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# DIAGONALIZING PROJECTIONS IN MULTIPLIER ALGEBRAS AND IN MATRICES OVER A $C^*$-ALGEBRA

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**Assume that $\mathscr{A}$ is a $C^*$-algebra with the FS property ([3] and [16]). We prove that every projection in $M_n(\mathscr{A})$ $(n \geq 1)$ or in $L(\mathscr{H}_{\mathscr{A}})$ is homotopic to a projection whose diagonal entries are projections of $\mathscr{A}$ and off-diagonal entries are zeros. This yields partial answers for Questions 7 and 8 raised by M. A. Rieffel in [18]. If $\mathscr{A}$ is $\sigma$-unital but non-unital, then every projection in the multiplier algebra $M(\mathscr{A})$ is unitarily equivalent to a diagonal projection, and homotopic to a block-diagonal projection with respect to an approximate identity of $\mathscr{A}$ consisting of an increasing sequence of projections. The unitary orbits of self-adjoint elements of $\mathscr{A}$ and $M(\mathscr{A})$ are also considered.**

**0. Introduction.** It is well known that a projection in $M_n(\mathbb{C})$ or in $L(\mathscr{H})$ is homotopic to a diagonal projection whose diagonal entries are either 1 or 0, where $M_n(\mathbb{C})$ is the algebra consisting of $n \times n$ scalar matrices and $L(\mathscr{H})$ is the algebra consisting of bounded operators on a separable Hilbert space $\mathscr{H}$. The following natural question comes up: if $\mathbb{C}$ is replaced by a $C^*$-algebra $\mathscr{A}$, is every projection in $M_n(\mathscr{A})$ or $L(\mathscr{H}_{\mathscr{A}})$ homotopic to a diagonal projection whose diagonal entries are projections of $\mathscr{A}$ and off-diagonal entries are zeros? Here $M_n(\mathscr{A})$ is the $C^*$-algebra of $n \times n$ matrices over $\mathscr{A}$ and $L(\mathscr{H}_{\mathscr{A}})$ can be regarded as bounded infinite matrices over $\mathscr{A}$ whose adjoints exist (see §1 for a more precise description). Certainly, diagonalizing projections of $M_n(\mathscr{A})$ for $n \geq 1$ would yield information about $K_0(\mathscr{A})$ (here diagonalizing projections in the sense of Murray-von Neumann is enough for this purpose).

Concerning the matrix algebra $M_n(\mathscr{A})$, R. V. Kadison proved (**[13]** and **[14]**) that if $\mathscr{A}$ is a von Neumann algebra, then every normal element in $M_n(\mathscr{A})$ is unitarily equivalent to a diagonal normal matrix over $\mathscr{A}$. Consequently, every projection in $M_n(\mathscr{A})$ is homotopic to a diagonal projection, since the unitary group of a von Neumann algebra is connected. In general, we certainly do not expect a positive answer for the question if $\mathscr{A}$ is an arbitrary $C^*$-algebra. K. Grove and

G. K. Pedersen have pointed out ([**11**, 1.3]) that if $\mathscr{A}$ is the algebra $C(S^2)$, the algebra of complex-valued continuous functions on $S^2$, then there exists a projection in $M_2(\mathscr{A})$ which is not unitarily equivalent to any diagonal projection. However, we do expect a positive answer for a large class of $C^*$-algebras.

The author has proved ([**22**]) that if $\mathscr{A}$ is a $C^*$-algebra with FS, then every projection in $M_n(\mathscr{A})$ or in $L(\mathscr{H}_{\mathscr{A}})$ is Murray-von Neumann equivalent to a diagonal projection. In this note, we will strengthen the previous results to unitary equivalence or homotopy. We prove that if $\mathscr{A}$ is a $C^*$-algebra with FS (not necessarily $\sigma$-unital), and if $p$ is a projection of the multiplier algebra $M(\mathscr{A})$, then every projection $q$ of $\mathscr{A}$ is homotopic to a projection $q' = p_1 + p_2$, where $p_1$ is a projection of $p\mathscr{A}p$ and $p_2$ is a projection of $(1-p)\mathscr{A}(1-p)$. As a special case, by induction we conclude that every projection in $M_n(\mathscr{A})$ is homotopic to a diagonal projection. This yields partial answers for Questions 7 and 8 raised by M. A. Rieffel in [**18**]. If $\mathscr{A}$ is $\sigma$-unital and $\{e_n\}$ is a fixed sequence of mutually orthogonal projections of $\mathscr{A}$ such that $\sum_{n=1}^{\infty} e_n = 1$, we prove that every projection in $M(\mathscr{A})$ is unitarily equivalent to a diagonal projection and homotopicto a block-diagonal projection with respect to the decomposition $\sum_{n=1}^{\infty} e_n = 1$. As a consequence, every projection in $L(\mathscr{H}_{\mathscr{A}})$ is unitarily equivalent (and hence homotopic) to a diagonal projection. In addition, the unitary orbits of self-adjoint elements of $\mathscr{A}$ or $M(\mathscr{A})$ are considered.

The class of $C^*$-algebras with FS includes many interesting subclasses of $C^*$-algebras. Obviously, AF algebras, the Calkin algebra, von Neumann algebras and $AW^*$-algebras have FS. The Bunce-Deddens algebras have FS ([**2**]). All purely infinite, simple $C^*$-algebras have FS ([**24**, Part I (1.3)] and [**25**]); in particular, the Cuntz algebras $\mathscr{O}_n$ and $\mathscr{O}_A$, where $2 \le n \le \infty$ and $A$ is an irreducible scalar matrix, have FS. Certain irrational rotation $C^*$-algebras have FS ([**9**]). Many corona and multiplier algebras have FS ([**5**], [**24**, Part I] and [**24**, Part IV]). L. G. Brown and G. K. Pedersen have recently proved ([**5**]) that a $C^*$-algebra $\mathscr{A}$ has FS if and only if $M_n(\mathscr{A})$ has FS for all $n \ge 1$; and $\mathscr{A}$ has FS if and only if $\mathscr{A}$ has real rank zero. In [**21**], [**22**], [**23**] and [**24**] the author has investigated the multiplier and corona algebras of $C^*$-algebras with FS from various angles.

**1. Notations.** If $\mathscr{A}$ is a $C^*$-algebra, we denote the Banach space double dual of $\mathscr{A}$ by $\mathscr{A}^{**}$ and the multiplier algebra of $\mathscr{A}$ by $M(\mathscr{A})$; where $M(\mathscr{A}) = \{m \in \mathscr{A}^{**} : xm, mx \in \mathscr{A}\ \forall x \in \mathscr{A}\}$ ([**1**], [**7**], [**15**], among others).

Let $\mathscr{H}_{\mathscr{A}} = \{\{a_i\} : a_i \in \mathscr{A} \text{ and } \sum_{i=1}^{\infty} a_i^* a_i \text{ converges in norm}\}$. Then $\mathscr{H}_{\mathscr{A}}$ becomes a Hilbert $\mathscr{A}$-module with the $\mathscr{A}$-valued inner product

$$\langle \{a_i\}, \{b_i\} \rangle = \sum_{i=1}^{\infty} a_i^* b_i \quad \text{for all } \{a_i\}, \{b_i\} \in \mathscr{H}_{\mathscr{A}}.$$

We denote by $L(\mathscr{H}_{\mathscr{A}})$ the set of all bounded module maps with an adjoint and by $K(\mathscr{H}_{\mathscr{A}})$ a closed ideal of $L(\mathscr{H}_{\mathscr{A}})$ called the "compact maps"; more precisely, $K(\mathscr{H}_{\mathscr{A}})$ is the norm closure of the set of all "finite rank" module maps, $\{\sum_{i=1}^{n} \theta_{x_i, y_i} : x_i, y_i \in \mathscr{H}_{\mathscr{A}} \text{ and } n \in \mathbb{N}\}$. Here for any pair of elements $x$ and $y$ in $\mathscr{H}_{\mathscr{A}}$, $\theta_{x,y}$ is defined by $\theta_{x,y}(a) = x\langle y, a\rangle \in \mathscr{H}_{\mathscr{A}}$ for all $a \in \mathscr{H}_{\mathscr{A}}$ (**[15]**). It was proved (**[15]**) that

$$L(\mathscr{H}_{\mathscr{A}}) \cong M(\mathscr{A} \otimes \mathscr{K}) \quad \text{and} \quad K(\mathscr{H}_{\mathscr{A}}) \cong \mathscr{A} \otimes \mathscr{K}$$

as $C^*$-algebras, where $\mathscr{K}$ is the algebra consisting of compact operators on $\mathscr{H}$. The formulation of $L(\mathscr{H}_{\mathscr{A}})$ and $K(\mathscr{H}_{\mathscr{A}})$ are closely analogous to those of $\mathrm{L}(\mathscr{H})$ and $\mathscr{K}$.

If $\mathscr{A}$ is a unital $C^*$-algebra, we will denote the unitary group of $M_n(\mathscr{A})$ by $U_n(\mathscr{A})$ and the path component of $U_n(\mathscr{A})$ containing the identity by $U_n^0(\mathscr{A})$. In particular, we will denote $U_1^0(\mathscr{A})$ by $U_0(\mathscr{A})$.

If $p$ and $q$ are projections in $\mathscr{A}$, $p \sim q$ means that $p$ and $q$ are equivalent in the sense of Murray-von Neumann, and $p \approx q$ means that $p$ and $q$ are homotopic, i.e., in the same norm path component of projections in $\mathscr{A}$. It is well known that $p \approx q$ if and only if there exists a unitary element $v$ in $U_0(\mathscr{A})$ such that $vpv^* = q$. We denote the matrix units of $\mathscr{K}$ by $\{e_{ij}\}$.

**2. Key Lemmas.** The following technical lemmas are the key of this paper:

2.1. LEMMA. *Suppose that $\mathscr{A}$ is a $C^*$-algebra with FS (not necessarily $\sigma$-unital) and $p$ is a projection in $M(\mathscr{A})$. If $q$ is a projection in $\mathscr{A}$, then for any $\varepsilon_0 > 0$ there exists a projection $q'$ in $\mathscr{A}$ such that both $pq'p$ and $(1-p)q'(1-p)$ have finite spectra and $\|q - q'\| < \varepsilon_0$. More precisely, the projection $q'$ has the following form:*

$$q' = \begin{pmatrix} f_0 & 0 & 0 \\ 0 & a_0 & b_0 \\ 0 & b_0^* & c_0 \end{pmatrix},$$

*where $f_0$ and the range of $a_0$ are mutually orthogonal subprojections of $p$. Consequently $q' \approx q$ if $\varepsilon_0 < 1$.*

*Proof.* Let $q = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be the decomposition of $q$ with respect to $p+(1-p) = 1$. It follows that $a-a^2 = bb^*$, $c-c^2 = b^*b$, $ab+bc = b$, $0 \le a \le p$ and $0 \le c \le 1-p$. (Actually these conditions are also sufficient for $q$ to be a projection.) We will start with the idea in [**6**] and then go further to construct a projection $q' = \begin{pmatrix} a' & b' \\ b'^* & c' \end{pmatrix}$ such that both $\sigma(a')$ and $\sigma(c')$ are finite sets, and $q'$ is close to $q$ in norm.

Let $0 < \delta < 1$ be a fixed positive number and $\varepsilon$ be another positive number such that $3\varepsilon < \delta$. Since $\mathscr{A}$ has FS, there exists a positive element $c_1$ in $(1-p)\mathscr{A}(1-p)$ with a finite spectrum such that

$$\|c - c_1\| < \varepsilon. \tag{1}$$

Set $e = \chi_{(\delta,\infty)}(c_1 - c_1^2)$. If $\delta_1$ is the smaller root of $t^2 - t + \delta = 0$, then $e = \chi_{(\delta_1, 1-\delta_1)}(c_1)$ which is a projection in $(1-p)\mathscr{A}(1-p)$.

Set $c_0 = c_1 e + \chi_{(1-\delta_1, 1]}(c_1)$. Then $\sigma(c_0)$ is a finite set, $c_0 - c_0^2 = e(c_1 - c_1^2)e \in e\mathscr{A}e$ and $\|c_0 - c_1\| \le \delta_1$. It follows that

$$\|c_0 - c\| \le \varepsilon + \delta_1 < \varepsilon + \sqrt{\delta}. \tag{2}$$

Set $v = (eb^*be)^{-1/2}(eb^*)$, of course where $(eb^*be)^{-1}$ is taken in $e\mathscr{A}e$. Since $e(c_1 - c_1^2)e \ge \delta e$ and hence $eb^*be \ge (\delta - 3\varepsilon)e$, $(eb^*be)^{-1/2}$ exists. It is clear that $vv^* = e$.

Set $b_0 = v^*(c_0 - c_0^2)^{1/2}$. Then $b_0^* b_0 = c_0 - c_0^2$.

Set $a_0 = v^*(e - c_0)v$. Then $a_0 - a_0^2 = b_0 b_0^*$ and $a_0 b_0 + b_0 c_0 = b_0$.

If we first fix $\delta$ small enough, then we choose $\varepsilon$ small enough and $c_1$ satisfying (1) such that $\|c - c_0\|$, $\|b - b_0\|$ and $\|(a-a^2)-(a_0-a_0^2)\|$ are all smaller than any preassigned positive number. However, $\|a - a_0\|$ can be equal to one. Here we give details for further reference.

It is obvious that

$$\|b^*b - (c_1 - c_1^2)\| \le 3\|c - c_1\| < 3\varepsilon. \tag{3}$$

Since $\|(1-e)b^*b(1-e) - (1-e)(c_1 - c_1^2)(1-e)\| \le 3\varepsilon$ and $\|(1-e)(c_1 - c_1^2)(1-e)\| \le \delta$, it is easily seen that

$$\|b(1-e)\| \le \sqrt{3\varepsilon + \delta}. \tag{4}$$

Since $eb^*be \ge (\delta - 3\varepsilon)e$, then

$$\|(eb^*be)^{-1}\| \le (\delta - 3\varepsilon)^{-1}. \tag{5}$$

By [**12**, 126] and (3), we can choose $\varepsilon$ small enough such that

$$\|(eb^*be)^{1/2} - [e(c_1 - c_2^2)e]^{1/2}\| < \delta. \tag{6}$$

By (4) and (6) we can choose $\varepsilon$ small enough such that

$$(7)\qquad \begin{aligned}\|b_0 - b\| &\le \|v^*(c_0 - c_0^2)^{1/2} - v^*(eb^*be)^{1/2}\| + \|b(1-e)\| \\ &\le \|[e(c_1 - c_1^2)e]^{1/2} - (eb^*be)^{1/2}\| + \sqrt{3\varepsilon + \delta} \\ &< \delta + \sqrt{3\varepsilon+\delta}.\end{aligned}$$

Consequently,

$$(8)\qquad \begin{aligned}\|(a - a^2) - (a_0 - a_0^2)\| &= \|bb^* - b_0 b_0^*\| \\ &\le 2\|b_0 - b\| < 2\delta + 2\sqrt{3\varepsilon + \delta}.\end{aligned}$$

It is clear from construction that $q_0 = \begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix}$ is a projection. By Lemma (2.4) of **[21]**, $\sigma(a_0)\backslash\{0, 1\} = \sigma(1 - c_0)\backslash\{0, 1\}$, and hence $\sigma(a_0)$ is also a finite set. The idea of constructing the projection $q_0$ is due L. G. Brown (**[6]**) for different purpose.

We will go further to adjust $q_0$ to a projection $q' = \begin{pmatrix} a' & b' \\ b'^* & c' \end{pmatrix}$ so that $\|a - a'\|$ is small, too. Set $f = v^*v$. Then $f$ is a subprojection of $p$ and $fa_0 = a_0 f = a_0$. We claim that $\|faf - a_0\|$ can be arbitrarily small if $\delta$, $\varepsilon$ and $c_1$ are properly chosen. To prove this claim, we need the following estimates.

$$(9)\qquad \begin{aligned}\|e(b^*b)^{1/2}(1-e)\| &= \|e[(b^*b)^{1/2} - (c_1 - c_1^2)^{1/2}](1-e)\| \\ &\le \|(b^*b)^{1/2} - (c_1 - c_1^2)^{1/2}\|.\end{aligned}$$

Then by **[12**, 126] and

$$[(eb^*be)^{1/2}]^2 = eb^*be = [e(b^*b)^{1/2}e]^2 + e(b^*b)^{1/2}(1-e)(b^*b)^{1/2}e,$$

for a fixed $\delta > 0$ we can choose $\varepsilon$ small enough (by (3)) such that

$$(10)\qquad \|(b^*b)^{1/2} - (c_1 - c_1^2)^{1/2}\| < \frac{\delta^2}{2} \quad \text{and}$$

$$(11)\qquad \|(eb^*be)^{1/2} - e(b^*b)^{1/2}e\| < \delta\sqrt{\frac{\delta}{2}}.$$

Since

$$\begin{aligned}f(a - a_0)f &= v^*ev(a - a_0)v^*ev \\ &= v^*e(vav^* - va_0v^*)ev \\ &= v^*[ec_0e - v(p-a)v^*]v,\end{aligned}$$

then

$$(12)\qquad \begin{aligned}\|f(a - a_0)f\| &\le \|ec_0e - ece\| + \|ece - v(p-a)v^*\| \\ &< \varepsilon + \|ece - v(p-a)v^*\|.\end{aligned}$$

Since $(1-a)b = bc$, $p(1-a)b = bp(c)$ for any polynomial $p(t)$. Approximating by polynomials, we obtain that $\sqrt{1-a}b = b\sqrt{c}$, and hence

$$b^*(1-a)b = c^2 - c^3 = (b^*b)^{1/2}c(b^*b)^{1/2}.$$

It follows that

$$\begin{aligned} v(p-a)v^* &= (eb^*be)^{-1/2}eb^*(p-a)be(eb^*be)^{-1/2} \\ &= (eb^*be)^{-1/2}e[b^*b - b^*ab]e(eb^*be)^{-1/2} \\ &= (eb^*be)^{-1/2}[e(b^*b)^{1/2}c(b^*b)^{1/2}e](eb^*be)^{-1/2} \\ &= (eb^*be)^{-1/2}[h_1 + h_2](eb^*be)^{-1/2}, \end{aligned}$$

where

$$\begin{aligned} h_1 &= e(b^*b)^{1/2}ece(b^*b)^{1/2}e \\ &= (eb^*be)^{1/2}c(eb^*be)^{1/2} + [e(b^*b)^{1/2}e - (eb^*be)^{1/2}]c(eb^*be)^{1/2} \\ &\quad + (eb^*be)^{1/2}c[e(b^*b)^{1/2}e - (eb^*be)^{1/2}] \\ &\quad + [e(b^*b)^{1/2}e - (eb^*be)^{1/2}]c[e(b^*b)^{1/2}e - (eb^*be)^{1/2}], \\ h_2 &= e(b^*b)^{1/2}(1-e)ce(b^*b)^{1/2}e \\ &\quad + e(b^*b)^{1/2}ec(1-e)(b^*b)^{1/2}e \\ &\quad + e(b^*b)^{1/2}(1-e)c(1-e)(b^*b)^{1/2}e. \end{aligned}$$

If $\delta$ is first fixed small enough, and $\varepsilon$ and $c_1$ can be chosen such that $6\varepsilon < \delta$ and

$$\begin{aligned} (13)\quad & \|(eb^*be)^{-1/2}h_1(eb^*be)^{-1/2} - ece\| \\ &\le 2\|(eb^*be)^{-1/2}\|\,\|e(b^*b)^{1/2}e - (eb^*be)^{1/2}\|\,\|c\| \\ &\quad + \|(eb^*be)^{-1/2}\|^2\|e(b^*b)^{1/2}e - (eb^*be)^{1/2}\|^2\|c\| \\ &\le 2\frac{\delta\sqrt{\delta/2}}{\sqrt{\delta-3\varepsilon}} + \left[\frac{\delta\sqrt{\delta/2}}{\sqrt{\delta-3\varepsilon}}\right]^2 < \delta^2 + 2\delta, \end{aligned}$$

(where using (5), (10) and (11)) and

$$\begin{aligned} (14)\quad & \|(eb^*be)^{-1/2}h_2(eb^*be)^{-1/2}\| \\ &\le 2\|(eb^*be)^{-1/2}\|^2\|e(b^*b)^{1/2}(1-e)\|\,\|c\|\,\|(b^*b)^{1/2}\| \\ &\quad + \|(eb^*be)^{-1/2}\|^2\|e(b^*b)^{1/2}(1-e)\|^2\|c\| \\ &< (\delta-3\varepsilon)^{-1}\left[\delta^2 + \frac{\delta^4}{4}\right] < 2\delta + \delta^2, \end{aligned}$$

where we used $\delta - 3\varepsilon > \delta/2$. Consequently,

$$\|v(p-a)v^* - ece\| \le 4\delta + 2\delta^2, \quad \text{and so}$$
$$\|f(a-a_0)f\| < \varepsilon + 4\delta + 2\delta^2 \quad \text{by (12)}.$$

If $\delta$ is fixed small enough and $\varepsilon$ is chosen small enough, then $\|faf - a_0\|$ can be arbitrarily small if $c_1$ satisfies (1).

Moreover, by properly choosing $\delta > 0$, $\varepsilon$ and $c_1$ in a similar way we can require that $\|(p-f)af\|$ is less than any preassigned positive number. This can be done as follows.

Since $a - a^2 = bb^*$ and the spectral mapping theorem, it is clear $\|b\| \le 1/2$. Since $(1-a)b = bc$, we have

$$\begin{aligned} -(1-f)av^* &= (1-f)(1-a)be(eb^*be)^{-1/2} \\ &= bce(eb^*be)^{-1/2} - be(eb^*be)^{-1}eb^*bce(eb^*be)^{-1/2} \\ &= bce(eb^*be)^{-1/2} - be(eb^*be)^{-1}eb^*bece(eb^*be)^{-1/2} \\ &\quad - be(eb^*be)^{-1}eb^*b(1-e)ce(eb^*be)^{-1/2} \\ &= b(1-e)ce(eb^*be)^{-1/2} \\ &\quad - be(eb^*be)^{-1}eb^*b(1-e)ce(eb^*be)^{-1/2}. \end{aligned}$$

It follows that

$$\begin{aligned} (15)\quad \|(1-f)af\| &\le \|(1-f)av^*\| \\ &\le \|b\|\,\|(1-e)ce\|\,\|(eb^*be)^{-1/2}\| \\ &\quad + \|b\|\,\|(eb^*be)^{-1}\|\,\|e(b^*b)(1-e)\|\,\|c\|\,\|(eb^*be)^{-1/2}\| \\ &< \frac{\varepsilon}{2}\left[\frac{1}{\sqrt{\delta - 3\varepsilon}}\right] + \frac{1}{2}\left[\frac{1}{\delta - 3\varepsilon}\right](3\varepsilon)\left[\frac{1}{\sqrt{\delta - 3\varepsilon}}\right] \\ &< \left[\frac{\varepsilon}{2}\right]\sqrt{\frac{2}{\delta}} + \left[\frac{3\varepsilon}{2}\right]\left[\frac{2}{\delta}\right]^{3/2}, \end{aligned}$$

where we use (1), (3), (5) and the facts:

$$\begin{aligned} \|(1-e)ce\| &= \|(1-e)(c-c_1)e\| \le \|c - c_1\|, \quad \text{and} \\ \|eb^*b(1-e)\| &= \|e[b^*b - (c_1 - c_1^2)](1-e)\| \le \|b^*b - (c_1 - c_1^2)\|. \end{aligned}$$

As a consequence of the last estimate and (8), for any $0 < \lambda < 1/2$, we can fix $\delta$ small enough and then choose $\varepsilon$ small enough such that $\sigma((p-f)a(p-f)) \subset [0, \lambda] \cup [1-\lambda, 1]$. This is because of the following estimates:

$$\begin{aligned} (p-f)[(a-a^2) - (a_0 - a_0^2)](p-f) &= (p-f)(a-a^2)(p-f) \\ &= (p-f)a(p-f) - [(p-f)a(p-f)]^2 - (p-f)afa(p-f), \end{aligned}$$

$$\begin{aligned} &\|(p-f)a(p-f) - [(p-f)a(p-f)]^2\| \\ &\quad \le \|(p-f)[(a-a^2) - (a_0 - a_0^2)](p-f)\| + \|(1-f)af\|^2 \\ &\quad \le \|(a-a^2) - (a_0 - a_0^2)\| + \|(p-f)af\|^2. \end{aligned}$$

Set $f_0 = \chi_{[1/2,1]}((p-f)a(p-f))$. Then $f_0$ is a projection in $(p-f)\mathscr{A}(p-f)$ such that $f_0a_0 = a_0f_0 = 0$ and $\|f_0-(p-f)a(p-f)\| \le \lambda$. Set $a' = a_0 + f_0$, $b' = b_0$ and $c' = c_0$. Then $q' = \left(\begin{smallmatrix} a' & b' \\ b'^* & c' \end{smallmatrix}\right)$ is a projection in $\mathscr{A}$ such that

$$
\begin{aligned}
(16)\ \|q' - q\| &\le \|(f_0 + a_0) - a\| + 2\|b_0 - b\| + \|c_0 - c\| \\
&\le \|f(a - a_0)f\| + 2\|fa(p-f)\| \\
&\quad + \|f_0 - (p-f)a(p-f)\| + 2\|b_0 - b\| + \|c_0 - c\|.
\end{aligned}
$$

Combining all above estimates, we first fix $\lambda$ small enough, then fix $\delta$ small enough, and then choose $\varepsilon$ small enough and $c_1$ satisfying (1) so that each term on the right-hand side of (16) is small. Then $\|q - q'\|$ is small. It is clear that $\sigma(pq'p) = \sigma(f_0 + a_0)$ is a finite set. The last sentence in the statement of this lemma is well known. □

2.2. Lemma. *Suppose that $\mathscr{A}$ is a $C^*$-algebra (not necessarily $\sigma$-unital) and $p$ is a projection in $M(\mathscr{A})$. If $q$ is a projection in $\mathscr{A}$ such that $\sigma(pqp) \neq [0, 1]$, then there exist two projections $q_1$ and $q_2$ in $\mathscr{A}$ such that $q_1 \le p$, $q_2 \le 1-p$ and $q \approx q_1 + q_2$.*

*Proof*. Let $q = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ be the composition of $q$ with respect to $p+(1-p) = 1$. Then $a = pqp$, $c = (1-p)q(1-p)$ and $b = pq(1-p)$. By [**21**, 2.4], $\sigma(a)\backslash\{0, 1\} = \sigma(1-c)\backslash\{0, 1\}$.

If $b = 0$, then $q_1 = a$ and $q_2 = c$ are as desired. Assume that $b \neq 0$. If $1 \notin \sigma(c)$, then $\|c\| < 1$. By the argument of [**8**, 1], $q$ is path connected to a subprojection $q_1$ of $p$. We can assume that $1 \in \sigma(c)$. Since $\sigma(c) \neq [0, 1]$ and $0$ is always in $\sigma(c)$, there is a $\lambda$ in $(0, 1)\backslash\sigma(c)$. Then there exists a positive number $\varepsilon$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(c) = \varnothing$. Since $b \neq 0$, we can assume that $\sigma(c) \cap (\lambda + \varepsilon, 1) \neq \varnothing$ (Otherwise, $\sigma(a) \cap (\lambda + \varepsilon, 1) \neq \varnothing$, we consider $a$ instead.) We will use a variation of [**8**, 1] to construct a path of projections for our purpose.

Define a family of continuous positive functions $\{f_t\}_{t\in[0,1]}$ from $[0, 1]$ to $[0, 1]$ with the following properties:

(1) $\lim_{t\to t_0} \|f_t - f_{t_0}\|_\infty = 0$ for any $t_0$ in $[0, 1]$;
(2) $f_1(s) = s$ for all $s$ in $[0, 1]$;
(3)

$$
f_0(s) = \begin{cases} 1, & \text{if } \lambda \le s \le 1, \\ \text{linear}, & \text{if } \lambda - \varepsilon < s < \lambda, \\ 0, & \text{if } 0 \le s \le \lambda - \varepsilon; \end{cases}
$$

(4) For all $t$ in $(0, 1)$, $f_t(s) \leq s$ if $s \in [0, \lambda - \varepsilon]$ and $f_t(s) \geq s$ if $s \in [\lambda, 1]$.

Since $q$ is a projection, $bc = (1-a)b$. Approximating by polynomials, we obtain that $bg(c) = g(1-a)b$ for any continuous function $g$ on $[0, 1]$. Set

$$c_t = f_t(c),$$
$$b_t = b\left[\frac{f_t(c) - f_t(c)^2}{c - c^2}\right]^{1/2},$$
$$a_t = p - f_t(p - a).$$

Then $b_t$ and $c_t$ are well defined elements in $\mathscr{A}$ by the properties of $f_t$. Although $p - a$ is not in $p\mathscr{A}p$ if $p$ is in $M(\mathscr{A})\backslash\mathscr{A}$, $p - f_t(p-a)$ is in $p\mathscr{A}p$ for $t \in [0, 1]$. To see this, first, $f_t(p-a)$ is well defined for each $t \in [0, 1]$ since $\sigma(p-a)\backslash\{0, 1\} = \sigma(c)\backslash\{0, 1\}$. Second, if we denote by $\pi$ the canonical map from $(p\mathscr{A}p)^+$ to $(p\mathscr{A}p)^+/p\mathscr{A}p$, where $(p\mathscr{A}p)^+$ is the $C^*$-algebra obtained by joining an identity to $p\mathscr{A}p$, then $p - f_t(p-a) \in p\mathscr{A}p$, since $\pi(p - f_t(p-a)) = \pi(p) - f_t(\pi(p)) = 0$. It is easily verified that

$$a_t - a_t^2 = b_t b_t^*,$$
$$a_t b_t = b_t(1 - c_t),$$
$$c_t - c_t^2 = b_t^* b_t.$$

Thus $q(t) = \left(\begin{smallmatrix} a_t & b_t \\ b_t^* & c_t \end{smallmatrix}\right)$ is a projection in $\mathscr{A}$ for each $t$ in $[0, 1]$. By the property (1) of $\{f_t\}$, $\{q(t)\}_{t\in[0,1]}$ is contained in the same path component of projections in $\mathscr{A}$. Then $q(0) \approx q(1) = q$. Since $(\lambda - \varepsilon, \lambda) \cap \sigma(c) = \varnothing$, $c_0 = f_0(c) = \chi_{[\lambda, 1]}(c)$ is a projection of $(1-p)\mathscr{A}(1-p)$. It is obvious that

$$q(0) = \left(\begin{smallmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_0 & 0 \\ 0 & c_0 \end{smallmatrix}\right).$$

Consequently, $a_0$ is a projection of $p\mathscr{A}p$. Set $q_1 = a_0$ and $q_2 = c_0$, as desired. □

Roughly speaking, with respect to a fixed sequential increasing approximate identity of $\mathscr{A}$ a block-diagonal projection of $M(\mathscr{A})$ whose blocks are with the same size is homotopic to a diagonal projection. More precisely, we have the following lemma:

2.3. LEMMA. *Suppose that $\mathscr{A}$ is a $\sigma$-unital, non-unital $C^*$-algebra with FS and $\sum_{i=1}^{\infty}(s_{i1} + s_{i2} + \cdots + s_{in}) = 1$, where $\{s_{ij}\colon i \geq 1, 1 \leq j \leq n\}$ are mutually orthogonal projections in $\mathscr{A}$ and the sum converges* *in the strict topology. If* $p$ *is a projection in* $M(\mathscr{A})$ *with the form* $\sum_{i=1}^{\infty} p_i$, *where* $p_i$ *is a projection in* $(s_{i1}+s_{i2}+\cdots+s_{in})\mathscr{A}(s_{i1}+s_{i2}+\cdots+s_{in})$ *for* $i \geq 1$, *then* $p \approx \sum_{i=1}^{\infty}(p_{i1}+p_{i2}+\cdots+p_{in})$, *where* $p_{ij}$ *is a projection in* $s_{ij}\mathscr{A}s_{ij}$ *for* $i \geq 1$ *and* $1 \leq j \leq n$.

*Proof*. It suffices to prove the case if $n = 2$. If $n > 2$, we simply employ the same proof recursively $n-1$ times by induction to reach the conclusion.

We write

$$p_i = \begin{pmatrix} a_i^* & b_i \\ b_i^* & c_i \end{pmatrix}$$

with respect to $s_{i1}+s_{i2}$. By Lemma (2.1), for each $i \geq 1$ we can find a projection

$$p_i' = \begin{pmatrix} f_i & 0 & 0 \\ 0 & a_i' & b_i' \\ 0 & b_i'^* & c_i' \end{pmatrix}$$

in $(s_{i1}+s_{i2})\mathscr{A}(s_{i1}+s_{i2})$ such that $\|p_i'-p_i\| < 1/4$, and both $a_i'$ and $c_i'$ have finite spectra. Here we use the proof of Lemma (2.1) to properly choose a positive number $\delta_i$ and a positive element $c'_{1i}$ in $s_{i2}\mathscr{A}s_{i2}$ with a finite spectrum, then we have that

$$e_i = \chi_{(\delta_i, 1-\delta_i)}(c'_{1i}), \qquad c_i' = c'_{1i}e_i + \chi_{(1-\delta_i, 1)}(c'_{1i}),$$
$$v_i = (e_i b_i^* b_i e_i)^{-1/2}(e_i b_i^*), \qquad b_i' = v_i^*(c_i' - c'^2_{ii})^{1/2},$$
$$a_i' = v_i^*(e_i - c'_{1i})v_i$$

and $f_i$ is a projection of $s_{i1}\mathscr{A}s_{i1}$ orthogonal to the range projection of $a_i'$.

Let $p' = \sum_{i=1}^{\infty} p_i'$. Then $\|p'-p\| < 1/4$, and hence $p \approx p'$.

Let $\sigma(c_i') = \{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{il_i}\}$ for each $i \geq 1$. It follows from the construction or [21, 2.4] that $\sigma(a_i') = \{1-\lambda_{i1}, 1-\lambda_{i2}, \ldots, 1-\lambda_{il_i}\}$. We can write $c_i' = \sum_{j=1}^{l_i} \lambda_{ij} r_{ij}$, where $\{r_{ij}\colon 1 \leq j \leq l_i\}$ is a set of mutually orthogonal projections in $s_{i2}\mathscr{A}s_{i2}$. Let $\lambda$ be any number in the open interval $(\frac{1}{2}, \frac{3}{4})$ but not in $\bigcup_{i=1}^{\infty}\sigma(c_i')$. Let $\varepsilon = \min\{\lambda - \frac{1}{2}, \frac{3}{4} - \lambda\}$. For $i \geq 1$, if $\lambda_{ij}$ is in the open interval $(\lambda-\varepsilon, \lambda)$, we replace $\lambda_{ij}$ by $\lambda'_{ij} = \lambda - \varepsilon$, and if $\lambda_{ij}$ is in $(\lambda, \lambda+\varepsilon)$, we replace $\lambda_{ij}$ by $\lambda'_{ij} = \lambda+\varepsilon$. If $\lambda_{ij}$ is not in $(\lambda-\varepsilon, \lambda+\varepsilon)$, then we let $\lambda'_{ij} = \lambda_{ij}$. Set $c_i'' = \sum_{j=1}^{l_i} \lambda'_{ij} r_{ij}$ for $i \geq 1$, and correspondingly set $b_i'' = v_i^*(c_i'' - c_i''^2)^{1/2}$ and $a_i'' = v_i^*(e_i - c_i'')v_i$. Then

$$\|a_i' - a_i''\| \leq \|c_i' - c_i''\| < \varepsilon \quad \text{and}$$
$$\|b_i' - b_i''\| \leq \|(c_i' - c_i'^2)^{1/2} - (c_i'' - c_i''^2)^{1/2}\| < \tfrac{1}{8}.$$

It follows that

$$p_i'' = \begin{pmatrix} f_i & 0 & 0 \\ 0 & a_i'' & b_i'' \\ 0 & b_i''^* & c_i'' \end{pmatrix}$$

is a projection in $(s_{i1}+s_{i2})\mathscr{A}(s_{i1}+s_{i2})$ such that $\|p_i'-p_i''\| \le 2\varepsilon+\frac{1}{4} < 1$. Define $p'' = \sum_{i=1}^{\infty} p_i''$. Then $\|p' - p''\| < 1$, and hence $p' \approx p''$. The remaining job is to prove that $p''$ is homotopic to a desired diagonal projection.

Let $\{f_t\}_{t\in[0,1]}$ be the family of continuous functions defined in the proof of Lemma (2.2). Since $\sigma(c_i'')$ does not intersect with the open interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ for $i \ge 1$, we can define

$$c_i(t) = f_t(c_i''),$$

$$b_i(t) = b_i'' \left[\frac{f_t(c_i'') - f_t(c_i'')^2}{c_i'' - c_i''^2}\right]^{1/2},$$

$$a_i(t) = p - f_t(p - a_i'' - f_i).$$

Then $a_i(t)$, $b_i(t)$ and $c_i(t)$ are well defined elements in $(s_{i1} + s_{i2})\mathscr{A}(s_{i1} + s_{i2})$ for each $t$ in $[0, 1]$ and $i \ge 1$ by the properties of $f_t$. Thus for each $t$ in $[0, 1]$

$$p_i(t) = \begin{pmatrix} a_i(t) & b_i(t) \\ b_i(t)^* & c_i(t) \end{pmatrix}$$

is a projection in $(s_{i1} + s_{i2})\mathscr{A}(s_{i1} + s_{i2})$. It is easily seen that

$$p_i(1) = p_i'' \quad \text{and} \quad p_i(0) = \begin{pmatrix} a_i(0) & 0 \\ 0 & c_i(0) \end{pmatrix},$$

where $a_i(0)$ is a projection of $s_{i1}\mathscr{A}s_{i1}$ and $c_i(0)$ is a projection of $s_{i2}\mathscr{A}s_{i2}$. Define $p(t) = \sum_{i=1}^{\infty} p_i(t)$ for each $t$ in $[0, 1]$. Then $\{p(t)\}_{t\in[0,1]}$ is a path of projection in $M(\mathscr{A})$. It is obvious that

$$p(1) = p'' \quad \text{and} \quad p(0) = \sum_{i=1}^{\infty} \begin{pmatrix} a_i(0) & 0 \\ 0 & c_i(0) \end{pmatrix}.$$

Since the choice of $\{f_t\}_{t\in[0,1]}$ does not depend on $i$, the path $\{p(t): t \in [0, 1]\}$ is continuous in the norm topology.

Set $p_{i1} = a_1(0)$, $p_{i2} = c_i(0)$ for $i \ge 1$. Then

$$p \approx p' \approx p'' \approx p(0) = \sum_{i=1}^{\infty}(p_{i1} + p_{i2}), \quad \text{as desired.} \qquad \square$$

**3. Diagonalizing projections in $\mathscr{A}$ and in $M_n(\mathscr{A})$.** Since we will frequently employ the following well-known fact in this paper, we state it as a lemma.

3.1. LEMMA. *If $\mathscr{A}$ is a $C^*$-algebra, and if $p$ and $q$ are two mutually orthogonal projections in $\mathscr{A}$, then $p \sim q$ if and only if $p \approx q$.*

*Proof*. Let $v$ be a partial isometry in $\mathscr{A}$ such that $vv^* = p$ and $v^*v = q$. Define $w = v + v^* + (1 - p - q)$. Then $w$ is a self-adjoint unitary in $M(\mathscr{A})$ such that $w^*pw = q$. It is well known that $w \in U_0(\mathscr{A})$. It follows that $p \approx q$. □

3.2. THEOREM. *Suppose that $\mathscr{A}$ is a $C^*$-algebra with FS and $p_1, p_2, \dots, p_n$ $(n \geq 1)$ are mutually orthogonal projections in $M(\mathscr{A})$ such that $\sum_{i=1}^n p_i = 1$. If $p$ is a projection in $\mathscr{A}$, then $p \approx \sum_{i=1}^n q_i$, where $q_i$ is a projection in $\mathscr{A}$ such that $q_i \leq p_i$ for $1 \leq i \leq n$.*

*Proof*. Recursively using Lemma (2.1) and Lemma (2.2), we reach the conclusion. □

The following theorem can be regarded as an analogue of the well-known fact: Every projection in $M_n(\mathbb{C})$ is homotopic to a diagonal projection whose entries are either 1 or 0.

3.3. THEOREM. *Assume that $\mathscr{A}$ is a $C^*$-algebra with FS and $n \geq 1$. If $p$ is a projection in $M_n(\mathscr{A})$, then $p \approx \sum_{i=1}^n p_i \otimes e_{ii}$, where $\{p_i\}$ is a set of projections in $\mathscr{A}$ such that*

$$p_1 \leq p_2 \leq \cdots \leq p_{n-1} \leq p_n.$$

*Proof*. It has been recently proved ([5]) that $\mathscr{A} \otimes \mathscr{K}$ has FS if and only if $\mathscr{A}$ has FS. By Theorem (3.2) we have $p \approx \sum_{i=1}^n p_i' \otimes e_{ii}$, where $\{p_i'\}$ is a set of projections in $\mathscr{A}$. The remaining work is to adjust $\{p_i'\}$. We use induction on $n$.

If $n = 2$, $p \approx p_1' \otimes e_{11} + p_2' \otimes e_{22}$, where $p_1'$ and $p_2'$ are projections in $\mathscr{A}$. Combining Lemma (2.1) and Lemma (2.2), we obtain that $p_1' \approx q_1 + q_2$ in $\mathscr{A}$, where $q_1$ and $q_2$ are two projections in $\mathscr{A}$ such that $q_1 \leq p_2'$ and $q_2 \leq 1 - p_2'$. It follows that $p \approx (q_1 + q_2) \otimes e_{11} + p_2' \otimes e_{22}$. Working in the hereditary $C^*$-subalgebra of $M_n(\mathscr{A})$ generated by $(1 - q_1) \otimes e_{11} + 1 \otimes e_{22}$, we have $q_2 \otimes e_{11} + p_2' \otimes e_{22} \approx (p_2' + q_2) \otimes e_{22}$ by Lemma (3.1). It follows that $p \approx q_1 \otimes e_{11} + (p_2' + q_2) \otimes e_{22}$. Let $p_1 = q_1$ and $p_2 = q_2 + p_2'$.

Assume that $p \approx \sum_{i=1}^n p_i' \otimes e_{ii}$ such that $p_2' \leq p_3' \leq \cdots \leq p_n'$. Applying Lemma (2.1) and Lemma (2.2) to $p_1'$, and $p_n'$, we have $p_1' \approx q_n + q_n'$, where $q_n$ and $q_n'$ are projections in $\mathscr{A}$ such that $q_n \leq 1 - p_n'$ and $q_n' \leq p_n'$. By the same argument as in the last paragraph we have that $p \approx q'_n \otimes e_{11} + \sum_{i=2}^{n-1} p'_i \otimes e_{ii} + (p'_n + q_n) \otimes e_{nn}$. Repeating this argument to $q'_n$ and $p'_{n-1}$, we have that $q'_n \approx q'_{n-1} + q_{n-1}$, where $q'_{n-1}$ and $q_{n-1}$ are two projections in $\mathscr{A}$ such that $q_{n-1} \le p'_n - p'_{n-1}$ and $q'_{n-1} \le p'_{n-1}$. It follows that $p \approx q'_{n-1} \otimes e_{11} + \sum_{i=2}^{n-2} p'_i \otimes e_{ii} + (p'_{n-1} + q_{n-1}) \otimes e_{n-1,n-1} + (p'_n + q_n) \otimes e_{nn}$.

Proceeding in this way, we write $p'_1 = \sum_{i=1}^n q_i$, where $\{q_i\}$ is a set of mutually orthogonal projections in $\mathscr{A}$ such that $q_i \le p'_{i+1} - p'_i$ for $2 \le i \le n$ (where $p'_{n+1} = 1$), $q_1 \le p'_2$, and $p \approx q_1 \otimes e_{11} + \sum_{i=2}^n (p'_i + q_i) \otimes e_{ii}$. Let $p_1 = q_1$ and $p_i = p'_i + q_i$ for $2 \le i \le n$. Then $p_1 \le p_2 \le \cdots \le p_n$ and $p \approx \sum_{i=1}^n p_i \otimes e_{ii}$. □

M. A. Rieffel raised a question in [**18**, 7]: If $\mathscr{A}$ is a unital $C^*$-algebra with cancellation, and if two projections $p$ and $q$ in $M_n(\mathscr{A})$ represent the same class in $K_0(\mathscr{A})$, are $p$ and $q$ in the same path component of projections in $M_n(\mathscr{A})$? Since $\mathscr{A}$ has cancellation, $[p] = [q]$ in $K_0(\mathscr{A})$ if and only if $p \sim q$ ([**3**] or [**4**]). Hence, Rieffel's question is equivalent to whether two Murray-von Neumann equivalent projections in $M_n(\mathscr{A})$ are in the same path component of projections in $M_n(\mathscr{A})$. The following corollary provides a partial answer for his question in the case that $\mathscr{A}$ has FS:

3.4. Corollary. *If $\mathscr{A}$ is a unital $C^*$-algebra with FS and cancellation, and if $p$ and $q$ are two projections in $M_n(\mathscr{A})$, then $p \sim q$ if and only if $p \approx q$.*

*Proof.* Of course we need only to show that $p \sim q$ implies $p \approx q$. Since $M_n(\mathscr{A})$ has FS, by Theorem (3.2) we have $p \approx q_1 + q_2$, where $q_1$ is a subprojection of $q$ and $q_2$ is a subprojection of $1 - q$. Since $\mathscr{A}$ has cancellation and $p \sim q$, $q_2 \sim q - q_1$. Working in $(1 - q_1)M_n(\mathscr{A})(1 - q_1)$, by Lemma (3.1) we can find a unitary $v$ in $U_0((1 - q_1)M_n(\mathscr{A})(1 - q_1))$ such that $vq_2v^* = q - q_1$. Set $u = q_1 + v$. Then $u$ is a unitary in $U_0(M_n(\mathscr{A}))$ such that $uq_1 = q_1u$. Thus $p \approx q_1 + q_2 \approx q$. □

Concerning the unitary orbit of elements in $M_n(\mathscr{A})$, we have the following corollary:

3.5. Corollary. *If $\mathscr{A}$ is a $C^*$-algebra with FS and $x$ is a normal element in $M_n(\mathscr{A})$ with finite spectrum, then there is a unitary element $u$ in $U_n^0(\mathscr{A})$ such that $uxu^* = \sum_{j=1}^n [\sum_{i=1}^n \lambda_i p_{ij}] \otimes e_{jj}$, where $\{p_{ij}\}$ is a set of projections in $\mathscr{A}$ such that $p_{i_1 j} \perp p_{i_2 j}$ in $\mathscr{A} \otimes e_{jj}$ if $i_1 \ne i_2$.*

*Proof*. By operator calculus we write $x = \sum_{i=1}^{m} \lambda_i p_i$, where $\{\lambda_i\}$ is a set of complex numbers and $\{p_i\}$ is a set of mutually orthogonal projections in $M_n(\mathscr{A})$. By Theorem (3.2) we can find a unitary element $u_1$ in $U_n^0(\mathscr{A})$ such that $u_1 p_1 u_1^* = \sum_{j=1}^{n} p_{1j} \otimes e_{jj}$ $(= q_1)$ for some projections $\{p_{1j}\}$ in $\mathscr{A}$. Working in $(I_n - q_1)M_n(\mathscr{A})(I_n - q_1)$ and repeating the same argument, we can find a unitary $u_2'$ in $U_0[(I_n - q_1)M_n(\mathscr{A})(I_n - q_1)]$ such that $u_2'(u_1 p_2 u_1^*)u_2'^2 = \sum_{j=1}^{n} p_{2j} \otimes e_{jj}$ for some projections $\{p_{2j}\}$ in $\mathscr{A}$. It follows from $p_1 p_2 = 0$ that $p_{1j} p_{2l} = 0$ for $1 \leq j < l \leq n$. Set $u_2 = q_1 + u_2'$. Then $u_2$ is a unitary in $U_n^0(\mathscr{A})$ and $u_2 u_1 (p_1 + p_2) u_1^* u_2^* = \sum_{i=1}^{2} \sum_{j=1}^{n} p_{ij} \otimes e_{jj} = \sum_{j=1}^{n} (\sum_{i=1}^{2} p_{ij}) \otimes e_{jj}$.

Proceeding in this way we can find unitary elements $\{u_i : 1 \leq i \leq m\}$ in $U_n^0(\mathscr{A})$ such that

$$\begin{aligned} u_m u_{m-1} \cdots u_1 (p_1 + p_2 + \cdots + p_m) u_1^* \cdots u_{m-1}^* u_m^* \\ = \sum_{i=1}^{m} \left[ \sum_{j=1}^{n} p_{ij} \otimes e_{jj} \right] = \sum_{j=1}^{n} \left[ \sum_{i=1}^{m} p_{ij} \right] \otimes e_{jj}. \end{aligned}$$

Let $u = u_m \cdots u_2 u_1$. It is obvious that $u$ is in $U_n^0(\mathscr{A})$ and

$$uxu^* = \sum_{j=1}^{n} \left[ \sum_{i=1}^{m} \lambda_i p_{ij} \right] \otimes e_{jj}. \qquad \square$$

It is well known that the unitary orbit of a self-adjoint matrix in $M_n(\mathbb{C})$ contains a diagonal self-adjoint matrix. If $\mathbb{C}$ is replaced by a unital $C^*$-algebra with FS, we have the following weaker analogue:

3.6. COROLLARY. *If $\mathscr{A}$ is a $C^*$-algebra with FS and $x$ is a self-adjoint element in $M_n(\mathscr{A})$ $(n \geq 1)$, then for any $\varepsilon > 0$ there exist a unitary element $u$ in $U_n^0(\mathscr{A})$ and elements $a_i$ in $\mathscr{A}$ with finite spectra such that*

$$\left\| uxu^* - \sum_{i=1}^{n} a_i \otimes e_{ii} \right\| < \varepsilon.$$

*Proof*. Since $M_n(\mathscr{A})$ has FS, there is a self-adjoint element $h$ in $M_n(\mathscr{A})$ with finite spectrum such that $\|x - h\| < \varepsilon$. By the same argument as in the proof of Corollary (3.5) we can find a unitary element $u$ in $U_n^0(\mathscr{A})$ such that $uhu^* = \sum_{i=1}^{n} a_i \otimes e_{ii}$, where $\{a_i\}$ is a set of self-adjoint elements in $\mathscr{A}$ with finite spectra. Therefore,

$$\left\| uxu^* - \sum_{i=1}^{n} a_i \otimes e_{ii} \right\| = \|x - h\| < \varepsilon. \qquad \square$$

3.7. REMARK. Concerning the computation of $K_0$-groups of a $C^*$-algebra, M. A. Rieffel raised a question in [**18**, 8]: What is the smallest $n$ such that the projections in $M_n(\mathscr{A})$ generate $K_0(\mathscr{A})$? Theorem (3.3) provides a partial answer for his question for the class of $C^*$-algebras with FS (actually it has been given in [**22**] although it was not mentioned there). In fact, if $\mathscr{A}$ is a $C^*$-algebra with FS, then the smallest such an integer is $n = 1$; in other words, $K_0(\mathscr{A})$ is generated by the set of Murray-von Neumann equivalence classes of projections in $\mathscr{A}$.

## 4. Diagonalizing projections in $M(\mathscr{A})$.

4.1. THEOREM. *Assume that $\mathscr{A}$ is a $\sigma$-unital $C^*$-algebra with FS and $\{e_n\}$ is a fixed increasing sequential approximate identity consisting of projections. If $p$ is a projection in $M(\mathscr{A})$, then the following hold:*

(i) *There is a unitary $u$ in $M(\mathscr{A})$ connected to the identity by a path of unitaries, where the path is continuous in the strict topology, such that $upu^* = \sum_{i=1}^{\infty} p_i$, where $p_i \leq e_i$ for $i \geq 1$; in other words, each strict path component of projections in $M(\mathscr{A})$ contains a diagonal projection with respect to $\{e_n\}$.*

(ii) *There exist a unitary $v$ in $U_0(M(\mathscr{A}))$ and a subsequence $\{e_{m_i}\}$ of $\{e_n\}$ such that $vpv^* = \sum_{i=1}^{\infty} p'_i$, where $p'_i$ is a projection of $(e_{m_i} - e_{m_{i-1}})\mathscr{A}(e_{m_i} - e_{m_{i-1}})$ for $i \geq 1$; in other words, each norm path component of projections in $M(\mathscr{A})$ contains a block-diagonal projection with respect to $\{e_n\}$.*

Before proving this theorem, we state the following corollary, which can be regarded as an analogue of the well known fact that a projection on a separable Hilbert space is unitarily equivalent to a diagonal projection whose diagonal entries are either 1 or 0.

4.2. COROLLARY. *If $\mathscr{A}$ is a $\sigma$-unital $C^*$-algebra with FS, and if $p$ is a projection in $L(\mathscr{H}_{\mathscr{A}})$, then there is a unitary $u$ in $L(\mathscr{H}_{\mathscr{A}})$ such that $upu^* = \sum_{i=1}^{\infty} p_i \otimes e_{ii}$, where $\{p_i\}$ is a sequence of projections in $\mathscr{A}$. Consequently, $p \approx \sum_{i=1}^{\infty} p_i \otimes e_{ii}$ (by* [**8**]).

*Proof of Theorem* (4.1).

*Case* 1. If $p$ is a projection of $\mathscr{A}$.

Choose $n \geq 1$ large enough such that $\|p(1 - e_n)p\|$ is small. Then Lemma (2.1) of [**10**] applies. We find a unitary $u$ in $U_0(M(\mathscr{A}))$ such that $upu^* \le e_n$. By Theorem (3.2), $p \approx upu^* \approx \sum_{i=1}^n p_i$, where $p_i \le e_i - e_{i-1}$ for $1 \le i \le n$. Hence both (i) and (ii) hold.

*Case* 2. If $p$ is a projection in $M(\mathscr{A})\backslash\mathscr{A}$.

Let $\{q_n\}$ and $\{q'_n\}$ be two increasing sequences of projections in $\mathscr{A}$ such that $q_n \nearrow p$ and $q'_n \nearrow 1-p$ in the strict topology. Set $f_n = q_n + q'_n$. Then $\{f_n\}$ is an increasing sequential approximate identity of $\mathscr{A}$ consisting of projections. By the argument of [**10**, 2.4] we find a unitary element $v$ in $U_0(M(\mathscr{A}))$ such that

$$e_{m_1} \le v f_{n_1} v^* \le e_{m_2} \le v f_{n_2} v^* \le e_{m_3} \le \cdots ,$$

where $\{n_i\}$ and $\{m_i\}$ are increasing sequences. It is clear that

$$vpv^* = \sum_{i=1}^{\infty} vp(f_{n_i} - f_{n_{i-1}})v^* = \sum_{i=1}^{\infty} v(q_{n_i} - q_{n_{i-1}})v^*$$

and $v(q_{n_i} - q_{n_{i-1}})v^* \le v(f_{n_i} - f_{n_{i-1}})v^* = (vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*)$ (where $q_{n_0} = 0$ and $f_{n_0} = 0$).

We first prove (i). By Theorem (3.2) we find a unitary $w_i$ in $U_0(\mathscr{A}_i)$, where $\mathscr{A}_i = [v(f_{n_i} - f_{n_{i-1}})v^*]\mathscr{A}[v(f_{n_i} - f_{n_{i-1}})v^*]$, such that $w_i v(q_{n_i} - q_{n_{i-1}})v^* w_i^* = r_i + r'_i$, where $r_i \le vf_{n_i}v^* - e_{m_i}$ and $r'_i \le e_{m_i} - vf_{n_{i-1}}v^*$. Set $w = \sum_{i=1}^{\infty} w_i$. Then $w$ is a unitary in $M(\mathscr{A})$ such that $w$ is path connected (in the strict topology) to the identity and

$$wvpv^*w^* = \sum_{i=1}^{\infty}(r_i + r'_i) \le \sum_{i=1}^{\infty}[(vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*)].$$

Since $r_i + r'_{i+1} \le e_{m_{i+1}} - e_{m_i}$, we can apply Theorem (3.2) again to get a unitary $w'_i$ in $U_0(\mathscr{B}_i)$, where $\mathscr{B}_i = (e_{m_{i+1}} - e_{m_i})M(\mathscr{A})(e_{m_{i+1}} - e_{m_i})$ such that

$$w'_i(r_i + r'_{i+1})w'^*_i = \sum_{j=m_i+1}^{m_{i+1}} p_j ,$$

where $p_j$ is in $(e_j - e_{j-1})\mathscr{A}(e_j - e_{j-1})$ for $m_i < j \le m_{i+1}$.

Define $w' = \sum_{i=1}^{\infty} w'_i$. Then $w'$ is a unitary in $M(\mathscr{A})$ such that $w'$ is path connected in the strict topology to the identity and $w'wvpv^*w^*w'^* = \sum_{i=1}^{\infty} p_i$. Set $u = w'wv$, as (i) desired.

To prove (ii), we start with $p \approx vpv^* = \sum_{i=1}^{\infty} v(q_{n_i} - q_{n_{i-1}})v^*$, where $s_i = v(q_{n_i} - q_{n_{i-1}})v^* \le v(f_{n_i} - f_{n_{i-1}})v^* = (vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*)$ for each $1 \ge 1$ and $q_{n_0} = 0$ and $f_{n_0} = 0$. With respect to

$$v(f_{n_i} - f_{n_{i-1}})v^* = (vf_{n_i}v^* - e_{m_i}) + (e_{m_i} - vf_{n_{i-1}}v^*),$$

we can write

$$s_i = \begin{pmatrix} a_i & b_i \\ b_i^* & c_i \end{pmatrix} \quad \text{for } i \geq 1.$$

By Lemma (2.3),

$$vpv^* \approx \sum_{i=1}^{\infty}(s_i + s_i'),$$

where $s_i$ is a projection in $(vf_{n_i}v^* - e_{m_i})\mathscr{A}(vf_{n_i}v^* - e_{m_i})$ and $s_i'$ is a projection in $(e_{m_i} - vf_{n_{i-1}}v^*)\mathscr{A}(e_{m_i} - vf_{n_{i-1}}v^*)$. Let $p_i' = s_i' + s_{i-1}$ for $i \geq 1$, where $s_0 = 0$, as desired. □

The following theorem asserts that the unitary orbit of each self-adjoint element of $M(\mathscr{A})$ contains an "almost" diagonal form, which is a natural analogue of the classical Weyl-von Neumann theorem.

4.3. THEOREM. *Assume that $\mathscr{A}$ is a $\sigma$-unital $C^*$-algebra with FS and also $M(\mathscr{A})$ has FS. If $\{e_n\}$ is a fixed increasing approximate identity of $\mathscr{A}$ consisting of projections and $h$ is a self-adjoint element in $M(\mathscr{A})$, then there exist a unitary $u$ in $M(\mathscr{A})$, an element $a$ in $\mathscr{A}$, some mutually orthogonal subprojection $p_{ij}$ $(1 \leq j \leq n_i)$ of $e_i - e_{i-1}$ for each $i \geq 1$ and a real bounded scalar sequence $\{\lambda_{ij}\}$ such that*

$$\sum_{ij} p_{ij} = 1, \quad \textit{and} \quad uhu^* = \sum_{i=1}^{\infty}\left[\sum_{j=1}^{l_i}\lambda_{ij}p_{ij}\right] + a,$$

*where $a$ can be chosen such that $\|a\|$ is arbitrarily small. Moreover, $u$ is connected to the identity by a path of unitaries in $M(\mathscr{A})$, where the path is continuous in the strict topology.*

4.4. COROLLARY. *If $\mathscr{A}$ is a unital $C^*$-algebra with FS and $L(\mathscr{H}_{\mathscr{A}})$ has FS also, then for any self-adjoint element $h$ in $L(\mathscr{H}_{\mathscr{A}})$ there are a unitary $u$ in $L(\mathscr{H}_{\mathscr{A}})$, an element $a$ in $K(\mathscr{H}_{\mathscr{A}})$, a sequence of projections $\{p_{ij}\}$ in $\mathscr{A}$ and a real bounded scalar sequence $\{\lambda_{ij}\}$ such that*

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{l_i}p_{ij}\right)\otimes e_{ii} = 1 \quad \textit{and} \quad uhu^* = \sum_{i=1}^{\infty}\left[\sum_{j=1}^{l_i}\lambda_{ij}p_{ij}\right]\otimes e_{ii} + a,$$

*where $p_{ij}$ $(i \leq j \leq l_i)$ are mutually orthogonal for each fixed $i$, and $a$ can be chosen with an arbitrarily small norm.*

*Proof of Theorem* (4.3).. Since $\mathscr{A}$ is $\sigma$-unital and both $\mathscr{A}$ and $M(\mathscr{A})$ have FS, by [**21**, 3.1] we can find mutually orthogonal projections $p_i$ in $\mathscr{A}$ with $\sum_{i=1}^{\infty} p_i = 1$, a real bounded scalar sequence

$\{\lambda_i\}$ and an element $b$ in $\mathscr{A}$ with arbitrarily small norm such that $h = \sum_{i=1}^{\infty} \lambda_i p_i + b$. Let $f_n = \sum_{i=1}^{n} p_i$. Then $\{f_n\}$ is an increasing approximate identity consisting of projections. By the same argument as in [**10**, 2.4] we can find a unitary $v$ in $M(\mathscr{A})$ such that $v \sim 1$, and

$$e_{m_1} \leq v f_{n_1} v^* \leq e_{m_2} \leq v f_{n_2} v^* \leq e_{m_3} \leq \cdots ,$$

where $\{n_i\}$ and $\{m_i\}$ are increasing sequences. Since

$$v\left(\sum_{j=n_{i-1}+1}^{n_i} p_i\right) v^* = (v f_{n_i} V^* - e_{m_i}) + (e_{m_i} - V f_{n_{i-1}} v^*)$$

(where $f_{n_0} = 0$), by the same arguments in the proof of Theorem (4.1) we can find a unitary $w_i$ of $[v(f_{n_i} - f_{n_{i-1}})v^*]M(\mathscr{A})[v(f_{n_i} - f_{n_{i-1}})v^*]$ path connected to the identity $v(f_{n_i} - f_{n_{i-1}})v^*$ such that

$$w_i v\left(\sum_{j=n_{i-1}+1}^{n_i} p_i\right) v^* w_i^* = \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i' v^* w_i^* + \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i'' v^* w_i^* ,$$

where

$$p_i' + p_i'' = p_i , \quad r_i = \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i' v^* w_i^* = v f_{n_i} v^* - e_{m_i} \quad \text{and}$$

$$r_i' = \sum_{j=n_{i-1}+1}^{n_i} w_i v p_i'' v^* w_i^* = e_{m_i} - v f_{n_{i-1}} v^*$$

Let $w = \sum_{i=1}^{\infty} w_i$. Then $w$ is a unitary in $M(\mathscr{A})$ such that $w$ is connected to the identity by a path of unitaries, where the path is continuous in the strict topology. Since $r_j + r_{j+1}' \leq e_{m_{j+1}} - e_{m_j}$, by the same arguments in the proof of Theorem (4.1), we obtain a unitary $w_j'$ of $(e_{m_{j+1}} - e_{m_j})M(\mathscr{A})(e_{m_{j+1}} - e_{m_j})$ path connected to the identity $e_{m_{j+1}} - e_{m_j}$ such that

$$w_j'(r_j + r_{j+1}')w_j'^* = \sum_{i=m_j+1}^{m_{j+1}} \sum_{j=1}^{l_i} p_{ij} ,$$

where $\{p_{ij} : 1 \leq j \leq l_i\}$ is a set of mutually orthogonal subprojections in $(e_i - e_{i-1})\mathscr{A}(e_i - e_{i-1})$.

Define $w' = \sum_{i=1}^{\infty} w_i'$. Then $w'$ is a unitary in $M(\mathscr{A})$ such that $w'$ is path connected to the identity, where the path is continuous in the strict topology. Set $u = w'wv$. Then $u$ is path connected to the identity, where the path is continuous in the strict topology. It is easily verified that $uhu^*$ has a desired form. (Notice that $\{\lambda_i\}$ is equal to $\{\lambda_{ij}\}$ as sets.) □

4.5. REMARKS. (i) The condition "$M(\mathscr{A})$ has FS" in the hypotheses of Theorem (4.3) and Corollary (4.4) has been studied in [**5**], [**21**] and [**24**]. Actually many multiplier algebras have the FS property.

(ii) Several applications of the results in this note have been given in the author's subsequent papers [**24**, Part II, III, IV].

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UNIVERSITY OF KANSAS
LAWRENCE, KS 66045-2142

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