Pacific Journal of Mathematics

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Vol. 146, No. 1 November 1990

PACIFIC JOURNAL OF MATHEMATICS
Vol. 146, No. 1, 1990

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Let G be a real reductive Lie group, θ a Cartan involution of G, and $B = N_0A_0M_0^1$ a fixed minimal parabolic subgroup of G. Fix H a θ-stable Cartan subgroup of G, and assume that if $H = H_+H_-$ is the decomposition of H relative to θ, then $H_- \subset A_0$. Let $\chi \in \widehat{H_+}$, then following Harish-Chandra, we introduce the functions of type $\mathrm{II}(\chi)$. It is known that the positive chamber $A_0(B)$ can be covered by sectors $A_0(B|Q)$, where Q varies over the maximal parabolic subgroups of G which are standard with respect to B. Let $Q = NM$ be such a maximal parabolic for which the split component of M can be conjugated by an element of G into H_-. We show that given a function $\varphi(x, \lambda)$ of type $\mathrm{II}(\chi)$ there exists an asymptotic expansion (along Q) for this function with the following properties: first the partial sums uniformly approximate the function as λ varies over $\widehat{H_-}$, and x varies over $A_0(B|Q)$, and second, the terms in this sum are essentially functions of type $\mathrm{II}(\chi)$ on an approximate Levi subgroup of M.

Introduction. For this introduction we shall let G be a real reductive Lie group satisfying some restrictive conditions which are explained in the next section; this class of groups contains all connected semisimple groups with finite center. Fix a maximal compact subgroup K of G, a Cartan involution θ fixing K pointwise, and a finite dimensional double unitary K-module (V, τ). One of the central problems in the harmonic analysis of reductive groups has been the characterization of the Fourier transforms of the Schwartz spaces of τ-spherical functions which are contained in $L^p(G)$, for $p \geq 1$. The inverse transform in these characterizations is defined by integrating a function on the Fourier transform side against the matrix elements of a parametrized family of representations, and showing that these so called "wavepackets" are rapidly decreasing on G. This analysis is therefore governed by the asymptotic behavior of the matrix elements. The estimates that one needs for these functions have to be uniform in the parameter so that one can transfer the asymptotic behavior of the matrix elements to their wavepackets.

For $p = 2$, the characterization goes back to Harish-Chandra [**1**], [**2**], [**5**], [**6**], [**7**]. One of the fundamental techniques he used was that of approximating the matrix elements by their so called constant term; specifically one considers series of unitary representations parametrized by the duals of certain abelian subgroups embedded in G. The τ-spherical matrix elements of these representations are functions on G which depend on a parameter in the dual group. The constant term roughly speaking is a function of the same type as the matrix element but defined on a reductive subgroup M of G having dimension strictly smaller than that of G, but of the same class as G; it represents the *leading term* describing the oscillations of the matrix elements at infinity on G, and the approximation is uniform in the parameter. If one wants to extend the Harish-Chandra theory and obtain the characterization of the Schwartz spaces corresponding to $L^p(G)$, $p \geq 1$, the technique of the constant term is no longer adequate; it becomes necessary to construct a *full* asymptotic expansion at infinity. One wants a *uniform* asymptotic expansion for the matrix elements, which is an exponential sum with the following properties; (i) they have partial sums approximating the matrix elements very closely (in a sense we describe more fully below) *uniformly* as the parameter varies over the dual group, and (ii) the coefficients of the series are functions of the same type as those being approximated, but defined on the subgroup M mentioned above. In Trombi and Varadarajan [**1**] such an expansion was obtained for the spherical principal series, thus extending the Harish-Chandra constant term (Harish-Chandra [**1**], [**2**]) for the case when $V = \mathbf{C}$, and τ is the trivial double representation. In Trombi [**1**] such an expansion was obtained for general (V, τ), but for G of real rank one. This paper is the higher-rank-extension of this last paper. We shall explain the exact nature of our results presently; first we shall make some conventions and introduce some concepts and notations which we hope will facilitate the exposition.

Let $B = N_0 M_0$ be a fixed minimal parabolic subgroup of G such that $G = BK$; A_0 will denote the split component of M_0. We adopt the convention used throughout the paper that subgroups of G will have Lie algebras denoted by the corresponding lower case German letter; subspaces $\mathfrak{u}$ of $\mathfrak{g}$ will have duals denoted by $\mathfrak{u}^*$, and complexifications denoted by $\mathfrak{u}_{\mathbf{C}}$. If $\mathfrak{u}$ is θ-invariant, then we shall write $\mathfrak{u}_\pm$ for the ± 1-θ-eigen subspaces of $\mathfrak{u}$. With these conventions, let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra such that $\mathfrak{h}_{0,-} = \mathfrak{a}_0$. Further, we shall denote the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$ by $U(\mathfrak{g}_{\mathbf{C}})$, and more generally for $\mathfrak{u}$ a subalgebra of $\mathfrak{g}$, we shall let $U(\mathfrak{u}_{\mathbf{C}})$ denote the subalgebra of $U(\mathfrak{g}_{\mathbf{C}})$ generated by $\mathfrak{u}_{\mathbf{C}}$ and 1. In this case we let $\mathfrak{z}(\mathfrak{u})$ denote the center of $U(\mathfrak{u}_{\mathbf{C}})$.

Now let us describe the unitary representations of concern to us. Fix P a cuspidal parabolic subgroup of G such that

$$P = N_P M_P \subset B, \quad \text{and} \quad G = PK.$$

Denote the split component of M_P by A_P so that $\mathfrak{a}_P \subset \mathfrak{a}_0$, and $A_P = \exp \mathfrak{a}_P$ is abelian. Suppose that σ is a discrete series representation of M_P, and $\lambda \in (-1)^{1/2}\mathfrak{a}_P^* \simeq \widehat{A}_P$. Then one can induce the representation $\sigma \otimes e^\lambda$ from P to G, the tensor product being trivially extended from M_P to P. The resulting unitary representation of G will be referred to as a *cuspidal principal series representation.*

Let $\mathscr{A}_\sigma(M_P, \tau_P)$ denote the finite dimensional Hilbert space of τ_P-spherical ($\tau_P = \tau|_{K\cap M_P}$) matrix elements of σ. For $\varphi \in \mathscr{A}_\sigma(M_P, \tau_P)$ we put

$$E_P(x, \varphi, \lambda) = \int_{K\cap M_P\backslash K} \tau(k^{-1})\varphi_P(kx)e^{(\lambda+\rho_P)H_P(kx)}\,dk,$$

where φ_P denotes the extension of φ to a right τ-spherical and left N_P-trivial function on G; recall the above equalities $G = PK = N_P M_P K$. These functions will be referred to as *Eisenstein integrals.* From their definition it is easy to see that they are τ-spherical and infinitely differentiable functions on G; they are the τ-spherical matrix elements of the cuspidal principal series of G induced from $\sigma \otimes e^\lambda$. These are the parametrized functions described above whose uniform approximation is the main concern of this paper.

Our expansions for an Eisenstein integral will be obtained on certain "sectors" of G which we now want to describe. Let $A_0(B)$ denote the positive chamber in A_0 corresponding to B. Then we have

$$G = K \operatorname{cl} A_0(B) K.$$

It follows that growth behaviour of τ-spherical functions on G is completely determined by their restrictions to the subset $\operatorname{cl} A_0(B)$ of the vector group A_0, $A_0(B)$ being the exponential of the positive cone $\mathfrak{a}_0(B)$ in $\mathfrak{a}_0$. Directions to infinity in $\mathfrak{a}_0(B)$ are parametrized by subsets of the simple roots, $\Sigma(B, A_0)$, of the pair (B, A_0). A subset $F \subset \Sigma(B, A_0)$ determines a subcone of $\mathfrak{a}_0(B)$ which can be described as follows; F defines a standard (with respect to B) parabolic subgroup, $P_F = N_F M_F$ of G. Let ρ_{P_F} denote half the sum of the roots of (B, A_0) which do not vanish on $\mathfrak{a}_F$, the split component of $\mathfrak{m}_F$;

where

$$\mathfrak{a}_F = \bigcap_{\alpha \in F} \ker(\alpha).$$

The subcone determined by F is defined by

$$\begin{aligned}&\mathfrak{a}_0(B|P_F)\\ &\quad = \{H \in \mathfrak{a}_0 | \alpha(H) \geq \mu\rho_F(H), \forall\alpha \text{ a root of } (B, A_0) \ni \alpha|_{\mathfrak{a}_F} \not\equiv 0\};\end{aligned}$$

μ is some sufficiently small positive real number. $A_0(B|P_F) = \exp \mathfrak{a}_0(B|P_F)$ is what we shall refer to as a sector of G. If $\beta \in \Sigma(B, A_0)$, and $F = \Sigma(B, A_0)\backslash\{\beta\}$ is a maximal proper subset of $\Sigma(B, A_0)$, then P_F is a maximal parabolic subgroup, and $\mathfrak{a}_0(B|P_F)$ is essentially (modulo the split component $\mathfrak{a}_G$ of $\mathfrak{g}$) a compact set crossed with a half line $\{tH : t > 0\}$ where

$$H \in \mathfrak{a}_F, \quad \text{and} \quad \beta(H) = 1.$$

We can choose a fixed μ above so that

$$A_0(B) = \bigcup_{Q \supset B} A_0(B|Q),$$

the union taken over all *maximal* standard parabolic subgroups Q.

Fix such a maximal parabolic Q; let $Q = NM$, with A denoting the split component of M, and H and β defined as above for $Q = P_F$. We are interested in producing a "uniform asymptotic expansion" for an Eisenstein integral $E_P(x, \varphi, \lambda)$ along the parabolic Q as described above. We can now be more specific; this means that we want an infinite exponential sum, with coefficients defined on the sector $A_0(B|Q)$, whose partial sums approximate the Eisenstein integral uniformly as the group variable ranges over $A_0(B|Q)$, and the parameter λ varies over $(-1)^{1/2}\mathfrak{a}_P^*$. Further we require that the coefficients of our expansion are given by Eisenstein integrals on the Levi factor M of Q.

Although our analysis shows that it is possible to determine an asymptotic expansion for every maximal parabolic subgroup Q, we can determine the leading exponents of this expansion, and make the identification of its coefficients, only in the case that the split component of Q is conjugate to a subgroup of the split component of P; otherwise, (5.2) is our best result. We should point out that in the exceptional case when Q fails the conjugacy condition with respect to P, we have been able to obtain uniform estimates for the decay of the Eisenstein integrals $E_P(x, \varphi, \lambda)$ along the sector $A_0(B|Q)$ in terms of the leading exponents of φ along $B \cap M$; this result will appear elsewhere.

Let us now formulate the principal results in this paper more precisely. *We assume for all that is to follow that P and Q satisfy the conjugacy condition on their split components.* In this case we prove the existence of an asymptotic expansion ("along Q") for the functions $E_P(x, \varphi, \lambda)$ which uniformly approximates on $A_0(B|Q) \times (-1)^{1/2}\mathfrak{a}_P^*$. If λ is restricted to an open dense subset $\mathscr{F}''(\chi) \subset (-1)^{1/2}\mathfrak{a}_P^*$, and $x = m \exp tH \in A_0(B|Q)$, then this expansion has the following form

$$\sum_s \sum_{n \geq 0} \Gamma_{s,n}(m, \lambda) e^{t(s\lambda - n\beta - \rho_Q)(H)}, \qquad m \in M;$$

where the first sum is over those elements of the little Weyl group which have the property that $s^{-1}\mathfrak{a} \subset \mathfrak{a}_P$. We also show that given any $n \in \mathbf{Z}_+$, there exists a partial sum of this expansion

$$\sum_s \sum_{n=0}^{p_s} \Gamma_{s,n}(m, \lambda) e^{t(s\lambda - n\beta - \rho_Q)(H)}, \qquad m \in M,$$

which uniformly approximates $\varphi(m \exp tH, \lambda)$ as λ varies over $\mathscr{F}''(\chi)$ and $m \exp tH$ varies over (essentially) $A_0(B|Q)$. The error estimate for the approximation is bounded by a constant times

$$|(m, \lambda)|^r d_Q(m)^{-1} \Xi_M(m) e^{-nt}$$

where the first term of the estimate is a polynomial factor in m and λ, and other unexplained notation is as in §3. These results are stated in a more precise form in Theorems 5.1 and 5.9. Theorem 5.9 is a refinement of Theorem 5.1 which identifies the leading exponents of our expansion along Q. This shows in particular that our expansion has as its leading terms just the Harish-Chandra constant term along Q (Harish-Chandra [3]) and hence, that our expansion is an extension of Harish-Chandra's constant term to a full asymptotic expansion.

The second principal result of the paper, Theorem 7.10, shows that the coefficients of our expansion along Q are given by linear combinations of derivatives of Eisenstein integrals on the reductive group M, the Levi factor of Q. Assuming that $Q \supset P$, $\Lambda \in \mathfrak{a}_P^*$, Λ perpendicular to $\mathfrak{a}^*$, and $\lambda \in \mathfrak{a}^*$, then we have by the principle of induction in stages that

$$E_Q(x, E_{P \cap M}(\Phi, \Lambda), \lambda) = E_P(x, \Phi, \Lambda + \lambda),$$

the equality holding for all τ-spherical matrix elements of σ. Applying Theorem 7.10 to this last equation can then be viewed as a generalization of Harish-Chandra's formula for the limit of an Eisenstein integral along an associated parabolic direction (cf. Harish-Chandra [**4**], Theorem 18.1).

As we pointed out above, the results of this paper generalize results of Trombi [**1**], which were obtained for groups of real rank one. They are also a generalization of the first part of Trombi and Varadarajan [**1**] which considered the case where (V, τ) is the trivial bi-K module. These earlier results were used in the investigation of the harmonic analysis of the L^p-Schwartz spaces $\mathscr{C}^p(G, \tau)$ (cf. Trombi and Varadarajan [**2**]) and we envision our results playing essentially the same role.

We shall now explain the mechanics for generating our uniform asymptotic expansion. The techniques used in Trombi and Varadarajan [**1**] and Trombi [**1**] are different from those used in this paper. In the general case treated here the iterative technique of those two papers will not work. We follow instead a method due to Wallach [**1**] but adapted to the task of getting uniform estimates.

Fix a maximal parabolic subgroup $Q = NM \supset B$; let the notations A, β, H, and ρ_Q be given the same meaning as above. We begin by observing that the Eisenstein integrals are eigenfunctions for $\mathfrak{z}(\mathfrak{g})$. The first step in the analysis is to show how to construct from an Eisenstein integral, a vector function whose derivatives by elements of $\mathfrak{z}(\mathfrak{m})$ are perturbations of eigen-equations. The definition of this vector function is best understood by first making some general observations which we can then apply to the derivatives of the Eisenstein integrals.

Let W be a Harish-Chandra module, and let $\overline{Q} = \overline{N}M$ be the parabolic opposite to Q. It is known that for any $k \in \mathbf{Z}_+$ the quotient

$$W/\bar{\mathfrak{n}}^k W$$

is locally $\mathfrak{a}$-finite. In the case $k = 1$, Harish-Chandra determined a basis for this $\mathfrak{a}$-action when the module W has infinitesimal character with Harish-Chandra parameter $\Lambda \in \mathfrak{h}^*_{0,\mathbf{C}}$. To describe this basis, observe that $\mathfrak{m}$ is a reductive subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}_0$. Let $\mathfrak{u}$ be a reductive subalgebra which contains $\mathfrak{h}_0$, and assume that $\mathfrak{a}_\mathfrak{u}$, the split component of $\mathfrak{u}$, is contained in $\mathfrak{h}_{0,-}$. We shall let

$$\mu_{\mathfrak{u}/\mathfrak{h}_0} : \mathfrak{z}(\mathfrak{u}) \mapsto S(\mathfrak{h}_{0,\mathbf{C}})^{W_0^\mathfrak{u}}$$

denote the Harish-Chandra isomorphism of the center of the enveloping algebra over $\mathfrak{u}_\mathbf{C}$ with the $W_0^\mathfrak{u}$-invariants in the symmetric algebra over $\mathfrak{h}_{0,\mathbf{C}}$; W_0 denoting the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h}_0)$, and $W_0^\mathfrak{u}$

denoting the subgroup which fixes $\mathfrak{a}_{\mathfrak{u}}$ pointwise; this subgroup can be identified with the Weyl group of the pair $(\mathfrak{u}, \mathfrak{h}_0)$. With this notation, we see that there exists an algebra injection

$$\mu : \mathfrak{z}(\mathfrak{g}) \mapsto \mathfrak{z}(\mathfrak{m})$$

which is essentially defined by the following equation (recall that $\mu_{\mathfrak{g}/\mathfrak{h}_0}$ and $\mu_{\mathfrak{m}/\mathfrak{h}_0}$ are isomorphisms, and clearly $S(\mathfrak{h}_0)^{W_0} \subset S(\mathfrak{h}_0)^{W_0^{\mathfrak{m}}}$),

$$\mu_{\mathfrak{g}/\mathfrak{h}_0}(z) = \mu_{\mathfrak{m}/\mathfrak{h}_0}(\mu(z)), \qquad z \in \mathfrak{z}(\mathfrak{g}).$$

Harish-Chandra showed that $\mathfrak{z}(\mathfrak{m})$ is a finite free-module over $\mu(\mathfrak{z}(\mathfrak{g}))$ of dimension $r = W_0/W_0^{\mathfrak{m}}$. Let $v_1 = 1, \ldots, v_r$ be a basis for this module, and recall that $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{m})$. Let $1 \leq i \leq r$, and H be as above. Then there exists unique $z_{H:ij} \in \mathfrak{z}(\mathfrak{g})$ such that

$$Hv_i = \sum_j \mu(z_{H:ij})v_j.$$

Moreover, if we twist elements in $U(\mathfrak{m}_{\mathbf{C}})$ by the quasicharacter $d_Q : M \mapsto \mathbf{R}$ which appears in the estimates above, say

$$'\zeta = d_Q^{-1} \circ \zeta \circ d_Q, \qquad \zeta \in U(\mathfrak{m}_{\mathbf{C}}),$$

then the above action can be written as

$$H\,'v_i = \sum_j {'\mu}(z_{H:ij})'v_j - \rho(H)'v_i,$$

and the twist is such that

$$z_{H:ij} - {'\mu}(z_{H:ij}) = \xi_{H:ij} \in \bar{\mathfrak{n}}U(\mathfrak{g}_{\mathbf{C}}).$$

In particular if $w \in W/\bar{\mathfrak{n}}W$, then

$$\begin{aligned} H\,'v_iw &= \sum_j z_{H:ij}\,'v_jw - \rho(H)\,'v_iw \\ &= \sum_j \mu_{\mathfrak{g}/\mathfrak{h}_0}(z_{H:ij} : \Lambda)\,'v_jw - \rho(H)\,'v_i, \end{aligned}$$

where $\mu_{\mathfrak{g}/\mathfrak{h}_0}(z_{H:ij})$ is considered as a polynomial on $\mathfrak{h}_{0,\mathbf{C}}^*$ and the notation

$$\mu_{\mathfrak{g}/\mathfrak{h}_0}(z_{H:ij} : \Lambda)$$

denotes the evaluation at Λ. Clearly,

$$'v_1w, \ldots, 'v_rw$$

is a basis for the local $\mathfrak{a}$-action. If Λ is a regular element then the matrix operator

$$B(\Lambda) = (\mu_{\mathfrak{g}/\mathfrak{h}_0}(z_{H:ij} : \Lambda) - \rho(H)\delta_{ij})$$

has eigenvalues $s_i\Lambda(H) - \rho(H)$ where $\{s_i | 1 \le i \le r\}$ is a complete set of representatives for $W_0/W_0^{\mathfrak{m}}$.

Now suppose that $\varphi(x, \lambda) = E_P(x, \varphi, \lambda)$ so that for all $z \in \mathfrak{z}(\mathfrak{g})$ we have

$$\varphi(x; z, \lambda) = \mu_{\mathfrak{g}/\mathfrak{h}_0}(z : X_\lambda)\varphi(x, \lambda), \qquad \lambda \in \mathfrak{a}^*_{P,\mathbf{C}},$$

where the notation "; z" denotes that z is differentiating the x variable, and X_λ denotes the Harish-Chandra parameter in $\mathfrak{h}^*_{0,\mathbf{C}}$ for the homomorphism of $\mathfrak{z}(\mathfrak{g})$ determined by φ. We can then reduce this system of equations satisfied by the function φ to a first order vector equation as follows; let $e_1, \dots, e_r$ denote the standard basis for $\mathbf{C}^r$ and put

$$\Phi(x, \lambda) = \sum_{i=1}^{r} \varphi(x; {}'v_i) \otimes e_i.$$

Now observe that for $1 \le i \le r$ we can write

$$\begin{aligned} \frac{d\varphi}{dt}(m \exp tH; {}'v_i, \lambda) &= \varphi(m \exp tH; H\,{}'v_i) \\ &= \sum_j \varphi(m \exp tH; (z_{H:ij} - \xi_{H:ij})\,{}'v_j - \rho(H)\,{}'v_i) \\ &= \sum_j \{\mu_{\mathfrak{g}/\mathfrak{h}_0}(z_{H:ij} : X_\lambda) - \rho(H)\delta_{ij})\varphi(m \exp tH; {}'v_j) \\ &\qquad - \varphi(m \exp tH; \xi_{H:ij}{}'v_j)\}. \end{aligned}$$

It is a simple matter now to show that Φ satisfies an equation of the following type:

$$\frac{d\Phi(m \exp tH, \lambda)}{dt} = \Gamma(X_\lambda)\Phi(m \exp tH, \lambda) + \Psi(m \exp tH, \lambda).$$

Here

$$\Gamma(X_\lambda) = 1 \otimes B(X_\lambda),$$

and the function Ψ, which will be referred to below as the *perturbation term*, is formed from the terms

$$\phi(m \exp tH; \xi_{H:ij}, \lambda) \otimes e_i.$$

The next step is to obtain estimates for the function Ψ.

The $\xi_{H:ij}$ belong to the ideal $\bar{\mathfrak{n}}U(\mathfrak{g}_{\mathbf{C}})$. If $Y \in \bar{\mathfrak{n}}$, and $\eta \in U(\mathfrak{g}_{\mathbf{C}})$, then

$$\varphi(x; Y\eta, \lambda) = \varphi(Y^x; x; \eta, \lambda),$$

where Y^x denotes the adjoint action of x on Y, and the notation "$Y^x;$" denotes the right G-invariant derivative determined by Y^x. If $x = m \exp tH \in A_0(B|Q)$, then it can be shown that

$$Y^x \mapsto 0, \quad \text{as } t \mapsto \infty,$$

the decay being of the order of e^{-t}. Combining this result with the a priori estimates for the functions $\varphi = E_P$, we easily show that $\|\Psi(m \exp tH, \lambda)\|$ is bounded by a constant times

$$|(m, \lambda, t)|^r e^{-(1+\rho(H))t} d_Q(m)^{-1} \Xi_M(m),$$

the estimate holding for $m \exp tH$ essentially varying in $A_0(B|Q)$, and $\lambda \in (-1)^{1/2}\mathfrak{a}_P^*$. We have a similar estimate for $\|\Phi(m \exp tH, \lambda)\|$ with exponential term

$$e^{-\rho(H)t};$$

so we pick up extra exponential decay for $\|\Psi(m \exp tH, \lambda)\|$ thanks to the right derivatives from the ideal $\bar{\mathfrak{n}}U(\mathfrak{g}_{\mathbf{C}})$.

The above differential equation is equivalent to the integral equation

$$\Phi(m \exp tH, \lambda) = \exp\{t\Gamma(X_\lambda)\}\Phi(m, \lambda) + \int_0^t \exp\{(t-u)\Gamma(X_\lambda)\}\Psi(m \exp uH, \lambda)\, du.$$

Let $E_0(\lambda)$ (resp. $E_\pm(\lambda)$) denote the projection which maps $V \otimes \mathbf{C}^r$ onto the eigen-subspaces of $\Gamma(X_\lambda)$ which have zero (resp. positive or negative) real part. Let $\Phi_0(m, \lambda)$ equal

$$E_0(\lambda)\exp\{t\Gamma(X_\lambda)\}\Phi(m, \lambda) + \int_0^\infty E_0(\lambda)\exp\{(t-u)\Gamma(X_\lambda)\}\Psi(m \exp uH, \lambda)\, du.$$

The integral in this expression converges thanks to the estimate of $\|\Psi(m \exp uH, \lambda)\|$. $\Phi_0(m, \lambda)$ is essentially Harish-Chandra's constant term; more precisely if we write

$$\Phi_0(m, \lambda) = \sum_{i=1}^r \Phi_{0,i}(m, \lambda) \otimes e_i$$

then the constant term of $\varphi(m, \lambda)$ along Q is the function $\varphi_0(m, \lambda) = \Phi_{0,1}(m, \lambda)$.

The error estimate $\|\varphi(m \exp tH, \lambda) - \varphi_0(m \exp tH, \lambda)\|$ is no worse than

$$\|\Phi(m \exp tH, \lambda) - \Phi_0(m \exp tH, \lambda)\|$$

which in turn is bounded by,

$$\|E_0(\lambda)\Phi(m \exp tH, \lambda) - \Phi_0(m \exp tH, \lambda)\| \\ + \|E_+(\lambda)\Phi(m \exp tH, \lambda)\| + \|E_-(\lambda)\Phi(m \exp tH, \lambda)\|.$$

Multiplying the above integral expression for $\Phi(m \exp tH, \lambda)$ by $E_0(\lambda)$ one can easily estimate the first term in this last expression and bound it by an error estimate of the type described above with $n = 0$. The remaining two terms also satisfy similar estimates but they require more work. What is important to point out about the estimates of these terms is that the exponential factor in their estimates is determined by two things: first the rate of decay of $\|\Psi(m \exp tH, \lambda)\|$ and second, the eigenvalues of $\Gamma(X_\lambda)$ whose real parts are negative; the positive eigenvalues cannot contribute to the limit because of a priori estimates satisfied by the functions $\varphi = E_P$.

As already mentioned above, φ_0 gives the leading terms of our expansion, that is, the terms corresponding to $n = 0$. The estimates we have described above lead to the desired "uniform estimates" for the partial sum

$$\sum_s \Gamma_{s,0}(m, \lambda) e^{t(s\lambda - \rho_Q)(H)}.$$

To this point we have taken $k = 1$ in $\bar{\mathfrak{n}}^k$. We now wish to obtain the coefficients $\Gamma_{s,n}$ for $n > 0$. To do this we must first obtain a basis for the local $\mathfrak{a}$-action on $W/\bar{\mathfrak{n}}^k W$ for any Harish-Chandra module W. We do this by extending the set of elements

$$\{'v_1, \dots, 'v_r\}$$

defined above. The elements we add are obtained as follows. First note that from the above we have

$$H'v_i = \sum_j \{z_{H:ij}'v_j - \xi_{H:ij}'v_j\} - \rho(H)'v_i.$$

Also recall that $\xi_{H:ij} \in \bar{\mathfrak{n}}U(\mathfrak{g}_\mathbf{C})$. Let $Y \in \bar{\mathfrak{n}}$ and $\eta \in U(\mathfrak{g}_\mathbf{C})$. Observe that we can compute the $\mathfrak{a}$-action on expressions of the form $Y'v_i\eta$

as follows:

$$\begin{aligned} HY\,'v_i\eta &= [H\,,\,Y]\,'v_i\eta + YH\,'v_i\eta \\ &\equiv \sum_{\substack{\alpha\in\Delta(B\,,\,A_0)\\ \alpha(H)\neq 0}} -\alpha(H)Y_\alpha\,'v_i\eta \\ &\quad + \sum_j (z_{H\,:\,ij} - \rho(H)\delta_{ij})Y\,'v_j\eta \pmod{\bar{\mathfrak{n}}^2 U(\mathfrak{g})}\,, \end{aligned}$$

where $Y_\alpha \in \mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [L\,,\,X] = \alpha(L)X\,,\ \forall L \in \mathfrak{a}_0\}$.

Fix a basis for $\bar{\mathfrak{n}}$, say

$$\{Y_1\,,\,\cdots\,,\,Y_n\}\,,$$

with Y_i belonging to some $\mathfrak{g}_\gamma$, $\gamma \in \Delta(B\,,\,A_0)$. We can then write

$$\xi_{H\,:\,ij} = \sum_{k=1}^n Y_k\eta_k(ij).$$

Note that

$$v_1 = 1 \Rightarrow {}'v_i = 1.$$

It should be clear from the above calculation and the last equality that if $w \in W/\bar{\mathfrak{n}}^2 W$ then the $\mathfrak{a}$-action on w is contained (assuming that W has infinitesimal character) in the subspace spanned by the vectors

$$'v_1 w\,,\,\ldots\,,\,'v_r w\,,\ \bigcup_{i\,,\,j\,,\,k\,,\,l} \{Y_k{}'v_l\eta_k(ij)w\}.$$

Hence we can continue to expand the set $\{'v_1\,,\,\ldots\,,\,'v_r\}$ for any k. In Lemma 2.1 we obtain these elements which we denote $u_{ij\,:\,M}$ with $1 \le i \le r$, $1 \le j \le d_M$, and $M \in \mathbf{Z}_+^n$. Following the above, we form the vector functions $\Phi(x\,,\,\lambda)$ and $\Psi(x\,,\,\lambda)$ using the derivatives $u_{ij\,:\,M}$ and their "error terms" $\xi_{ij\,:\,M} \in \bar{\mathfrak{n}}_{\mathbf{C}}^k U(\mathfrak{g}_{\mathbf{C}})$. These functions satisfy the same type of first order differential equation as before. Suppose then we fix k sufficiently large. The eigenvalues of $\Gamma(X_\lambda)$ can be linearly ordered by their real parts independent of $\lambda \in (-1)^{1/2}\mathfrak{a}_P^*$ (cf. §5). They are all of the form

$$s_i X_\lambda(H) - \rho(H) - \langle\beta\,,\,M\rangle\,, \qquad 1 \le i \le r\,,\, M \in \mathbf{Z}_+^n\,,\, o(M) \le k\,,$$

where undefined notation is as in §2. Choose $z_0(\lambda)$, one of these eigenvalues such that

$$-k - \rho(H) < \Re z_0(\lambda) \le -n - \rho(H)\,, \qquad \lambda \in (-1)^{1/2}\mathfrak{a}_P^*.$$

Let $E_0(\lambda)$ now denote the projection onto the eigenvalues of $\Gamma(X_\lambda)$ whose real part is greater than or equal to the real part of $z_0(\lambda)$, and

$E_1(\lambda)$ denote the projection onto the remaining eigenvalues of $\Gamma(X_\lambda)$; $E_1(\lambda)$ takes the place of $E_\pm(\lambda)$ above. Note that if $Y = Y_{i_1} \cdots Y_{i_k} \in \bar{\mathfrak{n}}^k$, $\eta \in U(\mathfrak{g}_C)$, and $x = m \exp tH \in A_0(B|Q)$, then

$$\varphi(x\,;\, Y\eta\,,\, \lambda) = e^{-\langle \beta\,, I\rangle(H)} \varphi(Y^m\,;\, x\,;\, \eta)\,, \qquad I = (i_1\,, \ldots\,, i_n).$$

Consequently we can estimate $\|\Psi(x\,,\lambda)\|$ as above with the exponential term now given by

$$e^{-((k+1)+\rho(H))t}.$$

Define $\Phi_0(x\,,\lambda)$ as above; the above estimate of $\|\Psi(x\,,\lambda)\|$ and the fact that the eigenvalues in the image of $E_0(\lambda)$ have real part greater than or equal to that of $\Re z_0(\lambda)$ where $\Re z_0(\lambda) > -\rho(H) - k$ allows us to easily show that the integral defining Φ_0 exists. These same estimates and bounds on eigenvalues also gives us "uniform estimates" for (Lemma 5.3)

$$\|E_0(\lambda)\Phi(m \exp tH\,,\lambda) - \Phi_0(m \exp tH\,,\lambda)\|$$

and (Lemma 5.2)

$$\|E_1\Phi(m \exp tH\,,\lambda)\|$$

of the type stated in the description of our principal results. Write the function Φ_0 as

$$\sum_{i=1}^{r} \Phi_{0\,,i}(x\,,\lambda) \otimes e_i.$$

Then

$$\varphi_0(x\,,\lambda) = \Phi_{0\,,1}(x\,,\lambda)$$

is the sum of the terms of our asymptotic expansion up to the exponent $z_0(\lambda)$. These results are developed in §§2 through 5.

Section 6 contains some facts concerning Eisenstein integrals. The material for this section is a slight regurgitation of several sections of Arthur [**1**].

The material in §7 was directed by the analogous results in my joint paper with V. S. Varadarajan [**1**]. One final remark; asymptotic expansions "along the walls of $A_0(B)$" for the matrix coefficients of a single representation have existed for some time; they were developed by Harish-Chandra around 1960, becoming available recently in his collected works (also see Casselman and Miličić [**1**] and Wallach [**1**]). The expansion (5.2) already appears in Wallach [**1**]; our contribution is to observe that using the techniques of Wallach [**1**], we are able to carry the dependence of the Eisenstein integrals on the λ-parameter along. In this way we are able to give the precise dependence of the coefficients and exponents of the asymptotic series on λ and make our error estimates uniform in both variables. For technical reasons and with a view to possible future applications, we have carried out our uniform asymptotic analysis not only for the Eisenstein integrals, but more generally for any family of eigenfunctions satisfying the same weak estimates as the Eisenstein integrals. These are the functions of type $\mathrm{II}(\chi)$ and in doing this we have followed Harish-Chandra [**6**].

1. Notation and assumptions. Let G be a reductive Lie group, and K a fixed maximal compact subgroup. Lie subgroups of G will be designated by upper case Roman letters, while their Lie algebras will be denoted by the corresponding lower case German letter. If $\mathfrak{m} \subset \mathfrak{g}$, then $U(\mathfrak{g}_{\mathbf{C}})$ will denote the universal enveloping over $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, and $U(\mathfrak{m}_{\mathbf{C}})$ will denote the subalgebra of $U(\mathfrak{g}_{\mathbf{C}})$ generated by 1 and $\mathfrak{m}_{\mathbf{C}}$. We assume that G and K satisfy the following general axioms: (1) $\mathrm{Ad}(G) \subset G_{\mathbf{C}}$, $G_{\mathbf{C}}$ the complex adjoint group of $\mathfrak{g}_{\mathbf{C}}$; (2) if G_1 is the analytic subgroup of G corresponding to $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, then the center of G_1 is finite; (3) if G^0 denotes the connected component of G, then $[G : G^0] < \infty$. In this case, G and $\mathfrak{g}$ are equipped with a Cartan involution which fixes the elements of K and $\mathfrak{k}$; we shall not distinguish between the two and simply designate it by θ. The corresponding decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces of θ will be written as

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}.$$

We have then that

$$G = K \exp \mathfrak{s}.$$

If $x \in G$ and $x = k \exp X$ with $k \in K$, and $X \in \mathfrak{s}$, then we shall write

$$\log x = X.$$

If P is a parabolic subgroup of G, then it has a decomposition $P = N_P M_P$, and $P = N_P A_P M_P^1$, where N_P is the unipotent radical of P and M_P is a reductive subgroup of G which is θ stable, and A_P is the split component of M_P. We shall call M_P the Levi component of P and say that a subgroup M of G is a Levi subgroup if $M = M_P$ for some parabolic P. Further we shall say a Levi subgroup is minimal (resp. maximal) if the corresponding parabolic is minimal (resp. maximal).

Suppose that $M_1 \subset M$ are two Levi subgroups of G. We shall denote the set of Levi subgroups of M which contain M_1 by $\mathscr{L}^M(M_1)$. Let us also write $\mathscr{F}^M(M_1)$ for the set of parabolic subgroups of M

which contain M_1, and $\mathscr{P}^M(M_1)$ for the set of groups in $\mathscr{F}^M(M_1)$ for which M_1 is the Levi component. Each of these sets is finite. If $M = G$ we shall usually drop the superscript G in the above notations. If $R \in \mathscr{P}^M(M_1)$ and $Q \in \mathscr{P}(M)$, then we shall denote by $Q(R)$ the unique subgroup in $\mathscr{P}(M_1)$ which is contained in Q.

We choose and fix a minimal Levi subgroup M_0. If $M \in \mathscr{L}(M_0)$, let $K_M = K \cap M$. For induction we note that the properties of the triple (G, K, M_0) are inherited by the triple (M, K_M, M_0).

If $P \in \mathscr{F}(M_0)$, then it is known that

$$G = PK = N_P(M_P \cap \exp \mathfrak{s})K = N_P(M_P^1 \cap \exp \mathfrak{s})A_P K.$$

If $x \in G$, then let $n_P(x)$, $m_P(x)$, $\kappa_P(x)$, $\mu_P(x)$, and $a_P(x)$ denote elements of N_P, $M_P \cap \exp \mathfrak{s}$, K, $M_P^1 \cap \exp \mathfrak{s}$, and A_P respectively such that

$$x = n_P(x)m_P(x)\kappa_P(x) = n_P(x)\mu_P(x)a_P(x)\kappa_P(x).$$

Let

$$H_P(x) = H_{M_P}(x) = \log a_P(x).$$

Suppose that $P \in \mathscr{P}(M)$. We shall sometimes write $A_P = A_M$ and $\mathfrak{a}_P = \mathfrak{a}_M$. Associated to P are various real quasicharacters on these two groups. One arises from the modular function δ_P of P. Its restriction to A_P equals

$$\delta_P(a) = a^{2\rho_P} = e^{2\rho_P(\log a)},$$

for a unique vector ρ_P in $\mathfrak{a}_M^*$. We shall also put

$$d_P(a) = a^{\rho_P} = e^{\rho_P(\log a)}.$$

Let $\Delta_P = \Delta(P, A_P)$ denote the roots of the pair (P, A_P), and $\Sigma_P = \Sigma(P, A_P)$ denote the set of simple roots contained in Δ_P. If $F \subset \Delta_P$ then we shall write $\mathbf{Z}(F)$ for the abelian subgroup of $\mathfrak{a}_P^*$ generated by F, and $\mathbf{Z}_+(F)$ for the subset consisting of nonnegative integral combinations of elements in F.

We fix a G-invariant, symmetric bilinear form $(\ ,\ )$ on $\mathfrak{g}$ such that the quadratic form

$$X \mapsto -(X, \theta(X)), \quad X \in \mathfrak{g},$$

is positive definite. Let (for any vector space V over $\mathbf{R}$ let us write $V_{\mathbf{C}}$ for $V \otimes_{\mathbf{R}} \mathbf{C}$)

$$\mathfrak{h}_{0,\mathbf{C}} = \mathfrak{h}_{+,\mathbf{C}} \oplus \mathfrak{a}_{0,\mathbf{C}}$$

here we subscript objects associated to M_0 by 0 rather than by M_0. Then $\mathfrak{h}_{0,\mathbf{C}} \cap \mathfrak{g}$ is a θ stable Cartan subalgebra of $\mathfrak{g}$, and $(\ ,\ )$ is nondegenerate on $\mathfrak{h}_{0,\mathbf{C}}$ and

$$H \mapsto -(H, \theta(H)), \quad H \in \mathfrak{h}_{0,\mathbf{C}} \cap \mathfrak{g},$$

extends to a Hermitian norm on $\mathfrak{h}_{0,\mathbf{C}}$. Then by restriction we obtain a bilinear form $(\ ,\ )$ and a Hermitian norm $\|\cdot\|$ on both $\mathfrak{a}_{0,\mathbf{C}}$ and $\mathfrak{a}^*_{0,\mathbf{C}}$. If $M \in \mathscr{L}(M_0)$, then there are imbeddings $\mathfrak{a}_{M,\mathbf{C}} \subset \mathfrak{a}_{0,\mathbf{C}}$ and $\mathfrak{a}^*_{M,\mathbf{C}} \subset \mathfrak{a}^*_{0,\mathbf{C}}$; hence we can apply $(\ ,\ )$ and $\|\cdot\|$ to these subspaces.

By a singular hyperplane in $\mathfrak{a}^*_M$, for $M \in \mathscr{L}(M_0)$, we mean a subspace of the form

$$\{\lambda \in \mathfrak{a}^*_M | (\lambda, \beta) = 0\}$$

for some root β of (G, A_M). If $P \in \mathscr{P}(M)$, we shall write

$$\mathfrak{a}^*_M(P) = \{\lambda \in \mathfrak{a}^*_M | (\beta, \lambda) > 0, \beta \in \Delta_P\},$$
$$\mathfrak{a}_M(P) = \{H \in \mathfrak{a}_M | \beta(H) > 0, \beta \in \Delta_P\},$$

and

$$A_M(P) = \{a \in A_M | a^\beta > 1, \beta \in \Delta_P\}.$$

Let $M \in \mathscr{L}(M_0)$ and let $W(A_M)$ denote the Weyl group of the pair (G, A_M). If $P_1, P_2 \in \mathscr{F}(M_0)$, then we shall denote by $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ the set of all distinct maps from $\mathfrak{a}_{P_1}$ into $\mathfrak{a}_{P_2}$ that are induced by elements of $W(A_0)$.

We shall call a subspace $\mathfrak{b}$ of $\mathfrak{a}^*_0$ a root subspace if it is of the form $\mathfrak{a}^*_M$ for some $M \in \mathscr{L}(M_0)$. If $\mathfrak{b}$ is such a space and $B \in \mathscr{P}(M_0)$, then put

$$\Delta_{\mathfrak{b}}(B, A_0) = \{\alpha \in \Delta(B, A_0) | \alpha \text{ not perpendicular to } \mathfrak{b}\},$$

and let us write $\mathrm{Chamb}(\mathfrak{b})$ for the open connected subsets of the set

$$\mathfrak{b}' = \{\lambda \in \mathfrak{b} | (\lambda, \alpha) \neq 0 \text{ for some } \alpha \in \Delta_{\mathfrak{b}}(B, A_0)\}.$$

It is known that the elements of $\mathrm{Chamb}(\mathfrak{b})$ are in one to one correspondence with the parabolic subgroups in $\mathscr{F}(M_0)$ whose split component is $\mathfrak{b}$.

Let $B \in \mathscr{P}(M_0)$, $F \subset \Sigma_B$. Then we shall use the notation B_F for the unique parabolic subgroup containing B whose Levi component is the centraliser in G of $\mathfrak{a}_{0,F}$ where

$$\mathfrak{a}_{0,F} = \{H \in \mathfrak{a}_0 | \alpha(H) = 0\ \forall \alpha \in F\}.$$

If $u = X_1 \cdots X_n \in U(\mathfrak{g}_{\mathbf{C}})$, with $X_1, \ldots, X_n \in \mathfrak{g}$, then we shall write for $f \in C^\infty(G)$,

$$f(u; x) = \left. \frac{\partial^n}{\partial s_n \cdots \partial s_1} \right|_{s_1 = \cdots = s_n = 0} f(\exp s_1 X_1 \cdots \exp s_n X_n x),$$

and

$$f(x; u) = \left. \frac{\partial^n}{\partial s_n \cdots \partial s_1} \right|_{s_1 = \cdots = s_n = 0} f(x \exp s_1 X_1 \cdots \exp s_n X_n).$$

If $L \subset \mathbf{R}$, then we use the notation L_+ for the nonnegative elements of L. If L is a finite set, we shall write $|L|$ for the number of elements of L.

Unless specified otherwise, the integrals over the unimodular subgroups of G will always be with respect to a fixed, but unnormalized, Haar measure. There will be two exceptions. On the compact group K_M, $M \in \mathscr{L}(M_0)$, we will always take the Haar measure for which the total volume is one. The second exception concerns the groups connected with the spaces $\mathfrak{a}_M$. On $\mathfrak{a}_M$ we will take the Euclidean measure with respect to the fixed norm $\|\cdot\|$. The exponential map will transform this measure to a fixed Haar measure which is dual to the measure we fixed on $\mathfrak{a}_M$.

2. Differential equations for eigenfunctions with regular infinitesimal character. In this section we shall determine elements $u_{ij:M}$, $\xi_{ij:M} \in U(\mathfrak{g})$ which will allow us to write a first order vector equation (Lemma 2.2) for a function $\Phi(x)$ constructed from a $\mathfrak{z}(\mathfrak{g})$-eigenfunction $\varphi(x)$. As the definitions of $u_{ij:M}$ and $\xi_{ij:M}$ are inductively given, we must first establish some notation and make some choices of basis.

Let $B \in \mathscr{P}(M_0)$, $B = N_0 M_0 = N_0 A_0 M_0^1$, be a fixed minimal parabolic subgroup; henceforth all objects associated with B will be subscripted by 0. We also fix $Q \in \mathscr{F}(M_0)$, $Q \supset B$, Q maximal, $Q = NM = NAM^1$. For ease of notation, we shall denote objects associated to Q with no subscript; for example we shall write ρ_0 for one half the sum of the positive roots (with multiplicities) for $\mathfrak{a}_0$ in $\mathfrak{b}$, and the similarly defined function for Q will be denoted by ρ. We shall fix $\alpha \in \Sigma(\mathfrak{q}, \mathfrak{a})$, and $H \in \mathfrak{a}$ such that

$$\alpha(H) = 1. \tag{2.1}$$

Let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra such that $\mathfrak{h}_0 \subset \mathfrak{m}_0$, and $\mathfrak{h}_{0\mathbf{R}} = \mathfrak{h}_0 \cap \mathfrak{s} = \mathfrak{a}_0$. Let μ_0 denote the Harish-Chandra homomorphism of $\mathfrak{z}(\mathfrak{g})$ onto $U(\mathfrak{h}_{0\mathbf{C}})^{W_0}$, where $W_0 = W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0\mathbf{C}})$, and the superscript

W_0 denotes the set of invariants. As $\mathfrak{a} \subset \mathfrak{a}_0 = \mathfrak{h}_{0\mathbf{R}}$ then $\mathfrak{m} \supset \mathfrak{h}_0$. Denote the Harish-Chandra homomorphism of $\mathfrak{m}$ (relative to $\mathfrak{h}_{0,\mathbf{C}}$) by μ_1 and let μ_{10} be the canonical injection of $\mathfrak{z}(\mathfrak{g})$ into $\mathfrak{z}(\mathfrak{m})$ such that $\mu_0(z) = \mu_1(\mu_{10}(z))$ for all $z \in \mathfrak{z}(\mathfrak{g})$. μ_1 maps $\mathfrak{z}(\mathfrak{m})$ into $U(\mathfrak{h}_{0,\mathbf{C}})^{W_1}$ where $W_1 = W(\mathfrak{m}, \mathfrak{h}_{0,\mathbf{C}})$. W_1 can be identified with the subgroup of W_0 generated by the reflections s_β such that $\beta(H) = 0, H$ as above. Let $r = \dim_{\mu_1(\mathfrak{z}(\mathfrak{g}))} \mathfrak{z}(\mathfrak{m}) = |W_0/W_1|$, and $v_1 = 1, v_2, \dots, v_r$ be a module basis for $\mathfrak{z}(\mathfrak{m})$ over $\mu_{10}(\mathfrak{z}(\mathfrak{g}))$. Define elements $z_{ij} \in \mathfrak{z}(\mathfrak{g})$ $(1 \le i, j \le r)$ by the equation

$$(2.2) \qquad Hv_i = \sum_{j=1}^{r} \mu_{10}(z_{ij}) v_j.$$

As in §1, let for $m \in M^1$, and $a \in A$,

$$(2.3) \qquad d(ma) = d_Q(ma) = e^{\rho(\log a)}.$$

If $\eta \in U(\mathfrak{m}_{\mathbf{C}})$ then we put

$$(2.4) \qquad {}'\eta = d^{-1} \circ \eta \circ d,$$

as differential operators on M.

Fix a basis $\{Y_1, \dots, Y_n\}$ for $\bar{\mathfrak{n}}$ such that

$$(2.5) \qquad [H, Y_i] = -\beta_i(H) Y_i,$$

where $\beta_i \in \mathfrak{a}^*$ is a positive integral multiple of α. If $M \in \mathbf{Z}_+^n$ (where $\mathbf{Z}_+ = \{n \in \mathbf{Z} | n \ge 0\}$) we shall use the notation

$$(2.6) \qquad Y^M = Y_1^{M_1} \cdots Y_n^{M_n} \qquad (M = (M_1, \dots, M_n))$$

where the product is taken in $U(\mathfrak{g}_{\mathbf{C}})$. We also use the notation

$$(2.7) \; \langle \beta, M \rangle (H) = \sum_{i=1}^{n} M_i \beta_i(H),$$

$$o(M) = \sum_{i=1}^{n} M_i, \qquad \omega(M) \in Z \ni \langle \beta, M \rangle (H) = \omega(M).$$

If V is a fixed finite dimensional complex vector space, $\Lambda \in \mathfrak{h}_{0,\mathbf{C}}^*$ then $\mathscr{A}(G : V : \Lambda)$ will denote the complex linear space of all $\varphi \in C^\infty(G, V)$ such that

$$(2.8) \qquad z\varphi = \mu_0(z : \Lambda)\varphi \qquad (z \in \mathfrak{z}(\mathfrak{g})).$$

Here $\mu_0(z) \in U(\mathfrak{h}_{0,\mathbf{C}})$ and we consider it as a polynomial function on $\mathfrak{h}_{0,\mathbf{C}}^*$; the notation $\mu_0(z : \Lambda)$ denotes the value of this function on Λ.

LEMMA 2.1. *For each* $M \in \mathbf{Z}_+^n$, *there exists* $d = d_M \in \mathbf{Z}_+$, *and for each* $0 \le j \le d$, *there exists* $u_{lj:M} \in U(\mathfrak{g}_{\mathbf{C}})$, $\xi_{lj:M} \in \bar{\mathfrak{n}}_{\mathbf{C}}^{(k+1)} U(\mathfrak{g}_{\mathbf{C}})$ $(1 \le l \le r$, *with* $k = o(M))$ *such that for* $\Lambda \in \mathfrak{h}_{0,\mathbf{C}}^*$, *if*

$$s_{\alpha\gamma} : M(\Lambda) = \mu_0(z_{\alpha\gamma} : \Lambda) - \{\langle \beta , M\rangle(H) + \rho(H)\}\delta_{\alpha\gamma} \qquad (\delta_{\alpha\gamma} = \textit{Kronecker delta})$$

and $\varphi \in \mathscr{A}(G : V : \Lambda)$, *then*

$$\frac{d}{dt}\varphi(m \exp tH ; u_{ij:M}) = \sum_{\alpha=1}^{r} s_{i\alpha:M}(\Lambda)\varphi(m\exp tH ; u_{\alpha j:M}) + \varphi(m\exp tH ; \xi_{ij:M}) \tag{2.9}$$

the equality holding for all $m \in M$ *and* $t \in \mathbf{R}$.

Proof. We shall proceed by induction on $k = o(M)$. For $k = 0$ let $v_1, \dots, v_r$ be as above a module basis for $\mathfrak{z}(\mathfrak{m})$ over $\mu_{10}(\mathfrak{z}(\mathfrak{g}))$. Set $d_0 = 0$ and

$$u_{i0:O} = {}'v_i , \quad \xi_{i0:O} = \sum_{j=1}^{r}({}'\mu_{10}(z_{ij}) - z_{ij})u_{j0:O}. \tag{2.10}$$

For H as in (2.1) we have

$$\begin{aligned} Hu_{i0:O} &= {}'\{(H - \rho(H))v_i\} \\ &= {}'\left\{\sum_{j=1}^{r}\mu_{10}(z_{ij})v_j - \rho(H)v_i\right\} \\ &= \sum_{j=1}^{r}{}'\mu_{10}(z_{ij})u_{j0:O} - \rho(H)u_{i0:O} \\ &= \sum_{j=1}^{r}\{z_{ij}u_{j0:O} + ({}'\mu_{10}(z_{ij}) - z_{ij})u_{j0:O}\} - \rho(H)u_{i0:O}. \end{aligned}$$

It is well known that $\xi_{i0:O} \in \bar{\mathfrak{n}}_{\mathbf{C}} U(\mathfrak{g}_{\mathbf{C}})$ and (2.9) holds for all $\varphi \in \mathscr{A}(G : V : \Lambda)$.

Assume the existence of d_M, $u_{ij:M}$, $\xi_{ij:M}$, for all $M \in \mathbf{Z}_+^n$ with $o(M) = k$, $k \ge 0$. Let $N \in \mathbf{Z}_+^n$, $o(N) = k + 1$. We wish to define the elements d_N, $u_{ij:N}$, $\xi_{ij:N}$. First, for every $M \in \mathbf{Z}_+^n$ such that $o(M) = k$ and i, j such that $1 \le i \le r$, $1 \le j \le d_M$ let us write

$$\xi_{ij:M} = \sum_{o(K)=k+1} Y^K \delta_K(i, j : M). \tag{2.11}$$

Let $\delta_{1K}, \ldots, \delta_{dK}$, $d = d_K$, be an enumeration of the distinct, nonzero elements of the set $\{\delta_K(i, j : M) | M \in \mathbf{Z}_+^n,\ o(M) = k,\ 1 \le j \le d_M,\ 1 \le i \le r\}$. We now use each of the terms δ_{jN} to "build" r additional terms as follows: put

$$(2.12)\qquad \begin{aligned} u_{ij:N} &= Y^N u_{i0:O}\delta_{jN} \\ \xi_{ij:N} &= Y^N \xi_{i0:O}\delta_{jN} \end{aligned} \qquad (1 \le i \le r,\ 1 \le j \le d_N).$$

Then for H as in (2.1) we have

$$\begin{aligned} Hu_{il:N} = HY^N u_{i0:O}\delta_{lN} &= -\langle\beta, N\rangle(H)u_{il:N} + Y^N H u_{i0:O}\delta_{lN} \\ &= -\langle\beta, N\rangle(H)u_{il:N} + \sum_{j=1}^{r} z_{ij}u_{jl:N} + \xi_{il:N} - \rho(H)u_{il:N}. \end{aligned}$$

This last line follows from the above calculation in the $k = 0$ case. As above we see that (2.9) holds for all $\varphi \in \mathscr{A}(G : V : \Lambda)$. □

Let us fix $k \in \mathbf{Z}_+$, and let $d_k = r\sum_{o(M)\le k} d_M$. We denote by T_k an arbitrary complex inner product space of dimension d_k. We choose an orthonormal basis for T_k which we can index as $\{e_{ij:M} | M \in \mathbf{Z}_+^n,\ o(M) \le k,\ 1 \le i \le r,\ 1 \le j \le d_M\}$. We shall identify linear operators on T_k with their matrices relative to this basis. In particular let us put with $M \in \mathbf{Z}_+^n$, $o(M) \le k$, $1 \le i \le r$, $1 \le j \le d_M$,

$$(2.13)\qquad S_k(\Lambda)e_{ij:M} = \sum_{\alpha=1}^{r} s_{\alpha j:M}(\Lambda)e_{\alpha j:M}.$$

Let

$$f_i = \mu_1(v_i) \qquad (1 \le i \le r).$$

Fix $\{s_1, \ldots, s_r\}$ a complete set of representatives for W_0/W_1. Let Δ_0^+ be a positive system of roots for $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}})$ and put

$$(2.14)\qquad \begin{aligned} &\Delta_1^+ = \Delta_0^+ \cap \Delta(\mathfrak{m}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}}), \\ &\overline{\omega}_j = \prod_{\alpha\in\Delta_j^+} \alpha \quad (j = 0, 1), \qquad \overline{\omega}_{01} = \overline{\omega}_0/\overline{\omega}_1. \end{aligned}$$

If $\Lambda \in \mathfrak{h}_{0,\mathbf{C}}^{*\prime}$, where

$$(2.15)\qquad \mathfrak{h}_{0,\mathbf{C}}^{*\prime} = \{\Lambda \in \mathfrak{h}_{0,\mathbf{C}}^{*} | \overline{\omega}_0(\Lambda) \ne 0\}$$

and if

$$(2.16)\qquad e_{ij:M}(\Lambda) = \sum_{\alpha=1}^{r} f_\alpha(s_i\Lambda)e_{\alpha j:M}$$

$$(M \in \mathbf{Z}_+^n,\ o(M) \le k,\ 1 \le i \le r,\ 1 \le j \le d_M),$$

then

$$T_k = \bigoplus \mathbf{C} e_{ij:M}(\Lambda) \tag{2.17}$$

and

$$S_k(\Lambda) e_{ij:M}(\Lambda) = X_M(s_i\Lambda) e_{ij:M}(\Lambda) \qquad \text{(for all } M, i, j \text{ as above)} \tag{2.18}$$

where

$$X_M(\Lambda) = \Lambda(H) - \langle \beta, M\rangle(H) - \rho(H). \tag{2.19}$$

We also define a nilpotent operator N_k on T_k; to this end let us recall in the definition of $\xi_{ij:M}$ that as $u_{10:O} = 1$ then by (2.11) for all $N \in \mathbf{Z}_+^N$, $o(N) < k$ we can write for $1 \le i \le r$, $1 \le l \le d_N$,

$$\xi_{il:N} = \sum_{o(K)=o(N)+1} \sum_{f=1}^{d_K} n_{fK}(i, l:N) u_{1f:K}, \tag{2.20}$$

where $n_{fK} = n_{fK}(i, l:N) \in \{1, 0\}$ and for each K only one n_{fK} $(1 \le f \le d_K)$ can be different from zero. Set

$$n_{jf:K}(i, l:N) = \delta_{1j} n_{fK}(i, l:N) \qquad (\delta_{1j} = \text{Kronecker delta}). \tag{2.21}$$

We now define N_k by the following equation:

$$N_k e_{jf:K} = (1 - \delta_{KO}) \sum_{o(N)=o(K)-1} \sum_{l=1}^{d_N} \sum_{i=1}^{r} n_{jf:K}(i, l:N) e_{il:N}. \tag{2.22}$$

In this last equation the notation δ_{KO} denotes the Kronecker delta function on the set of all n-tuples of nonnegative integers. Finally, set

$$B_k = B_k(\Lambda) = N_k + S_k(\Lambda). \tag{2.23}$$

LEMMA 2.2. *Let $\varphi \in \mathscr{A}(G:V:\Lambda)$, $a_t = \exp tH$, $m \in M$, and set*

$$\Phi_k(m) = \sum_{i,j,M} \varphi(m; u_{ij:M}) \otimes e_{ij:M} \qquad (o(M) \le k, 1 \le j \le d_M, 1 \le i \le r) \tag{2.24}$$

$$\Psi_k(m) = \sum_{i,j,N} \varphi(m; \xi_{ij:N}) \otimes e_{ij:N} \qquad (o(N) = k, 1 \le j \le d_N, 1 \le i \le r)$$

$$\Gamma_k(\Lambda) = 1 \otimes B_k(\Lambda) \qquad (1 = \textit{identity operator on } V).$$

Then Φ_k, $\Psi_k \in C^\infty(M : V \otimes T_k)$ *and (with* $\Phi = \Phi_k$, $\Psi = \Psi_k$, $\Gamma = \Gamma_k$*)*

$$\frac{d}{dt}\Phi(ma_t) = \Gamma(\Lambda)\Phi(ma_t) + \Psi(ma_t). \tag{2.25}$$

Equivalently,

$$\Phi(ma_t) = \mathrm{Exp}\{t\Gamma(\Lambda)\}\Phi(m) + \int_0^t \mathrm{Exp}\{(t-u)\Gamma(\Lambda)\}\Psi(ma_u)\,du. \tag{2.26}$$

Moreover, if $\eta_1, \eta_2, \in U(\mathfrak{m}_{\mathbf{C}})$ *then*

$$\Phi(\eta_1 ; ma_t ; \eta_2) = \mathrm{Exp}\{t\Gamma(\Lambda)\}\Phi(\eta_1 ; m ; \eta_2) + \int_0^t \mathrm{Exp}\{(t-u)\Gamma(\Lambda)\}\Psi(\eta_1 ; ma_u ; \eta_2)\,du \tag{2.27}$$

Proof. With the above notation we have from Lemma 2.1,

$$\begin{aligned}
&\sum_{i,j,M} \varphi(ma_t ; Hu_{ij:M}) \otimes e_{ij:M} \\
&= \sum_{\substack{i,j,M \\ o(M)<k}} \sum_{\alpha=1}^{r} s_{i\alpha:M}\varphi(ma_t ; u_{\alpha j:M}) \otimes e_{ij:M} \\
&\quad + \sum_{\substack{i,j,M \\ o(M)<k}} \sum_{o(K)=o(M)+1} \sum_{f=1}^{d_K} n_{fK}(i, j: M)\varphi(ma_t ; u_{1f:K}) \otimes e_{ij:M} \\
&\quad + \sum_{\substack{i,j,M \\ o(M)=k}} \left\{ \sum_{\alpha=1}^{r} s_{i\alpha:M}\varphi(ma_t ; u_{\alpha j:M}) \otimes e_{ij:M} + \varphi(ma_t ; \xi_{ij:M}) \otimes e_{ij:M} \right\} \\
&= \sum_{\alpha,j,M} \sum_{i=1}^{r} \varphi(ma_t ; u_{\alpha j:M}) \otimes s_{i\alpha:M}e_{ij:M} \\
&\quad + \sum_{\substack{i,j,M \\ o(M)<k}} \sum_{o(K)=o(M)+1} \sum_{f=1}^{d_K} \sum_{\beta=1}^{r} n_{\beta f:K}(i, j : M) \times \varphi(ma_t ; u_{\beta f:K}) \otimes e_{ij:K} \\
&\quad + \sum_{\substack{i,j,M \\ o(M)=k}} \varphi(ma_t ; \xi_{ij:M}) \otimes e_{ij:M}.
\end{aligned}$$

But we may rewrite the second sum occurring on the right-hand side of the equal sign in the last equality as follows:

$$\begin{aligned}
&\sum_{o(K)=1}^{k} \sum_{f=1}^{d_K} \sum_{\beta=1}^{r} \varphi(ma_t\,;\, u_{\beta f\,:\,K}) \\
&\qquad \otimes \sum_{i=1}^{r} \sum_{o(M)=o(K)-1} \sum_{j=1}^{d_M} n_{\beta f\,:\,K}(i\,,\, j : M) e_{ij\,:\,M} \\
&\quad = \sum_{o(K)=1}^{k} \sum_{f=1}^{d_K} \sum_{\beta=1}^{r} \varphi(ma_t\,;\, u_{\beta f\,:\,K}) \otimes N_k e_{\beta f\,:\,K}.
\end{aligned}$$

Observe that as $N_k(e_{\beta f\,:\,O}) = 0$ we may start the summation over K at $o(K) = 0$. It follows that

$$\begin{aligned}
&\sum_{i,j,M} \varphi(ma_t\,;\, Hu_{ij\,:\,M}) \otimes e_{ij\,:\,M} \\
&\qquad = \sum_{\alpha,j,M} \varphi(ma_t\,;\, u_{\alpha j\,:\,M}) \otimes S_k e_{\alpha j\,:\,M} \\
&\qquad\quad + \sum_{\beta,f,K} \varphi(ma_t\,;\, u_{\beta f\,:\,K}) \otimes N_k e_{\beta f\,:\,K} \\
&\qquad\quad + \sum_{\substack{i,j,M \\ o(M)=k}} \varphi(ma_t\,;\, \xi_{ij\,:\,M}) \otimes e_{ij\,:\,M}.
\end{aligned}$$

Recalling (2.24) we see (2.26) follows; (2.27) follows since $\Psi(m)$ is C^∞ on M. □

PROPOSITION 2.3. *Let* $\Lambda \in \mathfrak{h}^{*\prime}_{0,\mathbf{C}}$, $G = \mathrm{Gl}(d_k\,,\, \mathbf{C})$. *Then if* $B_k(\Lambda)^G$ *denotes the orbit of* $B_k(\Lambda)$ *under similarity transformations by* G *we have*

$$B_k(\Lambda)^G \ni S_k(\Lambda).$$

In particular, $B_k(\Lambda)$ *is semisimple and has eigenvalues* $X_M(s_j\Lambda)$ *with* $1 \le j \le r$, $M \in \mathbf{Z}^n_+$, *and* $o(M) \le k$.

Proof. We proceed by induction on k. For $k = 0$, $B_0 = S_O(\Lambda)$ and the result is trivial (the semisimplicity follows from (2.16) and (2.18)). Assume the result for $k \ge 0$. Let P_{k+1} denote the minimal polynomial of S_{k+1}; we wish to show that P_{k+1} is also the minimal polynomial of B_{k+1}. Observe that we can identify T_k with a subspace of T_{k+1} (which we again denote by T_k) defined by taking the span of the vectors $\{e_{ij:M} | o(M) \le k,\ 1 \le j \le d_M,\ 1 \le i \le r\}$; and that $T_{k+1} = T_k \oplus T_{kk+1}$ (orthogonal direct sum) where T_{kk+1} is the span of the remaining vectors $e_{ij:M}$ not in T_k. Further we have

$$(S_{k+1} + N_{k+1})T_k \subset T_k, \quad (S_k + N_k)|_{T_k} = (S_{k+1} + N_{k+1})|_{T_k}.$$

It follows that by the induction hypothesis that as $P_{k+1}(x) = P_{kk+1}(x)P_k(x)$, where $P_{kk+1}(x) = \prod(x - X_K(s_j\Lambda))$, the product taken over a distinct set of representatives of the set $\{X_K(s_j\Lambda) | o(K) = k+1,\ 1 \le j \le d_K\}$ then

$$P_{k+1}(S_{k+1} + N_{k+1})|_{T_k} = P_{kk+1}(S_k + N_k)|_{T_k} P_k(S_k + N_k)|_{T_k} \equiv 0.$$

From their definitions it is clear that

$$N_{k+1}(T_{kk+1}) \subset T_k, \quad P_{kk+1}(S_{k+1})|_{T_{kk+1}} \equiv 0$$

and

$$(S_{k+1} - X_M(s_j\Lambda)I)T_k \subset T_k, \quad (S_{k+1} - X_M(s_j\Lambda)I)T_{kk+1} \subset T_{kk+1}.$$

It follows that

$$\begin{aligned} P_{k+1}(S_{k+1} + N_{k+1})T_{kk+1} &= P_k(S_{k+1} + N_{k+1})P_{kk+1}(S_{k+1} + N_{k+1})T_{kk+1} \\ &\subset P_k(S_{k+1} + N_{k+1})T_k + P_k(S_{k+1} + N_{k+1})P_{kk+1}(S_{k+1})T_{kk+1} = 0. \end{aligned}$$

Therefore, as $T_{k+1} = T_k \oplus T_{kk+1}$ the minimal polynomial of B_{k+1} is just P_{k+1} which implies that B_{k+1} is semisimple, with eigenvalues precisely those of S_{k+1}. □

Let for any $n \in \mathbf{Z}$

$$X_n(\Lambda) = \Lambda(H) - n - \rho(H).$$

PROPOSITION 2.4. *Let $P_k(\Lambda : x)$ be the minimal polynomial of $B_k(\Lambda)$ $(\Lambda \in \mathfrak{h}^{*\prime}_{0,\mathbf{C}})$ and let for $n \in wt_k(A) = \{\omega(N) | N \in \mathbf{Z}^n_+ \ni o(N) \le k\}$*

$$P^n_k(\Lambda : x) = P_k(\Lambda : x)/(x - X_n(\Lambda)).$$

Then

$$\sum_{n,j} (P^n_k(s_j\Lambda : X_n(s_j\Lambda)))^{-1} P^n_k(s_j\Lambda : x) = 1, \tag{2.28}$$

here the sum is over a maximal set of pairs (n, j) *such that* $X_n(s_j\Lambda) \neq X_m(s_i\Lambda)$ *for* $(n, j) \neq (m, i)$ *with* $m, n \in wt_k(A)$ *and* $1 \leq i, j \leq r$. *Further, if*

$$E_n(\Lambda) = P_k^n(\Lambda\colon X_n(\Lambda))^{-1} P_k^n(\Lambda\colon B_k(\Lambda)) \tag{2.29}$$

then

$$\begin{aligned} B_k(\Lambda)E_n(s_j\Lambda) &= E_n(s_j\Lambda)B_k(\Lambda) \\ &= X_n(s_j\Lambda)E_n(s_j\Lambda) \qquad (\Lambda \in \mathfrak{h}_{0,\mathbf{C}}^{*\prime}) \end{aligned} \tag{2.30}$$

Proof. (2.28) follows easily from the partial fraction decomposition for polynomials with distinct roots applied to $1/P_k(\Lambda\colon x)$ on noting that as $B_k(s\Lambda) = B_k(\Lambda)$ $(s \in W_0)$, then $P_k(s\Lambda\colon x) = P_k(\Lambda\colon x)$. (2.29) follows from the Primary Decomposition Theorem. □

COROLLARY 2.5. $\Lambda \mapsto E_n(\Lambda)$ *is a rational map of* $\mathfrak{h}_{0,\mathbf{C}}^{*\prime}$ *into* $\mathrm{End}(T_k)$ *and the map*

$$\Lambda \mapsto P_n(\Lambda\colon X_n(\Lambda))E_n(\Lambda)$$

is polynomial.

3. Uniform estimates for functions of type $\mathrm{II}(\chi)$. We begin this section by introducing a class of functions which we are able to approximate by an asymptotic expansion. This class contains the matrix elements of cuspidal principal series representations, the so-called Eisenstein integrals. The introduction of this class follows Harish-Chandra [**6**]; it also is done for notational convenience; it's definition is a characterisation of those properties of Eisenstein integrals which we will need for later work.

We now fix a cuspidal parabolic subgroup $P = N_P M_P$, P standard with respect to B and we assume that $W(\mathfrak{a}, \mathfrak{a}_P) \neq \emptyset$. Let $\mathfrak{h} \subset \mathfrak{m}_P$ be a θ-stable Cartan subalgebra such that $\mathfrak{h}_{\mathbf{R}} = \mathfrak{a}_P$ $(\mathfrak{h}_{\mathbf{R}} = \mathfrak{h} \cap \mathfrak{s})$. An element $\chi \in (\mathfrak{h}_{\mathrm{I}})_{\mathbf{C}}^*$ $(\mathfrak{h}_{\mathrm{I}} = \mathfrak{h} \cap \mathfrak{t})$ is called *singular* if $\chi(H_\beta) = 0$ for some imaginary root $\beta \in \Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and *regular* otherwise. Put $\mathscr{F} = (-1)^{1/2}\mathfrak{a}_P^*$, and $\overline{\omega} = \overline{\omega}_{\mathfrak{g}/\mathfrak{h}} = \prod_{\beta>0} H_\beta$ where $\beta > 0$ means that the product is taken over some system of positive roots of $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Fix a regular $\chi \in (-1)^{1/2}\mathfrak{h}_{\mathrm{I}}^*$ and extend it trivially to $\mathfrak{h}$; similarly, extend any element of $\mathscr{F}_{\mathbf{C}} = \mathfrak{a}_{P\mathbf{C}}^*$ trivially to $\mathfrak{h}$ (note that our notation is nonstandard and nontypical for this paper as $\mathscr{F}_{\mathbf{C}}$ is not the complexification of $\mathscr{F}$; however, it avoids a lot of subscripting and will be mostly used in this section). Let $\mathscr{F}'_{\mathbf{C}}(\chi)$ denote the subset of $\mathscr{F}_{\mathbf{C}}$ of those elements such that $\overline{\omega}(\chi+\lambda) \neq 0$ and set $\mathscr{F}'(\chi) = \mathscr{F} \cap \mathscr{F}'_{\mathbf{C}}(\chi)$. Then $\mathscr{F}'(\chi)$ is an open dense subset of $\mathscr{F}$.

As we have assumed that P is standard with respect to B, then $\mathfrak{m}_P$ contains both $\mathfrak{h}$ and $\mathfrak{h}_0$. It follows that there exists $y \in G_{\mathbf{C}}$, the complex adjoint group of $\mathfrak{g}_{\mathbf{C}}$, such that y centralizes $\mathfrak{h}_{\mathbf{R}} = \mathfrak{a}$ and $\mathfrak{h}_{\mathbf{C}}^y = \mathfrak{h}_{0,\mathbf{C}}$. Put $X_\lambda = (\chi+\lambda)^y$ for $\lambda \in \mathscr{F}_{\mathbf{C}}$. Then if $\lambda \in \mathscr{F}'_{\mathbf{C}}(\chi)$ it is clear that $\overline{\omega}_0(X_\lambda) \neq 0$.

Let $\mathscr{D} = \mathscr{S}(\mathfrak{a}_{P\mathbf{C}}) \otimes \mathscr{S}(\mathscr{F}_{\mathbf{C}})$ (if W is a vector space we shall denote the symmetric algebra over W by $\mathscr{S}(W)$; we also consider elements of $\mathscr{S}(\mathfrak{a}_{P\mathbf{C}})$ as polynomial functions on $\mathfrak{a}^*_{P\mathbf{C}}$, and $\mathscr{S}(\mathscr{F}_{\mathbf{C}})$ as derivatives of functions defined on the Euclidean space $\mathscr{F}_{\mathbf{C}}$), $g_1 \otimes g_2 \in U(\mathfrak{g}_{\mathbf{C}})^{(2)} = U(\mathfrak{g}_{\mathbf{C}}) \otimes U(\mathfrak{g}_{\mathbf{C}})$, and let $\varphi \in C^\infty(G \times \mathscr{F} : V)$. We put for $p \otimes u \in \mathscr{D}$

$$\varphi(x, \lambda; p \otimes u) = p(\lambda)\varphi(x, \lambda; u),$$

$$(g_1 \otimes g_2 \otimes p \otimes u\varphi)(x, \lambda) = \varphi(g_1 ; x; g_2, \lambda; p \otimes u).$$

Further, we put $\widetilde{U}(\mathfrak{g}_{\mathbf{C}}) = U(\mathfrak{g}_{\mathbf{C}})^2 \otimes \mathscr{D}$, and for $D \in \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, $r \in \mathbf{R}$ set

$$S_{D,r}(\varphi) = \sup_{G \times \mathscr{F}} \|D\varphi(x, \lambda)\| \Xi^{-1}(x) |(x, \lambda)|^{-r},$$

where if $x = \kappa_0(x) \exp H_0(x) n_0(x) = k \exp X$, with k, $\kappa_0(x) \in K$, $H_0(x) \in \mathfrak{a}_0$, $n_0(x) \in \mathfrak{n}_0$, $X \in \mathfrak{s}$ we let

$$(3.1) \quad \Xi(x) = \int_K e^{-\rho_0(H_0(xk))}\, dk, \qquad |(x, \lambda)| = (1+\sigma(x))(1+\|\lambda\|)$$

and

$$\sigma(x) = \|X\|.$$

If $F \subset \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, $|F| < \infty$, let

$$S_{F,r}(\varphi) = \sum_{D \in F} S_{D,r}(\varphi).$$

A function $\varphi : G \times \mathscr{F} \mapsto V$ will be called *a function of* type $\mathrm{II}(\chi)$ if the following conditions hold:

(1) $\varphi \in C^\infty(G \times \mathscr{F} : V)$.

(2) for any $\lambda \in \mathscr{F}$, the function $\varphi_\lambda(x) = \varphi(x, \lambda)$ is τ-spherical.

(3) $z\varphi_\lambda = \mu_0(z : X_\lambda)\varphi_\lambda$ $(z \in \mathfrak{z}(\mathfrak{g}))$.

(4) For any $D \in \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, we can choose a number $r \geq 0$ such that $S_{D,r}(\varphi) < \infty$.

Assume now that V is a double unitary K-module, $\tau = (\tau_l, \tau_r)$, $\tau(k)v = \tau_l(k)v$, $v\tau(k) = v\tau_r(k)$. If $x \in G$, we can write $x = n_P(x)a_P(x)\mu_P(x)\kappa_P(x)$ where $\kappa_P(x) \in K$, $\mu_P(x) \in M_P^1 \cap \exp\mathfrak{s}$, $a_P(x) \in A_P$, $n_P(x) \in N_P$, and we put $H_P(x) = \log a_P(x)$. Then if $\psi \in C^\infty(M_P : V : \tau)$, where we use the obvious notation for the infinitely differentiable, V-valued functions on G which are τ-spherical, we define the *Eisenstein integral* $E_P(x, \psi, \lambda)$ by the equation,

$$E_P(x, \psi, \lambda) = \int_{K_{M_P}\backslash K} \tau(k^{-1})\psi_P(kx)e^{(\lambda+\rho_P)H_P(kx)}\, dk \tag{3.2}$$

where ψ_P is the function on G such that

$$\psi_P(nmk) = \psi(m)\tau(k) \qquad (n \in N_P,\, m \in M_P,\, k \in K).$$

We shall devote the next section to a more thorough investigation of the properties of these important functions. For the present however, we content ourselves with the following observation, indicative of the importance of functions of type $\mathrm{II}(\chi)$. If $\psi \in \mathscr{A}_{\mathrm{cusp}}(M_P, \tau)$ (the space of τ-spherical cusp forms on M_P (cf. §3)) and $\zeta\psi = \mu_{\mathfrak{m}_P/\mathfrak{h}}(\zeta\colon \chi)\psi$ ($\mu_{\mathfrak{m}_P/\mathfrak{h}}$ denoting the Harish-Chandra homomorphism for the pair $(\mathfrak{m}_P, \mathfrak{h})$) then $E_P(x, \psi, \lambda)$ is a function of type $\mathrm{II}(\chi)$. In fact we have the following lemma. Note that $\lambda \in \mathscr{F}_{\mathbf{C}}$ can be written as $\lambda = \lambda_{\mathbf{R}} + (-1)^{1/2}\lambda_{\mathbf{I}}$ with $\lambda_{\mathbf{R}}, \lambda_{\mathbf{I}} \in \mathfrak{a}_P^*$.

LEMMA 3.1. *Fix $D \in \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$. Then we can choose $r \geq 0$ and a finite set $F \subset U(\mathfrak{m}_{P,\mathbf{C}})^{(2)}$ with the property that*

$$\|(DE_P)(x, \psi, \lambda)\| \leq \nu_F(\psi)\Xi(x)|(x, \lambda)|^r \exp\{c_0\|\lambda_{\mathbf{R}}\|\sigma(x)\}$$

where $c_0 \geq 0$ and c_0 is independent of D, x, and λ and the inequality holds for all $\lambda \in \mathscr{F}_{\mathbf{C}}$, $\psi \in C^\infty(M_P : V : \tau)$, and $x \in G$. Here

$$\nu_F(\varphi) = \sum_{\delta\in F} \sup_{M_P} \|\delta\varphi\|\Xi_{M_P}^{-1},$$

and for any reductive subgroup $L \subset G$, Ξ_L denotes the corresponding elementary spherical function on L (cf. (3.1)).

We shall need one further standard result. For this result and the proof of Lemma 3.1 the reader is referred to Harish-Chandra [3]. Let

$$K_M = K \cap M, \quad \text{and} \quad {}^+M = K_M\, \mathrm{cl}(A_0(B))K_M. \tag{3.3}$$

LEMMA 3.2. (1) *There exist constants* C_0, $r_0 \geq 0$ *such that*

$$d(m)\Xi(m) \leq C_0 \Xi_M(m)(1+\sigma(m))^{r_0} \qquad (m \in {}^+M).$$

(2) *There exist constants* $D_0 > 0$, $q_0 \geq 0$ *such that*

$$e^{-\rho_0(\log h)} \leq \Xi(h) \leq D_0 e^{-\rho_0(\log h)}(1+\sigma(h))^{q_0} \qquad (h \in A_0(B)).$$

Let

$$M^+ = \{m \in M \mid \|\mathrm{Ad}_m|_{\bar{\mathfrak{n}}}\| < 1\} \tag{3.4}$$

where $\|\mathrm{Ad}_m|_{\bar{\mathfrak{n}}}\|$ denotes the Hilbert norm of the operator Ad_m restricted to $\bar{\mathfrak{n}}$. Observe that

$$M^+ \subset {}^+M.$$

Let

$$\gamma(m) = \gamma_Q(m) = \|\mathrm{Ad}_m|_{\bar{\mathfrak{n}}}\|.$$

LEMMA 3.3. *Let* $k \in \mathbf{Z}_+$, φ *a function of* type $\mathrm{II}(\chi)$, *and notation as in Lemma* 2.1. *Given* $\eta_1, \eta_2 \in U(\mathfrak{m}_\mathbf{C})$, $u_1, u_2 \in \mathcal{S}(\mathcal{F}_\mathbf{C})$ *there exists* $F_1 \subset \widetilde{U}(\mathfrak{g}_\mathbf{C})$, $|F_1| < \infty$, $r_1 \geq 0$ *such that* $S_{F_1, r_1}(\varphi) < \infty$, *and with* $a_t = \exp tH$ *we have*

$$\sum_{\substack{i,j,N \\ o(N)=k}} \|\varphi(\eta_1 ; ma_t; \eta_2\xi_{ij:N}, \lambda; u_1)\|$$
$$\leq S_{F_1, r_1}(\varphi)|(m, \lambda, t)|^{r_1} e^{-(k+1)t-\rho_0(H)t} d(m)^{-1}\Xi_M(m),$$

and

$$\sum_{i,j,M} \|\varphi(\eta_1 ; ma_t; \eta_2 u_{ij:M}, \lambda; u_2)\|$$
$$\leq S_{F_1, r_1}(\varphi)|(m, \lambda, t)|^{r_1} e^{-\rho_0(H)t} d(m)^{-1}\Xi_M(m),$$

where $|(m, \lambda, t)| = |(m, \lambda)|(1+|t|)$ *and the second sum is over* $M \in \mathbf{Z}_+^n$, $o(M) \leq k$, $1 \leq j \leq d_M$, $1 \leq i \leq r$. *The inequalities hold for all* $m \in M^+$, $t > 0$, *and* $\lambda \in \mathcal{F}$.

COROLLARY 3.4. *Fix* $k \in \mathbf{Z}_+$. *Given* $\eta_1, \eta_2 \in U(\mathfrak{m}_\mathbf{C})$, $u_1, u_2 \in \mathcal{S}(\mathcal{F}_\mathbf{C})$ *there exists* $S_1 \subset U(\mathfrak{m}_\mathbf{C})^{(2)}$, $|S_1| < \infty$, $s_1 \geq 0$ *such that*

$$\sum_{\substack{i,j,N \\ o(N)=k}} \|E_P(\eta_1 ; ma_t; \eta_2\xi_{ij:N}, \varphi, \lambda; u_1)\|$$
$$\leq \nu_{S_1, s_1}(\varphi)|(m, \lambda, t)|^{s_1} e^{-(k+1)t-\rho_0(H)t} d(m)^{-1}\Xi_M(m),$$

and

$$\sum_{i,j,M} \|E_P(\eta_1 \mathbin{:} ma_t; \eta_2 u_{ij:M}, \varphi, \lambda; u_2)\| \\ \leq \nu_{S_1,s_1}(\varphi)|(m, \lambda, t)|^{r_1} e^{-\rho_0(H)t} d(m)^{-1} \Xi_M(m),$$

the second sum is over $M \in \mathbf{Z}_+^n$, $o(M) \leq k$, $1 \leq j \leq d_M$, $1 \leq i \leq r$. *The inequalities hold for all* $m \in M^+$, $t > 0$, $\varphi \in \mathscr{A}_{\mathrm{cusp}}(M_P, \tau)$, *and* $\lambda \in \mathscr{F}$.

Proof of Corollary 3.4. The corollary follows easily from the lemma and Lemmas 3.1, and 3.2.

Proof of Lemma 3.3. First note that as $M^+ \subset {}^+M$ then by (1) of Lemma 3.2 and the fact that d is a homomorphism of M we have

$$\Xi(ma_t) \leq C_0(1 + \sigma(m))^{r_0}(1 + t)^{r_0} d(m)^{-1} \Xi_M(m) e^{-\rho_0(H)t}. \tag{3.5}$$

From this and property (4) of functions of type $\mathrm{II}(\chi)$ the second inequality of the lemma follows easily.

To prove the first inequality of the lemma let us observe for $M \in \mathbf{Z}_+^n$, $o(M) = k+1$, $\delta \in U(\mathfrak{g}_{\mathbf{C}})$, and $u \in \mathscr{S}(\mathscr{F}_{\mathbf{C}})$ we have the following,

$$\|\varphi(\eta_1 \mathbin{:} ma_t; Y^M \delta, \lambda)\| = \|\varphi(\eta_1 (Y^M)^{ma_t} \mathbin{:} ma_t; \delta, \lambda; u)\|.$$

But M^+ leaves $\bar{\mathfrak{n}}^{k+1}$ stable, hence we may write

$$(Y^M)^m = \sum_{\substack{N \\ o(N)=k+1}} c_{NM}(m) Y^N, \tag{3.6}$$

where $c_{NM} \in C(M)$, and $|c_{NM}(m)| \leq \gamma(m)$ and therefore c_{NM} remains bounded on M^+. As $(Y^M)^{a_t} = e^{-t\langle\beta, M\rangle(H)} Y^M$ and as $m \in M^+$, then we see that there exists a constant D_k depending only on k such that

$$\|\varphi(\eta_1 \mathbin{:} ma_t; Y^M \delta, \lambda; u)\| \\ \leq D_k \sum_{\substack{N \\ o(N)=k+1}} e^{-t\langle\beta, M\rangle(H)} \|\varphi(\eta_1 Y^N \mathbin{:} ma_t; \delta, \lambda; u)\|.$$

From (4) of the definition of functions of type $\mathrm{II}(\chi)$ and (3.5) one easily shows the existence of $\widetilde{F}_1 \subset \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, $\tilde{r}_1 \geq 0$ such that

$$\|\varphi(\eta_1 \mathbin{:} ma_t; Y^M \delta, \lambda; u)\| \\ \leq S_{\widetilde{F}_1, \tilde{r}_1}(\varphi)|(m, \lambda, t)|^{\tilde{r}_1} e^{-(k+1)t - \rho_0(H)t} d(m)^{-1} \Xi_M(m).$$

From this one can deduce the first inequality of the lemma for $\eta_2 = 1$. If $\eta_2 \neq 1$ one must observe that $\mathfrak{m}$ leaves $\bar{\mathfrak{n}}$ stable. Hence $\eta_2 Y^M \delta$ can be written as a sum of terms of the type $Y^N \delta'$ with $o(N) = k+1$, and $\delta' \in U(\mathfrak{g}_{\mathbf{C}})$, from which one can deduce the inequality from the above calculation. □

4. Spectral Analysis of $\Gamma_k(X_\lambda)$. Before applying our uniform estimates, we shall need some further refinements concerning the dependence of $\Gamma_k(X_\lambda)$ on $\lambda \in \mathscr{F}$. First we note that if

$$(4.1)\qquad \begin{aligned} \mathscr{E}_M(\lambda) &= \{X_M(s_j X_\lambda) | 1 \le j \le r\}, \qquad M \in \mathbf{Z}_+^n, \\ \mathscr{E}_k(\lambda) &= \bigcup_{o(M)\le k} \mathscr{E}_M(\lambda) \\ \mathscr{E}(\lambda) &= \bigcup_{k=0}^{\infty} \mathscr{E}_k(\lambda) \end{aligned}$$

then $\mathscr{E}_k(\lambda)$ is the set of eigenvalues of $B_k(X_\lambda)$ and hence of $\Gamma_k(X_\lambda)$. Identify $\mathfrak{h}_{0,\mathbf{C}}^*$ with $\mathbf{C}$ via the map $\Lambda \mapsto \Lambda(H)$. Observe that if $\mathfrak{h}_{\mathbf{C}}^y = \mathfrak{h}_{0,\mathbf{C}}$ then using the notation $(\mathfrak{b}_{\mathbf{C}})^{\mathbf{R}} = \sum_{\alpha>0} \mathbf{R} H_\alpha$ where $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$, $\sum_{\alpha>0}$ denotes the sum over any system of positive roots for $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$, we have $[(\mathfrak{h}_{\mathbf{C}})^{\mathbf{R}}]^y = (\mathfrak{h}_{0,\mathbf{C}})^{\mathbf{R}}$. It is known that if $s \in W_0$ then s leaves $(\mathfrak{h}_{0,\mathbf{C}})^{\mathbf{R}}$ invariant. Hence as $H \in \mathfrak{a} \subset \mathfrak{a}_0 \subset (\mathfrak{h}_{0,\mathbf{C}})^{\mathbf{R}}$ then there exists $H_{1j} \in (-1)^{1/2}\mathfrak{h}_{\mathbf{I}}$, $H_{2j} \in \mathfrak{h}_{\mathbf{R}} = \mathfrak{a}_P$ such that

$$(4.2)\qquad [s_j^{-1}(H)]^{y^{-1}} = H_{1j} + H_{2j}.$$

It follows that for all $\lambda \in \mathscr{F}$,

$$(4.3)\qquad \mathfrak{R} s_j X_\lambda(H) = \mathfrak{R}(\chi + \lambda)(H_{1j} + H_{2j}) = \chi(H_{1j}).$$

We may therefore, order the set $\mathscr{E}(\lambda)$ *independently of* $\lambda \in \mathscr{F}$ as follows:

$$(4.4)\qquad \mathscr{E}(\lambda) = \{z_1(\lambda), z_2(\lambda), \dots\},$$

where

$$(4.5)\qquad z_i(\lambda) \neq z_j(\lambda), \qquad i \neq j,$$

and

$$(4.6)\qquad \mathfrak{R} z_1(\lambda) \ge \mathfrak{R} z_2(\lambda) \ge \cdots \qquad (\lambda \in \mathscr{F}).$$

Note that if $\mathscr{E}_k(\lambda) = \{\zeta_1(\lambda), \dots, \zeta_p(\lambda)\}$, $\mathscr{E}_{k+1}(\lambda) = \{\mu_1(\lambda), \dots, \mu_q(\lambda)\}$ then although $\mathscr{E}_k(\lambda) \subset \mathscr{E}_{k+1}(\lambda)$ it may happen that for some

$1 \leq j \leq q$ and some $i \leq p$, $i < j$, $\Re\mu_j(\lambda) \leq \Re\zeta_i(\lambda)$. However, if we fix $l < p$ and let k increase, then we can guarantee that $\zeta_i(\lambda) = \mu_i(\lambda) = z_i(\lambda)$ if $1 \leq i \leq l$.

We can in fact say more about the numbers $z_i(\lambda)$. Let H^χ be any element of $(-1)^{1/2}\mathfrak{h}_{\mathrm{I}}$ such that

$$\chi(H^\chi) = 1.$$

Define a partial order on the set of integers $\{1, \ldots, r\}$ as follows. Let us write $i \preceq j$ for $1 \leq i, j \leq r$ if $\chi(H_{1j} - H_{1i}) \in \mathbf{Z}_+$. Let ${}^oQ \subset \{1, \ldots, r\}$ denote the maximal elements relative to this partial order; we also denote by Q^o a subset of one through r such that if $i, j \in Q^o$, $i \neq j$, then $H_{2i} \neq H_{2j}$, and Q^o is maximal with respect to this property. Then if $j \in \{1, \ldots, r\}$ we must have $j \preceq \alpha$ for some $\alpha \in {}^oQ$. It follows that

$$[s_j^{-1}(H)]y^{-1} = H_{1\alpha} - n_{\alpha j}H^\chi + {}_\chi H + H_{2j}$$

where $n_{\alpha j} \in \mathbf{Z}_+$ and ${}_\chi H \in \ker(\chi|\mathfrak{h}_{\mathrm{I}})$. We have then that for all $\lambda \in \mathscr{F}_{\mathbf{C}}$

$$s_jX_\lambda(H) = \chi(H_{1\alpha}) + \lambda(H_{2j}) - n_{\alpha j},$$

equivalently

$$X_m(s_jX_\lambda) = s_\alpha\chi^y(H) + s_j\lambda(H) - (n_{\alpha j} + m) - \rho(H).$$

We are now in a position to obtain a parametrization of the eigenvalues of $\Gamma_k(X_\lambda)$. For $\alpha \in {}^oQ$ let

$$Q_\alpha = \{j | 1 \leq j \leq r, \text{ and } j \preceq \alpha\}. \tag{4.7}$$

Further, let Q_α^o denote the subset of Q_α defined by the condition that for each $j \in Q_\alpha$ there exists $i \in Q_\alpha^o$ such that

$$i \leq j, \quad H_{2j} = H_{2i}, \tag{4.8}$$

and Q_α^o is the minimal set with this property. Then if $\alpha \in {}^oQ$, and $j \in Q_\alpha^o$, we let

$$\sigma_k(\alpha, j) = \{n \in \mathbb{Z} | s_\alpha\chi^y(H) + s_j\lambda(H) - n - \rho(H) \text{ is an eigenvalue of } \Gamma(X_\lambda)\}. \tag{4.9}$$

Let $\alpha_j \in {}^oQ$, $i_j \in Q_{\alpha_j}^o$, and $n_j \in \sigma(\alpha_j, i_j)$, with $j = 1, 2$. If

$$\begin{aligned} &s_{\alpha_1}\chi^y(H) + s_{i_1}\lambda(H) - n_1 - \rho(H) \\ &\quad = s_{\alpha_2}\chi^y(H) + s_{i_2}\lambda(H) - n_2 - \rho(H), \qquad \lambda \in \mathscr{F}_{\mathbf{C}}, \end{aligned}$$

then we must have $\alpha_1 = \alpha_2$ by the equality of the real parts of the last equation and the maximality of α_j, $j = 1, 2$. By the equality of the imaginary parts we must have as $i_1, i_2 \in Q^o_\alpha$, with $\alpha = \alpha_j$, $j = 1, 2$, that $i_1 = i_2$. It follows then that $n_1 = n_2$. Now let

$$\mathscr{F}''_{\mathbf{C}}(\chi) = \{\lambda \in \mathscr{F}'_{\mathbf{C}}(\chi) | \lambda(H_{2i} - H_{2j}) \neq 0, \, i, j \in Q^o \ni i \neq j\}.$$

Further we put

$$\mathscr{F}''(\chi) = \mathscr{F}''_{\mathbf{C}}(\chi) \cap \mathscr{F}.$$

Note that $\mathscr{F}''(\chi)$ is an open dense subset of $\mathscr{F}$. Let $\lambda \in \mathscr{F}''(\chi)$. It follows from the above that the set

$$\begin{aligned} \text{(4.10)} \qquad \operatorname{Spec}(\Gamma(X_\lambda)) &= \{(s_\alpha \chi^y + s_j \lambda)(H) - \rho(H) - n |\, \alpha \in {}^oQ, \\ &\qquad j \in Q^o_\alpha, \, n \in \sigma(\alpha, j)\} \end{aligned}$$

is exactly the set of eigenvalues of $\Gamma_k(X_\lambda)$ for all $\lambda \in \mathscr{F}''(\chi)$ and that the elements of this set are all distinct.

Let

$$\{z_{m_j} | j \geq 1\}$$

be the subsequence of the $z_m(\lambda)$ such that for all $\lambda \in \mathscr{F}$

(4.11) (1) $\Re z_{m_j}(\lambda) > \Re z_{m_{j+1}}(\lambda)$,

(2) for every $l \in \mathbf{Z}_+$ there exists j such that $\Re z_l(\lambda) = \Re z_{m_j}(\lambda)$,

(3) if $l > m_j$ then $\Re z_{m_j}(\lambda) > \Re z_l(\lambda)$.

Choose and fix $j_0 \geq 1$ such that

$$s_\alpha \chi^y(H) - \rho(H) \geq \Re z_{m_0}, \qquad \forall \alpha \in {}^oQ,$$

where for notational convenience we put

$$m_0 = m_{j_0}.$$

Set

$$\begin{aligned} \text{(4.12)} \qquad z_0(\lambda) &= z_{m_0}(\lambda) \\ &= s_{\alpha_0} \chi^y(H) + s_{i_0} \lambda(H) - \rho(H) - n_0, \\ &\qquad \alpha_0 \in {}^oQ, \, i_0 \in Q^0_{\alpha_0}, \, n_0 \in \sigma(\alpha_0, i_0), \end{aligned}$$

where, α_0, i_0, and n_0 are defined by the second equality. We shall (as is permissible by the above discussion) choose k so large that the following hold; let

$$\mathscr{E}_k(\lambda) = \{\zeta_1(\lambda), \ldots, \zeta_p(\lambda)\}.$$

Then

$$\text{(4.13)} \qquad \begin{aligned} &\text{(i)} \quad \zeta_i(\lambda) = z_i(\lambda), \qquad 1 \le i \le m_0, \; m_0 = m_{j_0}, \\ &\text{(ii)} \quad \Re z_0(\lambda) = s_{\alpha_0}\chi^y(H) - \rho(H) - n_0 \\ &\qquad\qquad \ge -\rho(H) - k, \qquad \lambda \in \mathscr{F}. \end{aligned}$$

Let $P_k(\lambda : x)$ denote the minimal polynomial for $\Gamma_k(X_\lambda)$ with $\lambda \in \mathscr{F}''(\chi)$. From the above we know that $P_k(\lambda\colon x)$ is just the product of all factors of the type $(x - \mu)$ with $\mu \in \operatorname{Spec}(\Gamma(X_\lambda))$. If $\alpha \in {}^oQ$, $j \in Q_\alpha^o$, $n \in \sigma_k(\alpha, j)$, put $P_k^{(\alpha j\,:\,n)}(\lambda\colon x)$ equal to the product of the factors $(x - \mu)$ with $\mu \in \operatorname{Spec}(\Gamma(X_\lambda))$ and $\mu \neq \{s_\alpha\chi^y(H) - s_j\lambda(H) - \rho(H) - n\}$. Let α, j, n, as above, and define meromorphic $\operatorname{End}(V \otimes T_k)$-valued functions on $\mathscr{F}_{\mathbf{C}}$ via

$$E_n(s_\alpha\chi^y + s_j\lambda) = P_k^{(\alpha j\,:\,n)}(\lambda\colon X_n(s_\alpha\chi^y + s_j\lambda))^{-1} P_k^{(\alpha j\,:\,n)}(\lambda\colon \Gamma(X_\lambda)).$$

Put $\sigma(\alpha) = \bigcup_{j \in Q_\alpha^o} \sigma(\alpha, j)$. If $n \in \sigma(\alpha)$ let

$$\text{(4.14)} \qquad E_{\alpha,n}(\lambda) = \sum_{\substack{j \in Q_\alpha^o \ni \\ n \in \sigma_k(\alpha, j)}} E_n(s_\alpha\chi^y + s_j\lambda).$$

Finally, set

$$\text{(4.15)} \qquad \begin{aligned} E_0(\lambda) &= \sum_{\alpha \in {}^oQ} \; \sum_{\substack{n \in \sigma(\alpha) \ni \\ (s_\alpha\chi^y - \rho)(H) - n \ge \Re z_0}} E_{\alpha,n}(\lambda), \\ E_1(\lambda) &= \sum_{\alpha \in {}^oQ} \; \sum_{\substack{n \in \sigma(\alpha) \ni \\ (s_\alpha\chi^y - \rho)(H) - n < \Re z_0}} E_{\alpha,n}(\lambda). \end{aligned}$$

For any linear operator A on $V \otimes T_k$ let $\|A\|$ denote its Hilbert-Schmidt norm.

LEMMA 4.1. (1) *The functions* $E_{\alpha,n}(\lambda)$, *and hence the functions* $E_0(\lambda)$, *and* $E_1(\lambda)$ *are defined and infinitely differentiable as maps of* $\mathscr{F}$ *into* $\operatorname{End}(V \otimes T_k)$.

(2) *For all* $\lambda \in \mathscr{F}$ *we have*

$$E_0(\lambda) + E_1(\lambda) = 1_{V \otimes T_k}$$

and

$$E_\iota(\lambda)\Gamma(X_\lambda) = \Gamma(X_\lambda)E_\iota(\lambda), \qquad \iota = 0, 1.$$

(3) *If* $u_1, u_2 \in \mathscr{S}(\mathscr{F}_{\mathbf{C}})$ *there exists* $A_0 = A_0(u_1, u_2)$, $a_0 = a_0(u_1, u_2) \geq 0$ *for which*

$$\|E_0(\lambda; u_1)\| + \|E_1(\lambda; u_2)\| \leq A_0(1 + \|\lambda\|)^{a_0} \qquad (\lambda \in \mathscr{F}_{\mathbf{C}}).$$

(4) *If* $u \in \mathscr{S}(\mathscr{F}_{\mathbf{C}})$, $\alpha \in {}^oQ$, $j \in Q^o_\alpha$, $n \in \sigma(\alpha, j)$ *then there exists* $A_0 = A_0(u)$, $a_0 = a_0(u)$ *such that in the above notation,*

$$\begin{aligned}&\|\partial(u_\lambda)\{P_k^{(\alpha j\,:\,n)}(\lambda\colon X_n(s_\alpha\chi^y + s_j\lambda))E_n(s_\alpha\chi^y + s_j\lambda)\}\| \\ &\qquad \leq A_0(1 + \|\lambda\|)^{a_0} \qquad (\lambda \in \mathscr{F}_{\mathbf{C}}).\end{aligned}$$

Proof. The statement in (4) of the lemma follows directly from the definition of $E_n(s_\alpha\chi^y + s_j\lambda)$ given above, and Corollary 2.5. From the above we obviously have for $\sigma(\alpha) = \bigcup_{j\in Q^o_\alpha} \sigma(\alpha, j)$

$$\sum_{\alpha\in{}^oQ}\sum_{n\in\sigma(\alpha)} E_{\alpha,n}(\lambda) = 1_{V\otimes T_k}, \qquad \lambda \in \mathscr{F}''(\chi).$$

It follows that for all $\lambda \in \mathscr{F}''(\chi)$,

$$\sum_{\alpha\in{}^oQ}\sum_{n\in\sigma(\alpha)} \operatorname{Exp}\{t\Gamma_k(X_\lambda)\}E_{\alpha,n}(\lambda) = \operatorname{Exp}\{t\Gamma(X_\lambda)\}.$$

Now let $\lambda_0 \in \mathscr{F}$, and suppose there exists a singular hyperplane $\mathscr{U}$ of some $E_n(s_\alpha\chi^y + s_j\lambda)$, $j \in Q^o_\alpha$, $n \in \sigma(\alpha, j)$ passing thru λ_0. For each α, j, n with $\alpha \in {}^oQ$, $j \in Q^o_\alpha$, $n \in \sigma(\alpha, j)$ there exists $f_{\alpha j\,:\,n} \in \mathscr{S}(\mathfrak{a}_P)$ of minimal degree such that

$$\lambda \mapsto f_{\alpha j\,:\,n}(\lambda)E_n(s_\alpha\chi^y + s_j\lambda)$$

is analytic on $\mathscr{F}$. It follows from the above that

$$P_k^{(\alpha j\,:\,n)}(\lambda\colon X_n(s_\alpha\chi^y + s_j\lambda)) = 0$$

for some $\lambda \in \mathscr{F}$ if and only if

$$s_j\lambda(H) = s_k\lambda(H) \quad \text{for some } k \in Q^o_\alpha, \quad k \neq j.$$

We may therefore, choose $f \in \mathscr{S}(\mathfrak{a}_P)$, $u \in \mathscr{S}(\mathfrak{a}^*_{PC})$ and some $\alpha \in {}^oQ$ for which the following holds for at least one $n \in \sigma(\alpha)$:

$$f(\lambda_0; u') = 0, \quad u' \in \mathscr{S}(\mathfrak{a}^*_{PC}) \ni \deg(u') \leq \deg(u),$$

but

$$\lim_{\lambda \mapsto \lambda_0} \partial(u_\lambda)\{f(\lambda)E_{\alpha,n}(\lambda)\} \neq 0,$$

and

$$f_{\alpha j:n} \text{ divides } f \text{ in } \mathscr{S}(\mathfrak{a}_P) \text{ for all } \alpha \in {}^oQ,\ j \in Q^o_\alpha, \text{ and } n \in \sigma(\alpha, j).$$

As $\mathscr{F}''(\chi)$ is dense in $\mathscr{F}$ then λ_0 is a limit point of $\mathscr{F}''(\chi)$. From the analyticity of the map $\lambda \mapsto \operatorname{Exp}\{t\Gamma(X_\lambda)\}$ we have for λ denoting a generic point in $\mathscr{F}''(\chi)$

$$\begin{aligned}
(4.16)\quad 0 &= \sum_{\alpha \in {}^oQ} \sum_{n \in \sigma(n)} \lim_{\lambda \mapsto \lambda_0} \partial(u_\lambda)\{f(\lambda)\operatorname{Exp}\{t\Gamma(X_\lambda)\}E_{\alpha,n}(\lambda)\} \\
&= \sum_\alpha \sum_n \sum_{\substack{j \\ n \in \sigma(\alpha, j)}} \lim_{\lambda \mapsto \lambda_0} \partial(u_\lambda)\{f(\lambda)\exp\{(s_\alpha \chi^y(H) \\
&\qquad + \lambda(H_{2j}) - \rho(H) - n)t\}E_n(s_\alpha \chi^y + s_j \lambda)\} \\
&= \sum_\alpha \sum_n \exp\{(s_\alpha \chi^y(H) - \rho(H) - n)t\}\widetilde{E}_{\alpha,n}(t),
\end{aligned}$$

where for each $\alpha \in {}^oQ$ there exists $q_\alpha < \deg(f)$ and for each j, l with $0 \le l \le q_\alpha$, $j \in Q^o_\alpha$ there exists $\Lambda^\alpha_{jl} \in \mathscr{S}(\mathfrak{a}^*_{PC})$ with $\Lambda^\alpha_{j0} = 1$, and $\widetilde{E}_{j,l}(\alpha, n) \in \operatorname{End}(V \otimes T_k)$ such that

$$\begin{aligned}
\widetilde{E}_{\alpha,n}(t) &= \sum_{j \ni n \in \sigma(\alpha, j)} \sum_{l=0}^{q_\alpha} e^{t\lambda_0(H_{2j})} t^l \Lambda^\alpha_{jl}(H_{2j}) \widetilde{E}_{j,l}(\alpha, n) \\
&= \sum_{j \in Q^o_\alpha} \sum_{l=0}^{q_\alpha} e^{t\lambda_0(H_{2j})} t^l \Lambda^\alpha_{jl}(H_{2j}) \widetilde{E}_{j,l}(\alpha, n),
\end{aligned}$$

here we define $\widetilde{E}_{j,l}(\alpha, n)$ to equal zero if $n \notin \sigma(\alpha, j)$. Define $Q^o_\alpha(\lambda_0)$ to be a maximal subset of Q^o_α such that $\lambda_0(H_{2i} - H_{2j}) \neq 0$ for $i, j \in Q^o_\alpha$ with $i \neq j$. Then we can write

$$\widetilde{E}_{\alpha,n}(t) = \sum_{r \in Q^o_\alpha(\lambda_0)} \sum_{l=0}^{q_\alpha} e^{t\lambda_0(H_{2r})} t^l \widetilde{\widetilde{E}}_{r,l}(\alpha, n),$$

where

$$\widetilde{\widetilde{E}}_{r,l}(\alpha, n) = \sum_j \Lambda^\alpha_{jl}(H_{2j}) \widetilde{E}_{j,l}(\alpha, n)$$

$$\text{sum over } j \in Q^o_\alpha \ni \lambda_0(H_{2j}) = \lambda_0(H_{2r}).$$

We have on letting $\lambda_{0r} = \lambda_0(H_{2r})$, that

$$\sum_{\alpha\in^o Q}\sum_{n\in\sigma(\alpha)} \exp\{(s_\alpha\chi^y(H)-\rho(H)-n)t\} \sum_{r\in Q^o_\alpha(\lambda_0)}\sum_{l=0}^{q_\alpha} e^{t\lambda_{0r}}t^l\tilde{\tilde{E}}_{r,l}(\alpha, n) = 0.$$

Let $\mu \in (V\otimes T_k)^*$, $v\in V\otimes T_k$. By this last equality we have

$$\sum_{\alpha\in^o Q}\sum_{n\in\sigma(\alpha)} \exp\{(s_\alpha\chi^y(H)-\rho(H)-n)t\}$$
$$\times \sum_{r\in Q^o_\alpha(\lambda_0)}\sum_{l=0}^{q_\alpha} e^{t\lambda_{0r}}t^l\mu(\tilde{\tilde{E}}_{r,l}(\alpha, n)v) = 0.$$

Let $z_1, \dots, z_s$ be an enumeration of the complex numbers $s_\alpha\chi^y(H)-\rho(H)-\lambda_{0r}$. Let

$$P_{in}(t) = \sum_{l=0}^{q_\alpha}\mu(\tilde{\tilde{E}}_{r,l}(\alpha, n)v)t^l \quad \text{if } z_i = s_\alpha\chi^y(H)-\rho(H)-\lambda_{0r}.$$

Then we have, on multiplying the last equation through by $e^{-z_k t}$ that

$$\sum_{i=1}^{s}\sum_{n\in\sigma(\alpha)} P_{in}(t)e^{((z_i-z_k)-n)t} = 0.$$

This sum can be written as

$$P_{k0}(t) + \sum_{\substack{i=1\\ i\neq k}}^{s} P_{i0}(t)e^{(z_i-z_k)t} + \sum_{i=1}^{s}\sum_{n\geq 1} P_{in}(t)e^{((z_i-z_k)-n)t} = 0.$$

As

$$\begin{aligned} z_i - z_k &= (s_\alpha\chi^y(H)-\rho(H)+\lambda_{0r}) - (s_{\alpha'}\chi^y(H)-\rho(H)+\lambda_{0r'}) \\ &= s_\alpha\chi^y(H) - s_{\alpha'}\chi^y(H) + \lambda_{0r} - \lambda_{0r'}, \end{aligned}$$

then

$$z_i - z_k = 0 \Leftrightarrow \alpha = \alpha' \quad \text{and} \quad r = r' \Leftrightarrow i = k.$$

We also have by the above that

$$z_i - z_k - n = 0 \Leftrightarrow s_\alpha\chi^y(H) - s_{\alpha'}\chi^y(H) = n \quad \text{and} \quad \lambda_{0r} = \lambda_{0r'}.$$

Equivalently,

$$\alpha \preceq \alpha' \text{ and } r = r' \Leftrightarrow \alpha = \alpha' \text{ and } r = r'.$$

It follows that $z_i - z_k - n \neq 0$ for all $i \neq k$ and n. Hence, by Lemma A 3.2.2 of Warner [1] we have

$$P_{k0}(t) \equiv 0.$$

As k was arbitrary we have

$$\sum_{i=1}^{s} \sum_{n \geq 1} P_{in}(t) e^{(z_i - n)t} = 0.$$

If we multiply this equality by e^t and re-index the sum, we obtain a sum of the original type. Arguing inductively we see that $P_{in}(t) \equiv 0$ for all i, n. As $P_{in}(t) = P_{in}(\mu, v\colon t) \equiv 0$ for all $\mu \in (V \otimes T_k)^*$, $v \in V \otimes T_k$ then we see that

$$\tilde{\tilde{E}}_{r,l}(\alpha, n) = 0 \qquad \forall \alpha, r, n, \text{ and } l.$$

Hence $\tilde{E}_{\alpha,n}(t) \equiv 0$ for all $t \in \mathbf{R}$, and all α, n. Putting $t = 0$ we obtain the contradiction that

$$\lim_{\lambda \mapsto \lambda_0} \partial(u_\lambda)\{f(\lambda) E_{\alpha,n}(\lambda)\} = 0 \qquad \forall \alpha, n.$$

As $E_{\alpha,n}$ is a rational function on $\mathscr{F}_{\mathbf{C}}$, we can write $E_{\alpha,n}(\lambda) = Q_{\alpha,n}(\lambda)^{-1} E'_{\alpha,n}(\lambda)$, where $Q_{\alpha,n}(\lambda)$ denotes the least common denominator of the matrix coefficients of $E_{\alpha,n}$ relative to the basis $\{e_{ij:M}\}$. It follows from the above that

$$\inf_{\lambda \in \mathscr{F}} |Q_{\alpha,n}(\lambda)| = m_{\alpha,n} > 0.$$

The lemma now follows easily from the fact that $Q_{\alpha,n}(\lambda) E_{\alpha,n}(\lambda)$ is a polynomial function. □

5. Uniform asymptotics of functions of type II(χ). We can now state an initial form of the first principal result of this paper. Let us recall the notation of §4; the eigenvalues of $\Gamma_k(X_\lambda)$ were denoted by $\zeta_1(\lambda), \ldots, \zeta_p(\lambda)$. We denoted the elements of $\mathscr{E}(\lambda)$ as $z_i(\lambda)$, and ordered the real parts of this sequence;

$$\Re z_1(\lambda) \geq \Re z_2(\lambda) \geq \cdots \geq \Re z_m(\lambda) \geq \cdots, \qquad \lambda \in (-1)^{1/2} \mathfrak{a}_P^*.$$

We let m_k be the indices where the jumps in the real parts of the $z_i(\lambda)$ occur so that

$$\Re z_{m_k}(\lambda) > \Re z_j(\lambda) \geq \Re z_{m_{k+1}}, \qquad k \geq 1,\ m_k < j \leq m_{k+1},\ \lambda \in (-1)^{1/2} \mathfrak{a}_P^*.$$

Next we chose $m_0 = m_{j_0}$ so that $z_0(\lambda) = z_{m_0}(\lambda)$, had real part smaller than all the prospective leading terms of our expansion. Finally we chose k so large that the first m_0 eigenvalues of $\Gamma_k(X_\lambda)$ had stabilized; that is so that

$$\zeta_i(\lambda) = z_i(\lambda), \qquad 1 \le i \le m_0,$$

and also so that

$$\Re z_0(\lambda) \ge -\rho(H) - k, \qquad \lambda \in (-1)^{1/2}\mathfrak{a}_P^*.$$

Further we recall that the $z_j(\lambda)$ are all of the form

$$X_M(s_i X_\lambda) = s_i X_\lambda(H) - \langle \beta, M \rangle(H) - \rho(H), \qquad 1 \le i \le r$$

where $M \in \mathbf{Z}_+$. The term $-\langle \beta, M \rangle(H)$ comes from our construction of the operators $u_{ij:M}$, and $\xi_{ij:M}$. Consequently we do not know which $M \in \mathbf{Z}_+$ parametrize an eigenvalue. It is clear however that the numbers

$$X_n(s_i X_\lambda) = s_i X_\lambda(H) - n - \rho(H)$$

contain all the eigenvalues as n varies over the nonnegative integers. We shall exploit this fact in the final version (Theorem 5.9) of the following theorem.

THEOREM 5.1. *Let φ be a function of type $\mathrm{II}(\chi)$. There exists $\varphi_0 \in C^\infty(M \times \mathscr{F} : V : \tau)$ and $\varepsilon > 0$ such that for any $\eta_1, \eta_2 \in U(\mathfrak{m}_{\mathbf{C}})$ there exists $F \subset \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, $r \ge 0$ such that $S_{F,r}(\varphi) < \infty$, and*

$$(5.1) \qquad \begin{aligned} &|\varphi(\eta_1 : ma_t; \eta_2, \lambda) - \varphi_0(\eta_1 : ma_t; \eta_2, \lambda)| \\ &\qquad \le S_{F,r}(\varphi)|(m, \lambda)|^r e^{(\Re z_0(\lambda) - \varepsilon)t} d(m)^{-1} \Xi_M(m), \end{aligned}$$

the inequality holding for all $m \in M^+$, $t \in \mathbf{R}^+$, $\lambda \in \mathscr{F}$. Further, there exists $P_l \in C^\infty(M \times \mathscr{F}''(\chi) : V : \tau)$ such that

$$(5.2) \qquad \varphi_0(ma_t, \lambda) = \sum_{l=1}^{m_0} P_l(m, \lambda) e^{z_l(\lambda)t},$$

for all $m \in M^+$, $t \ge 0$, $\lambda \in \mathscr{F}''(\chi)$ and $P_l \equiv 0$ if $z_l(\lambda)$ cannot be written in the form $X_n(s_\alpha \chi^\nu + s_j \lambda)$ with $s_\alpha^{-1}|_{\mathfrak{a}} \in W(\mathfrak{a}, \mathfrak{a}_P)$, $j \in Q_\alpha^o$, and $n \in \sigma(\alpha, j)$.

Proof. Let notation be as above and set for $m \in M$, $\lambda \in \mathscr{F}$,

$$(5.3) \qquad \begin{aligned} \Phi(m, \lambda) &= \sum_{i,M} \varphi(m; u_{ij:M}, \lambda) \otimes e_{ij:M}, \\ \Psi(m, \lambda) &= \sum_{\substack{i,j,N \\ o(N)=k}} \varphi(m; \xi_{ij:N}, \lambda) \otimes e_{ij:N}. \end{aligned}$$

It is clear from their definitions that $\Phi, \Psi \in C^\infty(M \times \mathscr{F} : V \otimes T)$. By (2.27) we have

$$(5.4) \quad \begin{aligned} &\Phi(\eta_1 \,¿\, ma_t; \eta_2, \lambda) \\ &\quad = \operatorname{Exp}\{t\Gamma(X_\lambda)\}\Phi(\eta_1 \,¿\, m; \eta_2, \lambda) \\ &\quad\quad + \int_0^t \operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\Psi(\eta_1 \,¿\, ma_u; \eta_2, \lambda)\, du \end{aligned}$$

We shall complete the proof of the theorem by a sequence of lemmas.

LEMMA 5.2. *There exists $F_2 \subset \widetilde{U}(\mathfrak{g}_\mathbf{C})$, $r_2 \geq 0$ such that $S_{F_2, r_2}(\varphi) < \infty$ and $\varepsilon > 0$ such that*

$$(5.5) \quad \begin{aligned} &\|E_1(\lambda)\Phi(\eta_1 \,¿\, ma_t; \eta_2, \lambda)\| \\ &\quad \leq S_{F_2, r_2}(\varphi)|(m, \lambda)|^{r_2} e^{t(\Re z_0(\lambda) - \varepsilon)} d(m)^{-1}\Xi_M(m) \end{aligned}$$

the inequality holding for all $m \in M^+$, $\lambda \in \mathscr{F}$, and $t > 0$.

Proof. By the continuity of $(m, t, \lambda) \mapsto E_1(\lambda)\Phi(\eta_1 \,¿\, ma_t; \eta_2, \lambda)$ it suffices to prove the inequality on an open dense set of $\mathscr{F}$; namely, we will show it on $\mathscr{F}'(\chi)$. From (5.4) we have

$$\begin{aligned} &\|E_1(\lambda)\Phi(\eta_1 \,¿\, ma_t; \eta_2)\| \\ &\quad \leq \|E_1(\lambda)\operatorname{Exp}\{t\Gamma(X_\lambda)\}\Phi(\eta_1 \,¿\, m; \eta_2, \lambda)\| \\ &\quad\quad + \left\| \int_0^t E_1(\lambda)\operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\Psi(\eta_1 \,¿\, ma_u; \eta_2, \lambda)\, du \right\|. \end{aligned}$$

By Lemma 3.3 there exists $F_1 \subset \widetilde{U}(\mathfrak{g}_\mathbf{C})$, $|F_1| < \infty$, $r_1 \geq 0$ such that $S_{F_1, r_1}(\varphi) < \infty$ and

$$\begin{aligned} &\|E_1(\lambda)\Phi(\eta_1 \,¿\, ma_t; \eta_2, \lambda)\| \\ &\quad \leq \|E_1(\lambda)\operatorname{Exp}\{t\Gamma(X_\lambda)\}\| S_{F_1, r_1}(\varphi)|(m, \lambda)|^{r_1} d(m)^{-1}\Xi_M(m) \\ &\quad\quad + S_{F_1, r_1}(\varphi) \int_0^t \|E_1(\lambda)\operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\| \\ &\quad\quad \cdot |(m, \lambda, u)|^{r_1} e^{-(k+1)u - \rho_0(H)u} d(m)^{-1}\Xi_M(m)\, du. \end{aligned}$$

By definition of the subsequence $z_{i_j}(\lambda)$, (4.13), and Lemma 4.1 we see that there exists $0 < \varepsilon_0 < 1$, and $D > 0$ depending only on j_0 such that

$$(5.6) \quad \|E_1(\lambda)\operatorname{Exp}\{s\Gamma(X_\lambda)\}\| \leq DA_0(1 + |\lambda|)^{a_0} e^{s(\Re z_0(\lambda) - \varepsilon_0)} \qquad (s > 0).$$

As $e^{-(1-\varepsilon_0)u}(1+u)^{r_1}$ remains bounded on $[0, \infty)$ then we can choose $D_1 > 0$ such that

$$\begin{aligned}
&\|E_1(\lambda)\Phi(\eta_1 \,\dot{;}\, ma_t ; \eta_2 , \lambda)\| \\
&\quad \le D \cdot A_0|(m, \lambda)|^{a_0+r_1} d(m)^{-1}\Xi_M(m) S_{F_1, r_1}(\varphi) \\
&\qquad \cdot \left[e^{(\Re z_0(\lambda)-\varepsilon_0)t} + D_1 \int_0^t e^{(t-u)(\Re z_0(\lambda)-\varepsilon_0)} e^{(\Re z_0(\lambda)-\varepsilon_0)u}\, du \right] \\
&\quad \le D \cdot A_0|(m, \lambda)|^{a_0+r_1} d(m)^{-1}\Xi_M(m) S_{F_1, r_1}(\varphi) \\
&\qquad \cdot e^{(\Re z_0(\lambda)-\varepsilon_0)t}(1 + D_1 t).
\end{aligned}$$

Again by the boundedness of $(1+D_1 t)e^{-\alpha t}$ on $[0, \infty)$ for any $\alpha > 0$ we see we can choose $D_2 > 0$, $\varepsilon_2 > 0$ such that the above inequality is less than or equal to

$$D_2 D A_0|(m, \lambda)|^{a_0+r_1} d(m)^{-1}\Xi_M(m) S_{F_1, r_1}(\varphi) \cdot e^{(\Re z_0(\lambda)-\varepsilon_2)t}.$$

From this it is clear that there exists $F_2 \subset \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, $r_2 > 0$, and $\varepsilon > 0$ as in the statement of the lemma. □

LEMMA 5.3. *Let $m \in M$, $t \in \mathbf{R}$, and $\lambda \in \mathscr{F}$. Set $\Phi_0(m, t, \lambda)$ equal to*

$$(5.7) \qquad \begin{aligned} &E_0(\lambda)\operatorname{Exp}\{t\Gamma(X_\lambda)\}\Phi(m, \lambda) \\ &\quad + \int_0^\infty E_0(\lambda)\operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\Psi(ma_u, \lambda)\, du, \end{aligned}$$

Then $\Phi_0 \in C^\infty(M \times \mathbf{R} \times \mathscr{F} : V \otimes T)$, and for every $\eta_1, \eta_2 \in U(\mathfrak{m}_{\mathbf{C}})$ we have $\Phi_0(\eta_1 \,\dot{;}\, m ; \eta_2 , t, \lambda)$ equals

$$(5.8) \qquad \begin{aligned} E_0(\lambda) \Big[&\operatorname{Exp}\{t\Gamma(X_\lambda)\}\Phi(\eta_2 \,\dot{;}\, m ; \eta_2 , \lambda) \\ &+ \int_0^\infty \operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\Psi(\eta_1 \,\dot{;}\, ma_u ; \eta_2 , \lambda)\, du \Big]. \end{aligned}$$

Moreover, there exists $F_3 \subset \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, $|F_3| < \infty$, $r_3 > 0$ such that $S_{F_3, r_3}(\varphi) < \infty$ and

$$\|E_0(\lambda)\Phi(\eta_1 \,\dot{;}\, ma_t ; \eta_2 , \lambda) - \Phi_0(\eta_1 \,\dot{;}\, m ; \eta_2 , t, \lambda)\|$$

is less than or equal to

$$(5.9) \qquad S_{F_3, r_3}(\varphi)|(\lambda, m)|^{r_3} d(m)^{-1}\Xi_M(m) e^{(\Re z_0(\lambda)-\varepsilon)t},$$

here ε is as in Lemma 2.12, $m \in M^+$, $t > 0$, $\lambda \in \mathscr{F}$.

We shall need the following technical result for the proof of the lemma (for a proof of this result cf. Harish-Chandra [1], Lemma 54).

LEMMA 5.4. *Let C be a compact subset of M. There exists $T_0 \geq 0$ such that $m \exp tH \in M^+$ for all $t \geq T_0$.*

Proof of Lemma 5.3. Let $C \subset M$ be compact and let T_0 be as in Lemma 5.4. Given $l \in \mathbf{Z}_+$, $u_1, u_2, u_3 \in \mathscr{S}(\mathscr{F}_{\mathbf{C}})$, $\eta_1, \eta_2 \in U(\mathfrak{m}_{\mathbf{C}})$ there exists F_1, r_1 as in Lemma 3.3 such that

$$(5.10)\quad \begin{aligned} &\|E_0(\lambda; u_1)\partial(u_{2\lambda}) \\ &\qquad \cdot \{\Gamma(X_\lambda)^l \operatorname{Exp}\{(t-u-T_0)\Gamma(X_\lambda)\}\}\Psi(\eta_1 \,;\, ma_{u+T_0}; \eta_2, \lambda; u_3)\| \\ &\quad \leq d(ma_{T_0})^{-1}\Xi_M(ma_{T_0})S_{F_1, r_1}(\varphi)|(ma_{T_0}, \lambda, u)|^{r_1} \\ &\qquad \cdot e^{-[(k+1)+\rho(H)]u}\|E_0(\lambda; u_1)\partial(u_{2\lambda}) \\ &\qquad\qquad \cdot \{\Gamma(X_\lambda)^l \operatorname{Exp}\{(t-u-T_0)\Gamma(X_\lambda)\}\}\|. \end{aligned}$$

From (4.13) and the ordering of $\mathscr{E}_k(\lambda) = \{\zeta_1(\lambda), \dots, \zeta_p(\lambda)\}$ we have

$$(5.11)\qquad -\Re\zeta_i(\lambda) \leq -\Re z_0(\lambda) < \rho(H) + k \qquad (1 \leq i \leq m_0).$$

Taking into account Lemma 4.1 and (5.11) we see that for all $\lambda \in \mathscr{F}'(\chi)$, and then by continuity in λ, for all $\lambda \in \mathscr{F}$, there exists $b_0 \geq 0$, $B_0 > 0$, such that

$$(5.12)\quad \begin{aligned} &\|E_0(\lambda; u_1)\partial(u_{2\lambda})\{\Gamma(X_\lambda)^l \operatorname{Exp}\{(t-u-T_0)\Gamma(X_\lambda)\}\}\| \\ &\qquad \leq m_0 A_0 B_0 (1+|\lambda|)^{a_0+b_0} \\ &\qquad\quad \cdot (1+|t-u-T_0|)^{b_0} e^{(t-u-T_0)\Re z_0(\lambda)} \qquad (u \geq t-T_0). \end{aligned}$$

Hence the right-hand side of (5.10) is less than or equal to

$$(5.13)\quad \begin{aligned} &(d^{-1}\Xi_M)(ma_{T_0})S_{F_1, r_1}(\varphi)|(ma_{T_0}, \lambda, u)|^{r_1} \\ &\qquad \cdot m_0 A_0 \cdot (1+|\lambda|)^{a_0+b_0}(1+|t-u-T_0|)^{b_0} e^{(t-T_0)\Re z_0(\lambda)-u}, \end{aligned}$$

the inequality holding for $\lambda \in \mathscr{F}$, and $u \geq t - T_0$. This estimate shows that the integral in the definition of Φ_0, when the variables (m, t, λ) are differentiated, converges uniformly on C, and on compact subsets of $\mathbf{R}$ and $\mathscr{F}$; here we must recall that $a_{T_0} \in \operatorname{Center}(M)$. It remains to show (5.9).

From (5.4) and (5.8) we have for all $m \in M^+$, $t > 0$, $\lambda \in \mathscr{F}$,

$$\begin{aligned} &\|E_0(\lambda)\Phi(\eta_1 \,;\, ma_t; \eta_2, \lambda) - \Phi_0(\eta_1 \,;\, m; \eta_2, t, \lambda)\| \\ &\qquad \leq \int_t^\infty \|E_0(\lambda) \operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\| \cdot \|\Psi(\eta_1 \,;\, ma_u; \eta_2, \lambda)\| \, du. \end{aligned}$$

Appealing to Lemma 3.3 once again, we see that the above is less than or equal to

$$\int_t^\infty \|E_0(\lambda)\operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\| \cdot S_{F_1,r_1}(\varphi)d(m)^{-1}\Xi_M(m)|(m,\lambda,u)|^{r_1}e^{-[k+1+\rho(H)]u}\,du.$$

From (5.12) with $u_1=u_2=1$ we have constants A_0', a_0', such that the above integral is less than or equal to

$$m_0A_0'(1+|\lambda|)^{a_0'}S_{F_1,r_1}(\varphi)|(m,\lambda)|^{r_1}\cdot d(m)^{-1}\Xi_M(m) \cdot \int_t^\infty (1+u)^{r_1}e^{-(u-t)\Re z_0(\lambda)-(k+1)u-\rho(H)u}\,du.$$

From (4.13) and the boundedness of $(1+u)^{r_1}e^{-u}$ on $[0,\infty)$ there exists D_3 such that the above is

$$\begin{aligned}&\le m_0A_0'D_3(1+|\lambda|)^{a_0'}S_{F_1,r_1}(\varphi)d(m)^{-1}\Xi_M(m)|(m,\lambda)|^{r_1}\\ &\quad\cdot e^{-[k+\rho(H)]t}\int_t^\infty e^{-(u-t)(\Re z_0(\lambda)+k+\rho(H))}\,du\\ &\le m_0A_0'D_3|(m,\lambda)|^{r_1+a_0'}\\ &\quad\cdot S_{F_1,r_1}(\varphi)d(m)^{-1}\Xi_M(m)e^{-[k+\rho(H)]t}(\Re z_0(\lambda)+k+\rho(H))^{-1}.\end{aligned}$$

Recalling that $\Re z_0(\lambda)$ depends only on j_0 not on λ we see that there exists $D_4>0$ depending only on k and j_0 such that the above is

$$\le D_4|(m,\lambda)|^{r_1+a_0}S_{F_1,r_1}(\varphi)d(m)^{-1}\Xi_M(m)e^{-[k+\rho(H)]t}.$$

By the choice of ε_0 in (5.6) and ε_2 in the proof of Lemma 5.2 we see that for ε as in Lemma 5.2 the above is

$$\le D_4|(m,\lambda)|^{r_1+a_0}S_{F_1,r_1}(\varphi)d(m)^{-1}\Xi_M(m)e^{(\Re z_0(\lambda)-\varepsilon)t}. \qquad \square$$

We shall need to extend this last result for φ an Eisenstein integral. The extended result will only be used in §7 *not in the remainder of the proof of Theorem* 5.1. Recall that (cf. (4.3)) for $\lambda\in\mathscr{F}''(\chi)$,

$$z_j(\lambda)=s_\alpha\chi^y(H)+s_i\lambda(H)-\rho(\lambda)-n,\quad \alpha\in{}^oQ,\ i\in Q_\alpha^o,\ n\in\sigma(\alpha,i).$$

Considering then $z_j(\lambda)$ as a function on $\mathscr{F}_{\mathbf{C}}$, we have for $\lambda\in\mathscr{F}_{\mathbf{C}}$

$$\Re z_j(\lambda)=\chi(H_{1\alpha})+\lambda_{\mathbf{R}}(H_{2i})-\rho(H)-n=\Re z_j(\lambda_{\mathbf{I}})+\lambda_{\mathbf{R}}(H_{2i}).$$

Let

$$\mathscr{F}_{\mathbf{C},\delta}=\{\lambda\in\mathscr{F}_{\mathbf{C}}|\lambda=\lambda_{\mathbf{R}}+(-1)^{1/2}\lambda_{\mathbf{I}}\text{ and }\|\lambda_{\mathbf{R}}\|<\delta\}.$$

Note that as no singular hyperplane of the projections $E_{\alpha,n}(\lambda)$ intersects $\mathscr{F}$, then we may choose δ so small that none of the singular hyperplanes intersect $\mathscr{F}_{\mathbf{C},\delta}$ as well. It follows that there exists $m = m_{\alpha,n,\delta} > 0$ such that,

$$\inf_{\lambda\in\mathscr{F}_{\mathbf{C},\delta}} |Q_{\alpha:n}(\lambda)| > m,$$

where as above $Q_{\alpha,n}$ is a polynomial function on $\mathscr{F}_{\mathbf{C}}$ such that $Q_{\alpha,n}(\lambda)E_{\alpha,n}(\lambda)$ is a polynomial function from $\mathscr{F}_{\mathbf{C}}$ into $\mathrm{End}(V\otimes T_k)$. It is evident from the expression for $z_j(\lambda)$ given above that we may choose $\delta > 0$ such that the set of eigenvalues of $\Gamma(X_\lambda)$ is still the set $\mathrm{Spec}(\Gamma(X_\lambda))$, and the elements of this set are distinct if $\lambda\in\mathscr{F}_{\mathbf{C},\delta}$. Further, if $\lambda\in\mathscr{F}_{\mathbf{C},\delta}$, we have

$$|e^{z_j(\lambda)}| \le e^{\Re z_j(\lambda_{\mathrm{I}})+\delta\|H\|}.$$

As $\Re z_j(\lambda_{\mathbf{I}}) \ge \Re z_{j+1}(\lambda_{\mathbf{I}})$ it follows that

$$|e^{-z_j(\lambda)}| \le e^{-\Re z_0(\lambda_{\mathrm{I}})+\delta\|H\|}, \qquad 1\le j\le m_0,\ \lambda\in\mathscr{F}_{\mathbf{C},\delta}. \tag{5.14}$$

Let (cf. (4.13))

$$0 < \alpha_0 = \Re z_0(\lambda) + \rho(H) + k, \qquad \lambda\in\mathscr{F}.$$

The above shows that α_0 does not depend on $\lambda\in\mathscr{F}$.

COROLLARY 5.5. *Let $\varphi(x,\lambda) = E_P(x,\psi,\lambda)$, with $\psi\in\mathscr{A}_\omega(M_P,\tau)$, $\zeta\psi = \mu_0(\zeta: X_\lambda)\psi$, for $\zeta\in\mathfrak{z}(\mathfrak{m}_P)$. If $\delta(c_0+1)\|H\| < \alpha_0$, with c_0 as in Lemma* 3.1, *then $\Phi_0\in C^\infty(M\times\mathbf{R}\times\mathscr{F}_{\mathbf{C},\delta} : V\otimes T)$ and for each $m\in M$, $t\in\mathbf{R}$, $\Phi_0(m,t,\cdot)$ is holomorphic on $\lambda\in\mathscr{F}_{\mathbf{C},\delta}$, where Φ_0 is defined as in Lemma* 5.3. *Furthermore,* (5.8) *remains true if $\lambda\in\mathscr{F}_{\mathbf{C},\delta}$.*

Proof. Let $\eta_1, \eta_2\in U(\mathfrak{m}_{\mathbf{C}})$, $u\in\mathscr{S}(\mathscr{F}_{\mathbf{C}})$. By Lemma 3.1 the estimate (5.10) in the proof of the last lemma would be changed by a factor of $\exp\{c_0\|\lambda_{\mathbf{R}}\|\sigma(ma_t)\}$, if $\lambda\in\mathscr{F}_{\mathbf{C}}$. Also if $\lambda\in\mathscr{F}_{\mathbf{C},\delta}$, then by (5.14) the estimate in (5.12) can be replaced by

$$\begin{aligned} &m_0A_0B_0(1+|\lambda|)^{a_0+b_0} \\ &\quad\cdot(1+|t-u-T_0|)^{b_0}e^{(t-u-T_0)(\Re z_0(\lambda_{\mathrm{I}})-\delta\|H\|)}, \qquad u > t-T_0. \end{aligned}$$

It follows then that the left-hand side of (5.10) is less than or equal to

$$\begin{aligned} &d(ma_{T_0})^{-1}\Xi_M(ma_{T_0})\nu_{S_1,s_1}(\psi)|(ma_{T_0},\lambda,u)|^{r_1} \\ &\quad\cdot e^{[c_0\sigma(m)\delta+uc_0\|H\|\delta-u(k+1+\rho(H))]} \\ &\quad\cdot m_0A_0\cdot(1+|\lambda|)^{a_0+b_0}(1+|t-u-T_0|)^{b_0}e^{(t-u-T_0)(\Re z_0(\lambda_{\mathrm{I}})-\delta\|H\|)}. \end{aligned}$$

As
$$c_0\|H\|\delta + \|H\|\delta - \Re z_0(\lambda) - (k+\rho(H)) < 0$$
by choice of δ, we have finally that the left-hand side of (5.10), for $u \geq t - T_0$, $\lambda \in \mathscr{F}_{\mathbf{C},\delta}$ is less than or equal to
$$d(ma_{T_0})^{-1}\Xi_M(ma_{T_0})\nu_{S_1,s_1}(\psi)|(ma_{T_0}, \lambda, u)|^{r_1}e^{c_0\sigma(m)\delta-u}$$
$$\cdot\, m_0A_0\cdot(1+|\lambda|)^{a_0+b_0}(1+|t-u-T_0|)^{b_0}e^{(t-T_0)\Re z_0(\lambda_1)}.$$

This last estimate, as in the proof of the previous lemma, shows that $\Phi_0 \in C^\infty$, and also proves the holomorphy statement, if we recall that $E_0(\lambda)$ is rational and everywhere defined on $\mathscr{F}_{\mathbf{C},\delta}$. □

LEMMA 5.6. *Let Φ_0 be as in (5.7). There exists*
$$Q_l \in C^\infty(M\times\mathscr{F}''(\chi) : V\otimes T), \qquad 1\leq l\leq m_0$$
(m_0 as in (4.11)), such that

$$\Phi_0(m, t, \lambda) = \sum_{l=1}^{m_0} Q_l(m, \lambda)e^{z_l(\lambda)t}. \tag{5.15}$$

Further, if we let
$$\Phi_0(m, t, \lambda, k) = \Phi_0(m, t, \lambda), \quad Q_l(m, \lambda, k) = Q_l(m, \lambda)$$
showing the dependence on k, then if $k' > k$ we have
$$Q_l(m, \lambda, k) = Q_l(m, \lambda, k') \qquad (1\leq l\leq m_0). \tag{5.16}$$

Proof. The equality (5.15) just follows from the definition of Φ_0 and the semisimplicity of $\Gamma(X_\lambda)$ for $\lambda \in \mathscr{F}''(\chi)$. The equality (5.16) just expresses the fact that if $\mathscr{E}_k(\lambda) = \{\zeta_1(\lambda), \ldots, \zeta_p(\lambda)\}$, and $\mathscr{E}_{k'}(\lambda) = \{\mu_1(\lambda), \ldots, \mu_{p'}\}$ then if $k' > k$
$$\zeta_j(\lambda) = z_j(\lambda) = \mu_j(\lambda) \qquad (1\leq j\leq m_0),$$
and $\Gamma_{k'}(X_\lambda)$ restricted to $T_k \subset T_{k'}$ is equal to $\Gamma_k(X_\lambda)$. The fact that $Q_l \in C^\infty(M\times\mathscr{F}''(\chi) : V\otimes T)$ follows from (5.13) with λ varying in a compact subset, $\Omega \subset \mathscr{F}''(\chi)$; one has to observe that there exists $m = m_{\alpha,n,j,\Omega} > 0$ such that
$$\inf_{\lambda\in\Omega}|P_k^{(\alpha,j:n)}(\lambda)| > m.$$
As before, the lemma follows quickly from this. □

Let us now write,
$$\Phi_0(m, t, \lambda) = \sum \Phi_{ij:M}(m, t, \lambda)\otimes e_{ij:M}. \tag{5.17}$$

Here the sum is over $M \in \mathbf{Z}_+^n$, $o(M) \leq k$, $1 \leq j \leq d_M$, $1 \leq i \leq r$, and $\Phi_{ij:M}(m, t, \lambda) \in V$. Put

$$\varphi_0(ma_t, \lambda) = \Phi_{10:O}(m, t, \lambda) \qquad (m \in M, t \in \mathbf{R}). \tag{5.18}$$

We also let

$$Q_l(m, \lambda) = \sum Q_{ij:M}^l(m, \lambda) \otimes e_{ij:M},$$

M, j, i, as above, and set

$$P_l(m, \lambda) = Q_{10:O}^l(m, \lambda). \tag{5.19}$$

LEMMA 5.7. *The function φ_0 is well defined on $M \times \mathscr{F}$.*

Proof. From (5.4) with $\eta_1 = \eta_2 = 1$, we have for any $s, t \in \mathbf{R}$

$$\begin{aligned} \Phi(a_s a_t, \lambda) &= \operatorname{Exp}\{t\Gamma(X_\lambda)\}\Phi(a_s, \lambda) \\ &\quad + \int_0^t \operatorname{Exp}\{(t-u)\Gamma(X_\lambda)\}\Psi(a_{s+u}, \lambda)\, du \\ &= \operatorname{Exp}\{t\Gamma(X_\lambda)\}\Phi(a_s, \lambda) \\ &\quad + \int_s^{s+t} \operatorname{Exp}\{(s+t-u)\Gamma(X_\lambda)\}\Psi(a_u, \lambda)\, du. \end{aligned} \tag{5.20}$$

The last line by change of variables. On the other hand

$$\begin{aligned} \Phi(a_s a_t, \lambda) &= \Phi(a_{s+t}, \lambda) \\ &= \operatorname{Exp}\{(s+t)\Gamma(X_\lambda)\}\Phi(1, \lambda) \\ &\quad + \int_0^{s+t} \operatorname{Exp}\{(s+t-u)\Gamma(X_\lambda)\}\Psi(a_u, \lambda)\, du. \end{aligned} \tag{5.21}$$

Subtracting (5.21) from (5.20) we see that the following expression is equal to zero;

$$\begin{aligned} &\operatorname{Exp}\{t\Gamma(X_\lambda)\}\Phi(a_s, \lambda) - \operatorname{Exp}\{(s+t)\Gamma(X_\lambda)\}\Phi(1, \lambda) \\ &\quad + \int_s^0 \operatorname{Exp}\{(s+t-u)\Gamma(X_\lambda)\}\Psi(a_u, \lambda)\, du. \end{aligned}$$

An easy computation similar to the above now shows that

$$\Phi_0(a_s, t, \lambda) = \Phi_0(1, s+t, \lambda)$$

which shows that φ_0 is well defined. □

From Lemma 5.6 and (5.19) it follows that

$$\varphi_0(ma_t, \lambda) = \sum_{l=1}^{m_0} P_l(m, \lambda) e^{z_l(\lambda)t} \qquad (\lambda \in \mathscr{F}''(\chi), m \in M^+, t > 0).$$

One easily deduces that $\varphi_0 \in C^\infty(M \times \mathscr{F} : V : \tau)$. Recall that $u_{10:O} = 1$. If ε is defined as in Lemma 5.2 then as

$$\begin{aligned}\Phi(\eta_1 \,\dot{;}\, ma_t; \eta_2, \lambda) \\ &= E_0(\lambda)\Phi(\eta_1 \,\dot{;}\, ma_t; \eta_2, \lambda) + E_1(\lambda)\Phi(\eta_1 \,\dot{;}\, ma_t; \eta_2, \lambda),\end{aligned}$$

one obtains the estimate in Theorem 5.1 by combining Lemmas 5.2 and 5.3.

All that remains to be shown in the proof of Theorem 5.1 is that $P_l \equiv 0$ if $z_l(\lambda)$ cannot be written in the form $X_n(s_\alpha \chi^y + s_j\lambda)$ with $s_\alpha^{-1}|_{\mathfrak{a}} \in W(\mathfrak{a}, \mathfrak{a}_P)$. To see this, let $\alpha \in {}^oQ$. We call $X_n(s_\alpha(\chi + \lambda))$ a *leading exponent* of φ along Q if the corresponding coefficient of this eigenvalue is nonzero in the sum (5.2). From Harish-Chandra [3], §6 all leading exponents satisfy the requisite property. As all other exponents are integrally equivalent to a leading one, then all exponents (i.e. those exponential terms in (5.2) which appear with a nonzero coefficient) satisfy the required property. Hence Theorem 5.1 is completely proved. □

COROLLARY 5.8. *Let $\varphi(x, \lambda) = E_P(x, \psi, \lambda)$ with the notation and assumptions of Corollary 5.5. Let $\varphi_0(x, \lambda)$ be the function of Theorem 5.1, and let δ be as in Corollary 5.5. Then for every $m \in M$, $\varphi_0(m, \cdot)$ is holomorphic on $\mathscr{F}_{\mathbf{C},\delta}$.*

It will be necessary for applications to have Theorem 5.1 written in a slightly different form. First, if $\alpha \in {}^oQ$, and $s_\alpha^{-1}|_{\mathfrak{a}} \in W(\mathfrak{a}, \mathfrak{a}_P)$, then we have for all $j \in Q_\alpha^o$, and $n \in \sigma(\alpha, j)$ (as y centralizes $\mathfrak{a}_P$)

$$\begin{aligned}s_\alpha\chi^y(H) + s_j\lambda(H) - \rho(H) - n \\ &= \chi(s_\alpha^{-1}H) + \lambda(s_j^{-1}H) - \rho(H) - n = s_j\lambda(H) - \rho(H) - n.\end{aligned}$$

THEOREM 5.9. *Let φ be a function of type $\mathrm{II}(\chi)$. For every $s^{-1} \in W(\mathfrak{a}, \mathfrak{a}_P)$, $n \in \mathbf{Z}_+$, there exists $\Gamma_{s,n} \in C^\infty(M \times \mathscr{F}''(\chi) : V : \tau)$ such that the series*

$$\sum_{s^{-1} \in W(\mathfrak{a}, \mathfrak{a}_p)} \sum_{n=0}^{\infty} \Gamma_{s,n}(m, \lambda) a^{(s\lambda - n\alpha - \rho)}, \tag{5.22}$$

is an asymptotic series approximating φ uniformly for m, a, and λ varying in M^+, $A(Q)$, and $\mathscr{F}''(\chi)$ respectively. More precisely, there exists $\varepsilon > 0$ such that for any $\eta_1, \eta_2 \in U(\mathfrak{m}_{\mathbf{C}}^1)$, $u \in \mathscr{S}(\mathfrak{a}_{\mathbf{C}})$ there exists

$F \subset \widetilde{U}(\mathfrak{g}_{\mathbf{C}})$, $r \geq 0$ such that $S_{F,r}(\varphi) < \infty$ and

$$\begin{aligned}(2.64)\qquad &\Bigg\| \varphi(\eta_1 : ma_t ; \eta_2 u) \\ &\quad - \sum_{s^{-1} \in W(\mathfrak{a}, \mathfrak{a}_P)} \sum_{n=0}^{p_s} \Gamma_{s,n}(\eta_1 : m ; \eta_2, \lambda) \\ &\qquad \cdot u(s\lambda - n\alpha - \rho) a^{(s\lambda - n\alpha - \rho)} \Bigg\| \\ &\leq S_{F,r}(\varphi) |(m, \lambda)|^r d(m)^{-1} \Xi_M(m) e^{(\Re z_0(\lambda) - \varepsilon)t}, \\ &\qquad \lambda \in \mathscr{F}''(\chi),\ m \in M^+,\ t > 0,\end{aligned}$$

where $p_s \in \mathbf{Z}_+$ is chosen so that $-(p_s\alpha + \rho)(H) \geq \Re z_0(\lambda)$, and $-((p_s + 1)\alpha + \rho)(H) < \Re z_0(\lambda)$ for all $\lambda \in \mathscr{F}$.

Proof. The existence of the series follows easily from Theorem 5.1. The estimate (5.23) follows from Theorem 5.1 on noting that the function φ_0 equals the sum on $\mathscr{F}''(\chi)$. □

6. Eisenstein integrals. In this section we shall recall some standard facts concerning Eisenstein integrals. Most of the section is taken from Arthur [**1**]. We shall therefore only state the results which we find necessary, and refer the reader to Arthur for their proofs.

As before V will denote a fixed finite dimensional double unitary K-module. Fix a Levi subgroup M in $\mathscr{L}(M_0)$. Let $\mathscr{A}_{\text{cusp}}(M, \tau)$ be the space of τ-spherical functions on M/A_M which are square integrable, and $\mathfrak{z}(\mathfrak{m})$-finite. Equivalently, we may consider elements of $\mathscr{A}_{\text{cusp}}(M, \tau)$ as functions defined on M which are right invariant under A_M, i.e. $\varphi(ma) = \varphi(m)$, $m \in M$, and $a \in A_M$, and which are $\mathfrak{z}(\mathfrak{m})$-finite and square integrable on M/A_M. The space $\mathscr{A}_{\text{cusp}}(M, \tau)$ is finite dimensional, and equals $\{0\}$ unless M/A_M has discrete series. If ω is an equivalence class of square integrable representations of M/A_M, let $\mathscr{A}_\omega(M, \tau)$ be the space of functions $\varphi \in \mathscr{A}_{\text{cusp}}(M, \tau)$ such that for any $\xi^* \in V^*$, the function

$$m \mapsto \xi^*(\varphi(m)), \qquad m \in M/A_M,$$

is a sum of matrix coefficients of ω. Then

$$\mathscr{A}_{\text{cusp}}(M, \tau) = \bigoplus_\omega \mathscr{A}_\omega(M, \tau).$$

Let $\varphi \in \mathscr{A}_{\mathrm{cusp}}(M, \tau)$, $P \in \mathscr{P}(M)$, $x \in G$, and $\lambda \in \mathfrak{a}_{M,\mathbf{C}}$. As in §3 we define the Eisenstein integral by the formula

$$E_P(x, \varphi, \lambda) = \int_{K_M\backslash K} \tau(k^{-1})\varphi_P(kx)e^{(\lambda+\rho_P)(H_P(kx))}\, dk,$$

where the function φ_P is the function on G such that

$$\varphi_P(nmk) = \varphi(m)\tau(k), \qquad n \in N_P,\ m \in M,\ k \in K.$$

Then the function

$$E_P(\varphi, \lambda): x \mapsto E_P(x, \varphi, \lambda)$$

depends analytically on λ, and is a $\mathfrak{z}(\mathfrak{g})$-finite, τ-spherical function on G.

Let $W(A_0)$ be the Weyl group of (G, A_0). It is a finite group which acts on the vector spaces $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$. Suppose that M_1 is another Levi subgroup in $\mathscr{L}(M_0)$. As is customary, we will write $W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ for the set of distinct isomorphisms from $\mathfrak{a}_M$ into $\mathfrak{a}_{M_1}$, which are induced by $W(A_0)$. Recall that any two groups $P \in \mathscr{P}(M)$ and $P_1 \in \mathscr{P}(M_1)$ are said to be *associated if* $\dim \mathfrak{a}_M = \dim \mathfrak{a}_{M_1}$ *and this set is not empty.* If $t \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ we shall denote by w_t some representative of t in K. Now, suppose that Σ is any subset of M such that $K_M \Sigma K_M = \Sigma$, and that φ is a τ-spherical function from Σ to V. If $t \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$, define a τ-spherical function on $\Sigma_1 = w_t \Sigma w_t^{-1}$ by

$$(t\varphi)(m_1) = \tau(w_t)\varphi(w_t^{-1} m_1 w_t)\tau(w_t), \qquad m_1 \in \Sigma_1.$$

If $P \in \mathscr{P}(M)$ let tP be the group $w_t P w_t^{-1}$ in $\mathscr{P}(M_1)$. Then if $\varphi \in \mathscr{A}_{\mathrm{cusp}}(M, \tau)$, it is easily shown that

$$E_P(\varphi, \lambda) = E_{tP}(t\varphi, t\lambda). \tag{6.1}$$

More generally, if L is any Levi subgroup which contains both M and M_1, and $R \in \mathscr{P}^L(M)$, there is an identity

$$tE_R(\varphi, \lambda) = E_{tR}(t\varphi, t\lambda) \tag{6.2}$$

for Eisenstein integrals on L and $tL = w_t L w_t^{-1}$.

Suppose that $P \in \mathscr{P}(M)$. If T is a Cartan subgroup of M, let $\Delta_P(G, T)$ be the set of roots of (G, T) whose restriction to A_P belongs to $\Delta(P, A_P)$. Suppose that P' also belongs to $\mathscr{P}(M)$. Then the number

$$\beta_{P'|P} = \prod \left(\frac{(\alpha, \alpha)}{2}\right)^{1/2} \quad \text{(product over } \alpha \in \Delta_{\overline{P}}(G, T) \cap \Delta_{P'}(G, T))$$

is independent of T. (As usual, $\overline{P}$ stands for the group in $\mathscr{P}(M)$ opposite to P.) Let dX be the Euclidean measure on $\mathfrak{n}_{P'}$ associated to the norm

$$\|X\|^2 = -(X, \theta X), \qquad X \in \mathfrak{n}_{P'}.$$

We can normalize a Haar measure dn' on $N_{P'}$ by

$$\int_{N_{P'}} \phi(n')\, dn' = \int_{\mathfrak{n}_{P'}} \varphi(\exp X)\, dX, \qquad \varphi \in C_c^\infty(N_{P'}).$$

The same prescription gives a Haar measure on the subgroup $N_{P'} \cap N_P$. From these two measures we then obtain an invariant quotient measure on the coset space $N_{P'} \cap N_P \backslash N_{P'}$. Now if $\varphi \in \mathscr{A}_{\mathrm{cusp}}(M, \tau)$, $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$ and $m \in M$, define

$$(J^l_{P'|P}(\lambda)\varphi)(m) = \beta_{P'|P} \int_{N_{P'} \cap N_P \backslash N_{P'}} \tau(K_P(n))\varphi(M_P(n)m)e^{(\lambda+\rho_P)(H_P(n))}\, dn,$$

and

$$(J^r_{P'|P}(\lambda)\varphi)(m) = \beta_{P'|P} \int_{N_{P'} \cap N_P \backslash N_{P'}} \varphi(mM_P(n))\tau(K_P(n))e^{(\lambda+\rho_P)(H_P(n))}\, dn.$$

The integrals converge if

$$(\Re\lambda + \rho_P, \alpha) > 0$$

for each root $\alpha \in \Delta_P \cap \Delta_{\overline{P}'}$. Because of the factor $\beta_{P'|P}$, the integrals are independent of the measure on $N_{P'}$ and of the form $(\ ,\ )$.

Both $J^l_{P'|P}(\lambda)$ and $J^r_{P'|P}(\lambda)$ can be continued as meromorphic functions from $\mathfrak{a}_{M,\mathbf{C}}$ to the finite dimensional space of endomorphisms of $\mathscr{A}_{\mathrm{cusp}}(M, \tau)$. They satisfy all the usual properties of intertwining operators. In particular, let $d(P', P)$ be the number of singular hyperplanes which lie between the chambers $\mathfrak{a}_M(P)$ and $\mathfrak{a}_M(P')$. If P'' is a third group in $\mathscr{P}(M)$ such that

$$d(P'', P) = d(P'', P') + d(P', P),$$

one has

$$J^\iota_{P''|P}(\lambda) = J^\iota_{P''|P'}(\lambda) J^\iota_{P'|P}(\lambda), \qquad \iota = l, r. \tag{6.3}$$

Suppose that M_* is a Levi subgroup which is contained in M, and that $R, R' \in \mathscr{P}^M(M_*)$. The functions

$$J^\iota_{R'|R}(\Lambda), \qquad \Lambda \in \mathfrak{a}^*_{M_*,\mathbf{C}}, \iota = l, r,$$

associated to M and M_* (instead of G and M) can certainly be defined. They depend only on the projection of Λ onto the orthogonal complement of $\mathfrak{a}^*_{M,\mathbf{C}}$ in $\mathfrak{a}^*_{M_*,\mathbf{C}}$. We have the formula

$$J^{\iota}_{P(R')|P(R)}(\Lambda) = J^{\iota}_{R'|R}(\Lambda), \qquad \iota = l, r. \tag{6.4}$$

If $\lambda \in \mathfrak{a}_{M,\mathbf{C}}$ and $P_1, P_1' \in \mathscr{P}(M)$, the operators $J^{l}_{P_1'|P_1}(\lambda)$ and $J^{r}_{P'|P}(\lambda)$ commute. They also both commute with $\mathfrak{z}(\mathfrak{m})$.

Suppose that $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$. Let $P \in \mathscr{P}(M)$ and $P_1 \in \mathscr{P}(M_1)$. The groups $s^{-1}P_1$ and $s^{-1}\overline{P}_1$ both belong to $\mathscr{P}(M)$. Define

$$c_{P_1|P}(s, \lambda) = sJ^{l}_{s^{-1}P_1|P}(\lambda)J^{r}_{s^{-1}\overline{P}_1|P}(\lambda), \tag{6.5}$$

$$c^0_{P_1|P}(s, \lambda) = c_{P_1|P}(s, \lambda)c_{P|P}(1, \lambda)^{-1}, \tag{6.6}$$

and

$${}^0c_{P_1|P}(s, \lambda) = c_{P_1|P_1}(1, s\lambda)^{-1}c_{P_1|P}(s, \lambda). \tag{6.7}$$

These are all meromorphic functions on $\mathfrak{a}_{M,\mathbf{C}}$ with values in the space of linear maps from $\mathscr{A}_{\text{cusp}}(M, \tau)$ to $\mathscr{A}_{\text{cusp}}(M_1, \tau)$. We have the following functional equations:

$$c^0_{P_2|P}(s_1s, \lambda) = c^0_{P_2|P_1}(s_1, s\lambda)c^0_{P_1|P}(s, \lambda), \tag{6.8}$$

$${}^oc_{P_2|P}(s_1s, \lambda) = {}^oc_{P_2|P_1}(s_1, s\lambda)^0c_{P_1|P}(s, \lambda), \tag{6.9}$$

$$\begin{aligned} c_{P_2|P}(s_1s, \lambda) &= c^0_{P_2|P_1}(s_1, s\lambda)c_{P_1|P}(s, \lambda) \\ &= c_{P_2|P_1}(s_1, s\lambda)^0c_{P_1|P}(s, \lambda), \end{aligned} \tag{6.10}$$

$$E_P(x, \varphi, \lambda) = E_{P_1}(x, {}^oc_{P_1|P}(s, \lambda)\varphi, s\lambda), \tag{6.11}$$

the equalities holding for $s_1 \in W(\mathfrak{a}_{M_1}, \mathfrak{a}_{M_2})$ and $P_2 \in \mathscr{P}(M_2)$. Suppose that $t \in W(A_0)$. If $M' \in \mathscr{L}(M_0)$,

$$tM' = w_tM'w_t^{-1}$$

is another Levi subgroup; if $P \in \mathscr{P}(M)$, then $tP \in \mathscr{P}(tM')$. The restriction of t to $\mathfrak{a}_{M'}$ defines an element in $W(\mathfrak{a}_{M'}, \mathfrak{a}_{M'})$, which we will also denote by t. We then have the following identities:

$$tJ^{\iota}_{P'|P}(\lambda)t^{-1} = J^{\iota}_{tP'|tP}(t\lambda), \qquad \iota = l, r. \tag{6.12}$$

$$tc_{P_1|P}(s, \lambda) = c_{tP_1|P}(ts, \lambda), \tag{6.13}$$

$$c_{P_1|P}(s, \lambda)t^{-1} = c_{P_1|tP}(st^{-1}, t\lambda), \tag{6.14}$$

and similar formulas for ${}^0c_{P_1|P}$ and $c^0_{P_1|P}$. One also has the formulas:

$$c^0_{P_1|P}(s\lambda) = sJ^l_{\overline{P}|s^{-1}P_1}(\lambda)^{-1}J^r_{s^{-1}\overline{P}_1|P}(\lambda), \tag{6.15}$$

and

$${}^0c_{P_1|P}(s, \lambda) = sJ^l_{s^{-1}P_1|P}(\lambda)J^r_{P|s^{-1}P_1}(\lambda)^{-1}. \tag{6.16}$$

One can show that

$$J^l_{P|P'}(\lambda)J^l_{P'|P}(\lambda) = J^r_{P'|P}(\lambda)J^r_{P|P'}(\lambda).$$

Let $\mu_{P'|P}(\lambda)$ be the inverse of this operator. It is a meromorphic function of λ with values in the space of endomorphisms of $\mathscr{A}_{\text{cusp}}(M, \tau)$. For any λ, $\mu_{P'|P}(\lambda)$ commutes with any of the operators

$$J^\iota_{P'_1|P_1(\lambda_1)}, \qquad \lambda_1 \in \mathfrak{a}^*_{M,\mathbf{C}}, P_1, P'_1 \in \mathscr{P}(M), \iota = l, r.$$

Therefore $\mu_{P'|P}(\lambda)$ also commutes with $\mu_{P'_1|P_1}(\lambda_1)$. One can then show that

$$\mu_{P''|P}(\lambda) = \mu_{P''|P'}(\lambda)\mu_{P'|P}(\lambda) \tag{6.17}$$

if

$$d(P'', P) = d(P'', P') + d(P', P),$$

and

$$\mu_{P(R')|P(R)}(\Lambda) = \mu_{R'|R}(\Lambda), \tag{6.18}$$

if Λ, P, R, and R' are as in (6.4). For any $P \in \mathscr{P}(M)$, define

$$\mu_P(\lambda) = \mu_{\overline{P}|P}(\lambda). \tag{6.19}$$

It follows from the above properties that $\mu_P(\lambda)$ depends only on M and G, and not the group P. Moreover, for any $t \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$ and $P_1 \in \mathscr{P}(M_1)$,

$$\mu_P(\lambda) = \mu_{P_1}(t\lambda). \tag{6.20}$$

LEMMA 6.1 [*Arthur* [**1**, *Lemma* 2.1]. *Suppose that* $M_* \in \mathscr{L}^M(M_0)$, $R \in \mathscr{P}^M(M_*)$ *and* $P \in \mathscr{P}(M)$. *Then if* $\Lambda \in \mathfrak{a}^*_{M_*,\mathbf{C}}$ *and* $\lambda \in \mathfrak{a}^*_{M,\mathbf{C}}$,

$$\mu_{\overline{P}(R)|P(R)}(\Lambda + \lambda) = \mu_{P(R)}(\Lambda + \lambda)\mu_R(\Lambda)^{-1}.$$

We shall now quickly review the relation between Eisenstein integrals and induced representations. Fix $M \in \mathscr{L}(M_0)$. Let ω be an equivalence class of irreducible square integrable representations of

M/A_M, and let (σ, U_σ) be a representation in the class of ω (U_σ is the Hilbert space on which σ acts). Let $P \in \mathscr{P}(M)$ and define $\mathscr{H}(\sigma)$ to be the Hilbert space of measurable functions

$$\psi : K \mapsto U_\sigma$$

such that

(i) $\psi(mk) = \sigma(m)\psi(k)$, $m \in K_M$, $k \in K$,

(ii) $\|\psi\|_2^2 = \int_{K_M \backslash K} \|\psi\|^2 \, dk < \infty$.

If $\lambda \in \mathfrak{a}_{M,\mathbf{C}}$, there is the usual induced representation

$$(I_P(\sigma, \lambda, x)\psi)(k) = e^{(\lambda + \rho P)(H_P(kx))}\sigma(M_P(kx))\psi(K_P(kx)),$$

$\psi \in \mathscr{H}(\sigma)$, $x \in G$, which acts on $\mathscr{H}(\sigma)$.

Let now (V, τ) be an irreducible representation of K, instead of a double representation. Let $\mathscr{H}(\sigma)_\tau$ be the finite dimensional subspace of vectors in $\mathscr{H}(\sigma)$ under which the restriction of $I_P(\sigma, \lambda)$ to K is equivalent to a multiple of τ. Suppose that $S \in \mathrm{Hom}_{K_M}(V, U_\sigma)$; that is, S is a map from V to U_σ such that

$$S(\tau(m)\xi) = \sigma(m)S(\xi), \qquad \xi \in V, \, m \in K_M.$$

If $\xi \in V$, the function

$$\Psi_S(\xi) : k \mapsto S(\tau(k)\xi), \qquad k \in K,$$

belongs to $\mathscr{H}(\sigma)$. By Frobenius reciprocity, the map

$$S \mapsto \Psi_S$$

is an isomorphism from $\mathrm{Hom}_{K_M}(V, U_\sigma)$ onto $\mathrm{Hom}_K(V, \mathscr{H}(\sigma))$. Notice that $\mathscr{H}(\sigma)_\tau$ is the space spanned by

$$\{\Psi_S(\xi) | S \in \mathrm{Hom}_{K_M}(V, U_\sigma), \, \xi \in V\}.$$

It is isomorphic to $\mathrm{Hom}_{K_M}(V, U_\sigma) \otimes V$. If $S \in \mathrm{Hom}_{K_M}(V, U_\sigma)$, and P' is another group in $\mathscr{P}(M)$, set

$$\begin{aligned} &J_{P'|P}(\sigma, \lambda)S \\ &\quad = \beta_{P'|P} \int_{N_{P'} \cap N_P \backslash N_{P'}} \sigma(M_P(n))S\tau(K_P(n))e^{(\lambda + \rho_P)(H_P(n))} \, dn, \end{aligned}$$

defined a priori only for those $\lambda \in \mathfrak{a}_{M,\mathbf{C}}^*$ for which the integral converges, $J_{P'|P}(\sigma, \lambda)$ can be continued as a meromorphic function from $\mathfrak{a}_{M,\mathbf{C}}^*$ to the space of endomorphisms of $\mathrm{Hom}_{K_M}(V, U_\sigma)$. The map

$$\Psi_S(\xi) \mapsto \Psi_{J_{P'|P}(\sigma, \lambda)S}(\xi), \qquad S \in \mathrm{Hom}_{K_M}(V, U_\sigma), \, \xi \in V,$$

which we can also denote by $J_{P'|P}(\sigma, \lambda)$, is just the restriction to $\mathscr{H}(\sigma)_\tau$ of the usual intertwining operator from $I_P(\sigma, \lambda)$ to $I_{P'}(\sigma, \tau)$.

There is a canonical isomorphism

$$\Psi^*: \operatorname{Hom}_{K_M}(U_\sigma, V) \mapsto \operatorname{Hom}_K(\mathscr{H}(\sigma), V).$$

This map is given by

$$\Psi^*_{S^*}(\psi) = \int_{K_M\backslash K} \tau(k)^{-1} S^*(\psi(k))\, dk,$$

for $S^* \in \operatorname{Hom}_{K_M}(U_\sigma, V)$ and $\psi \in \mathscr{H}(\sigma)$. Notice that $\mathscr{H}(\sigma)^*_\tau$ is the space spanned by

$$\{\xi^*\Psi^*_{S^*} | S^* \in \operatorname{Hom}_{K_M}(U_\sigma, V), \xi \in V^*\}.$$

It is isomorphic to $\operatorname{Hom}_{K_M}(U_\sigma, V) \otimes V^*$.

Suppose that

$$(\tau_i, V_i), \qquad i = 1, 2,$$

is a pair of irreducible representations of K. We shall now take τ to be the *double* representation of K on

$$V = \operatorname{Hom}_{\mathbf{C}}(V_1, V_2),$$

defined by

$$\tau(k_2)X\tau(k_1) = \tau_2(k_2) \circ X \circ \tau_1(k_1),$$

for $k_1, k_2 \in K$, and $X \in \operatorname{Hom}_{\mathbf{C}}(V_1, V_2)$. Any double representation of K will be a direct sum of representations of this form. If $S_1 \in \operatorname{Hom}_{K_M}(V_1, U_\sigma)$, and $S_2^* \in \operatorname{Hom}_{K_M}(U_\sigma, V_2)$, then

$$\varphi(m) = S_2^*\sigma(m)S_1, \qquad m \in M,$$

is a function in $\mathscr{A}_{\text{cusp}}(M, \tau)$. The J functions defined above are related to those defined previously by the formulas

$$(6.21) \qquad (J^l_{P'|P}(\lambda)\varphi)(m) = (J_{P|P'}(\sigma, \lambda)^* S_2^*)\sigma(m)S_1,$$

and

$$(6.22) \qquad (J^r_{P'|P}(\lambda)\varphi)(m) = S_2^*\sigma(m)(J_{P'|P}(\sigma, \lambda)S_1).$$

We also have

$$(6.23) \qquad \Psi^*_{S_2^*} I_P(\sigma, \lambda, x)\Psi_{S_1} = E_P(x, \varphi, \lambda).$$

We now will recall some facts concerning the Harish-Chandra asymptotic expansion of the Eisenstein integrals. Let $B, B' \in \mathscr{P}(M_0)$ and consider the expansions of the functions

$$E_B(x, \Phi, \Lambda), \qquad \Phi \in \mathscr{A}_0, \Lambda \in \mathfrak{a}_{0,\mathbf{C}},$$

along the chamber $A_0(B')$. (Here $\mathscr{A}_0$ is the finite dimensional space of τ-spherical functions on M_0^1.) If ε is a small positive number and $a \in A_0^\varepsilon(B')$, (cf. (6.27) below) then $E_B(a, \Phi, \Lambda)$ can be written

$$\sum_{s \in W(A_0)} \sum_{\zeta \in \mathbf{Z}_+(\Sigma_{B'})} (c_{B'|B,\zeta}(s, \Lambda)\Phi)(1)a^{(s\Lambda - \zeta - \rho_{B'})},$$

where $c_{B'|B,\zeta}(s, \Lambda)$ is a meromorphic function of Λ with values in $\mathrm{End}(\mathscr{A}_0)$, the space of endomorphisms of $\mathscr{A}_0$. The functional equations for the Eisenstein integral give rise to the formulae

$$tc_{B'|B,\zeta}(s, \Lambda) = c_{tB'|B,t\zeta}(ts, \Lambda), \tag{6.24}$$

$$c_{B'|B,\zeta}(s, \Lambda)t^{-1} = c_{B'|tB,\zeta}(st^{-1}, t\Lambda), \tag{6.25}$$

$$c_{B'|B,\zeta}(s_1 s, \Lambda) = c_{B'|B_1,\zeta}(s_1, s\Lambda)^0 c_{B_1|B}(s, \Lambda), \tag{6.26}$$

for elements t, $s_1 \in W(A_0)$ and $B_1 \in \mathscr{P}(M_0)$. Suppose that x is an element of

$$G_\varepsilon = K \cdot A_0^\varepsilon(B) \cdot K;$$

where

$$A_0^\varepsilon(B) = \{a \in A_0(B) | \alpha(H_0(a)) > \varepsilon, \alpha \in \Sigma_B\}. \tag{6.27}$$

Then

$$x = k_1 a k_2, \quad k_1, k_2 \in K, \quad a \in A_0^\varepsilon(B').$$

Define

$$\begin{aligned}(6.28)\quad & E_{B'|B,s}(x, \Phi, \Lambda) \\ &= \tau(k_1) \sum_{\zeta \in \mathbf{Z}_+(\Sigma_{B'})} (c_{B'|B,\zeta}(s, \Lambda)\Phi)(1)a^{(s\Lambda - \zeta - \rho_{B'})}\tau(k_2).\end{aligned}$$

Then $E_{B'|B,s}(\Phi, \Lambda)$ is a τ-spherical function on G_ε. It is meromorphic in Λ, and

$$E_B(x, \Phi, \Lambda) = \sum_{s \in W(A_0)} E_{B'|B,s}(x, \Phi, \Lambda). \tag{6.29}$$

From the three functional equations above, one obtains

$$E_{B'|B,s}(x, \Phi, \Lambda) = E_{tB'|B,ts}(x, \Phi, \Lambda), \tag{6.30}$$

$$E_{B'|B,s}(x, \Phi, \Lambda) = E_{B'|tB,st^{-1}}(x, t\Phi, t\Lambda), \tag{6.31}$$

$$E_{B'|B,s_1 s}(x, \Phi, \Lambda) = E_{B'|B_1,s_1}(x, {}^0c_{B_1|B}(s, \Lambda)\Phi, s\Lambda). \tag{6.32}$$

These functions are all $\mathfrak{z}(\mathfrak{g})$-finite. Indeed, if $z \in \mathfrak{z}(\mathfrak{g})$ let $\mu_{10}(z:\Lambda)$ be the differential operator on M_0^1 obtained by evaluating $\mu_{10}(z)$ at Λ. Then the equation

$$E_{B'|B,s}(x; z, \Phi, \Lambda) = E_{B'|B,s}(x, \mu_{10}(z:\Lambda)\Phi, \Lambda) \tag{6.33}$$

follows from the analogous formula for Eisenstein integrals. Since Φ is $\mathfrak{z}(\mathfrak{m})$-finite, then $E_{B'|B,s}(x, \Phi, \Lambda)$ must be $\mathfrak{z}(\mathfrak{g})$-finite.

Let $M \in \mathscr{L}(M_0)$, $r \in W(A_0)^M$, $R, R' \in \mathscr{P}^M(M_0)$, where $W(A_0)^M$ denotes the subgroup of elements of $W(A_0)$ which leave $\mathfrak{a}_M^*$ pointwise fixed; it can be identified with the Weyl group of the pair (M, A_0). Then we can define the functions

$$E_{R'|R,r}(\Phi, \Lambda)$$

on a neighborhood of infinity in M. Let $P, L \in \mathscr{P}(M)$. Let $B = P(R)$. There is a unique coset s in $W(A_0)/W(A_0)^M$ such that the group $P_1 = sL$ contains B. If $M_1 = w_s M w_s^{-1}$ then P_1 belongs to $\mathscr{P}(M_1)$. Let s_B be the unique representative of s in $W(A_0)$ such that $s_B(\alpha) \in \Delta(B, A_0)$ for every $\alpha \in \Delta(R, A_0)$.

LEMMA 6.2 [*Arthur* [**1**], *Lemma* I.4.2]. *If Λ is in general positive in $\mathfrak{a}_0^*$, then*

$$E_{R|R,1}(J^l_{L(R)|P(R)}(\Lambda)J^r_{\overline{L}(R)|P(R)}(\Lambda)\Phi, \Lambda) = s^{-1}E^{P_1}(E_{B|B,s_B}(\Phi, \Lambda));$$

where

$$E^{P_1}(a, E_{B|B,s_B}(\Phi, \Lambda)) = \sum_{\zeta \in \mathbf{Z}_+(\Sigma_R)} (c_{B|B,\zeta}(s_B, \Lambda)\Phi)(1)a^{(s_B(\Lambda-\rho_R)-\zeta)},$$

here we consider Σ_R as a subset of Σ_B, and the equality holds on M_ε.

We shall also need two results of Harish-Chandra's. Fix $M \in \mathscr{P}(M_0)$, M cuspidal, T a Cartan subgroup contained in M, and $P \in \mathscr{P}(M)$. Let notation be as in §2; in particular $\mathscr{F} = (-1)^{-1/2}\mathfrak{a}^*_{M,\mathbf{C}}$. Further let $\mathscr{F}'$ denote the set of $\lambda \in \mathscr{F}$ such that $\pi(\lambda) \neq 0$, where

$$\pi(\lambda) = \prod(\alpha, \lambda) \text{ (product over } \alpha \in \Delta_P(G, T)).$$

For $\delta > 0$ let

$$\mathscr{F}_{\mathbf{C},\delta} = \{\lambda \in \mathscr{F}_{\mathbf{C}} \mid \|\lambda_{\mathbf{R}}\| < \delta\}. \tag{6.34}$$

LEMMA 6.3 [*Harish-Chandra* [4], *Theorem* 18.1]. *Fix $\lambda \in \mathscr{F}'$ and $P_1, P_2 \in \mathscr{P}(M)$. Then the constant term of $E_{P_1}(ma, \psi, \lambda)$ along the parabolic P_2 is given by the following expression:*

$$\sum_{s \in W(A)} (c_{P_2|P_1}(s, \lambda)\psi)(m)a^{s\lambda}$$

the equality holding for all $\psi \in \mathscr{A}_{\mathrm{cusp}}(M, \tau)|$, $m \in M^1$, $a \in A_M$. Moreover, we can choose $\delta > 0$ such that for every $s \in W(A)$, $\pi(\lambda)c_{P_2|P_1}(s, \lambda)$ extends to a holomorphic function of λ on $\mathscr{F}_{\mathbf{C},\delta}$.

LEMMA 6.4 [*Harish-Chandra* [5], *Lemma* 17.3, *Lemma* 19.2]. (1) *Let $s \in W(A)$, $P_1, P_2 \in \mathscr{P}(M)$, then ${}^0c_{P_1|P_2}(s, \lambda)$ maps $\mathscr{A}_\omega(M, \tau)$ into $\mathscr{A}_{s\omega}(M, \tau)$, ω, an equivalence class of square integrable representations on M. Moreover, if $\lambda \in \mathscr{F}'$, then ${}^0c_{P_1|P_2}(s, \lambda)$ is unitary.*

(2) *Fix $P, Q \in \mathscr{P}(A)$ and $s \in W(A)$. Then*

$$\nu \mapsto {}^0c_{Q|P}(s, \nu)$$

defines a rational map of $\mathfrak{a}^*_{\mathbf{C}}$ into $\operatorname{End} L$.

7. Continuation of the asymptotic expansion. Suppose that $\varphi(x, \lambda)$ equals $E_P(x, \Phi, \lambda)$, where P is some cuspidal parabolic subgroup, $\Phi \in \mathscr{A}_{\mathrm{cusp}}(M_P, \tau)$. We will now show that the coefficients of the asymptotic expansion (5.22) of φ along Q, where $W(\mathfrak{a}_Q, \mathfrak{a}_P) \neq \varnothing$, can be expressed in terms of Eisenstein integrals defined on M. This shows that the asymptotic expansion has coefficients which have meromorphic extensions to $\mathfrak{a}_{P,\mathbf{C}}$.

Let $B \subset P$, $B \subset Q$, with B minimal, Q maximal, and P cuspidal. We shall continue to use the notational conventions of §2. Namely, we shall *not* subscript objects associated with Q (so we shall write $\mathfrak{a}$ for $\mathfrak{a}_Q = \mathfrak{a}_M$ where $Q = NM$), and subscript objects associated to B with $_0$ (e.g. $A_0 = A_B$). Further, we let $\mathscr{F} = (-1)^{1/2}\mathfrak{a}^*_P$, and we recall the nonstandard notation, $\mathscr{F}_{\mathbf{C}} = \mathfrak{a}_{P,\mathbf{C}}$. Let $L \in \mathscr{F}(M_0)$ and put,

$$\mathfrak{a}^*_{L,\mathbf{C}}(L) = \{\lambda \in \mathfrak{a}^*_{L,\mathbf{C}} | \Re(\lambda, \alpha) > 0 \ \forall \alpha \in \Delta(L, A_L)\}. \tag{7.1}$$

LEMMA 7.1. *Let $t \in W(A_0)$, $t \notin W(A_0)^M$, and $\Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}(B)$. Then*

$$\Re t\Lambda(H) < \Re\Lambda(H)$$

where $H \in \mathfrak{a}$ is as in (2.1).

Proof. It suffices to consider the case, $\Lambda \in \mathfrak{a}^*_0(B) = \mathfrak{a}^*_{0,\mathbf{C}}(B) \cap \mathfrak{a}^*_0$. Let $\Sigma(B, A_0) = \{\alpha_1, \dots, \alpha_q\}$. Denote by β_i, $1 \le i \le q$, the fundamental dominant weights dual to $\check{\alpha}_i$ where $\check{\alpha}_i = 2/(\alpha_i, \alpha_i)\cdot\alpha_i$. As Q

is a maximal parabolic, standard with respect to B, then there exists k, $1 \leq k \leq q$, such that $\alpha_i(H) = 0$ for all $i \neq k$. Define $\lambda_H \in \mathfrak{a}_{0,\mathbf{C}}^*$ by the equation

$$(\lambda_H, \mu) = \mu(H), \qquad \mu \in \mathfrak{a}_{0,\mathbf{C}}^*.$$

Then

$$(\lambda_H, \check{\alpha}_i) = \begin{cases} 0, & \text{if } i \neq k, \\ c_k = 2/(\alpha_k, \alpha_k), & \text{if } i = k. \end{cases}$$

It follows that

$$\lambda_H = c_k \beta_k.$$

Suppose then that $\Lambda \in \mathfrak{a}_0^*(B)$. We have for $s \in W(A_0)$

$$(s\Lambda - \Lambda)(H) = \Lambda(s^{-1}H - H) = (\Lambda, s\lambda_H - \lambda_H).$$

But

$$\Lambda = \sum_{i=1}^{a} (\Lambda, \check{\alpha}_i)\beta_i,$$

and

$$s\lambda_H - \lambda_H = \sum_{i=1}^{q} -m_i\check{\alpha}_i \qquad m_i \geq 0.$$

This last equality since λ_H is a positive multiple of a dominant weight. Hence

$$\Lambda(s^{-1}H - H) = \sum_{i=1}^{q} -(\Lambda, \check{\alpha}_i)m_i.$$

This shows that $\Lambda(s^{-1}H - H) \leq 0$ as $(\Lambda, \check{\alpha}_i) > 0$ by definition of $\mathfrak{a}_0^*(B)$. Further, we see that $\Lambda(s^{-1}H - H) = 0$ if and only if $m_i = 0$ for all i, which is equivalent to $sH - H = 0$ which would imply that $s \in W(A_0)^M$. □

For any $b \in U(\mathfrak{g}_\mathbf{C})$ let $\nu_Q(b)$ be that unique element of $\lambda(\mathscr{S}(\mathfrak{m} \cap \mathfrak{s}))U(\mathfrak{t})$ such that $b - \nu_Q(b) \in \theta(\mathfrak{n}_Q)U(\mathfrak{g}_\mathbf{C})$. Then by Lemma 4.3 of Trombi and Varadarajan [2] we have

$$b = \nu_q(b) + \sum_{i=1}^{s} g_i(m)\xi_i^{m^{-1}}\eta_i\zeta_i, \qquad m \in M^+, \tag{7.2}$$

where ξ_i, $\zeta_i \in U(\mathfrak{t}_\mathbf{C})$, $\eta_i \in U(\mathfrak{m}_\mathbf{C})$ and $g_i(m \exp tH) \mapsto 0$ as $t \mapsto \infty$, H as in (2.1).

Let $R = B \cap M$. Note that R is a minimal parabolic subgroup of M. Put

$$c_{R|B}(\lambda) = c_{R|R}(1, \lambda)^{-1} c_{B|B}(1, \lambda). \tag{7.3}$$

It is known (cf. Cohn [1], Theorems 4 and 5) that for a parabolic subgroup P, the singularities of $c_{P|P}(1, \lambda)$ and $c_{P|P}(1, \lambda)^{-1}$ are all of the form $\mathscr{U}_{\alpha, r} = \{\lambda \in \mathfrak{a}_{0,\mathbf{C}} | (\lambda, \alpha) = r\}$, where $r \in \mathbf{C}$, $\alpha \in \mathbf{Z}_+(\Delta(P, A_P))$. We shall say that $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}$ is in *general position* if λ does not lie on any of the singular hyperplanes $\mathscr{U}_{\alpha, r}$.

LEMMA 7.2. *If $\lambda \in \mathfrak{a}^*_{0,\mathbf{C}}(B)$, λ in general position $a_t = \exp tH$, with H as in (2.1), then*

$$\lim_{t \mapsto \infty} a_t^{-(\lambda-\rho)} E_B(ma_t, \Phi, \lambda) = E_R(m, c_{R|B}(\lambda)\Phi, \lambda),$$

where the equality holds for all $m \in K_M A_0(R) K_M$, and $K_M = K \cap M$. Furthermore, if $\eta \in U(\mathfrak{m}_\mathbf{C})$, $b \in U(\mathfrak{g}_\mathbf{C})$ then

$$\lim_{t \mapsto \infty} a^{-(\lambda-\rho)} E_B(\eta ; ma_t; b, \Phi, \lambda) = E_R(\eta ; m; \nu_Q(b), c_{R|B}(\lambda)\Phi, \lambda),$$

with λ, m as above.

Proof. By (6.29) we have

$$\begin{aligned} E_B(ma_t, \Phi, \lambda) &= \sum_{s \in W(A_0)} E_{B|B,s}(ma_t, \Phi, \lambda) \\ &= \sum_{t \in W(A_0)/W(A_0)^M} \sum_{s \in W(A_0)^M} E_{B|B,ts}(ma_t, \Phi, \lambda). \end{aligned}$$

Here, the first sum is over a complete set of representatives of the cosets of $W(A_0)^M$ in $W(A_0)$, where we choose the element 1 to represent the identity coset. As λ is assumed to be in general position we have by (6.32) that the above equals

$$\sum_{w \in W(A_0)/W(A_0)^M} \sum_{s \in W(A_0)^M} E_{B|B,w}(ma_t, {}^0c_{B|B}(s, \lambda)\Phi, s\lambda).$$

Observe that for w, s as in the last sum we have with $\lambda_Q = \lambda|_{\mathfrak{a}}$, $\lambda_{^*R} = \lambda|_{\mathfrak{a}_{^*R}}$, where $^*R = R \cap M^1$, that

$$ws\lambda(H) = ws(\lambda_Q + \lambda_{^*R})(H) = w(\lambda_Q + s\lambda_{^*R})(H), \tag{7.4}$$

and,

$$\rho_0 = \rho + \rho_R.$$

For all $w \neq 1$, $ws \notin W(A_0)^M$ for all $s \in W(A_0)^M$. Hence from

Lemma 7.1 we see that

$$\begin{aligned}
&\lim_{t\mapsto\infty} a_t^{-(\lambda-\rho)} E_B(ma_t, \Phi, \lambda) \\
&\quad = \sum_{s\in W(A_0)^M} \lim_{t\mapsto\infty} a^{-(\lambda-\rho)} E_{B|B,1}(ma_t, {}^0c_{B|B}(s,\lambda)\Phi, s\lambda) \\
&\quad = \sum_{s\in W(A_0)^M} E_{R|R,1}(m, J^l_{B|B}(s\lambda) J^r_{\overline{Q(R)}|B}(s\lambda)^0 c_{B|B}(s,\lambda)\Phi, s\lambda).
\end{aligned}$$

This last equality follows from Lemma 6.2; here we need to observe that if λ is in general position, then so is $s\lambda$ for all $s \in W(A_0)^M$. Since,

$$J^l_{R|R}(s\lambda) = J^l_{B|B}(s\lambda) = 1,$$

then we have

$$c_{R|R}(1, s\lambda) = J^r_{\overline{R}|R}(s\lambda), \quad \text{and} \quad c_{B|B}(1, s\lambda) = J^r_{\overline{B}|B}(s\lambda).$$

Hence, if $m = k_1 a k_2$, with $k_1, k_2 \in K_M$, and $a \in M_\varepsilon$, then

$$\begin{aligned}
&E_{R|R,1}(m, J^r_{\overline{Q(R)}|B}(s\lambda)^0 c_{B|B}(s,\lambda)\Phi, s\lambda) \\
&\quad = \tau(k_1) \sum_{\zeta\in \mathbf{Z}_+(\Sigma_R)} c_{R|R,\zeta}(1, s\lambda) J^r_{\overline{Q(R)}|B}(s\lambda)^0 c_{B|B}(s,\lambda) \\
&\qquad\qquad \cdot \Phi(1) a^{(s\lambda-\rho_R-\zeta)} \tau(k_2),
\end{aligned}$$

and

$$c_{R|R,\zeta}(t,\mu) = \Gamma_{R,\zeta}(t\mu - \rho_R) c_{R|R}(t,\mu).$$

As we have observed $c_{R|R}(1, s\lambda) = J^r_{\overline{R}|R}(s\lambda)$ and this in turn equals $J^r_{\overline{Q(\overline{R})}|\overline{Q(R)}}(s\lambda)$ by (6.4). From (6.3) we have

$$J^r_{\overline{Q(\overline{R})}|(\overline{Q(R)}}(s\lambda) J^r_{\overline{Q(R)}|B}(s\lambda) = J^r_{\overline{Q(\overline{R})}|B}(s\lambda) = J^r_{\overline{B}|B}(s\lambda) = c_{B|B}(1, s\lambda).$$

By (6.7) we have that the right side of the above equality equals

$$\begin{aligned}
&\tau(k_1) \sum_\zeta \Gamma_{R,\zeta}(s\lambda - \rho_R) c_{R|R}(s,\lambda) c_{R|R}(s,\lambda)^{-1} c_{B|B}(s,\lambda) \\
&\qquad\qquad \cdot \Phi(1) a^{(s\lambda - \rho R - \zeta)} \tau(k_2).
\end{aligned}$$

But by (6.10) we have

$$c_{R|R}(s,\lambda) = c^0_{R|R}(s,\lambda) c_{R|R}(1,\lambda),$$

and by Theorem 7 of Harish-Chandra **[4]** we have

$$c^0_{R|R}(s,\lambda) = c^0_{B|B}(s,\lambda).$$

Hence,

$$c_{R|R}(s, \lambda)^{-1} c_{B|B}(s, \lambda)^{-1} = c_{R|R}(1, \lambda)^{-1} c_{B|B}(1, \lambda) = c_{R|B}(\lambda).$$

The first equality now follows.

For the second equality of the lemma, let us take $\eta_1, \eta_2 \in U(\mathfrak{m}_{\mathbf{C}})$, and prove the equality with b and $\nu_Q(b)$ replaced by η_2. Granting this case we see that the general case with $b \in U(\mathfrak{g}_{\mathbf{C}})$ follows from (7.2) on noting that the functions $g_i(m \exp tH) \mapsto 0$ as $t \mapsto \infty$. For the proof of the special case we note that for any $\varepsilon > 0$, $x \in G_\varepsilon$ that

$$E_B(\eta_1 ; x ; \eta_2, \Phi, \lambda) = \sum_s E_{B|B,s}(\eta_1 ; x ; \eta_2, \Phi, \lambda),$$

and $E_{B|B,s}(\eta_1 ; x ; \eta_2, \Phi, \lambda)$ can be expressed as an infinite series of the same type as the undifferentiated series. The proof of Lemma 3.3 given in Arthur [**1**] can easily be adjusted so that it applies to this series. Hence the proof in the special case proceeds just as in the first case. □

We now wish to use the subquotient theorem to show that the identities of the last lemma can be extended to arbitrary Eisenstein integrals. First note that by Lemma I.7.2 of Arthur [**1**], that given $\Phi \in \mathscr{A}_{\mathrm{cusp}}(M_P, \tau)$ there exists $\Psi_i \in \mathscr{A}_{\mathrm{cusp}}(M_0, \tau)$, $\Lambda_i \in \mathfrak{a}^*_{0,\mathbf{C}}$ $(1 \leq i \leq s)$ such that if $R_1 = M_P \cap B$ then

$$\Phi = \sum_{i=1}^{s} E_{R_1}(x, \Psi_i, \Lambda_i). \tag{7.5}$$

We shall assume without loss of generality (for example by restricting $\Phi \in \mathscr{A}_\omega(M_P, \tau)$, with $\omega \in \widehat{M_P^2}$), that $s = 1$. By Lemma 17.5 of Harish-Chandra [7] we have

$$E_P(x, E_{R_1}(\Psi, \Lambda), \lambda) = E_B(x, \Psi, \Lambda + \lambda). \tag{7.6}$$

Here we may by the functional equation (6.11) assume that $\Lambda \in {}^*\mathfrak{a}^*_{0,\mathbf{C}}(R_1 \cap M_P^1)$ where ${}^*\mathfrak{a}^*_{0,\mathbf{C}}$ equals the orthogonal complement of $\mathfrak{a}^*_{P,\mathbf{C}}$ in $\mathfrak{a}^*_{0,\mathbf{C}}$ and Λ is extended by zero to $\mathfrak{a}_{P,\mathbf{C}}$. We have then that

$$E_P(x, \Phi, \lambda) = E_P(x, E_{R_1}(\Psi, \Lambda), \lambda) = E_B(x, \Psi, \Lambda + \lambda). \tag{7.7}$$

Let $s^{-1} \in W(\mathfrak{a}, \mathfrak{a}_P)$. As $s^{-1}\mathfrak{a} \subset \mathfrak{a}_P$ then $\mathfrak{a} \subset s\mathfrak{a}_P$. Choose $\mathfrak{b}_s \in \mathrm{Chamb}(s\mathfrak{a}_P^{*\prime})$ and $\mathfrak{a}^*_{0,s} \in \mathrm{Chamb}(\mathfrak{a}_0^{*\prime})$ such that $\mathfrak{a}^*(Q) \subset \mathrm{cl}\,\mathfrak{b}_s$, and $\mathfrak{b}_s \subset \mathrm{cl}\,\mathfrak{a}^*_{0,s}$. As each element of $\mathrm{Chamb}(s\mathfrak{a}_P^{*\prime})$ (resp. $\mathrm{Chamb}(\mathfrak{a}_0^{*\prime})$) corresponds to an element of $\mathscr{P}(sM_P)$ (resp. $\mathscr{P}(M_0)$) then there exists $P_s \in \mathscr{P}(sM_P)$ (resp. $B_s \in \mathscr{P}(M_0)$) such that $P_s = M_s N_s$ where $M_s = sM_P$ and if $\mathfrak{a}_s = s\mathfrak{a}_P$ then

$$\mathfrak{a}(Q) \subset \operatorname{cl} \mathfrak{a}_s(P_s) \subset \operatorname{cl} \mathfrak{a}_0(B_s) \quad \text{and} \quad B_s \subset P_s \subset Q.$$

For notational convenience let us put

$$B_s(M) = B_s \cap M \quad \text{and} \quad P_s(M) = P_s \cap M. \tag{7.8}$$

Let s_B denote the unique representative of the coset $sW(A_0)^M$ such that $s_B\alpha \in \Delta(B, A_0)$, for all $\alpha \in \Delta(B \cap M, A_0)$.

LEMMA 7.3. *Let $\mu \in \mathfrak{a}^*_{s,\mathbf{C}}(P_s)$. Then $\mu = s\lambda$ for some unique $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}(P)$. We shall assume μ is in general position in $\mathfrak{a}^*_{s,\mathbf{C}}$. Let $\Phi \in \mathscr{A}_{\text{cusp}}(M_P, \tau)$, and $m \in K_M A_0(B_s(M))K_M$. Then*

$$\lim_{t \mapsto \infty} e^{-t(s_B\lambda - \rho)(H)} E_P(ma_t, \Phi, \lambda)$$

exists and equals

$$E_{P_s(M)}(m, c_{P_s(M)|P_s}(s_B\lambda)^0 c_{P_s|P}(s_B, \lambda)\Phi, s_B\lambda), \tag{7.9}$$

where

$$c_{P_s(M)|P_s}(\lambda) = c_{P_s(M)|P_s(M)}(1, \lambda)^{-1} c_{P_s|P_s}(1, \lambda).$$

Furthermore, if $\eta \in U(\mathfrak{m}_\mathbf{C})$, $b \in U(\mathfrak{g}_\mathbf{C})$, then

$$\lim_{t \mapsto \infty} e^{-t(s_B\lambda - \rho)(H)} E_P(\eta \,;\, ma_t; b, \Phi, \lambda)$$

exists and equals

$$E_{P_s(M)}(\eta \,;\, m; \nu_Q(b), c_{P_s(M)|P_s}(s_B\lambda)^0 c_{P_s|P}(s_B, \lambda)\Phi, s_B\lambda). \tag{7.10}$$

Proof. First by the functional equation (6.11) we have

$$\begin{aligned} &\lim_{t \mapsto \infty} e^{-t(s\lambda - \rho)(H)} E_P(ma_t, \Phi, \lambda) \\ &\quad = \lim_{t \mapsto \infty} e^{-t(s\lambda - \rho)(H)} E_{P_s}(ma_t, {}^0c_{P_s|P}(s_B, \lambda)\Phi, s_B\lambda). \end{aligned}$$

Now suppose that Φ and Ψ are related as in (7.7), and $\Lambda \in \mathfrak{a}^*_{0,\mathbf{C}}(B \cap M_P)$, with Λ perpendicular to $\mathfrak{a}^*_P$. Observe that $s_B\Lambda(H) = \Lambda(s_B^{-1}H) = 0$ as Λ vanishes on $\mathfrak{a}_P$. Then by (6.16) and Lemma I.7.3 of Arthur **[1]** we have the above limit equals

$$\lim_{t \mapsto \infty} e^{-t(s_B(\Lambda + \lambda) - \rho)(H)} E_{B_s}(ma_t, {}^0c_{B_s|B}(s_B, \Lambda + \lambda)\Psi, s_B(\Lambda + \lambda)).$$

But $\mu \in \mathfrak{a}^*_{s,\mathbf{C}}(P_s)$ implies that $\lambda \in \mathfrak{a}^*_{P,\mathbf{C}}(P)$; further, letting $\Sigma(B, A_0) = \{\alpha_1, \dots, \alpha_d\}$, and $\alpha_1, \dots, \alpha_p$ denoting those simple roots which vanish on $\mathfrak{a}_P$, then we can define $H_i \in \mathfrak{a}_0$ by the condition $\alpha_i(H_j) = \delta_{ij}$, and $\beta_i \in \mathfrak{a}^*_{0,\mathbf{C}}$ by $(\lambda, \beta_i) = \lambda(H_i)$ for all $\lambda \in \mathfrak{a}^*_{0,\mathbf{C}}$. Then $\Lambda_{\mathbf{R}} = \sum_{i=1}^{p} m_i\beta_i$ with $m_i > 0$. Hence,

$$\mathfrak{R}(\Lambda + \lambda, \alpha_j) = \begin{cases} \mathfrak{R}(\Lambda, \alpha_j) = m_j, & \text{if } 1 \le j \le p, \\ \mathfrak{R}(\lambda, \alpha_j) > 0, & \text{if } p+1 \le j \le d. \end{cases}$$

The second equality follows since $\mathfrak{R}(\Lambda, \alpha_j) = \sum_{i=1}^{p} m_i(\beta_i, \alpha_j) = 0$ as $i \neq j$. But $(\lambda, \alpha_j) = (\lambda, \alpha_j^0)$ where α_j^0 denotes the restriction of α_j to $\mathfrak{a}_P$. Clearly α_j^0 belongs to $\Delta(P, A_P)$ since $B \subset P$. It follows tht $\mathfrak{R}(\lambda, \alpha_j) > 0$, and hence that $s_B(\Lambda+\lambda) \in \mathfrak{a}^*_{0,\mathbf{C}}(B_s)$. From Lemma 7.2 the above limit equals

$$E_{B_s(M)}(m, c_{B_s(M)|B_s}(s_B(\Lambda+\lambda))^0c_{B_s|B}(s_B, \lambda)\Psi, s_B(\Lambda+\lambda)).$$

As $B_s \subset P_s \subset Q$ then $B_s(M) \subset P_s(M)$. Let

$$\tilde{\Phi} = {}^0c_{P_s|P}(s_B, \lambda)\Phi = E_{B_s\cap M_s}(\tilde{\Psi}, s_B(\Lambda+\lambda)),$$

where

$$\tilde{\Psi} = {}^0c_{B_s|B}(s_B, \Lambda+\lambda)\Psi.$$

From Lemma I.7.3 of Arthur [**1**] we have

$$c_{P_s|P_s}(1, s_B\lambda)\tilde{\Phi} = E_{B_s\cap M_s}(c_{B_s|B_s}(1, s_B(\Lambda+\lambda))\tilde{\Psi}, s_B\Lambda).$$

Applying Lemma I.7.3 of Arthur [**1**] once again but this time to M replacing G we obtain

$$c_{P_s(M)|P_s}(s_B\lambda)\tilde{\Phi} = E_{B_s\cap M_s}(c_{B_s(M)|B_s}(s_B(\Lambda+\lambda))\tilde{\Psi}, s_B\Lambda).$$

Hence,

$$\begin{aligned} &E_{P_s(M)}(c_{P_s(M)|P_s}(s_B\lambda)\tilde{\Phi}, s_B\lambda) \\ &\quad = E_{B_s(M)}(c_{B_s(M)|B_s}(s_B(\Lambda+\lambda))\tilde{\Psi}, s_B(\Lambda+\lambda)). \end{aligned}$$

This proves (7.9). The proof of (7.10) follows from the second equality of Lemma 7.2. □

PROPOSITION 7.4. *Let notation and assumptions be as in Corollary 5.5. Then for* $\alpha \in {}^0Q$, $j \in Q^0_\alpha$, $(s_\alpha\chi^y - \rho)(H) - n \ge \mathfrak{R}z_0$, $\lambda \in \mathscr{F}_{\mathbf{C},\delta} \cap \mathscr{F}''_{\mathbf{C}}(\chi)$, $n \in \sigma(\alpha, j)$ *and* $m \in M^+$, *we have*

$$\begin{aligned} &\operatorname{Exp}\{-s\Gamma(X_\lambda)\}E_n(s_\alpha\chi^y + s_j\lambda)\Phi_0(m, s, \lambda) \\ &\quad = \lim_{t\mapsto\infty} \operatorname{Exp}\{-t\Gamma(X_\lambda)\}E_n(s_\alpha\chi^y + s_j\lambda)\Phi(ma_t, \lambda). \end{aligned}$$

The equality holds for all $s \in \mathbf{R}$.

Proof. By (5.7) we have

$$\begin{aligned}
&\operatorname{Exp}\{-s\Gamma(X_\lambda)\}E_n(s_\alpha\chi^y + s_j\lambda)\Phi_0(m, s, \lambda)\\
&\quad = E_n(s_\alpha\chi^y + s_j\lambda)\Phi(m, \lambda)\\
&\qquad + \int_0^\infty E_n(s_\alpha\chi^y + s_j\lambda)\operatorname{Exp}\{-u\Gamma(X_\lambda)\}\Psi(ma_u, \lambda)\, du.
\end{aligned}$$

On the other hand we have from (5.4) that

$$\begin{aligned}
&\operatorname{Exp}\{-t\Gamma(X_\lambda)\}E_n(s_\alpha\chi^y + s_j\lambda)\Phi(ma_t, \lambda)\\
&\quad = E_n(s_\alpha\chi^y + s_j\lambda)\Phi(m, \lambda)\\
&\qquad + \int_0^t E_n(s_\alpha\chi^y + s_j\lambda)\operatorname{Exp}\{-u\Gamma(X_\lambda)\}\Psi(ma_u, \lambda)\, du.
\end{aligned}$$

It follows from the estimates in the proof of Corollary 5.5 that the limit of this last expression as t tends to infinity equals the first expression. □

Recall that $T_k = \bigoplus_{i,j,M} \mathbf{C}e_{ij:M}(X_\lambda)$ where $e_{ij:M}(X_\lambda)$ is defined in (2.16). For any $\alpha \in {}^oQ$, $j \in Q_\alpha^o$, $n \in \sigma(\alpha, j)$, let $T_k^{(\alpha j:n)}$ denote the linear space spanned by the following set of vectors:

$$\{e_{kl:K}(X_\lambda) | k \in Q_\alpha^o,\ n \in \sigma(\alpha, k),\ \text{and}\ K \in \mathbf{Z}_+^n,\ \ni n_{\alpha k} + w(K) = n,\ 1 \le l \le d_K\}.$$

LEMMA 7.5. *Let $\lambda \in \mathscr{F}_{\mathbf{C}}''(\chi) \cap \mathscr{F}_{\mathbf{C},\delta}$ with δ as in the discussion preceding Corollary 5.5, and $v \in V$. Then*

$$E_n(s_\alpha\chi^y + s_j\lambda)v \otimes e_{kl:M}(X_\lambda) = 0, \quad \text{if } e_{kl:M}(X_\lambda) \notin T_k^{(\alpha j:n)}; \tag{7.11}$$

Proof. To see this note that $e_{kl:M}(X_\lambda) \notin T_k^{(\alpha j:n)}$ implies that either $k \not\preceq \alpha$, or $k \preceq \alpha$ but $n_{\alpha k} + w(M) \neq n$. But $E_n(s_\alpha\chi^y + s_j\lambda)$ is just a polynomial (with rational coefficients) in $\Gamma(X_\lambda)$. Hence, from (2.13) and (2.22) we see that there exists a rational function $p(\lambda)$ such that for all $v \in V$,

$$\begin{aligned}
&E_n(s_\alpha\chi^y + s_j\lambda)v \otimes e_{kl:M}(X_\lambda)\\
&\quad = p(\lambda)v \otimes e_{kl:M}(X_\lambda) + \sum_{\substack{r,u,N\\ o(N)<o(M)}} d_{r,u,N}(\lambda)e_{ru:N}(X_\lambda).
\end{aligned} \tag{7.12}$$

Let

$$\mathscr{F}_{\mathbf{C}}'''(\chi) = \{\lambda \in \mathscr{F}_{\mathbf{C}}''(\chi) | p(\lambda) \neq 0\},$$

then $\mathscr{F}_{\mathbf{C}}'''(\chi)$ is dense in $\mathscr{F}_{\mathbf{C}}''(\chi)$. Suppose that $E_n(s_\alpha\chi^y + s_j\lambda_0)v \otimes e_{kl:M}(X_\lambda) \neq 0$, for $\lambda = \lambda_0 \in \mathscr{F}_{\mathbf{C}}''(\chi) \cap \mathscr{F}_{\mathbf{C},\delta}$. Then there exists a neighborhood $N(\lambda_0)$ in $\mathscr{F}_{\mathbf{C}}''(\chi)$ on which $E_n(s_\alpha\chi^y + s_j\lambda)v \otimes e_{kl:M}(X_\lambda) \neq 0$. It follows from the above mentioned density that there exists $\mu_0 \in N(\lambda_0) \cap \mathscr{F}_{\mathbf{C}}'''(\chi) \cap \mathscr{F}_{\mathbf{C},\delta}$. Applying $\Gamma_k(X_\lambda)$ to the equation (7.12) above with λ replaced by μ_0 we obtain on the one hand that $p(\mu_0)X_n(s_\alpha\chi^y + s_j\mu_0)e_{kl:M}(X_{\mu_0})$ equals

$$p(\mu_0)X_M(s_kX_{\mu_0})e_{kl:M}(X_{\mu_0}) + \sum_{\substack{r,u,N\\ o(N)<o(M)}} D_{ru:N}(\mu_0)e_{ru:N}(X_{\mu_0}).$$

On the other hand it also equals by (7.12)

$$p(\mu_0)X_n(s_\alpha\chi^y + s_j\mu_0)e_{kl:M}(X_{\mu_0}) \\ + \sum_{\substack{r,u,N\\ o(N)<o(M)}} X_n(s_\alpha\chi^y + s_j\mu_0)d_{ru:N}(\lambda)e_{ru:N}(X_{\mu_0}).$$

Subtracting this equality from the one above we obtain (recall that the $e_{ij:M}(X_{\mu_0})$ form a basis)

$$p(\mu_0)(X_M(s_kX_{\mu_0}) - X_n(s_\alpha\chi^y + s_j\mu_0)) = 0.$$

As $p(\mu_0) \neq 0$ then we must have $X_M(s_kX_{\mu_0}) = X_n(s_\alpha\chi^y + s_j\mu_0)$ which is a contradiction as this equality implies that $k \preceq \alpha$ and $n = \omega(M) + n_{\alpha k}$. □

Let $m \in M$. Put $m^\dagger = \theta(m^{-1})$, and set

(7.13) $$M' = \{m \in M \mid \det[(\mathrm{Ad}_{m^{-1}} - \mathrm{Ad}_{m^\dagger})|_{\mathfrak{n}}] \neq 0\}.$$

For $m \in M'$ let

(7.14) $$b(m) = (\mathrm{Ad}_{m^{-1}} - \mathrm{Ad}_{m^\dagger})|_{\mathfrak{n}}, \qquad c(m) = (\mathrm{Ad}_{m^{-1}})_{\mathfrak{n}}b(m).$$

Note that M' is an open subset of M, and $b(m)$, $c(m)$ are C^∞-functions on M'. Also recalling (3.4) we see that $M^+ \subset M'$, and for all $m \in M^+$,

(7.15) $$c(m) = -\sum_{r\geq 1}(\mathrm{Ad}_{m^\dagger m}|_{\mathfrak{n}})^{-r}, \\ b(m) = -\mathrm{Ad}_{\theta(m)}\sum_{r\geq 0}(\mathrm{Ad}_{m^\dagger m}|_{\mathfrak{n}})^{-r}.$$

Let $\mathscr{S}_Q$ be the algebra generated (without 1) by the matrix coefficients of $c(m)$ and $b(m)$.

LEMMA 7.6. *Let* $o(M) \geq 1$, *and* $b \in Y^M U(\mathfrak{g}_C)$. *Then there exists* $f_{iM} \in \mathscr{S}_Q$, ξ_{iM}, $\zeta_{iM} \in U(\mathfrak{k}_C)$, $\eta_{iM} \in U(\mathfrak{M}_C)$, $1 \leq i \leq p$, *such that*

(1) $b = \sum_{i=1}^p f_{iM}(m)\xi_{iM}^{m^{-1}}\eta_{iM}\zeta_{iM}$, $m \in M'$,

(2) *if* $m \in M$,

$$\lim_{\substack{a \mapsto \infty \\ Q}} a^{\langle \beta, M \rangle} f_{iM}(ma),$$

exists, and if $\overline{f}_{iM}(m)$ *denotes the limit, then* $\overline{f}_{iM}$ *is continuous on* M *and bounded on* M^+.

Proof. (1) of the lemma is just the statement of Lemma 4.3 of Trombi and Varadarajan **[2]**, modulo an observation made in the proof of that lemma, namely, that $\nu_F(b) = \nu_Q(b) = 0$.

For (2), let notation be as in (2.5) and put $X_i = \theta(Y_i)$. We observe that from Lemma 4.1 of Trombi and Varadarajan **[ibid.]** we have,

$$Y_i = -2\,\mathrm{Ad}_{m^{-1}} Eb(m)X_i + 2Ec(m)X_i,$$

where E is the projection of $\mathfrak{g}$ onto $\mathfrak{k}$ along $\mathfrak{s}$. As Ad_m leaves the A-weight of $X \in \bar{\mathfrak{n}}$ invariant, then on letting $Z_i = -(X_i + Y_i) \in \mathfrak{k}$ we can write,

$$Y_i = \sum_{\substack{j \ni \\ \beta_j(H) = \beta_i(H)}} \{b_{ji}(m)Z_j^{m^{-1}} - c_{ji}(m)Z_j\}, \tag{7.16}$$

where b_{ji} and c_{ji} are the matrix elements of b and c relative to the basis $\{X_1, \ldots, X_n\}$. We note that by (7.15)

$$\begin{aligned} c(ma)X_i &= \left(\sum_{r \geq 1} (\mathrm{Ad}_{m^\dagger m})^{-r}\,\mathrm{Ad}_{a^2}^{(-r+1)}\right) \mathrm{Ad}_{a^2}^{-1} X_i, \\ &= a^{-2\beta_i} \sum_{r \geq 1} a^{-2(r-1)\beta_i} (\mathrm{Ad}_{m^\dagger m})^{-r} X_i. \end{aligned}$$

By comparison to a geometric series we see that the series converges absolutely and uniformly for m varying in a compact subset of M and for all a such that $a^\alpha \gg 1$, where we use the notation of (2.1). It follows that

$$\lim_{\substack{a \mapsto \infty \\ Q}} a^{2\beta_i} c(ma)X_i = (\mathrm{Ad}_{m^\dagger m})^{-1} X_i, \tag{7.17}$$

from which we observe that,

$$\lim_{\substack{a \mapsto \infty \\ Q}} a^{\beta_i} c_{ij}(ma) = 0. \tag{7.18}$$

A similar calculation shows that

$$\lim_{\substack{a\mapsto\infty \\ Q}} a^{\beta_i} b(ma)X_i = -\operatorname{Ad}_{\theta(m)} X_i. \tag{7.19}$$

We can now proceed to prove (2) of the lemma by induction on $o(M)$. If $o(M) = 1$, then

$$b = Y_i'b$$

for some $1 \leq i \leq n$. Let us write,

$$'b = \nu_Q('b) + \sum_{k=1}^{'s} {'g_k(m)}\, {'\xi_k^{m^{-1}}}\, {'\eta_k}\, {'\zeta_k}, \tag{7.20}$$

where

$$\nu_Q('b) = \sum_{l=1}^{q} u_l v_l$$

and u_l, $\eta_k \in U(\mathfrak{m}_{\mathbf{C}}) \cap \lambda(\mathscr{S}(\mathfrak{m}_{\mathbf{C}} \cap \mathfrak{s}_{\mathbf{C}}))$, v_1, $'\xi_k$, $'\zeta_k \in U(\mathfrak{t}_{\mathbf{C}})$, $'g_k \in \mathscr{S}_Q$. Hence,

$$\begin{aligned} b = & \sum_{\substack{j\ni \\ \beta_j(H)=\beta_i(H)}} -c_{ji}(m)[Z_j, 'b] \\ & + \sum_{\substack{j\ni \\ \beta_j(H)=\beta_i(H)}} \sum_{l=1}^{q} \{b_{ji}(m) Z_j^{m^{-1}} u_l v_l - c_{ji}(m) u_l v_l Z_j\} \\ & + \sum_{\substack{j\ni \\ \beta_j(H)=\beta_i(H)}} \sum_{k=1}^{'s} {'g_k(m)} \{b_{ji}(m)(Z_j{'\xi_k})^{m^{-1}} {'\eta_k}{'\zeta_k} \\ & \qquad - c_{ji}(m){'\xi_k^{m^{-1}}} {'\eta_k}{'\zeta_k} Z_j\}. \end{aligned}$$

We can express $[Z_j, \overline{b}]$ in the same form as (7.20). Recalling that the $\lim_{\substack{a\to\infty \\ Q}} {'g_k(ma)} = 0$, and (7.17) we see that (2) holds for all $M \in \mathbf{Z}_+^n$ such that $o(M) = 1$. Now assume (2) holds for all $M \in \mathbf{Z}_+^n$ such that $o(M) = p$ with $p \geq 1$ and let $M \in \mathbf{Z}_+^n$ be such that $o(M) = p+1$. If $b \in Y^M U(\mathfrak{g}_{\mathbf{C}})$ then we can write

$$b = Y_i'b, \quad 'b \in Y'^M U(\mathfrak{g}_{\mathbf{C}}), \quad o('M) = p.$$

We apply the induction hypothesis to $'b$. Hence, we can write

$$'b = \sum_{k=1}^{'s} {'g_k(m)}{'\xi_k^{m^{-1}}}{'\eta_k}{'\zeta_k},$$

where the functions satisfy the properties listed in (2) for M replaced by $'M$. As above we can write

$$b = \sum_{\substack{j \ni \\ \beta_j(H)=\beta_i(H)}} \left\{ - c_{ji}(m)[Z_j, \overline{b}] \right.$$

$$+ \sum_{k=1}^{'s} {}'g_k(m)\{b_{ji}(m)(Z_j\, {}'\xi_k)^{m^{-1}}{}'\eta' \zeta_k$$

$$\left. - c_{ji}(m)\, {}'\xi_k^{m^{-1}}{}'\eta_k{}'\zeta_k Z_j\} \right\}.$$

Now, $\langle \beta, M\rangle(H) = \langle \beta, 'M\rangle(H) + \beta_i(H)$. It follows from (7.17), (7.19) and the induction hypothesis that

$$\lim_{\substack{a \mapsto \infty \\ Q}} a^{\langle \beta, M\rangle} \overline{g}_k(ma)\mu_{ji}(ma)$$

exists for all i, j where $\mu_{ji} = c_{ji}$ or b_{ji}; and the results of (2) hold for the limits. We observe that the terms $c_{ji}(m)[Z_j, 'b]$ can be written as $-c_{ji}(m)(ad_{X_j}\overline{b} + ad_{Y_j}\overline{b})$. Recalling the definition of $'b$, we see that the terms $\mathrm{ad}_{X_j}{}'b$ are either equal to zero or can be written as a sum of nonzero terms of the form $Y^{N'}b_N$ where $o(N) \le p$ and $\omega(N) = \omega('M) - \beta_j(H) = \omega(\overline{M}) - \beta_i(H) = \omega(M) - 2\beta_i(H)$. The terms of the form $\mathrm{ad}_{Y_j}{}'b$ are either zero or can be written as a sum of nonzero terms of the form $Y^{N'}b_N$ where $o(N) = p$ and $\omega(N) = \omega('M) + \beta_j(H) = \omega('M) + \beta_i(H) = \omega(M)$. Applying (1) of the lemma to these terms, using the induction hypothesis, and recalling that c_{ji} multiplies each of these terms, then by (7.18) we have (2) in this case. □

Let notation be as in Corollary 5.5. We shall denote by $\mathscr{A}_Q(M)$ the algebra generated by the functions f, f^0, f^θ, where f is a matrix coefficient of $\mathrm{Ad}_m|_{\mathfrak{n}}$, and $f^0(m) = f(m^{-1})$, $(f^\theta(m) = f(\theta(m)))$. Observe that $\mu\mathscr{A}_Q(M) \subset \mathscr{A}_Q(M)$ for all $\mu \in U(\mathfrak{m}_{\mathbf{C}})$. Let $\mathscr{L}_Q(M)$ denote the space of all complex valued functions η defined on M such that $\eta \mapsto \eta_n$ is a map of $\mathbf{Z}_+$ into $\mathscr{A}_Q(M)$ such that there exists $C_\eta > 0$, $s_\eta \in \mathbf{Z}_+$ for which

$$\|\eta_n\| \le C_\eta n^{s_\eta} \gamma_Q(m)^2, \qquad m \in M^+, n \in \mathbf{Z}_+,$$
$$\eta(m \exp tH) = \sum_{n \ge 1} e^{-nt}\eta_n(m), \qquad m \in M^+.$$

Note that $\eta_n(m \exp tH) = e^{tn}\eta_n(m)$. Set

$$\mathscr{A}_{Q,n}(M) = \{f \in \mathscr{A}_Q(M) | f(m \exp tH) = e^{-nt} f(m)\}.$$

LEMMA 7.7. *Let* $\eta \in U(\mathfrak{m}_{\mathbf{C}})$, $f \in \mathscr{S}_Q$. *Then* $\eta f \in \mathscr{A}_Q(M)$.

Proof. The proof follows from Lemma 2.2.2 of Trombi and Varadarajan [2], on noting that $b(m) = (\mathrm{Ad}_m|_{\mathfrak{n}})c(m)$, and $\mathscr{A}_Q(M) = \mathscr{L}_1$ in the notation of Lemma 2.2.2. □

LEMMA 7.8. *Let* f_{iM} *be as in Lemma* 4.6. *Then* $\overline{f}_{iM} \in \mathscr{A}_{Q,n}(M)$, *where* $n = \omega(M)$.

Proof. This just follows from the series representation for $c(m)$ and $b(m)$. □

If $s^{-1} \in W(\mathfrak{a}, \mathfrak{a}_P)$, $s_i|\mathfrak{a} = s$, with s_i as in §2, and $n \geq 0$ then put

$$\mathscr{Q}_{s,n} = \mathscr{A}_{Q,n}(M) \otimes \mathbf{Q}(\mathscr{S}(s_B \mathfrak{a}_{P,\mathbf{C}})) \otimes U(\mathfrak{t}_{\mathbf{C}}) \otimes U(\mathfrak{t}_{\mathbf{C}}) \otimes U(\mathfrak{m}^{\perp}_{\mathbf{C}}) \otimes U(\mathfrak{m}^{\perp}_{\mathbf{C}}),$$

where $\mathfrak{m}^1$ denotes the orthogonal complement of $\mathfrak{a}_M$ in $\mathfrak{m}$, and $\mathfrak{m}^{\perp}_{\mathbf{C}}$ denotes its complexification. If $X \in \mathscr{Q}_{s,n}$, say $X = f \otimes P \otimes \xi \otimes \xi' \otimes \eta \otimes \eta'$, and $\Phi \in \mathscr{A}_{sw}(sM, \tau)$, $\mu \in s\mathscr{F}_{\mathbf{C}}$, then set

$$\theta_X(m, \Phi, \mu) = f(m)P(\mu)\tau(\xi)E_{P_s(M)}(\eta ; m; \eta', \Phi, \mu)\tau(\xi').$$

PROPOSITION 7.9. *Let notation and assumptions be as in Corollary* 5.5. *Let* $\mu_1, \mu_2 \in U(\mathfrak{m}_{\mathbf{C}})$, $s \in W(A_0)$ *such that* $s^{-1} \in W(\mathfrak{a}, \mathfrak{a}_P)$. *Choose* $\alpha \in {}^oQ$ *such that* $s_{\alpha|\mathfrak{a}} = s$. *Let* $j \in Q^o_\alpha$, *and* $n \in \sigma(\alpha, j)$. *Then there exists* $X_{ij:M} = X_{ij:M}(s, \mu_1, \mu_2) \in \mathscr{Q}_{s,n}$ *where* $n = wt(M) + n_{\alpha i}$ *such that for all* $\lambda \in \mathscr{F}_{\mathbf{C},\delta} \cap \mathscr{F}''_{\mathbf{C}}(\chi)$,

$$E_n(s_\alpha \chi^y_- + s_j\lambda)\Phi_0(\mu_1 ; m; \mu_2, t, \lambda)$$

equals

$$\sum_{i,j,M} e^{ts\lambda(H)}\theta_{X_{ij.M}}(m, c_{P_s(M)|P_s}(s_B\lambda)^0 c_{P_s|P}(s_B, \lambda)\Phi, s_B\lambda) \otimes e_{ij:M}.$$

Proof. First note that for $\lambda \in \mathscr{F}''_{\mathbf{C}}(\chi) \cap \mathscr{F}_{\mathbf{C},\delta}$ that there exists for i, j, M as in the statement of the proposition, rational functions of λ such that

$$e_{ij:M} = \sum_{l=1}^{r} c^l_{ij:M}(\lambda) e_{lj:M}(X_\lambda).$$

Hence, if $u_{lj:M}(\lambda) = \sum_{i=1}^{r} c^{l}_{ij:M}(\lambda) u_{ij:M}$,

$$\begin{aligned}\Phi(ma_t, \lambda) &= \sum_{i,j,M} \sum_{l=1}^{r} c^{l}_{ij:M}(\lambda) E_P(ma_t; u_{ij:M}, \Phi, \lambda) \otimes e_{lj:M}(X_\lambda) \\ &= \sum_{l,j,M} E_P(ma_t; u_{lj:M}(\lambda), \Phi, \lambda) \otimes e_{lj:M}(X_\lambda).\end{aligned}$$

It follows from (7.11) that

$$E_n(s_\alpha \chi^y + s_j\lambda)\Phi(ma_t, \lambda) = \sum{}' E_P(ma_t; u_{lj:M}(\lambda), \Phi, \lambda) \otimes e_{lj:M}(X_\lambda),$$

where $\sum'$ denotes the sum over all those l, j, M for which $e_{lj:M}(X_\lambda) \in T_k^{(\alpha j:n)}$. By Proposition 7.4 this implies that

$$\begin{aligned}&e^{-t(s\lambda(H)+\rho(H)+n)} E_n(s_\alpha \chi^y + s_j\lambda)\Phi_0(\mu_1 \,;\, m; \mu_2, t, \lambda) \\ &\quad = \sum{}' \lim_{u\mapsto\infty} e^{-u(s\lambda(H)+\rho(H)+n)} E_P(\mu_1 \,;\, ma_u; \mu_2 u_{lj:M}(\lambda), \Phi, \lambda) \\ &\qquad\qquad \otimes e_{lj:M}(X_\lambda).\end{aligned}$$

But ad_{μ_2} leaves the A-weight of Y^M invariant. Applying Lemmas 7.3 and 7.6 we obtain the result upon noting that any element of $\mathfrak{m}_{\mathbf{C}}$ can be written as a sum of terms of the form vu with $v \in U(\mathfrak{m}^1_{\mathbf{C}})$, and $u \in U(\mathfrak{a}_{\mathbf{C}})$; hence if $u_1, u_2 \in U(\mathfrak{a}_{\mathbf{C}})$, $v_1, v_2 \in U(\mathfrak{m}^1_{\mathbf{C}})$, then one can easily prove that

$$\begin{aligned}&E_{P_s(M)}(u_1 v_1 \,;\, m; v_2 u_2, \Phi, \lambda) \\ &\quad = (u_1 \cdot u_2)(\lambda - \rho_{P_s(M)}) E_{P_s(M)}(v_1 \,;\, m; v_2, \Phi, \lambda). \qquad \square\end{aligned}$$

We can now combine Theorem 5.9 with the results of this section.

THEOREM 7.10. *Let notation and assumptions be as in Corollary 5.5. Further, let $m \in M^+$, $\mu_1, \mu_2 \in U(\mathfrak{m}_{\mathbf{C}})$, $t > 0$, and $\lambda \in \mathscr{F}''(\chi)$. Then we can write*

$$\begin{aligned}(7.21)\qquad &E_P(\mu_1 \,;\, ma_t; \mu_2, \Phi, \lambda) \\ &\quad = \sum_{s^{-1}\in W(\mathfrak{a}, \mathfrak{a}_P)} \sum_{n\geq 0} \Gamma^Q_{s,n}(\mu_1 \,;\, ma_t; \mu_2, \Phi, \lambda),\end{aligned}$$

where,

$$(7.22)\qquad \Gamma^Q_{s,n}(ma_t, \Phi, \lambda) = a_t^{(s_B\lambda - n\alpha - \rho)} \Gamma^Q_{s,n}(m, \Phi, \lambda),$$

and,

$$\begin{aligned}(7.23)\quad &\Gamma^Q_{s,0}(m, \Phi, \lambda) \\ &\quad = E_{P_s(M)}(m, c_{P_s(M)|P_s}(s_B\lambda)^0 c_{P_s|P}(s_B, \lambda)\Phi, s_B\lambda);\end{aligned}$$

if $n \geq 1$, *and* s, *and* μ_1, μ_2, *are as above, then there exists a unique* $X = X_{s,n,\mu_1,\mu_2} \in \mathscr{Q}_{s,n}$ *such that*

$$\begin{aligned}(7.24)\qquad \Gamma^{Q}_{s,n}(\mu_1 \colon m; \mu_2, \Phi, \lambda) \\ = \theta_X(m, c_{P_s(M)|P_s}(s_B\lambda)^0 c_{P_s|P}(s_B, \lambda)\Phi, s_B\lambda).\end{aligned}$$

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Received October 26, 1988 and in revised form August 25, 1989. Partically supported by NSF grant DNS8700829.

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Pacific Journal of Mathematics

Vol. 146, No. 1 November, 1990