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TRIANGLE IDENTITIES AND SYMMETRIES OF A SUBSHIFT OF FINITE TYPE

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We prove the group $\mathrm{Aut}(\sigma_A)$ of symmetries of a subshift of finite type is isomorphic to the fundamental group of the space $\mathrm{RS}(\mathscr{E})$ of strong shift equivalences built from the algebraic RS Triangle Identities for zero-one matrices which arise from triangles in the contractable simplicial complex of Markov partitions. Moreover, we show the higher homotopy groups of $\mathrm{RS}(\mathscr{E})$ are zero. $\mathrm{RS}(\mathscr{E})$ is therefore homotopy equivalent to the classifying space of $\mathrm{Aut}(\sigma_A)$.

1. Introduction and statement of results. First we briefly review Williams' strong shift equivalence criterion for conjugacy of subshifts of finite type. See [**3**, **4**, **8**]. Let $A\colon \mathscr{S}\times\mathscr{S}\to\{0,1\}$ and $B\colon\mathscr{T}\times\mathscr{T}\to\{0,1\}$ be zero-one matrices on the finite state spaces $\mathscr{S}$ and $\mathscr{T}$. An *elementary strong shift equivalence*

$$(R, S)\colon A\to B$$

is a pair of zero-one matrices $R\colon\mathscr{S}\times\mathscr{T}\to\{0,1\}$ and $S\colon\mathscr{T}\times\mathscr{S}\to\{0,1\}$ satisfying

$$RS = A \quad\text{and}\quad SR = B.$$

Let (X_A, σ_A) and (X_B, σ_B) be the subshifts of finite type (SFT) constructed from A and B respectively. The strong shift equivalence (R, S) gives rise to an *elementary symbolic conjugacy*

$$c(R, S)\colon X_A\to X_B$$

defined as follows: Let $x=\{x_n\}$ be in X_A. Then $y=c(R,S)(x)$ is the unique point $y=\{y_n\}$ in X_B such that $1=A(x_n, x_{n+1})=R(x_n, y_n)S(y_n, x_{n+1})$ for all n. Similarly, one has

$$c(S, R)\colon X_B\to X_A$$

and it is easy to verify the identities

$$c(S, R)c(R, S)=\sigma_A \quad\text{and}\quad c(R, S)c(S, R)=\sigma_B$$

which show that $c(R, S)$ and $c(S, R)$ are conjugacies. More generally, let $\mathscr{E}$ denote the set of zero-one matrices on finite state spaces. We shall assume that any matrix in $\mathscr{E}$ has at least one non-zero entry in each row and in each column. Williams defined A and B to be *strong shift equivalent in* $\mathscr{E}$ provided there is a chain of elementary strong shift equivalences from A to B through intermediate matrices in $\mathscr{E}$. The composition of the corresponding elementary conjugacies gives a conjugacy from (X_A, σ_A) to (X_B, σ_B). Williams' proof of the converse that conjugacy implies strong shift equivalence brings in the set P_A of Markov partitions for σ_A on X_A. In [**8**] P_A is given the structure of a locally finite simplicial complex in which each Markov partition is a vertex, and the key step in Williams' argument really amounts to showing that P_A is connected. In fact, P_A turns out to be contractable [**8**], and in this paper we make use of the fact that it is simply connected to prove (1.5) and contractability to prove (1.13) below.

The definition of Markov partition used in [**8**] for P_A is the one in [**5**]. A similar theory goes through using the Markov partitions by rectangles as presented, say, in [**4**, p. 100]. Let $U = \{U_i\}$ be in P_A and let $M = M(U) = \{M(i, j)\}$ be the zero-one matrix where

$$M(i, j) = 1 \quad \text{iff} \quad U_i \cap \sigma_A^{-1}(U_j) \neq \varnothing.$$

For example, $A = M(U^A)$ where $U^A = \{U_i^A\}$ is the "standard" Markov partition with U_i^A equal to the cylinder set of those $x = \{x_n\}$ such that $x_0 = i$. Let $V = \{V_k\}$ also be in P_A. Define zero-one matrices $R = R(U, V) = \{R(i, k)\}$ and $S(V, U) = \{S(k, i)\}$ by the formulas

$$R(i, k) = 1 \quad \text{if and only if} \quad U_i \cap V_k \neq \varnothing,$$
$$S(k, i) = 1 \quad \text{if and only if} \quad V_k \cap \sigma_A^{-1}(U_i) \neq \varnothing.$$

Write $U < V$ to mean that V refines U. As in [**8**] we write $U \underset{+}{\rightarrow} V$ provided $U < V < U \cap \sigma_A^{-1}(U)$, and we write $U \underset{-}{\rightarrow} V$ provided $U < V < \sigma_A(U) \cap U$. Finally, we write $U \rightarrow V$ iff $U \underset{+}{\rightarrow} U \cap V \underset{-}{\leftarrow} V$. An (ordered) 1-simplex of P_A is a pair $[U, V]$ where $U \neq V$ and $U \rightarrow V$. Let $P = M(U)$, $Q = M(V)$, and R and S be as above. It was shown in [**8**] that

$$P = RS \quad \text{and} \quad Q = SR$$

whenever $U \rightarrow V$; that is, $(R, S)\colon P \rightarrow Q$. In particular, connectivity of P_A implies the transition matrices $M(U)$ and $M(V)$ of any two Markov partitions U and V in P_A are strong shift equivalent in $\mathscr{E}$. To finish off the outline of Williams' proof, let $\alpha\colon (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ be a topological conjugacy. Let $U = \{U_i\}$ be in P_A where

i runs through an indexing set I. Then $\alpha(U) = \{\alpha(U_i)\}$ is in P_B. In fact, let $V = \{V_k\}$ be in P_B where k is in the index set K, and suppose we have $V = \alpha(U)$; that is, each $V_k = \alpha(U_i)$ for exactly one U_i. Then α gives a bijection $\alpha\colon I \to K$. Throughout this paper we will follow the convention that α also denotes the $K \times I$ permutation matrix with $\alpha(k, i) = 1$ iff $\alpha(i) = k$. Let $P = M(U)$ and $Q = M(V)$. Since $U_i \cap \sigma_A^{-1}(U_j) \neq \varnothing$ iff $\alpha(U_i) \cap \sigma_A^{-1}(\alpha(U_j)) \neq \varnothing$, we have $Q = \alpha P \alpha^{-1}$. Therefore

$$(\alpha^{-1}, \alpha P)\colon P \to Q.$$

Hence A is strong shift equivalent to B.

Let $\mathrm{Aut}(\sigma_A)$ be the group of symmetries of (X_A, σ_A). By definition, this is the group of homeomorphisms of X_A which commute with σ_A. It is discrete in the topology of uniform convergence, because σ_A is expansive. See **[1, 2, 8]** for some recent information about this group. The preceding discussion suggests that elements of $\mathrm{Aut}(\sigma_A)$ can be described as products of various elementary conjugacies $c(R, S)$ modulo certain relations. This turns out to be the case, and a very natural set of relations which do work come from triangles in P_A.

By definition a triangle in P_A is an ordered triple $[U, V, W]$ of Markov partitions such that $U \to V$, $V \to W$, and $U \to W$. Let

$$M = M(U), \qquad P = M(V), \qquad Q = M(W),$$

$$\begin{aligned} &R_1 = R(U, V), && S_1 = S(V, U), \\ &R_2 = R(V, W), && S_2 = S(W, V), \\ &R_3 = R(U, W), && S_3 = S(W, U). \end{aligned} \tag{1.1}$$

In §2 we will verify the *RS Triangle Identities*:

$$R_1 R_2 = R_3, \qquad R_2 S_3 = S_1, \qquad S_3 R_1 = S_2. \tag{1.2}$$

Upon either multiplying the second equation on the left by S_2 or by multiplying the third equation on the right by S_1, we derive the *SS Triangle Identities* found in **[8]**:

$$\begin{aligned} R_1 R_2 &= R_3, \\ S_2 S_1 &= Q S_3 = S_3 M. \end{aligned} \tag{1.3}$$

DEFINITION 1.4 The *space $RS(\mathscr{E})$ of strong shift equivalence in $\mathscr{E}$ based on the RS Triangle Identities* is the geometric realization of the simplicial set where an n-simplex consists of the following data:

(a) an $(n+1)$-tuple $\langle A_0, \dots, A_n \rangle$ of square matrices in $\mathscr{E}$ and

(b) for each $i < j$ a strong shift equivalence $(R_{ij}, S_{ji})\colon A_i \to A_j$ in

$\mathscr{E}$ such that the RS Triangle Identities hold for $i < j < k$; that is,

$$R_{ij}R_{jk} = R_{ik}, \qquad R_{jk}S_{ki} = S_{ji}, \qquad S_{ki}R_{ij} = S_{kj}.$$

The face operators are the usual forgetful ones and the degeneracies insert the strong shift equivalence $(1, A_i)$ from A_i to itself. See **[6]** or [7] for background on simplicial sets and CW complexes. It is immediate from the definition that the set of components $\pi_0(RS(\mathscr{E}))$ is exactly the set of strong shift equivalences in $\mathscr{E}$.

THEOREM 1.5. *There is an isomorphism*

$$\Phi_A\colon \mathrm{Aut}(\sigma_A) \to \pi_1(\mathrm{RS}(\mathscr{E}), A).$$

The explicit formula for Φ_A is given in (3.5) and (3.8) below.

For any strong shift equivalence $(R, S)\colon P \to Q$, let $\gamma(R, S)$ denote the corresponding homotopy class of paths from P to Q in $\mathrm{RS}(\mathscr{E})$. Elementary arguments in algebraic topology show that elements γ in $\pi_1(\mathrm{RS}(\mathscr{E}), A)$ can be represented as products

$$\gamma = \prod_{i=1}^{n} \gamma(R_i, S_i)^{\varepsilon_i} \tag{1.6}$$

where $\varepsilon_i = \pm 1$. The defining relations are

$$\gamma(R, S)\gamma(R, S)^{-1} = 1$$

and

$$\gamma(R_1, S_1)\gamma(R_2, S_2) = \gamma(R_3, S_3)$$

whenever the RS Triangle Identites hold; that is,

(1.7)
$$\begin{tikzcd} & P \arrow[dr, "{(R_2,S_2)}"] & \\ M \arrow[ur, "{(R_1,S_1)}"] \arrow[rr, "{(R_3,S_3)}"'] & & Q \end{tikzcd}$$

is a triangle in $\mathrm{RS}(\mathscr{E})$. Given this presentation of $\pi_1(\mathrm{RS}(\mathscr{E}), A)$, the inverse

$$\Theta_A\colon \pi_1(\mathrm{RS}(\mathscr{E}), A) \to \mathrm{Aut}(\sigma_A)$$

of Φ_A is easy to describe: namely, Θ_A takes the product (1.6) to

$$\Theta_A(\gamma) = \prod_{i=1}^{n} c(R_i, S_i)^{-\varepsilon_i} \tag{1.8}$$

where composition is read from right to left. That this is well defined follows immediately from

LEMMA 1.9. *Suppose* $(R_1, S_1)\colon M \to P$, $(R_2, S_2)\colon P \to Q$, *and* $(R_3, S_3)\colon M \to Q$. *Assume that* $R_1R_2 = R_3$. *Then*

$$c(R_2, S_2)c(R_1, S_1) = c(R_3, S_3)$$

if and only if $R_2S_3 = S_1$ *and* $S_3R_1 = S_2$.

The first equation of the RS Triangle Identities is very reasonable. This lemma shows that the last equations, which were harder to guess, are exactly what is needed to make composition of elementary conjugacies behave well around a triangle.

There is also the *space* $\mathrm{SS}(\mathscr{E})$ *of strong shift equivalence in* $\mathscr{E}$ *based on the* SS *Triangle Identities* with the same face and degeneracy operators as used for $\mathrm{RS}(\mathscr{E})$. Since the RS Identities imply the SS identities, there is a natural continuous map

$$\mathrm{RS}(\mathscr{E}) \to \mathrm{SS}(\mathscr{E}).$$

The set of components $\pi_0(\mathrm{SS}(\mathscr{E}))$ is also the set of strong shift equivalence in $\mathscr{E}$ and the induced map $\pi_0(\mathrm{RS}(\mathscr{E})) \to \pi_0(\mathrm{SS}(\mathscr{E}))$ is a bijection. The commutative diagram (4.30) of **[8]** expands to

(1.10)

$$\begin{tikzcd}
& \pi_1(\mathrm{RS}(\mathscr{E}), A) \arrow[dr] \arrow[dd, "\Psi_A"] & \\
\mathrm{Aut}(\sigma_A) \arrow[ur, "\Phi_A"] \arrow[rr] \arrow[dr, "\Psi_A"'] & & \pi_1(\mathrm{SS}(\mathscr{E}), A) \arrow[dl] \\
& \pi_1(S(\overline{\mathscr{E}}), A) &
\end{tikzcd}$$

where $S(\overline{\mathscr{E}})$ is the space of shift equivalences of non-negative integral matrices. It was shown in **[8]** there is an isomorphism between $\pi_1(S(\overline{\mathscr{E}}), A)$ and the group $\mathrm{Aut}(s_A) = \mathrm{Aut}(G(A), G(A)_+, s_A)$ of order preserving automorphisms of the dimension group $G(A)$ which commute with the automorphism s_A induced on $G(A)$ by A. The Finite

Order Generation Conjecture (FOG) can therefore be reformulated to state

The kernel of

$$(1.11) \qquad \delta_A \colon \pi_1(\mathrm{RS}(\mathscr{E}), A) \to \pi_1(S(\overline{\mathscr{E}}), A)$$

is generated by elements of finite order.

In **[9]** we use the spaces $\mathrm{RS}(\mathscr{E})$ and $\mathrm{S}(\overline{\mathscr{E}})$ together with methods of this paper to prove the following eventual version of this conjecture:

Eventual fog 1.12. Let α be in $\ker(\delta_A)$. There is an integer $k_0 \geq 1$ such that if $k \geq k_0$, then α is a product of homeomorphisms of X_A of finite order which commute with σ_A^k.

In the proof of (1.12) given in **[9]**, the integer k_0 seems to depend on α and the product expressions are possibly different for different k. Of course, FOG conjectures that we can take $k_0 = 1$, in which case the product expression for $k = 1$ works for all k. M. Boyle has shown α in (1.12) is a product of just two finite order elements commuting with σ_A^k for large k, although in practice one must take k larger than k_0 to do this.

In §4 we prove

THEOREM 1.13. $\pi_n(\mathrm{RS}(\mathscr{E}), A) = 0$ *for* $n \geq 2$.

For simplicity, we assume throughout this paper that all the matrices in $\mathscr{E}$ are finite. However, (1.5) and (1.13) generalize without change when $\mathscr{E}$ consists of infinite matrices in one of the three classes considered in (4.1) of **[8]**. There is also a stochastic version for symmetries of (X_A, σ_A) which preserve a σ_A-invariant Markov measure.

2. Proof of the RS triangle identities. Consider the triangle

(2.1)
$$\begin{tikzcd} & V \arrow[dr] & \\ U \arrow[ur] \arrow[rr] & & W \end{tikzcd}$$

in P_A with the corresponding matrices (1.1). The equation $R_1 R_2 = R_3$ was verified in (3.3) of **[8]**. So it remains to prove the last two equations in (1.2). As in §3 of **[8]** this subdivides into 9 triangles

(2.2)
$$\begin{tikzcd}
 & & V \arrow[dl, "-"'] \arrow[dr, "+"] \arrow[ddddrr] & & \\
 & U \cap V \arrow[rr] \arrow[dr, "+"'] & & V \cap W \arrow[dl, "-"] & \\
 & & U \cap V \cap W & & \\
 & & U \cap W \arrow[u, "+", "-"'] & & \\
U \arrow[uuuurr] \arrow[uuur, "+"] \arrow[uurr, "+"] \arrow[urr, "+"] \arrow[rrrr] & & & & W \arrow[uuul, "-"'] \arrow[uull, "-"'] \arrow[ull, "-"']
\end{tikzcd}$$

As in (3.9) of **[8]** we have $U \cap W = U \cap V \cap W$ and the diagram

(2.3)
$$\begin{tikzcd}
 & & V \arrow[dl, "-"'] \arrow[dr, "+"] \arrow[dddrr] \arrow[d, phantom, "\#2"] & & \\
 & U \cap V \arrow[rr] \arrow[dr, "+"'] \arrow[dl, phantom, "\#1"] \arrow[d, phantom, "\#5"] & & V \cap W \arrow[dl, "-"] \arrow[dr, phantom, "\#3"] \arrow[d, phantom, "\#6"] & \\
 & & U \cap W \arrow[u, phantom, "\#4"] \arrow[d, phantom, "\#7"] & & \\
U \arrow[uuurr] \arrow[uur, "+"] \arrow[urr, "+"] \arrow[rrrr] & & & & W \arrow[uul, "-"'] \arrow[ull, "-"']
\end{tikzcd}$$

The first step is to verify the RS Triangle Identities for each triangle in (2.3). In all that follows, "LHS" is the left hand side of an equation and "RHS" is the right hand side. Let $\sigma = \sigma_A$.

Triangles #1, #3, #4, #7:

These are all of the form

$$\begin{tikzcd}
 & U \cap W & \\
U \arrow[ur, "+"] \arrow[rr] & & W \arrow[ul, "-"']
\end{tikzcd}$$

(I) $S_1 = R_2 S_3$.
LHS: $S_1((i, p), j) = 1$ iff $U_i \cap W_p \cap \sigma^{-1}(U_j) \neq 0$.
RHS: The term $R_2((i, p), q) S_3(q, j) = 1$ iff $U_i \cap W_p \cap W_q \neq \varnothing$ and $W_q \cap \sigma^{-1}(U_j) \neq \varnothing$.

In particular $q = p$; so there is at most one non-zero term and the RHS is a zero-one matrix.

Write

$$U_i = \bigcup \sigma(W_a) \cap W_b, \qquad U_j = \bigcup \sigma(W_c) \cap W_d.$$

Assume RHS $\neq 0$. Then $U_i \cap W_p \neq \varnothing$ implies some $b = p$ and $W_q \cap \sigma^{-1}(U_j) \neq \varnothing$ implies some $c = q = p$. Therefore,

$$U_i \cap \sigma^{-1}(U_j) \supset \sigma(W_a) \cap W_p \cap \sigma^{-1}(W_d) \neq \varnothing$$

for some a and some d. This shows $U_i \cap W_p \cap \sigma^{-1}(U_j) \neq 0$ implying LHS $\neq 0$. Conversely, suppose that LHS $\neq 0$. Then $U_i \cap W_p \cap \sigma^{-1}(U_j) \neq 0$. Let $q = p$. Then $U_i \cap W_p \cap W_q \neq \varnothing$ and $W_q \cap \sigma^{-1}(U_j) \neq \varnothing$ implying RHS $\neq 0$.

(II) $S_2 = S_3 R_1$.
LHS: $S_2(p, (j, q)) = 1$ iff $W_p \cap \sigma^{-1}(U_j) \cap \sigma^{-1}(W_q) \neq \varnothing$.
RHS: Consider a single term $S_3(p, i) R_1(i, (j, q))$. This is equal to 1 iff $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ and $U_i \cap U_j \cap W_q \neq \varnothing$; which implies $i = j$ and $U_i \supset U_j \cap W_q$.

Thus there is at most one i giving a non-zero term on the RHS, and so $S_3 R_1$ is a zero-one matrix.

Now write

$$W_p = \bigcup U_a \cap \sigma^{-1}(U_b), \qquad W_q = \bigcup U_e \cap \sigma^{-1}(U_f).$$

Assume RHS $\neq 0$. Then $i = j$ and

$$W_p \cap \sigma^{-1}(U_i) = \bigcup U_a \cap \sigma^{-1}(U_b) \cap \sigma^{-1}(U_j)$$

implying some $b = j$. Similarly,

$$U_j \cap W_q = \bigcup U_j \cap U_e \cap \sigma^{-1}(U_f)$$

implying some $e = j$. Therefore,

$$W_p \cap \sigma^{-1}(U_j) \cap \sigma^{-1}(W_q) \supset U_a \cap \sigma^{-1}(U_j) \cap \sigma^{-2}(U_f) \neq \varnothing$$

for some a and some f. Hence LHS $\neq 0$. Conversely, suppose LHS $\neq 0$. Then $W_p \cap \sigma^{-1}(U_j) \neq \varnothing$ and $U_j \cap U_j \cap W_q \neq \varnothing$ implying RHS $\neq 0$.

Triangles #5, #6:
These are of the form

$$\begin{tikzcd} & V \arrow[dr, "\varepsilon"] & \\ U \arrow[ur, "\varepsilon"] \arrow[rr, "\varepsilon"'] & & W \end{tikzcd}$$

where all $\varepsilon = \pm 1$ are equal.

First, we give the argument for $\varepsilon = +1$.

(I) $S_1 = R_2 S_3$.
LHS: $S_1(k, i) \neq 0$ iff $V_k \cap \sigma^{-1}(U_i) \neq \varnothing$.
RHS: $R_2(k, p)S_3(p, i) \neq 0$ iff $V_k \cap W_p \neq \varnothing$ and $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$.

We claim that RHS is a zero-one matrix. Given k, choose some j such that $R_1(j, k) \neq 0$. Then $R_1(j, k)R_2(k, p)S_3(p, i) = R_3(j, p)S_3(p, i)$. Since $R_3 S_3 = Q$ is a zero-one matrix, there can be at most one p for which this term is non-zero.

Assume LHS $\neq 0$. We have $W < U \cap \sigma^{-1}(U) < V \cap \sigma^{-1}(U)$. So choose the unique index p with $W_p \supset V_k \cap \sigma^{-1}(U_i)$. Then $V_k \cap W_p \neq \varnothing$ and $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ implying RHS $\neq 0$. Conversely, let RHS $\neq 0$. Since $V < W$, $V_k \cap W_p \neq \varnothing$ implies $V_k \supset W_p$. Thus $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ implies $V_k \cap \sigma^{-1}(U_i) \neq \varnothing$. This shows LHS $\neq 0$.

(II) $S_2 = S_3 R_1$.
LHS: $S_2(p, k) \neq 0$ iff $W_p \cap \sigma^{-1}(V_k) \neq \varnothing$.
RHS: $S_3(p, i)R_1(i, k) \neq 0$ iff $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ and $U_i \cap V_k \neq \varnothing$.

Since $U < V$, we have $U_i \supset V_k$ and thus i is determined by k. This shows RHS is a zero-one matrix.

Assume LHS $\neq 0$. Let i be such that $U_i \supset V_k$. Then $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ and $U_i \cap V_k \neq \varnothing$ showing that RHS $\neq 0$. Conversely, suppose RHS $\neq 0$. Then $U_i \supset V_k$ as remarked above. Write

$$W_p = \bigcup U_a \cap \sigma^{-1}(U_b), \qquad V_k = \bigcup U_i \cap \sigma^{-1}(U_d).$$

The condition $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ implies some $b = i$. Hence,

$$W_p \cap \sigma^{-1}(V_k) \supset U_a \cap \sigma^{-1}(U_i) \cap \sigma^{-2}(U_d) \neq \varnothing$$

for some a and some d. Thus LHS $\neq 0$.

Next, we give the argument for $\varepsilon = -1$. Remember that now $R_1 = R(W, V)$, $R_2 = R(V, U)$, $R_3 = R(W, U)$, etc.

(I) $S_1 = R_2 S_3$,
LHS: $S_1(k, p) \neq 0$ iff $V_k \cap \sigma^{-1}(W_p) \neq \varnothing$.
RHS: $R_2(k, i)S_3(i, p) \neq 0$ iff $V_k \cap U_i \neq \varnothing$ and $U_i \cap \sigma^{-1}(W_p) \neq \varnothing$.

We claim that RHS is a zero-one matrix. Given k and p, choose some q such that $R_1(q, k) \neq 0$. Then

$$R_1(q, k)R_2(k, i)S_3(i, p) = R_3(q, i)S_3(i, p).$$

Since $R_3 S_3 = Q$ is a zero-one matrix, there can be at most one i for which this term is non-zero.

Assume LHS $\neq 0$. We have $V < U \cap \sigma^{-1}(U) < V \cap \sigma^{-1}(W)$. So choose the unique index i with $U_i \supset V_k \cap \sigma^{-1}(W_p) \neq \varnothing$. Then $V_k \cap U_i \neq \varnothing$ and $U_i \cap \sigma^{-1}(W_p) \neq \varnothing$ implying RHS $\neq 0$. Conversely, the RHS $\neq 0$. Write

$$W_p = \bigcup \sigma(U_a) \cap U_b, \qquad V_k = \bigcup \sigma(U_c) \cap U_d.$$

Since $V_k \cap U_i \neq \varnothing$, some $d = i$. Also $U_i \cap \sigma^{-1}(W_p) \neq \varnothing$ implies some $a = i$. Therefore

$$V_k \cap \sigma^{-1}(W_p) \supset \sigma(U_c) \cap U_i \cap \sigma^{-1}(U_b) \neq \varnothing$$

for some c and some b. Thus LHS $\neq 0$.

(II) $S_2 = S_3 R_1$.
LHS: $S_2(i, k) \neq 0$ iff $U_i \cap \sigma^{-1}(V_k) \neq \varnothing$.
RHS: $S_3(i, p)R_1(p, k) \neq 0$ iff $U_i \cap \sigma^{-1}(W_p) \neq \varnothing$ and $W_p \cap V_k \neq \varnothing$.

We claim that RHS is a zero-one matrix. Given i and k, choose some j such that $R_2(k, j) \neq 0$. Then

$$S_3(i, p)R_1(p, k)R_2(k, j) = S_3(i, p)R_3(p, j).$$

Since $S_3 R_3 = M$ is a zero-one matrix, there can be at most one p for which this term is non-zero.

Assume LHS $\neq 0$. We have $W < \sigma(U) \cap U < \sigma(U) \cap V$. Let p be the unique index where $W_p \supset \sigma(U_i) \cap V_k \neq \varnothing$. Then $U_i \cap \sigma^{-1}(W_p) \neq \varnothing$

and $W_p \cap V_k \neq \varnothing$. This says RHS $\neq 0$. Conversely, assume RHS $\neq 0$. Since $V < W$, we must have $W_p \subset V_k$. Hence $U_i \cap \sigma^{-1}(V_k) \neq \varnothing$ and LHS $\neq 0$.

Triangle #2:
This triangle is of the form

$$\begin{tikzcd} & V \arrow[dl, "-"'] \arrow[dr, "+"] & \\ U \arrow[rr] & & W \end{tikzcd}$$

(I) $S_1 = R_2 S_3$.
LHS: $S_1(k, i) \neq 0$ iff $V_k \cap \sigma^{-1}(U_i) \neq \varnothing$.
RHS: $R_2(k, p) S_3(p, i) \neq 0$ iff $V_k \cap W_p \neq \varnothing$ and $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$.

The RHS is a zero-one matrix. To see this, choose j so that $R_1(j, k)$
$= 1$. Then $R_1(j, k) R_2(k, p) S_3(p, i) = R_3(j, p) S_3(p, i)$. Since $R_3 S_3 = M$ is a zero-one matrix, there is at most one p giving a non-zero term.

Assume LHS $\neq 0$. We have $W < V \cap \sigma^{-1}(V) < V \cap \sigma^{-1}(U)$. So choose p with $W_p \supset V_k \cap \sigma^{-1}(U_i) \neq \varnothing$. Then $V_k \cap W_p \neq \varnothing$ and $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ and RHS $\neq 0$. Conversely, suppose RHS $\neq 0$. Then $V_k \supset W_p$ and $V_k \cap \sigma^{-1}(U_i) \neq \varnothing$. This gives LHS $\neq 0$.

(II) $S_2 = S_3 R_1$.
LHS: $S_2(p, k) \neq 0$ iff $W_p \cap \sigma^{-1}(V_k) \neq \varnothing$.
RHS: $S_3(p, i) R_1(i, k) \neq 0$ iff $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ and $U_i \cap V_k \neq \varnothing$.

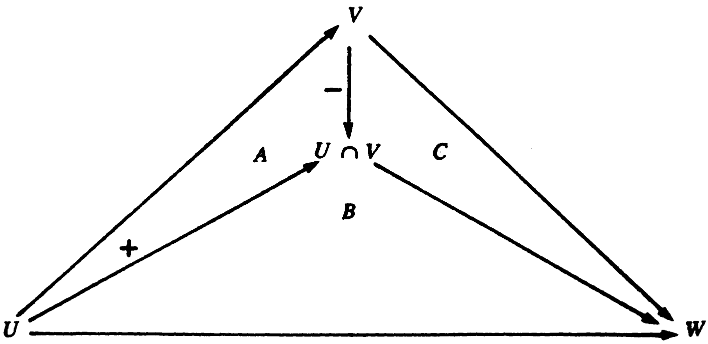
The RHS is a zero-one matrix. To see this, let q be any index where $R_2(k, q) \neq 0$. Then $S_3(p, i) R_1(i, k) R_2(k, q) = S_3(p, i) R_3(i, q)$. Since $S_3 R_3 = Q$ is zero-one, there is at most one i for which the term is non-zero.

Assume LHS $\neq 0$. We have $U < \sigma(V) \cap V < \sigma(W) \cap V$. Let i be the index for which $U_i \supset \sigma(W_p) \cap V_k \neq \varnothing$. Then $W_p \cap \sigma^{-1}(U_i) \neq \varnothing$ and $U_i \cap V_k \neq \varnothing$. This shows RHS $\neq 0$. Finally, let RHS $\neq 0$. Then $U_i \subset V_k$, because $U > V$. Therefore $W_p \cap \sigma^{-1}(V_k) \neq \varnothing$ and LHS $\neq 0$.

This completes the argument for the triangles in (2.3). We now show how these combine to prove the RS Identities for (2.1). From

(2.2) we have the diagram of three triangles

(2.4)



Step 1. We will show the RS Identities for Triangles A, B, and C in (2.4) imply the RS Identities for (2.1).

(I) $S(V, U) = R(V, W)S(W, U)$?

Since $U < U \cap V$, the matrix $R(U, U \cap V)$ has exactly one non-zero term in each column. Therefore, the above equation holds provided it holds when multiplied on the right by $R(U, U \cap V)$. We have

$$S(V, U)R(U, U \cap V) = S(V, U \cap V)$$

and

$$\begin{aligned} R(V, W)S(W, U)R(U, U \cap V) &= R(V, W)S(W, U \cap V) \\ &= S(V, U \cap V) \end{aligned}$$

(II) $S(W, V) = S(W, U)R(U, V)$?

$$\begin{aligned} S(W, V) &= S(W, U \cap V)R(U \cap V, V) \\ &= S(W, U)R(U, U \cap V)R(U \cap V, V) \\ &= S(W, U)R(U, V). \end{aligned}$$

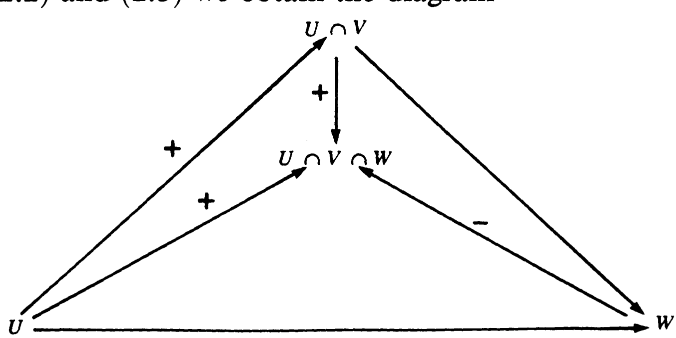
It remains to check the RS Identities for each of the triangles A, B, C.

Step 2. Triangle A is just Triangle #1 of (2.3).

Step 3. Triangle B.

From (2.2) and (2.3) we obtain the diagram

(2.5)



Recall that $U \cap W = U \cap V \cap W$ and that the RS Identities for each of the subtriangles was verified above for (2.3).

(I) $S(U \cap V, U) = R(U \cap V, W)S(W, U)$?

$$\begin{aligned} S(U \cap V, U) &= R(U \cap V, U \cap V \cap W)S(U \cap V \cap W, U) \\ &= R(U \cap V, U \cap V \cap W)R(U \cap V \cap W, W)S(W, U) \\ &= R(U \cap V, W)S(W, U). \end{aligned}$$

(II) $S(W, U \cap V) = S(W, U)R(U, U \cap V)$?

Similarly to (I) of Step 1 this equation holds iff it holds when multiplied on the right by $R(U \cap V, U \cap V \cap W)$. We have

$$S(W, U \cap V)R(U \cap V, U \cap V \cap W) = S(W, U \cap V \cap W)$$

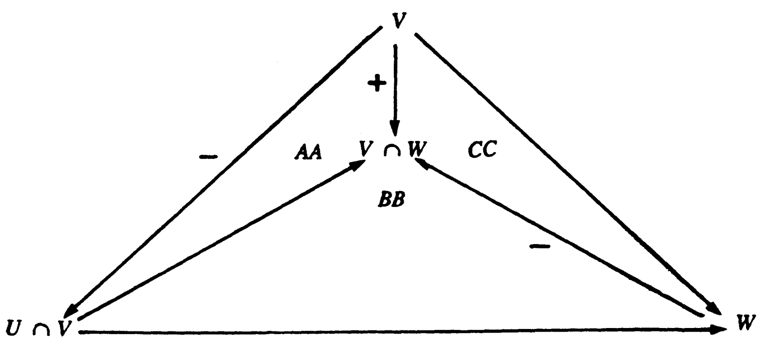
and

$$\begin{aligned} &S(W, U)R(U, U \cap V)R(U \cap V, U \cap V \cap W) \\ &\quad = S(W, U)R(U, U \cap V \cap W) = S(W, U \cap V \cap W). \end{aligned}$$

Step 4. Triangle C.

From (2.2) and (2.3) we get

(2.6)



Assuming the RS Identities hold for the Triangles AA, BB, and CC, we now verify the RS Identities for Triangle C. In Step 5 below the RS Identities for AA, BB, CC are checked.

(I) $S(V, U \cap V) = R(V, W)S(W, U \cap V)$?

$$\begin{aligned} S(V, U \cap V) &= R(V, V \cap W)S(V \cap W, U \cap V) \\ &= R(V, V \cap W)R(V \cap W, W)S(W, U \cap V) \\ &= R(V, W)S(W, U \cap V) \end{aligned}$$

(II) $S(W, V) = S(W, U \cap V)R(U \cap V, V)$?

Similarly to above remarks, this equation holds iff it holds when multiplied on the right by $R(V, V \cap W)$. We have

$$S(W, V)R(V, V \cap W) = S(W, V \cap W)$$

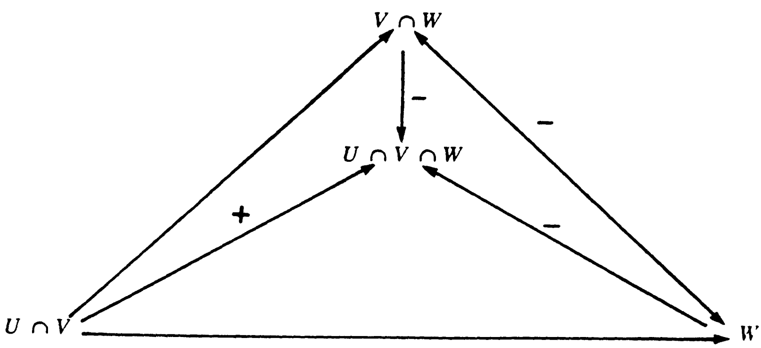
and

$$\begin{aligned} &S(W, U \cap V)R(U \cap V, V)R(V, V \cap W) \\ &\quad = S(W, U \cap V)R(U \cap V, V \cap W) = S(W, V \cap W). \end{aligned}$$

Step 5. Triangles AA, BB, CC

Triangles AA and CC are just like Triangles #2 and #3 in (2.3). From diagram (2.2) we see that Triangle BB subdivides into

(2.7)



Each of the subtriangles in (2.7) satisfies the RS Identities by Step 1. So proceed as before:

(I) $S(V \cap W, U \cap W) = R(V \cap W, W)S(W, U \cap V)$?

The matrix $R(U \cap V \cap W, V \cap W)$ has the property that each row has exactly one non-zero entry. Therefore the above equation holds iff it holds when multiplied on the left by $R(U \cap V \cap W, V \cap W)$. We have

$$\begin{aligned} &R(U \cap V \cap W, V \cap W)S(V \cap W, U \cap V) \\ &\quad = S(U \cap V \cap W, U \cap V) \end{aligned}$$

and

$$\begin{aligned} &R(U \cap V \cap W, V \cap W)R(V \cap W, W)S(W, U \cap V) \\ &\quad = R(U \cap V \cap W, W)S(W, U \cap V) = S(U \cap V \cap W, U \cap V). \end{aligned}$$

(II) $S(W, V \cap W) = S(W, U \cap V)R(U \cap V, V \cap W)$?

$$\begin{aligned} S(W, V \cap W) &= S(W, U \cap V \cap W)R(U \cap V \cap W, V \cap W) \\ &= S(W, U \cap V)R(U \cap V, U \cap V \cap W)R(U \cap V \cap W, V \cap W) \\ &= S(W, U \cap V)R(U \cap V, V \cap W). \end{aligned}$$

This finally completes the proof of the RS Triangle Identities.

3. The isomorphisms Φ_A and Θ_A. Our first goal is to give a formula in (3.5) and (3.8) for the homomorphism

$$\Phi_A\colon \mathrm{Aut}(\sigma_A) \to \pi_1(\mathrm{RS}(\mathscr{E}), A).$$

If the square matrices A and B lie in the same component of $\mathrm{RS}(\mathscr{E})$, let

$$\pi_1(\mathrm{RS}(\mathscr{E}); A, B)$$

denote the set of homotopy classes of paths starting at A and ending at B. Concatenation of paths gives a pairing

$$\pi_1(\mathrm{RS}(\mathscr{E}); A, B) \times \pi_1(\mathrm{RS}(\mathscr{E}); B, C) \to \pi_1(\mathrm{RS}(\mathscr{E}); A, C)$$

denoted by "$*$". When $A = B$ we just have the fundamental group $\pi_1(\mathrm{RS}(\mathscr{E}), A)$.

Let $\mathrm{Isom}(\sigma_A, \sigma_B)$ denote the set of conjugacies $\alpha\colon (X_A, \sigma_A) \to (X_B, \sigma_B)$. Suppose $U \to V$ in P_A with $P = M(U)$ and $Q = M(V)$, and let R and S be defined as in §1 giving a strong shift equivalence $(R, S)\colon P \to Q$. Let α be in $\mathrm{Isom}(\sigma_A, \sigma_B)$. Let $U' = \alpha(U)$ and $V' = \alpha(V)$ in P_B. Let $P' = M(U')$ and $Q' = M(V')$. We have $U' \to V'$ in P_B. Let R' and S' be the corresponding matrices giving a strong shift equivalence from P' to Q'. These matrices satisfy the identities

$$P' = \alpha P \alpha^{-1}, \qquad Q' = \alpha Q \alpha^{-1},$$
$$R' = \alpha R \alpha^{-1}, \qquad S' = \alpha S \alpha^{-1},$$

which translate into the following diagrams of triangles in $\mathrm{RS}(\mathscr{E})$:

(3.1)
$$\begin{tikzcd}
P \arrow[r, "{(R,S)}"] \arrow[d, "{(\alpha^{-1},\alpha P)}"'] & Q \arrow[d, "{(\alpha^{-1},\alpha Q)}"] \\
P' \arrow[r, "{(R',S')}"'] \arrow[ur, "{(\alpha R,S\alpha^{-1})}"'] & Q'
\end{tikzcd}$$

(3.2)
$$\begin{tikzcd}
P \arrow[r, "{(R,S)}"] \arrow[d, "{(\alpha^{-1},\alpha P)}"'] \arrow[dr, "{(R\alpha^{-1},\alpha S)}"] & Q \arrow[d, "{(\alpha^{-1},\alpha Q)}"] \\
P' \arrow[r, "{(R',S')}"'] & Q'
\end{tikzcd}$$

Incidentally, these two diagrams form the boundary of the tetrahedron $\langle P, P', Q, Q' \rangle$ in $\mathrm{RS}(\mathscr{E})$.

LEMMA 3.3. (a) $\gamma(1, A) = 1$ *in* $\pi_1(RS(\mathscr{E}), A)$.
(b) *If* $\gamma \in \pi_1(RS(\mathscr{E}); A, B)$, *then* $\gamma(A, 1) * \gamma = \gamma * \gamma(B, 1)$.
(c) $\gamma(R, S) * \gamma(S, R) = \gamma(A, 1)$.
(d) $\gamma(\alpha^{-1}, \alpha P)^{-1} = \gamma(\alpha, P\alpha^{-1})$.

Proof of (a). The triangle

$$\begin{tikzcd}
& A \arrow[dr, "{(1,A)}"] & \\
A \arrow[ur, "{(1,A)}"] \arrow[rr, "{(1,A)}"'] & & A
\end{tikzcd}$$

shows that $\gamma(1, A) * \gamma(1, A) = \gamma(1, A)$. Now cancel.

Proof of (b) *and* (c). Since γ is a product of paths $\gamma(R, S)$ and their inverses, it suffices to consider the case $\gamma = \gamma(R, S)$. The formula (b) then follows from the diagram

$$\begin{tikzcd}
A \arrow[r, "{(A,1)}"] \arrow[d, "{(R,S)}"'] & A \arrow[d, "{(R,S)}"] \\
B \arrow[ur, "{(S,R)}"] \arrow[r, "{(B,1)}"'] & B
\end{tikzcd}$$

The formula (c) is a consequence of the top triangle in this diagram.

Proof of (d). This follows from the triangle

$$\begin{tikzcd}
& \alpha P \alpha^{-1} \arrow[dr, "{(\alpha, P\alpha^{-1})}"] & \\
P \arrow[ur, "{(\alpha^{-1}, \alpha P)}"] \arrow[rr, "{(1,P)}"'] & & P
\end{tikzcd}$$

PROPOSITION 3.4. *Let U and V be in P_A with $P = M(U)$ and $Q = M(V)$. Then there is a well defined path $\Gamma(U, V)$ in $\pi_1(\mathrm{RS}(\mathscr{E}); P, Q)$ such that*

$$\Gamma(U, U) = 1,$$
$$\Gamma(U, W) = \Gamma(U, V) * \Gamma(V, W).$$

Moreover, if $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$, then

$$\begin{aligned}\Gamma(\alpha(U), \alpha(V)) &= \gamma(\alpha^{-1}, \alpha P)^{-1} * \Gamma(U, V) * \gamma(\alpha^{-1}, \alpha Q)\\ &= \gamma(\alpha, P\alpha^{-1}) * \Gamma(U, V) * \gamma(\alpha^{-1}, \alpha Q).\end{aligned}$$

Proof. For the special case $U \to V$ in P_A, define $\Gamma(U, V) = \gamma(R, S)$. In general, choose a path from U to V in P_A which is concatenation of edges $\langle U_{i-1}, U_i\rangle^{\varepsilon(i)}$ for $i = 1, \dots, n$ where $\varepsilon(i) = \pm 1$. Then define

$$\Gamma(U, V) = \Gamma(U_0, U_1)^{\varepsilon(1)} * \Gamma(U_1, U_2)^{\varepsilon(2)} * \cdots * \Gamma(U_{n-1}, U_n)^{\varepsilon(n)}. \tag{3.5}$$

From the definition of Γ, we see that $\Gamma(U, U) = \gamma(1, A) = 1$ by (a) of (3.3). It is also clear that $\Gamma(U, V)*\Gamma(V, W) = \Gamma(U, W)$ provided Γ is independent of the path chosen in P_A from U to V. But this follows immediately from the RS Triangle Identities (1.2) and simple connectivity of P_A proved in **[8]**. The formula for $\Gamma(\alpha(U), \alpha(V))$ follows from the diagram (3.1) and (d) of (3.3).

PROPOSITION 3.6. *There is a map $\Phi = \Phi(A, B)$ from* $\mathrm{Isom}(\sigma_A, \sigma_B)$ *to $\pi_1(RS(\mathscr{E}); A, B)$ such that if $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$ and $\beta \in \mathrm{Isom}(\sigma_B, \sigma_C)$ then*

$$\Phi(\beta\alpha) = \Phi(\alpha) * \Phi(\beta).$$

Considering $\sigma_A \in \mathrm{Aut}(\sigma_A) = \mathrm{Isom}(\sigma_A, \sigma_A)$ we have

$$\Phi(\sigma_A) = \gamma(A, 1).$$

From this result we then obtain the homomorphism

$$\Phi_A\colon \mathrm{Aut}(\sigma_A) \to \pi_1(\mathrm{RS}(\mathscr{E}), A) \tag{3.7}$$

by taking $A = B$ and setting $\Phi_A(\alpha) = \Phi(\alpha^{-1})$.

Proof. Let $\alpha \in \mathrm{Isom}(\alpha_A, \sigma_B)$. We then have

$$(\alpha^{-1}, \alpha A)\colon A = M(U^A) \to M(\alpha(U^A)) = \alpha A \alpha^{-1}.$$

Define

$$\Phi(\alpha) = \gamma(\alpha^{-1}, \alpha A) * \Gamma(\alpha(U^A), U^B). \tag{3.8}$$

Now let $\alpha \in \operatorname{Isom}(\sigma_A, \sigma_B)$ and $\beta \in \operatorname{Isom}(\sigma_B, \sigma_C)$. Then

$$\begin{aligned}\Phi(\beta\alpha) &= \gamma(\alpha^{-1}\beta^{-1}, \alpha\beta A) * \Gamma(\beta\alpha(U^A), U^C) \\ &= \gamma(\alpha^{-1}, \alpha A) * \gamma(\beta^{-1}, \alpha\beta A\alpha^{-1}) * \Gamma(\beta\alpha(U^A), \beta(U^B)) \\ &\quad * \Gamma(\beta(U^B), U^C).\end{aligned}$$

From (3.4) we see that

$$\begin{aligned}&\Gamma(\beta\alpha(U^A), \beta(U^B)) \\ &\qquad = \gamma(\beta^{-1}, \alpha\beta A\alpha^{-1})^{-1} * \Gamma(\alpha(U^A), U^B) * \gamma(\beta^{-1}, \beta B).\end{aligned}$$

Substituting and simplifying gives

$$\begin{aligned}\Phi(\beta\alpha) &= \gamma(\alpha^{-1}, \alpha A) * \Gamma(\alpha(U^A), U^B) * \gamma(\beta^{-1}, \beta B) * \Gamma(\beta(U^B), U^C) \\ &= \Phi(\alpha) * \Phi(\beta).\end{aligned}$$

To compute $\Phi(\sigma_A)$, recall that $\sigma_A(U^A) \to U^A$. We then have the triangle

$$\begin{tikzcd} A \arrow[rr, "{(A,1)}"] \arrow[dr, "{(\alpha^{-1},\alpha A)}"'] & & A \\ & \alpha A\alpha^{-1} \arrow[ur, "{(\alpha A,\alpha^{-1})}"'] & \end{tikzcd}$$

where α is the bijection between the states U^A and the states of $\sigma_A(U^A)$ induced by σ_A. This shows

$$\Phi(\sigma_A) = \gamma(\alpha^{-1}, \alpha A) * \Gamma(\sigma_A(U^A), U^A) = \gamma(A, 1).$$

This completes the construction of Φ_A. Next consider the homomorphism

$$\Theta_A\colon \pi_1(\mathrm{RS}(\mathscr{E}), A) \to \operatorname{Aut}(\sigma_A).$$

The formula for Θ_A was given in (1.8), and as discussed in §1 it only remains to give the

Proof of (1.9). Suppose that $(R, S)\colon A \to B$. The equation $c(R, S)c(S, R) = \sigma_A$ implies $c(R, S)^{-1} = \sigma_A^{-1}c(S, R)$. If $y = \{y_n\} \in X_B$, we can therefore characterize $x = c(R, S)^{-1}(y)$ to be the unique point $x = \{x_n\} \in X_A$ such that

$$S(y_{n-1}, x_n)R(x_n, y_n) = 1$$

for all n.

First we will show that the RS Triangle Identities imply

$$c(R_2, S_2)c(R_1, S_1) = c(R_3, S_3).$$

Let $x = \{x_n\} \in X_M$ with $y = c(R_1, S_1)(x) = \{y_n\}$ and $z = c(R_2, S_2)(y) = \{z_n\}$. Let $w = c(R_3, S_3)^{-1}(z) = \{w_n\}$. We want to show that $x_n = w_n$ for all n. We know that w_n is the unique state satisfying

$$S_3(z_{n-1}, w_n)R_3(w_n, z_n) = 1.$$

Let k be the unique state such that

$$S_3(z_{n-1}, w_n)R_1(w_n, k)R_2(k, z_n) = S_2(z_{n-1}, k)R_2(k, z_n) = 1.$$

Then $k = y_n$. We also have

$$R_3(x_{n-1}, z_{n-1}) = R_1(x_{n-1}, y_{n-1})R_2(y_{n-1}, z_{n-1}) = 1.$$

Thus

$$R_2(y_{n-1}, z_{n-1})S_3(z_{n-1}, w_n) = S_1(y_{n-1}, w_n) = 1$$

and

$$S_1(y_{n-1}, w_n)R_1(w_n, y_n) = 1.$$

We conclude that $w_n = x_n$.

For the converse, remember that each matrix in $\mathscr{E}$ has at least one non-zero entry in each row and in each column.

The equation $S_3R_1 = S_2$: The left hand side LHS is a zero-one matrix, because $S_3R_1R_2 = S_3R_3 = Q$ is zero-one. Fix a pair of indices (p, l) such that $S_2(p, l) = 1$. Choose points $z = \{z_n\} \in X_Q$ and $y = \{y_n\} \in X_P$ such that $z_0 = p$, $y_1 = l$, and $z = c(R_2, S_2)(y)$. Then choose $x = \{x_n\} \in X_M$ with $y = c(R_1, S_1)(x)$. In particular, we have $z = c(R_2, S_2)c(R_1, S_1)(x) = c(R_3, S_3)(x)$. Then

$$1 = S_3(z_0, x_1)R_3(x_1, z_1) = S_3(z_0, x_1)R_1(x_1, y_1)R_2(y_1, z_1)$$

and $S_3(p, x_1)R_1(x_1, l) = 1$. Thus LHS $\neq 0$. On the other hand, suppose for an index i that $S_3(p, i)R_1(i, l) = 1$. We must show $S_2(p, l) = 1$. Choose a point z in X_Q with $z_0 = p$ and $R_2(l, z_1) = 1$. Then choose a point x in X_M with $z = c(R_3, S_3)(x)$ and $x_1 = i$. Consider $y = c(R_1, S_1)(x)$. Then $z = c(R_2, S_2)(y)$. We claim that $y_1 = l$. Recall $R_1(x_1, y_1)R_2(y_1, z_1) = 1$. But l is the unique state with $R_3(i, z_1) = R_1(i, l)R_2(l, z_1) = 1$. Hence $y_1 = l$. Since $y = c(R_2, S_2)^{-1}(z)$, we must have $S_2(p, l)R_2(l, z_1) = 1$ and so $S_2(p, l) = 1$.

The equation $R_2S_3 = S_1$: The left hand side LHS is a zero-one matrix, because $R_1R_2S_3 = R_3S_3 = M$ is zero-one. Suppose RHS $\neq 0$ and fix a pair of indices (k, i) such that $S_1(k, i) = 1$. Choose x in X_M and y in X_P with $x_1 = i$, $y_0 = k$, and $y = c(R_1, S_1)(x)$. Let $z = c(R_2, S_2)(y) = c(R_3, S_3)(x)$. Then $R_3(x_0, z_0) = R_1(x_0, y_0)R_2(y_0, z_0) = 1$ and $S_3(z_0, x_1) = 1$. Therefore $R_2(k, z_0)S_3(z_0, i) = 1$, which says LHS $\neq 0$. Conversely, assume LHS $\neq 0$ and choose a state p with $R_2(k, p)S_3(p, i) = 1$. Choose an x in X_M such that $x_1 = i$ and $R_1(x_0, k) = 1$. Then $R_1(x_0, k)R_2(k, p)S_3(p, i) = R_3(x_0, p)S_3(p, i) = 1$. Let $y = c(R_1, S_1)(x)$ and $z = c(R_2, S_2)(y) = c(R_3, S_3)(x)$. Then $z_0 = p$ and y_0 must satisfy $R_1(x_0, y_0)S_1(y_0, x_1) = 1$. We also have $R_1(x_0, y_0)R_2(y_0, z_0)S_3(z_0, x_1) = 1$. Thus $R_1(x_0, y_0)R_2(y_0, z_0) = 1$, so $y_0 = k$. This gives $S_1(k, i) = 1$.

Proof that Φ_A and Θ_A are isomorphisms.

Step 1. $\Theta_A\Phi_A = \mathrm{Id}$.

Let $U = \{U_k\}$ be Markov partition in P_A with $P = M(U)$. We have the well-known isomorphism $I = I(U, U^A)$ in $\mathrm{Isom}(\sigma_A, \sigma_P)$ defined on x in X_A by the condition

$$I(x)_n = k \quad \text{if and only if} \quad \sigma_A^n(x) \in U_k. \tag{3.9}$$

See [8] for example. Now consider an isomorphism $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$ and let $U \in P_A$ and $V \in P_B$ where $\alpha(U) = V$. Let $U = \{U_e\}$ and $V = \{V_f\}$ where e and f run through indexing sets E and F respectively. The homeomorphism α induces a bijection from E to F. Let $P = M(U)$ and $Q = M(V)$. Then $Q = \alpha P\alpha^{-1}$ and $(\alpha^{-1}, \alpha P): P \to Q$.

LEMMA 3.10. $I(V, U^B)\alpha = c(\alpha^{-1}, \alpha P)I(U, U^A)$.

Proof. Let $I_U = I(U, U^A)$, $I_V = I(V, U^B)$, and $c = c(\alpha^{-1}, \alpha P)$. Remember that c is the one-block map taking $y = \{y_n\}$ with $y_n \in E$ to $\alpha(y) = \{\alpha(y_n)\}$ with $\alpha(y_n) \in F$. Since α, c, I_U, and I_V are shift commuting, it suffices to show that

$$I_V(\alpha(x))_0 = c(I_U(x))_0$$

for all $x \in X_A$. Let $y = I_U(x) = \{y_n\}$ with $y_n \in E$. Let $e = y_0$. Then $c(I_U(x))_0 = f$ if and only if $\alpha(e) = f$. That is, if and only if $\alpha(U_e) = V_f$ where $x \in U_e$. On the other hand, suppose

$h = I_V(\alpha(x))_0$. Then $\alpha(x) \in V_h$. Let $g \in E$ be the unique index where $x \in U_g$ with $\alpha(U_g) = V_h$. Then $e = g$ and so $f = h$.

Let $U \to V$ in P_A. Let $R = R(U, V)$ and $S = S(V, U)$.

LEMMA 3.11. $I(V, U^A) = c(R, S)I(U, U^A)$.

Since $I(U^A, U^A) = 1$, we get $I(U, U^A) = c(R, S)$ whenever $U^A \to U$ in P_A.

Proof. Let $I = I(U, U^A)$, $J = I(V, U^A)$, and $c = c(R, S)$. It suffices to show that

$$J(x)_0 = cI(x)_0$$

for all $x \in X_A$. Let $y = I(x) = \{y_n\}$ with $y_0 = k$ and $y_1 = l$. This means $x \in U_k$ and $\sigma_A(x) \in U_l$, and consequently $U_k \cap \sigma_A^{-1}(U_l) \neq \varnothing$. Using $V < U \cap \sigma_A^{-1}(U)$, let V_b be the unique set in V with $V_b \supset U_k \cap \sigma_A^{-1}(U_l)$. Then $J(x_0) = b$. But also we have $U_k \cap V_b \neq \varnothing$. Equivalently, $R(k, b)S(b, l) = 1$, which implies $c(y)_0 = b$.

We are now ready to prove $\Theta_A\Phi_A = \mathrm{Id}$. Recall that $\Phi_A(\alpha) = \Phi(\alpha^{-1})$. We will show more generally for $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$ that $\Theta\Phi(\alpha) = \alpha^{-1}$ where

$$\Phi\colon \mathrm{Isom}(\sigma_A, \sigma_B) \to \pi_1(\mathrm{RS}(\mathscr{E}); A, B)$$

is defined by (3.8), and

$$\Theta\colon \pi_1(\mathrm{RS}(\mathscr{E}); A, B) \to \mathrm{Isom}(\sigma_B, \sigma_A)$$

is defined on

$$\gamma = \prod_{i=1}^{n} \gamma(R_i, S_i)^{\varepsilon_i}$$

by the formula

$$\Theta(\gamma) = \prod_{i=1}^{n} c(R_i, S_i)^{-\varepsilon_i}. \tag{3.12}$$

Let $\alpha \in \mathrm{Isom}(\sigma_A, \sigma_B)$ and let

$$\alpha(U^A) = V_0 \to V_1 \leftarrow \cdots \to V_{n-1} \leftarrow V_n = U^B$$

be the path in P_B as in the definition (3.8). For $0 \leq k \leq n$, let $A_k = M(V_k)$. For $1 \leq k \leq n$, $R_k = R(V_{k-1}, V_k)$, $S_k = S(V_k, V_{k-1})$, and $\varepsilon_k = +1$ when $V_{k-1} \to V_k$, and let $R_k = R(V_k, V_{k-1})$, $S_k = S(V_{k-1}, V_k)$, and $\varepsilon_k = -1$ when $V_k \to V_{k-1}$. Then (3.10) and (3.11) show that

$$\begin{aligned}\alpha^{-1} &= c(\alpha^{-1}, \alpha A)^{-1} \prod_{k=1}^{n} c(R_k, S_k)^{-\varepsilon_k} \\ &= \Theta\left(\gamma(\alpha^{-1}, \alpha A) \prod_{k=1}^{n} \gamma(R_k, S_k)^{\varepsilon_k}\right) = \Theta\Phi(\alpha).\end{aligned}$$

Step II: $\Phi_A \Theta_A = \mathrm{Id}$.

Assume $(R, S)\colon A \to B$ and let $c = c(R, S)\colon X_A \to X_B$. Let $U^A = \{U_i^A\}$ and $U^B = \{U_k^B\}$ be the standard Markov partitions for σ_A and σ_B respectively. Let $U = \{U_k\} = \{c^{-1}(U_k^B)\} = c^{-1}(U^B)$ and $V = \{V_i\} = \{c(U_i^A)\} = c(U^A)$.

LEMMA 3.13. *$U^A \to U$ in P_A and $V \to U^B$ in P_B.*

Proof. It clearly suffices to verify $U^A \to U$. The definition of $c(R, S)$ shows that U_k consists of those points x in X_A such that $R(x_0, k)S(k, x_1) = 1$. In other words,

$$U_k = \bigcup U_i^A \cap \sigma_A^{-1}(U_j^A)$$

where the pairs (i, j) run over those states with $R(i, k)S(k, j) = 1$. For a given set $U_i^A \cap \sigma_A^{-1}(U_j^A)$ in $U^A \cap \sigma_A^{-1}(U^A)$, let k be the unique state such that $R(i, k)S(k, j) = 1$. Then $U_i^A \cap \sigma_A^{-1}(U_j^A) \subset U_k$. This shows that $U^A \underset{+}{\to} U^A \cap U$. Next we verify $U \underset{-}{\to} U^A \cap U$. Fix a pair of states (k, l) and write

$$U_k = \bigcup U_a^A \cap \sigma_A^{-1}(U_b^A), \qquad U_l = \bigcup U_c^A \cap \sigma_A^{-1}(U_d^A).$$

Then

$$\sigma_A(U_k) \cap U_l = \bigcup \sigma_A(U_a^A) \cap U_b^A \cap U_c^A \cap \sigma_A^{-1}(U_d^A)$$

where each non-empty term must have $b = c$, $R(a, k)S(k, b) = 1$, and $R(c, l)S(l, d) = 1$. Then $R(k, b)S(b, l) = 1$, and so only one

b can occur. The expression becomes

$$\sigma_A(U_k) \cap U_l = \bigcup \sigma_A(U_a^A) \cap U_b^A \cap \sigma_A^{-1}(U_d^A) \subset U_b^A \cap U_l.$$

This completes the proof of (3.13).

Let $\alpha \in \operatorname{Isom}(\sigma_A, \sigma_B)$ be of the form $\alpha = c(R, S)$ corresponding to $(R, S)\colon A \to B$ and let $f \in \operatorname{Isom}(\sigma_B, \sigma_C)$. Let $U = f\alpha(U^A)$ and $V = f(U^B)$. From (3.13) we see that $U \to V$ in P_B. Let $A' = M(U)$, $B' = M(V)$, $R' = R(U, V)$, and $S' = S(V, U)$. Then

$$\begin{aligned} A' &= f\alpha A\alpha^{-1}f^{-1}, &\qquad B' &= fBf^{-1}, \\ R' &= f\alpha Rf^{-1}, &\qquad S' &= fS\alpha^{-1}f^{-1}, \end{aligned}$$

and moreover there is the following diagram of triangles in $\mathrm{RS}(\mathscr{E})$:

(3.14)
$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "{(R,S)}"] \arrow[d, "{(\alpha^{-1}f^{-1}, f\alpha A)}"'] \arrow[dr, "{(Rf^{-1}, fS)}"] & B \arrow[d, "{(f^{-1}, fB)}"] \\
f\alpha A\alpha^{-1}f^{-1} \arrow[r, "{(f\alpha Rf^{-1}, fS\alpha^{-1}f^{-1})}"'] & fBf^{-1}
\end{tikzcd}$$

Similarly to Step I we will show that $\Phi\Theta(\gamma^{-1}) = \gamma$ for any path $\gamma = \prod \gamma(R_k, S_k)^{\varepsilon_k}$ from $A = A_0$ to $B = A_n$ in $\pi_i(\mathrm{RS}(\mathscr{E}); A, B)$. For $0 \le k \le n-1$, let $\alpha_k \in \operatorname{Isom}(\sigma_{A_k}, \sigma_B)$ be given by the formula

$$\alpha_k = c(R_n, S_n)^{\varepsilon_n} \cdots c(R_{k+1}, S_{k+1})^{\varepsilon_{k+1}}.$$

Let $\alpha_n = \mathrm{id}$. Remember that our convention is to read composition of homeomorphisms between spaces from right to left. Let $V_k = \alpha_k(U^{A_k})$ in P_B for $0 \le k \le n-1$ and let $V_n = U^B$. From (3.1) we know that $V_{k-1} \to V_k$ if $\varepsilon_k = +1$ and $V_k \to V_{k-1}$ if $\varepsilon_k = -1$. Let $B_k = M(V_k)$. Then $B_k = \alpha_k A_k \alpha_k^{-1}$. From (3.14) we have the following two diagrams corresponding respectfully to the parities $\varepsilon_k = +1$ and $\varepsilon_k = -1$:

$$\begin{tikzcd}
A_{k-1} \arrow[rr, "{(R_k, S_k)}"] \arrow[d, "{(\alpha_{k-1}^{-1}, \alpha_{k-1}A_{k-1})}"'] \arrow[drr, "{(R_k\alpha_k^{-1}, \alpha_k S_k)}"] & & A_k \arrow[d, "{(\alpha_k^{-1}, \alpha_k A_k)}"] \\
\alpha_{k-1}A_{k-1}\alpha_{k-1}^{-1} \arrow[rr, "{(\alpha_{k-1}R_k\alpha_k^{-1}, \alpha_k S_k\alpha_{k-1}^{-1})}"'] & & \alpha_k A_k \alpha_k^{-1}
\end{tikzcd}$$

$$\begin{tikzcd}
A_{k-1} \arrow[d, "{(\alpha_{k-1}^{-1}, \alpha_{k-1}A_{k-1})}"'] & & A_k \arrow[ll, "{(R_k, S_k)}"'] \arrow[d, "{(\alpha_k^{-1}, \alpha_k A_k)}"] \arrow[dll, "{(R_k\alpha_{k-1}^{-1}, \alpha_{k-1} S_k)}"'] \\
\alpha_{k-1}A_{k-1}\alpha_{k-1}^{-1} & & \alpha_k A_k \alpha_k^{-1} \arrow[ll, "{(\alpha_{k-1}R_k\alpha_k^{-1}, \alpha_k S_k\alpha_{k-1}^{-1})}"]
\end{tikzcd}$$

These fit together to provide a homotopy in $\mathrm{RS}(\mathscr{E})$ with end points fixed between the original path γ and the path

$$\gamma(\alpha_0^{-1}, \alpha_0 A) * \Gamma(\alpha_0(U^A), U^B) = \Phi(\alpha_0).$$

But $\alpha_0 = \Theta(\gamma^{-1})$. So $\Phi\Theta(\gamma^{-1}) = \gamma$. This finally completes the proof that Φ and Θ are isomorphisms.

REMARK 3.15. The CW-complex $\mathrm{RS}(\mathscr{E})$ is locally compact. This follows from (3.13), the observation in [**8**] that P_A is a locally finite simplicial complex, and the fact that there are only finitely many $(R, S)\colon P \to P$ for a given square zero-one matrix P.

4. $\mathrm{RS}(\mathscr{E})$ **is aspherical.** We now give the proof of (1.13). Let $\mathrm{RS}(\mathscr{E})_A$ denote the component of $\mathrm{RS}(\mathscr{E})$ containing A. The universal cover $\widetilde{\mathrm{RS}}_A$ of $\mathrm{RS}(\mathscr{E})_A$ is the realization of the following simplicial set: The k-simplices are pairs (γ, Δ) where

(i) Δ is a k-simplex of $\mathrm{RS}(\mathscr{E})_A$ given by the data $\langle A_0, \ldots, A_n\rangle$ and $(R_{ij}, S_{ji})\colon A_i \to A_j$ as in (1.4) and

(ii) γ is a homotopy class of paths from the base point A to A_0.

The ith face operator of $\widetilde{\mathrm{RS}}$ acts on Δ just as it does in RS. For $1 \le i \le n$, it leaves γ unchanged, and for $i = 0$ it changes γ to

$\gamma * \gamma(R_1, S_1)$. Similarly for the degeneracy operators. The covering map

$$\widetilde{\mathrm{RS}}(\mathscr{E})_A \to \mathrm{RS}(\mathscr{E})_A$$

is induced by the map of simplicial sets taking (γ, Δ) to Δ.

For each k-simplex (γ, Δ), let $\alpha_0 = \Theta(\gamma)$ and for $1 \leq k \leq n$, let $\alpha_k = \Theta(\gamma * \gamma(R_{0k}, S_{k0}))$. Let $U_k = \alpha_k(U^{A_k}) \in P_A$. The discussion in §3 shows $\langle U_0, \dots, U_k \rangle$ is a k-simplex in P_A. Moreover, the correspondence taking (γ, Δ) to $\langle U_0, \dots, U_k \rangle$ is map of simplicial sets. Using (3.14) we obtain a homotopy commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{RS}}(\mathscr{E})_A \arrow[rr] \arrow[dr] & & P_A \arrow[dl] \\ & \mathrm{RS}(\mathscr{E})_A & \end{tikzcd}$$

Since P_A is contractable **[8]**, we see that

$$\pi_n(\widetilde{\mathrm{RS}}(\mathscr{E})_A, A) \to \pi_n(\mathrm{RS}(\mathscr{E})_A, A)$$

is the zero homomorphism. On the other hand, it is also an isomorphism for $n \geq 2$ because $\widetilde{\mathrm{RS}}(\mathscr{E})_A$ is the universal cover.

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