

Pacific Journal of Mathematics

RADON-NIKODÝM PROBLEM FOR THE VARIATION OF A VECTOR MEASURE

LILIANA JANICKA

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We consider the problem of representing the variation $|m|$ of a vector measure m as an integral in the Dinculeanu sense with respect to M .

Throughout this paper (S, Σ) denotes a measurable space. If X is a Banach space, we write X^* for the dual space and K_X for the closed unit ball of X . We use brackets $\langle \cdot, \cdot \rangle$ for the pairing between a Banach space and its dual. Let $m: \Sigma \rightarrow X$ be a vector measure with finite variation $|m|$. Recall that a strongly measurable function $f: S \rightarrow X^*$ is said to be integrable in Dinculeanu's sense if there exists a sequence $\{f_n\}_{n \geq 1}$ of simple functions converging $|m|$ -a.e. to f such that

$$\lim_{n, p \rightarrow \infty} \int \|f_n - f_p\| d|m| = 0,$$

i.e., the function $\|f\|$ is $|m|$ -integrable. Further, $D\text{-}\int_A f dm$ denotes the Dinculeanu integral of the function f with respect to m over the set A .

It was proved in [2] that for every $\varepsilon > 0$ there exists an X^* -valued strongly measurable function f defined on the set S such that $\|f\| \leq 1 + \varepsilon |m|$ -a.e. and $|m|(A) = D\text{-}\int_A f dm$ for each $A \in \Sigma$. We are interested in the following question: For which Banach spaces may we obtain the preceding equality when we insist that $\|f\| = 1$ a.e. $|m|$?

We begin our investigation by introducing the following property of Banach spaces. The Banach space X has property (DV) if for every equivalent norm on x , for every measurable space (S, Σ) for every equivalent norm on X and every vector measure $m: \Sigma \rightarrow X$ with finite variation $|m|$ there exists a strongly measurable function $f: S \rightarrow X^*$ with $\|f\| = 1$ $|m|$ -a.e. such that $|m|(A) = D\text{-}\int_A f dm$ for each $A \in \Sigma$.

THEOREM 1. *If both X and X^* have the Radon-Nikodym Property, then X has property (DV).*

Proof. Let (S, Σ) be a measurable space and $m: \Sigma \rightarrow X$ be a measure with finite variation $|m|$. Since X has RNP, there exists a strongly measurable function $f: S \rightarrow X$ such that $m(A) = \text{B-} \int_A f dm$ for each $A \in \Sigma$. ($\text{B-} \int_A f dm$ denotes the Bochner integral of f with respect to m over the set A .) For every $x \in X$ let

$$G(x) = \{x^* \in K_{X^*}: \|x^*\| = 1 \text{ and } \langle x, x^* \rangle = \|x\|\}.$$

Then G is a set-valued mapping, and $G(x)$ is non-empty and w^* -compact for every $x \in X$. We now see that G is upper semi-continuous if X is endowed with the norm topology and K_{X^*} is endowed with the w^* -topology. Indeed, let H be a w^* -closed subset of K_{X^*} . It suffices to show that

$$\{x \in X: G(x) \cap H \neq \emptyset\}$$

is norm closed in X . Let $\|x_n - x\| \rightarrow 0$, and suppose that $G(x_n) \cap H \neq \emptyset$, i.e., for every n there exists $x_n^* \in H$ such that $\|x_n^*\| = 1$ and $\|x_n\| = \langle x_n, x_n^* \rangle$. Let x^* be any w^* -cluster point of $\{x_n^*\}$. It is not difficult to see that for every $\varepsilon > 0$ we have $|\|x\| - \langle x, x^* \rangle| < \varepsilon$; i.e. the set is norm closed. Following [7, Theorem 8], we see that the set-valued mapping G has a selector which is of the first Baire class when X^* is equipped with the norm topology. Then using [1, Lemma 4.11.13] we see that the function $h: S \rightarrow X^*$ defined by $h = g \circ f$ is strongly measurable. (The preceding lemma and the fact that f is strongly measurable ensures that h has essentially separable range; the strong measurability of f and the fact that g belongs to the first Baire class ensures that $h^{-1}(u)$ is an element of the $|m|$ -completion of Σ for every set u which is open in the norm topology on X^* .) But for every $A \in \Sigma$ we have

$$|m|(A) = \int_A \|f\| d|m|.$$

Therefore following [4, Theorem 3.4.II], we have

$$\begin{aligned} |m|(A) &= \int_A \|f\| d|m| = \int_A \langle f(s), h(s) \rangle d|m|(s) \\ \S &= \text{D-} \int_A h df|m| = \text{D-} \int_A h dm. \end{aligned}$$

PROPOSITION 2. *If X has property (DV), then every subspace Y of X has property (DV).*

Proof. Let $m: \Sigma \rightarrow Y$ be a vector measure with $|m| < \infty$. Since X has property (DV), there exists a strongly measurable function $f: S \rightarrow X^*$ with $\|f(x)\| = 1$ $|m|$ -a.e. such that $|m|(A) = D\text{-}\int_A f dm$ for each $A \in \Sigma$. Define $g: S \rightarrow Y^*$ by $g(s) = f(s)|_{Y^*}$ (the restriction of $f(s)$ to Y). Of course g is strongly measurable and $\|g(s)\| \leq \|f(s)\| = 1$. For every $A \in \Sigma$ we have $D\text{-}\int_A g dm = D\text{-}\int_A f dm$ since m takes its values in Y . But

$$|m|(A) = D\text{-}\int_A f dm = D\text{-}\int_A g dm \leq \int_A \|g\| d|m| \leq |m|(A);$$

therefore $\|g(s)\| = 1$ $|m|$ -a.e.

PROPOSITION 3. *Banach spaces l_1 and c_0 do not have property (DV).*

Proof. Let (I, \mathcal{B}) be the unit interval with the Borel σ -algebra.

(1) For $A \in \mathcal{B}$ define m by $m(A) = (\int_A (1/2^n) r_n(t) dt)_{n=1}^\infty$, where r_n denotes the n th Rademacher function. Then m is a vector measure with values in l_1 such that $|m| = \lambda$, where λ is Lebesgue measure. (It is enough to verify this last equality on intervals of the form $[1/2^i, 1/2^{i-1})$.) Suppose there exists a strongly measurable function $f: I \rightarrow l_\infty$, $f(t) = (f_n(t))$, such that $\|f(t)\| = 1$ λ -a.e. and $|m|(A) = D\text{-}\int_A f dm$ for each A . Because of the definition of m , we have

$$|m|(A) = \int_A \sum_{n=1}^\infty f_n(t) (1/2^n) r_n(t) dt.$$

In particular, for $A = [0, 1]$ we have $\sum_{n=1}^\infty f_n(t) (1/2^n) r_n(t) = 1$ λ -a.e. Further, it is easy to see that $(f_n(t)) = (r_n(t))$ is the unique element of l_∞ which satisfies the preceding equality. But the function $t \rightarrow (r_n(t))$ from I to l_∞ is not weakly measurable [9].

(2) For $A \in \mathcal{B}$ define m by $m(A) = (\int_A (n/n+1) r_n(t) dt)_{n=1}^\infty$. It is easy to verify that m is a vector measure with values in c_0 and $|m| = \lambda$. (The last statement follows from the equality $\sup_n (n/n+1) r_n(t) = 1$.) Assume there exists a strongly measurable function $f: I \rightarrow l_1$, $f(t) = (f_n(t))$ with $\|f(t)\| = \sum_{n=1}^\infty |f_n(t)| = 1$ λ -a.e. such that $|m|(A) = D\text{-}\int_A f dm$ for every $A \in \mathcal{B}$. Then for $A = [0, 1]$ we have

$$1 = \int_0^1 \sum_{n=1}^\infty f_n(t) (n/n+1) r_n(t) dt,$$

i.e., $\sum_{n=1}^{\infty} f_n(t)(n/n+1)r_n(t) = 1$ λ -a.e. But this is impossible since for every n we have

$$f_n(t)(n/n+1)r_n(t) \leq |f_n(t)(n/n+1)r_n(t)| < |f_n(t)|.$$

REMARK 1. Propositions 2 and 3 show that none of the assumptions in Theorem 1 can be omitted. Namely, l_1 has RNP, c_0 does not have RNP, and c_0 does not have (DV). Similarly, l_1 has RNP, l_{∞} does not have RNP, and l_1 does not have (DV).

REMARK 2. Since c_0 does not have property (DV) and l_1 has RNP, we note that (1) and (2) of the theorem in [3] are, in fact, not equivalent. The difficulty with the proof of this equivalence occurs when the author concludes that the w^* -cluster point of a sequence of strongly measurable functions is w^* -measurable. Indeed, it is well known that every pointwise cluster point of the sequence of Rademacher functions is not Lebesgue measurable. We note that there is also a difficulty with the proof that (3) \Rightarrow (1) in [3]. The author makes strong use of this Lemma 1 in this proof, and in the proof of Lemma 1 he concludes that if X^* is not separable, then $\bigcap \ker\{x_j^*\} \neq \{\theta\}$ when the intersection is taken over a countable set of indices. However, if $X = l_1$, then X^* is not separable, but it does have a countable total subset. In fact, we note that this formulation of Lemma 1 is incorrect. To see this, let X be separable and let B be a countable subset of smooth points of the unit sphere which is dense in the unit sphere (Mazur's theorem provides us with the set B). If there exist nets $\{x_{\alpha}\}_{\alpha < \Omega} \subset B$ and $\{x_{\alpha}^*\}_{\alpha < \Omega} \subset S(X^*)$, with $\langle x_{\alpha}, x_{\alpha}^* \rangle = 1$ and $\|x_{\alpha} - x_{\beta}\| > 0$ as required in Lemma 1 of [3], then we contradict the smoothness of x_{α} for some α . Further, Theorem 5.6 of [8] shows that Lemma 2 is also incorrect as stated.

We are able to deduce a weaker version of Debieve's conjecture, however. Using the fact that X^* has the weak RNP whenever l_1 does not embed in X [6]—and the results of this paper—we obtain the following result.

COROLLARY. *If X has property (DV), then X^* has the weak RNP.*

Unfortunately, we are not able to decide if X^* must have RNP whenever X has property (DV).

Acknowledgment. I am very grateful to Professor C. Ryll-Nardzewski and to my colleagues participating in his seminar for helpful comments.

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Received September 19, 1988.

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