi_j * f)_{j \geq 0}\|_{L^p(l^r_s)} < \infty.$$

6.3. Remark. The sequence $(\Phi_j)_{j \geq 0}$ used in Definition 6.2 and given as in 6.1 is unique. In fact, if $(\psi_j)_{j \geq 0}$ is a sequence of elements of $S(\mathbb{K})$ such that sup $\hat{\psi}_j \subset A^{-j}$ for $j \geq 1$, supp $\hat{\psi}_0 \subset \mathbb{D}$ and $\sum_j \hat{\psi}_j = 1$, then $\hat{\psi}_j$ is the characteristic function of A^{-j} for $j \geq 1$, and $\hat{\psi}_0$ is the characteristic function of $\mathbb{D}$, that is, $\psi_j = \Phi_j$ for $j \geq 0$.

6.4. Remark. As in the Euclidean case, there is another way to define the spaces $B^s_{pr}(\mathbb{K})$ and $F^s_{pr}(\mathbb{K})$ (see [11]). We can say that the distribution f is in $B^s_{pr}(\mathbb{K})$ ($F^s_{pr}(\mathbb{K})$, respectively) if there is a sequence $(a_j)_{j \geq 0}$ of elements of $S'(\mathbb{K})$ such that $\sum_j a_j$ converges in $S'(\mathbb{K})$ to f, supp $\hat{a}_j \subset A^{-j}$ for $j \geq 1$, supp $\hat{a}_0 \subset \mathbb{D}$ and

$$\|(a_j)_{j \geq 0}\|_{l^r_s(L^p)} < \infty \qquad (\|(a_j)_{j \geq 0}\|_{L^p(l^r_s)} < \infty, \text{ respectively}).$$

But this definition is trivial because there is only one sequence $(a_j)_{j \geq 0}$ for each f, namely, the sequence $(\Phi_j * f)_{j \geq 0}$. In fact,

$$(\Phi_j * f)\widehat{\ } = \widehat{\Phi}_j \hat{f} = \widehat{\Phi}_j \sum_{k=0}^{\infty} \hat{a}_k = \sum_{k=0}^{\infty} \widehat{\Phi}_j \hat{a}_k = \widehat{\Phi}_j \hat{a}_j = \hat{a}_j,$$

and hence $a_j = \Phi_j * f$ for $j \geq 0$.

If $s \in \mathbb{R}$ and $f \in S'(\mathbb{K})$, the Bessel potential of order s of f is defined by

$$(J^s f)\widehat{\ } = (\max\{1, |x|\})^s \hat{f}.$$

For α, $\beta \in \mathbb{R}$, the map $f \mapsto J^\alpha f$ is a homeomorphism from $S'(\mathbb{K})$ onto $S'(\mathbb{K})$, $(J^\alpha)^{-1} = J^{-\alpha}$ and $J^{\alpha+\beta} f = J^\alpha(j^\beta f)$ for $f \in S'(\mathbb{K})$ (see **[9]**, p. 137).

The next theorem shows that J^s is an isometry on F^t_{pr} and B^t_{pr}.

6.5. THEOREM. *Let* $s, t \in \mathbb{R}$ *and* $1 < p < \infty$. *Then*

(1) $$\|J^s f\|_{F^{t-s}_{pr}} = \|f\|_{F^t_{pr}}, \qquad f \in F^t_{pr}(\mathbb{K}), \quad 1 < r < \infty;$$

(2) $$\|J^s f\|_{B^{t-s}_{pr}} = \|f\|_{B^t_{pr}}, \qquad f \in B^t_{pr}(\mathbb{K}), \quad 1 \le r \le \infty.$$

Proof. We can easily verify that $J^s \Phi_j = q^{sj}\Phi_j$ for $j \ge 0$. Then, for $j \ge 0$, $s \in \mathbb{R}$ and $f \in S'(\mathbb{K})$ we have

(3) $$J^s(\Phi_j * f) = (J^s\Phi_j) * f = q^{sj}(\Phi_j * f).$$

For $f \in F^t_{pr}(\mathbb{K})$ and $1 < r < \infty$, it follows from (3) that

$$\begin{aligned} \|J^s f\|_{F^{t-s}_{pr}} &= \|(q^{sj}\{\Phi_j * f\})_{j\ge 0}\|_{L^p(l^r_{t-s})} \\ &= \|(q^{tj}\{\Phi_j * f\})_{j\ge 0}\|_{L^p(l^r)} \\ &= \|f\|_{F^t_{pr}}. \end{aligned}$$

Now, for $f \in B^t_{pr}(\mathbb{K})$ and $1 \le r \le \infty$, it also follows from (3) that

$$\begin{aligned} \|J^s f\|_{B^{t-s}_{pr}} &= \|(q^{sj}\{\Phi_j * f\})_{j\ge 0}\|_{l^r_{t-s}(L^p)} \\ &= \|(q^{tj}\{\Phi_j * f\})_{j\ge 0}\|_{l^r(L^p)} \\ &= \|f\|_{B^t_{pr}}. \end{aligned}$$

Now, we will give a theorem of the Littlewood-Paley type. It is a variant of Taibleson's theorem (see **[9]**, pp. 200 and 202), but our proof makes use of vector singular integral operators.

6.6. THEOREM. *For each* $1 < p < \infty$, *there are constants* A_p *and* B_p, *depending only on* p, *such that, for all* $f \in L^p(\mathbb{K})$ *we have*

(1) $$A_p\|f\|_p \le \|(\Phi_j * f)_{j\ge 0}\|_{L^p(l^2)} \le B_p\|f\|_p.$$

Proof (Sketch). Let us consider the operator T from $L^\infty_c(\mathbb{K})$ into $L^0(\mathbb{K}, l^2)$ defined by

(2) $$Tf = (\Phi_j * f)_{j\ge 0},$$

and S from $L^\infty_c(\mathbb{K}, l^2)$ into $L^0(\mathbb{K})$ defined by

(3) $$S(\alpha_j)_{j\ge 0} = \sum_{j=0}^{\infty} \Phi_j * \alpha_j.$$

We can show that

$$\|Tf\|_{L^2(l^2)} = \|f\|_2$$

and

$$\|S(\alpha_j)_{j\geq 0}\|_2 \leq \|(\alpha_j)_{j\geq 0}\|_{L^2(l^2)}.$$

Therefore we can conclude that T has a bounded extension from $L^2(\mathbb{K})$ into $L^2(\mathbb{K}, l^2)$ and S has a bounded extension from $L^2(\mathbb{K}, l^2)$ into $L^2(\mathbb{K})$.

Let K_1 and K_2 be the kernels defined by

$$K_1(x)\lambda = (\Phi_j(x)\lambda)_{j\geq 0}, \qquad x \in \mathbb{K}, \quad \lambda \in \mathbb{C}; \tag{4}$$

$$K_2(x)(\lambda_j)_{j\geq 0} = \sum_{j=0}^{\infty} \Phi_j(x)\lambda_j, \qquad x \in \mathbb{K}, \quad (\lambda_j)_{j\geq 0} \in l^2. \tag{5}$$

We have that

$$\|K_2(x)\|_{L(l^2,\mathbb{C})} \leq \|K_1(x)\|_{L(\mathbb{C},l^2)} = \|(\Phi_j(x))_{j\geq 0}\|_{l^2},$$

therefore, showing that $x \mapsto \|(\Phi_j(x))_{j\geq 0}\|_{l^2}$ is locally integrable we can conclude that K_1 and K_2 are locally integrable. Since

$$\|K_1(x-y) - K_1(x)\|_{L(\mathbb{C},l^2)} = \|K_2(x-y) - K_2(x)\|_{L(l^2,\mathbb{C})} = 0$$

for $|x| > |y|$, we have that K_1 and K_2 satisfy the conditions (H_1) and (H_1') of Theorem 3.4. We can easily verify that

$$Tf(x) = \int_{\mathbb{K}} K_1(x-y)f(y)\,dy$$

and

$$S\alpha(x) = \int_{\mathbb{K}} K_2(x-y)\alpha(y)\,dy,$$

for all $x \in \mathbb{K}$, $f \in L_c^\infty(\mathbb{K})$ and $\alpha \in L_c^\infty(\mathbb{K}, l^2)$. Then, it follows from 3.4 that T and S are singular integral operators of the strong type (p, p) for $1 < p < \infty$, and consequently we have the inequalities 6.6(1).

In Taibleson [9] the space of Bessel potentials $L_s^p(\mathbb{K})$ is defined for $s \in \mathbb{R}$ and $1 \leq p < \infty$, as the set of all distributions $f \in S'(\mathbb{K})$ such that

$$\|f\|_{L_s^p} = \|J^s f\|_p < \infty.$$

The next theorem is a consequence of Theorem 6.6.

6.7. THEOREM. *If* $s \in \mathbb{R}$ *and* $1 < p < \infty$, *then the spaces* $L_s^p(\mathbb{K})$ *and* $F_{p2}^s(\mathbb{K})$ *are isomorphic.*

Proof. If $f \in S'(\mathbb{K})$, it follows from 6.6(1) and 6.5(1) that

$$\|f\|_{L_s^p} = \|J^s f\|_p \approx \|J^s f\|_{F_{p2}^0} = \|f\|_{F_{p2}^s}.$$

6.8. To close this section we will show that $B_{pr}^s(\mathbb{K})$ ($F_{pr}^s(\mathbb{K})$, respectively) is a retract of $l_s^r(L^p(\mathbb{K}))$ ($L^p(\mathbb{K}, l_s^r)$, respectively). Let us consider mappings $\mathscr{I}$ and $\mathscr{P}$ given as follows. The mapping $\mathscr{I}$ is defined on the elements of $S'(\mathbb{K})$ by

$$\mathscr{I}f = (\Phi_j * f)_{j\geq 0}. \tag{1}$$

The mapping $\mathscr{P}$ is defined for sequences $\alpha = (\alpha_j)_{j\geq 0}$ of elements of $S'(\mathbb{K})$ by

$$\mathscr{P}\alpha = \sum_{j=0}^{\infty} \Phi_j * \alpha_j, \tag{2}$$

where the convergence is considered in $S'(\mathbb{K})$. We are not saying that $\mathscr{P}$ is defined on all sequences $\alpha = (\alpha_j)_{j\geq 0}$ of elements of $S'(\mathbb{K})$, but only on those sequences for which the series defining $\mathscr{P}\alpha$ converge in $S'(\mathbb{K})$. It follows from the property 6.1(1) that $\mathscr{P}\mathscr{I}f = f$ for all $f \in B_{pr}^s(\mathbb{K}) \cup F_{pr}^s(\mathbb{K})$.

6.9. THEOREM. *The space* $B_{pr}^s(\mathbb{K})$ *is a retract of* $l_s^r(L^p(\mathbb{K}))$ *and* $F_{pr}^s(\mathbb{K})$ *is a retract of* $L^p(\mathbb{K}, l_s^r)$, *for* $s \in \mathbb{R}$ *and* $1 < p$, $r < \infty$.

Proof. First we note that

$$\|f\|_{B_{pr}^s} = \|\mathscr{I}f\|_{l_s^r(L^p)} \quad \text{and} \quad \|f\|_{f_{pr}^s} = \|\mathscr{I}f\|_{L^p(l_s^r)}.$$

Since $\widehat{\Phi}_j(x+y) = \widehat{\Phi}_j(x)$ for $|x| > |y|$, it follows that $\{\widehat{\Phi}_j : j \geq 0\}$ is a family of scalar multipliers uniformly bounded on $L^p(\mathbb{K})$, $1 < p < \infty$ (see **[9]**, p. 218). Thus, using properties of the functions Φ_j we obtain for $\alpha = (\alpha_j)_{j\geq 0} \in S(\mathbb{K}, l_0^\infty)$,

$$\begin{aligned} \|\mathscr{P}\alpha\|_{B_{pr}^s} &= \|(\Phi_j * \mathscr{P}\alpha)_{j\geq 0}\|_{l_s^r(L^p)} \\ &= \|(\Phi_j * \alpha_j)_{j\geq 0}\|_{l_s^r(L^p)} \\ &= \|(q^{sj}\|\Phi_j * \alpha_j\|_p)_{j\geq 0}\|_{l^r} \\ &\leq C\|(q^{sj}\|\alpha_j\|_p)_{j\geq 0}\|_{l^r} = C\|\alpha\|_{l_s^r(L^p)}. \end{aligned}$$

On the other hand, since $\widehat{\Phi}_j(x+y) = \widehat{\Phi}_j(x)$ for $|x| > |y|$, it follows from 5.3 that $(\widehat{\Phi}_j)_{j\geq 0}$ is a multiplier on $L^p(\mathbb{K}, l^r)$, $1 < p, r < \infty$. Consequently, by the properties of the function Φ_j we have for $\alpha = (\alpha_j)_{j\geq 0} \in S(\mathbb{K}, l_0^\infty)$,

$$\begin{aligned} \|\mathscr{P}\alpha\|_{F^s_{pr}} &= \|(\Phi_j * \mathscr{P}\alpha)_{j\geq 0}\|_{L^p(l^r_s)} \\ &= \|(\Phi_j * \alpha_j)_{j\geq 0}\|_{L^p(l^r_s)} \\ &= \|(\Phi_j * \{q^{sj}\alpha_j\})_{j\geq 0}\|_{L^p(l^r)} \\ &\leq C\|(q^{sj}\alpha_j)_{j\geq 0}\|_{L^p(l^r)} = C\|\alpha\|_{L^p(l^r_s)}. \end{aligned}$$

Hence, $\mathscr{J}$ is bounded from $B^s_{pr}(\mathbb{K})$ into $L^r_s(L^p(\mathbb{K}))$ and from $F^s_{pr}(\mathbb{K})$ into $L^p(\mathbb{K}, l^r_s)$, and $\mathscr{P}$ is bounded from $l^r_s(L^p(\mathbb{K}))$ into $B^w_{pr}(\mathbb{K})$ and from $L^p(\mathbb{K}, l^r_s)$ into $F^s_{pr}(\mathbb{K})$, for $s \in \mathbb{R}$ and $1 < p, r < \infty$.

6.10. Remark. Due to Theorem 6.9 it is possible to obtain interpolation theorems for the spaces $L^p_s(\mathbb{K})$, $B^s_{pr}(\mathbb{K})$ and $Fpr^s(\mathbb{K})$ as in the Euclidean case. For instance, we have (see **[1]**, p. 153) that

$$(L^p_{s_0}(\mathbb{K}), L^p_{s_1}(\mathbb{K}))_{\theta, r} = B^s_{pr}(\mathbb{K}),$$

where $s = (1-\theta)s_0 + \theta_{s_1}$, $0 < \theta < 1$, $s_0 \neq s_1$, $1 < p < \infty$, $1 \leq r \leq \infty$.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin, 1976.

[2] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math., **1** (1971), 107–115.

[3] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[4] U. Neri, *Some properties of functions with bounded mean oscillation*, Studia Math., **61** (1977), 63–75.

[5] K. Phillips and M. H. Taibleson, *Singular integrals in several variables over a local field*, Pacific J. Math., **30** (1969), 209–231.

[6] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Advances in Math., **62** (1986), 7–48.

[7] L. Saloff-Coste, *Opérateurs pseudo-différentiels sur un corps local*, C.R. Acad. Sci. Paris, **297** (1983).

[8] M. H. Taibleson, *Harmonic analysis on n-dimensional vector spaces over local fields*, III. *Multipliers*, Math. Ann., **187** (1970), 259–271.

[9] ——, *Fourier Analysis on Local Fields*, Princeton University Press, Princeton, 1975.

[10] ——, *The existence of natural field structures for finite dimensional vector spaces over local fields*, Pacific J. Math., **63** (1976), 545–551.

[11] H. Triebel, *Spaces of distributions of Besov type on Euclidean n-spaces. Duality, interpolation*, Arkiv Math., **11** (1973), 13–64.
[12] F. Zó, *A note on approximation of the identity*, Studia Math., **55** (1976), 11–122.

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