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## **A COMPARISON ALGEBRA ON A CYLINDER WITH SEMI-PERIODIC MULTIPLICATIONS**

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## A COMPARISON ALGEBRA ON A CYLINDER WITH SEMI-PERIODIC MULTIPLICATIONS

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**A necessary and sufficient Fredholm criterion is found for a  $C^*$ -algebra of bounded operators on a cylinder, which contains operators of the form  $L\Lambda^M$ , where  $\Lambda = (1 - \Delta)^{-1/2}$  and  $L$  is an  $M$ th order differential operator whose coefficients are periodic at infinity.**

**0. Introduction.** Let  $\Omega$  denote the cylinder  $\mathbb{R} \times \mathbb{B}$ , where  $\mathbb{B}$  is a compact Riemannian manifold,  $\Delta_\Omega$  its Laplacian and  $\mathcal{H}$  the Hilbert space  $L^2(\Omega)$ . Cordes [3] found a necessary and sufficient Fredholm criterion for operators in the  $C^*$ -subalgebra of  $\mathcal{L}(\mathcal{H})$  generated by: (i) multiplications by functions that extend continuously to  $[-\infty, +\infty] \times \mathbb{B}$ , (ii)  $\Lambda = (1 - \Delta_\Omega)^{-1/2}$  and (iii) operators of the form  $D\Lambda$ , where  $D$  is either  $\partial/\partial t$ ,  $t \in \mathbb{R}$ , or a first order differential operator on  $\mathbb{B}$  with smooth coefficients. Here we extend this algebra by adjoining the multiplications by  $2\pi$ -periodic continuous functions to the generators, and a similar Fredholm criterion is obtained.

The commutator ideal  $\mathcal{E}_\mathcal{F}$  of the extended algebra  $\mathcal{E}_\mathcal{F}$  is proven to be  $*$ -isomorphic to  $\mathcal{S}\mathcal{L} \otimes \mathcal{K}_\mathbb{Z} \otimes \mathcal{K}_\mathbb{B}$ , where  $\mathcal{S}\mathcal{L}$  denotes the algebra of singular integral operators on the circle and  $\mathcal{K}_\mathbb{Z}$  and  $\mathcal{K}_\mathbb{B}$  denote the algebras of compact operators on  $L^2(\mathbb{Z})$  and  $L^2(\mathbb{B})$ , respectively. This allows us to define on  $\mathcal{E}_\mathcal{F}$  an operator-valued symbol, the “ $\gamma$ -symbol”, such that  $\ker \gamma \cap \ker \sigma$  equals the compact ideal of  $\mathcal{L}(\mathcal{H})$ . Here  $\sigma$  denotes the complex-valued symbol on  $\mathcal{E}_\mathcal{F}$  that arises from the Gelfand map of the commutative  $C^*$ -algebra  $\mathcal{E}_\mathcal{F}/\mathcal{E}_\mathcal{F}$ . We prove that  $A \in \mathcal{E}_\mathcal{F}$  is Fredholm if and only if  $\gamma_A$  and  $\sigma_A$  are invertible.

The simpler case when the compact manifold reduces to a point is considered in [5]. There, a unitary map  $W$  from  $L^2(\mathbb{R})$  onto  $L^2(S^1) \otimes L^2(\mathbb{Z})$  is defined, such that the conjugate  $W\mathcal{E}W^{-1}$  of the commutator ideal equals  $\mathcal{S}\mathcal{L} \otimes \mathcal{K}_\mathbb{Z}$ . Here, we conjugate  $\mathcal{E}_\mathcal{F}$  with  $W \otimes I_\mathbb{B}$ , where  $I_\mathbb{B}$  denotes the identity operator on  $L^2(\mathbb{B})$ , and obtain  $\mathcal{S}\mathcal{L} \otimes \mathcal{K}_\mathbb{Z} \otimes \mathcal{K}_\mathbb{B}$ .

If  $L$  is a differential operator on  $\Omega$  whose coefficients are continuous and approach periodic functions at infinity, the operator  $A = L\Lambda^M$  belongs to  $\mathcal{E}_\mathcal{F}$ , where  $M$  is the order of  $L$ . We can apply

the criterion above to  $A$  and prove that  $L$  is a Fredholm operator if and only if it is uniformly elliptic and a certain family of elliptic differential operators on the compact manifold  $S^1 \times \mathbb{B}$  is invertible. This applies also for matrices of such operators.

These results can be extended in a standard way to non-compact manifolds with cylindrical ends (cf. [2], VIII-3,4). Fredholm properties of elliptic-differential operators on such manifolds have been studied, for example, by Lockhart-McOwen [6] and Taubes [8]. The case where the coefficients are periodic on the ends is included in Taubes' results.

**1. Definition of the algebra  $\mathcal{E}_{\mathcal{F}}$  and a description of its commutator ideal.** Let  $\Omega$  denote the Riemannian manifold  $\mathbb{R} \times \mathbb{B}$ , where  $\mathbb{B}$  denotes an  $n$ -dimensional compact manifold with metric tensor locally given by  $h_{jk}$ , and let  $\mathcal{H}$  denote the Hilbert space  $L^2(\Omega)$ , with  $\Omega$  being given the surface measure

$$dS = \sqrt{h} dt dx^1 \cdots dx^n,$$

where  $h$  is the determinant of the  $n \times n$ -matrix  $((h_{jk}))_{1 \leq j, k \leq n}$ . The metric on  $\Omega$  is given by  $ds^2 = dt^2 + h_{jk} dx^j dx^k$ , and the Laplace operator is locally given by

$$\Delta_{\Omega} = \Delta_{\mathbb{R}} + \Delta_{\mathbb{B}} = \frac{d^2}{dt^2} + \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^j} \sqrt{h} h^{jk} \frac{\partial}{\partial x^k},$$

where  $((h^{jk})) = ((h_{jk}))^{-1}$ , and the summation convention from 1 to  $n$  is being used.

The symmetric operator  $\Delta_{\Omega}$  with domain  $C_0^\infty(\Omega)$  is essentially self-adjoint, since  $\Omega$  is complete (cf. [2], IV). We denote by  $H$  the closure of  $1 - \Delta_{\Omega}$  and by  $\Lambda$  its inverse square root,  $\Lambda = H^{-1/2}$ . Since  $H \geq 1$ , we have  $\Lambda \in \mathcal{L}(\mathcal{H})$ . The algebra  $\mathcal{E}_{\mathcal{F}}$  is defined as the smallest  $C^*$ -subalgebra of  $\mathcal{L}(\mathcal{H})$  containing the following operators (or classes of operators):

$$(1) \quad a \in C^\infty(\mathbb{B}); \quad b \in \text{CS}(\mathbb{R});$$

$$e^{ijt}, j \in \mathbb{Z}; \quad \Lambda; \quad \frac{1}{i} \frac{\partial}{\partial t} \Lambda \quad \text{and} \quad D_x \Lambda,$$

$D_x$  being a first order differential operator on  $\mathbb{B}$ , locally given by  $-ib^j(x)\partial/\partial x^j$ , where  $b^j(x)$ ,  $j = 1, \dots, n$ , are the components of a smooth vector field on  $\mathbb{B}$ . The operators  $\frac{\partial}{\partial t} \Lambda$  and  $D_x \Lambda$ , defined on the dense subspace  $\Lambda^{-1}(C_0^\infty(\Omega))$ , can be extended to bounded operators of  $\mathcal{L}(\mathcal{H})$  (cf. [2], for example). Bounded functions on  $\Omega$

have been identified with the corresponding multiplication operators in  $\mathcal{L}(\mathcal{H})$  and  $\mathbf{CS}(\mathbb{R})$  denotes the class of continuous functions on  $\mathbb{R}$  with limits at  $+\infty$  and  $-\infty$ .

Our first objective is to obtain a necessary and sufficient criterion for an operator in  $\mathcal{E}_{\mathcal{F}}$  to be Fredholm. Such a criterion has been found by Cordes [3] for the algebra generated by the operators in (1) except  $e^{ijt}$ ,  $j \in \mathbb{Z}$ .

Taking advantage of the tensor product structure of  $\mathcal{H}$ ,

$$\mathcal{H} = L^2(\mathbb{R}) \otimes L^2(\mathbb{B}),$$

we consider the conjugate of  $\mathcal{E}_{\mathcal{F}}$  with respect to the unitary operator  $F \otimes I_{\mathbb{B}}$ , where  $I_{\mathbb{B}}$  denotes the identity operator on  $L^2(\mathbb{B})$  and  $F$  the Fourier transform on  $L^2(\mathbb{R})$ ,

$$(Fu)(\tau) = \frac{1}{\sqrt{2\pi}} \int e^{-i\tau t} u(t) dt.$$

In order to simplify notation,  $A \otimes I_{\mathbb{B}}$  is denoted by  $A$  and  $I_{\mathbb{R}} \otimes B$  by  $B$ , whenever  $A \in \mathcal{L}(L^2(\mathbb{R}))$  or  $B \in \mathcal{L}(L^2(\mathbb{B}))$ .

We seek to describe what are  $B_k := F^{-1}A_kF$ , where  $A_k$ ,  $k = 1, \dots, 6$ , denote the operators listed in (1), in that order. The operator-valued functions  $\tilde{\Lambda}(\tau) := (1 + \tau^2 - \Delta_{\mathbb{B}})^{-1/2}$ ,  $\tau\tilde{\Lambda}(\tau)$  and  $D_x\tilde{\Lambda}(\tau)$ ,  $\tau \in \mathbb{R}$ , are all in  $\mathbf{CB}(\mathbb{R}, \mathcal{L}_{\mathbb{B}})$ , as proven in [3], page 220, and thus determine operators in  $\mathcal{L}(\mathcal{H})$  by multiplication in the real variable. Here  $\mathcal{L}_{\mathbb{B}}$  denotes the algebra of bounded operators on  $L^2(\mathbb{B})$  and  $\mathbf{CB}(\mathbb{R}, \mathcal{L}_{\mathbb{B}})$  the bounded continuous  $\mathcal{L}_{\mathbb{B}}$ -valued functions on  $\mathbb{R}$ . With this interpretation, we get  $B_k$ ,  $k = 1, \dots, 6$ , respectively given by

$$(2) \quad a \in C^\infty(\mathbb{B}); \quad b(D), b \in \mathbf{CS}(\mathbb{R}); \quad T_j, j \in \mathbb{Z};$$

$$\tilde{\Lambda}(\tau); \quad -\tau\tilde{\Lambda}(\tau) \quad \text{and} \quad D_x\tilde{\Lambda}(\tau),$$

where  $b(D) := F^{-1}bF$  and  $T_j$  denotes the translation  $(T_ju)(\tau) = u(\tau + j)$ .

Let  $\mathcal{K}_{\mathbb{B}}$  denote the ideal of compact operators on  $L^2(\mathbb{B})$  and  $\mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$  denote the  $\mathcal{K}_{\mathbb{B}}$ -valued continuous functions on  $\mathbb{R}$  that vanish at infinity. All commutators  $[B_k, B_l]$ ,  $k, l \neq 3$ , are contained in the algebra

$$\mathcal{EH} := \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}}) + \mathcal{K}(\mathcal{H}),$$

where  $\mathcal{K}(\mathcal{H})$  denotes the ideal of compact operators of  $\mathcal{L}(\mathcal{H})$ , as proven in [3], Proposition 1.2. Next we investigate what are the

commutators  $[B_3, B_k], k = 1, \dots, 6$ . We easily get  $[B_3, B_1] = [B_3, B_2] = 0$ . It is also clear that, for any  $K(\tau) \in \mathbf{CB}(\mathbb{R}, \mathcal{L}_{\mathbb{B}})$ , we have

$$(3) \quad [T_k, K(\tau)] = (K(\tau + k) - K(\tau))T_k, \quad k \in \mathbb{Z}.$$

**PROPOSITION 1.1.** *The commutators of the generators in (2)—and of their adjoints—of the algebra  $\widehat{\mathcal{C}}_{\varphi} := F^{-1}\mathcal{E}_{\varphi}F$  are contained in*

$$\mathcal{E}\mathcal{H} = \left\{ \sum_{j=-N}^N K_j(\tau)T_j + K; N \in \mathbb{N}, K_j \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}}), K \in \mathcal{H}(\mathcal{H}) \right\}.$$

*Proof.* Let us first prove that  $K(\tau + j) - K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$ , for  $K(\tau) = \tilde{\Lambda}(\tau), \tau\tilde{\Lambda}(\tau)$  or  $D_x\tilde{\Lambda}(\tau)$ . It follows from the fact that  $-\Delta_{\mathbb{B}}$  on  $L^2(\mathbb{B})$  has an orthonormal basis of eigenfunctions, with eigenvalues  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots, \lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ , that, for each  $\tau \in \mathbb{R}, \tilde{\Lambda}(\tau)$  is unitarily equivalent to the multiplication operator  $(1 + \tau^2 + \lambda_n)^{-1/2}$  on  $L^2(\mathbb{N})$ . Hence:  $\tilde{\Lambda}(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$ ,

$$\|\tau[\tilde{\Lambda}(\tau + j) - \tilde{\Lambda}(\tau)]\|_{L^2(\mathbb{B})} \leq \max_{s \in [1, \infty)} |\tau[(s + (\tau + j)^2)^{-1/2} - (s + \tau^2)^{-1/2}]|$$

and

$$\|\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1\|_{L^2(\mathbb{R})} \leq \max_{s \in [1, \infty)} |(\tau^2 + s)^{1/2}((\tau + j)^2 + s)^{-1/2} - 1|.$$

Note that the right-hand sides of the two previous inequalities go to zero as  $\tau \rightarrow \pm\infty$ . Furthermore, as

$$\lim_{n \rightarrow \infty} (1 + \tau^2 + \lambda_n)^{1/2}(1 + (\tau + j)^2 + \lambda_n)^{-1/2} - 1 = 0,$$

we have that  $\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1 \in \mathcal{K}_{\mathbb{B}}$ , for each  $\tau \in \mathbb{R}$ . We then get:

$$(\tau + j)\tilde{\Lambda}(\tau + j) - \tau\tilde{\Lambda}(\tau) = \tau(\tilde{\Lambda}(\tau + j) - \tilde{\Lambda}(\tau)) + j\tilde{\Lambda}(\tau + j) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$$

and

$$D_x\tilde{\Lambda}(\tau + j) - D_x\tilde{\Lambda}(\tau) = D_x\tilde{\Lambda}(\tau)[\tilde{\Lambda}(\tau)^{-1}\tilde{\Lambda}(\tau + j) - 1] \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}}).$$

By the remarks preceding the statement of the proposition, this proves that the commutators of the generators (2) are indeed contained in  $\mathcal{E}\mathcal{H}$ . Concerning the adjoints, let us note that the classes of  $B_k$ 's,  $k = 1, \dots, 5$ , are self-adjoint and that, as proven in [3],  $D_x\tilde{\Lambda} - \tilde{\Lambda}D_x \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$ . Hence

$$(4) \quad (D_x\tilde{\Lambda})^* - D_x^*\tilde{\Lambda} = \tilde{\Lambda}D_x^* - D_x^*\tilde{\Lambda} \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}}).$$

Here,  $D_x^*$  denotes the formal adjoint of  $D_x$ . The commutators of any  $K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$  with the generators  $B_k$ ,  $k = 1, 3, 4, 5, 6$ , are clearly contained in  $\mathcal{E}\mathcal{H}$ . For  $K(\tau)$  of the special form  $K(\tau) = a(\tau)\tilde{K}$ ,  $a \in \mathbf{CO}(\mathbb{R})$  and  $\tilde{K} \in \mathcal{K}_{\mathbb{B}}$ , the commutator  $[b(D), K(\tau)] = [b(D), a(\tau)] \otimes \tilde{K}$  is compact, since  $[b(D), a(\tau)]$  is compact (cf. [4], Chapter III, for example), for  $b \in \mathbf{CS}(\mathbb{R})$ . The vector space generated by all  $K(\tau) = a(\tau)\tilde{K}$  as above is dense in  $\mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$  and thus we have

$$(5) \quad [b(D), K(\tau)] \in \mathcal{K}(\mathcal{H}), \quad \text{for } b \in \mathbf{CS}(\mathbb{R}), K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}}).$$

This concludes the proof. □

Denoting by  $\mathcal{E}_{\mathcal{P}}$  the commutator ideal of  $\mathcal{E}_{\mathcal{P}}$  and by  $\widehat{\mathcal{E}}_{\mathcal{P}}$  the commutator ideal of  $\widehat{\mathcal{E}}_{\mathcal{P}}$ , it is obvious that  $\widehat{\mathcal{E}}_{\mathcal{P}} = F^{-1}\mathcal{E}_{\mathcal{P}}F$ .

**PROPOSITION 1.2.** *The commutator ideal  $\widehat{\mathcal{E}}_{\mathcal{P}}$  of the algebra  $\widehat{\mathcal{E}}_{\mathcal{P}}$  is obtained by closing the set of operators:*

$$\widehat{\mathcal{E}}_{\mathcal{P},0} := \left\{ \sum_{j=-N}^N b_j(D)K_j(\tau)T_j + K; \right. \\ \left. b_j \in \mathbf{CS}(\mathbb{R}), N \in \mathbb{N}, K_j \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}}), K \in \mathcal{K}(\mathcal{H}) \right\}.$$

*Proof.* The algebra  $\mathcal{E}_{\mathcal{P}}$  is a ‘‘Comparison Algebra’’, in the sense of [2], Chapter V, with ‘‘generating classes’’:

$$(6) \quad \mathcal{A}^{\#} := \mathbf{C}_0^{\infty}(\Omega) \cup \mathbf{C}^{\infty}(\mathbb{B}) \cup \{e^{ijt}; j \in \mathbb{Z}\} \cup \{s(t) = t(1+t^2)^{-1/2}\}$$

and  $\mathcal{D}^{\#}$  equal to the vector space generated by the first order linear partial differential expressions on  $\mathbb{B}$  with smooth coefficients and by the expression  $\partial/\partial t$ . Indeed,  $\mathcal{E}_{\mathcal{P}}$  can alternatively be defined as the  $C^*$ -algebra generated by all multiplications by functions in  $\mathcal{A}^{\#}$  and by all  $D\Lambda$ ,  $D \in \mathcal{D}^{\#}$ . It follows then from Lemma V-1-1 of [2] that  $\mathcal{K}(\mathcal{H}) \subset \mathcal{E}_{\mathcal{P}}$  and therefore  $\mathcal{K}(\mathcal{H}) \subset \widehat{\mathcal{E}}_{\mathcal{P}}$ . Moreover, it was proven in [3], Proposition 1.5, that  $\mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$  is contained in the commutator ideal of the  $C^*$ -algebra generated by  $B_4$ ,  $B_5$  and  $B_6$ . Thus we get  $\widehat{\mathcal{E}}_{\mathcal{P},0} \subset \widehat{\mathcal{E}}_{\mathcal{P}}$ .

All commutators of the generators (2) and their adjoints are contained in  $\widehat{\mathcal{E}}_{\mathcal{P},0}$ , by Proposition 1.1. Again using (3), (4) and (5), it is easy to verify that  $\widehat{\mathcal{E}}_{\mathcal{P},0}$  is invariant under right or left multiplication by the operators in (2) and their adjoints. Two facts then follow:

(i) all commutators of the algebra (finitely) generated by the operators in (2) and their adjoints are contained in  $\widehat{\mathcal{E}}_{P,0}$  and therefore all commutators of  $\widehat{\mathcal{E}}_\varphi$  are contained in the closure of  $\widehat{\mathcal{E}}_{P,0}$ , and (ii) the closure of  $\widehat{\mathcal{E}}_{P,0}$  is an ideal of  $\widehat{\mathcal{E}}_\varphi$ . By definition of commutator ideal,  $\widehat{\mathcal{E}}_\varphi$  is contained in the closure of  $\widehat{\mathcal{E}}_{P,0}$ .  $\square$

Let  $\mathbf{CO}(\mathbb{R})$  denote the set of continuous functions on  $\mathbb{R}$  vanishing at infinity and let  $\widehat{\mathcal{E}}_0$  denote the set of bounded operators on  $L^2(\mathbb{R})$

$$\widehat{\mathcal{E}}_0 := \left\{ \sum_{j=-N}^N b_j(D)a_j(\tau)T_j + K; N \in \mathbb{N}, b_j \in \mathbf{CS}(\mathbb{R}), \right. \\ \left. a_j \in \mathbf{CO}(\mathbb{R}), K \in \mathcal{K}_{\mathbb{R}} \right\}.$$

**COROLLARY 1.3.** *With  $\widehat{\mathcal{E}}$  denoting the closure of  $\widehat{\mathcal{E}}_0$  defined above, we have:*

$$\widehat{\mathcal{E}}_\varphi = \widehat{\mathcal{E}} \overline{\otimes} \mathcal{K}_{\mathbb{B}}$$

where  $\overline{\otimes}$  denotes the operator-norm closure of the algebraic tensor product.

*Proof.* The vector-space generated by

$$\{(b(D)a(\tau)T_j + K) \otimes \widetilde{K}; b \in \mathbf{CS}(\mathbb{R}), a \in \mathbf{CO}(\mathbb{R}), j \in \mathbb{Z}, \\ K \in \mathcal{K}_{\mathbb{R}}, \widetilde{K} \in \mathcal{K}_{\mathbb{B}}\}$$

is dense in  $\widehat{\mathcal{E}}_{P,0}$  and in  $\widehat{\mathcal{E}} \overline{\otimes} \mathcal{K}_{\mathbb{B}}$ .  $\square$

In the rest of this section, we define a unitary map

$$W: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{S}^1; L^2(\mathbb{Z}))$$

and find a useful description for  $(W \otimes I_{\mathbb{B}})\widehat{\mathcal{E}}_\varphi(W \otimes I_{\mathbb{B}})^{-1}$ .

Given  $u \in L^2(\mathbb{R})$ , denote:

$$u^\diamond(\varphi) := (u(\varphi - j))_{j \in \mathbb{Z}},$$

for each  $\varphi \in \mathbb{R}$ . The sequence  $u^\diamond(\varphi)$  belongs to  $L^2(\mathbb{Z})$  for almost every  $\varphi$ , by Fubini's Theorem, since  $L^2(\mathbb{R})$  can be identified with  $L^2([0, 1) \times \mathbb{Z})$ . Let

$$F_d: L^2(\mathbb{S}^1, d\theta) \rightarrow L^2(\mathbb{Z}), \quad \mathbb{S}^1 = \{e^{i\theta}; \theta \in \mathbb{R}\},$$

denote the discrete Fourier transform:

$$(7) \quad (F_d u)_j = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(\theta)e^{-ij\theta} d\theta, \quad j \in \mathbb{Z}.$$

For each  $\varphi \in \mathbb{R}$ , define

$$(8) \quad Y_\varphi := F_d e^{-i\varphi\theta} F_d^{-1}.$$

The operators  $Y_\varphi$  define a smooth function on  $\mathbb{R}$ , taking values in the unitary operators on  $L^2(\mathbb{Z})$  and satisfying  $(Y_k u)_j = u_{j+k}$ , for  $k \in \mathbb{Z}$  and  $u \in L^2(\mathbb{Z})$ , and  $Y_\varphi Y_\omega = Y_{\varphi+\omega}$ , for  $\varphi, \omega \in \mathbb{R}$ .

We now define the map (with  $\mathbb{S}^1 = \{e^{2\pi i\varphi}; \varphi \in \mathbb{R}\}$ )

$$(9) \quad \begin{aligned} W: L^2(\mathbb{R}) &\rightarrow L^2(\mathbb{S}^1, d\varphi; L^2(\mathbb{Z})), \\ u &\mapsto (Wu)(\varphi) = Y_\varphi u^\diamond(\varphi). \end{aligned}$$

Let  $\mathbf{CS}(\mathbb{Z})$  denote the set of sequences  $b(j)$ ,  $j \in \mathbb{Z}$ , with limits as  $j \rightarrow +\infty$  and  $j \rightarrow -\infty$  and let  $b(D_\theta)$  denote  $F_d^{-1} b(M) F_d$ , where  $b(M)$  denotes the operator multiplication by  $b$  on  $L^2(\mathbb{Z})$ . We then denote by  $\mathcal{S}\mathcal{L}$  the  $C^*$ -subalgebra of  $\mathcal{L}_{\mathbb{S}^1} := \mathcal{L}(L^2(\mathbb{S}^1))$  generated by  $b(D_\theta)$ ,  $b \in \mathcal{E}\mathcal{S}(\mathbb{Z})$ , and by the multiplications by smooth functions on  $\mathbb{S}^1$ . It is easy to check that, with  $\Lambda_{\mathbb{S}^1} := (1 - \Delta_{\mathbb{S}^1})^{-1/2}$ ,

$$\frac{1}{i} \frac{d}{d\theta} \Lambda_{\mathbb{S}^1} = s(D_\theta), \quad s(j) = (1 + j^2)^{-1/2}.$$

Since the polynomials in  $s$  are dense in  $\mathbf{CS}(\mathbb{Z})$ ,  $\mathcal{S}\mathcal{L}$  coincides with the  $C^*$ -subalgebra of  $\mathcal{L}_{\mathbb{S}^1}$  generated by  $-i \frac{d}{d\theta} \Lambda_{\mathbb{S}^1}$  and  $C^\infty(\mathbb{S}^1)$ . In other words,  $\mathcal{S}\mathcal{L}$  is the unique comparison algebra over  $\mathbb{S}^1$ . It therefore contains the compact ideal  $\mathcal{K}_{\mathbb{S}^1}$  and all its commutators are compact (cf. [2], Chapters V and VI).

The following theorem was proven in [5] (Theorem 2.6). See also [7], Theorem 1.2.

**THEOREM 1.4.** *With the above notation, we have:*

$$(10) \quad W \widehat{\mathcal{E}} W^{-1} = \mathcal{S}\mathcal{L} \overline{\otimes} \mathcal{K}_{\mathbb{Z}},$$

where  $\mathcal{K}_{\mathbb{Z}}$  denotes the set of compact operators on  $L^2(\mathbb{Z})$ . Furthermore, for  $b \in \mathbf{CS}(\mathbb{R})$ ,  $a \in \mathbf{CO}(\mathbb{R})$  and  $j \in \mathbb{Z}$ , we have:

$$A^\diamond(e^{2\pi i\varphi}) := Y_\varphi a(\varphi - M) Y_{-\varphi} \in \mathbf{C}(\mathbb{S}^1, \mathcal{K}_{\mathbb{Z}})$$

and

$$(11) \quad W(b(D) a T_j) W^{-1} = b(D_\theta) Y_\varphi a(\varphi - M) Y_{-\varphi-j} + K, \quad K \in \mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z}}.$$

**PROPOSITION 1.5.** *The map*

$$\begin{aligned} \widehat{\mathcal{E}}_\varphi &\rightarrow \mathcal{S}\mathcal{L} \overline{\otimes} \mathcal{K}_{\mathbb{Z}} \overline{\otimes} \mathcal{K}_{\mathbb{B}}, \\ A &\mapsto W A W^{-1} \end{aligned}$$



is an onto  $*$ -isomorphism. For  $A \in \widehat{\mathcal{E}}_\varphi$  of the form  $A = b(D)K(\tau)T_j$ , with  $b \in \mathbf{CS}(\mathbb{R})$ ,  $K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_\mathbb{B})$  and  $j \in \mathbb{Z}$ , we have:

$$(12) \quad WAW^{-1} = b(D_\theta)Y_\varphi K(\varphi - M)Y_{-\varphi-j} + K, \quad \text{with } K \in \mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbb{B}}.$$

For each  $\varphi \in \mathbb{R}$  here,  $K(\varphi - M)$  denotes the compact operator on  $L^2(\mathbb{Z}) \overline{\otimes} L^2(\mathbb{B})$  defined by the sequence  $K(\varphi - j) \in \mathcal{K}_\mathbb{B}$ ,  $j \in \mathbb{Z}$ . The first term of the right-hand side of (12) defines therefore a  $\mathcal{K}_{\mathbb{Z} \times \mathbb{B}}$ -valued continuous function on  $\mathbb{S}^1 = \{e^{2\pi i\varphi}; \varphi \in \mathbb{R}\}$ .

*Proof.* By Corollary 1.3 and (10),

$$W\widehat{\mathcal{E}}_\varphi W^{-1} = \mathcal{L} \overline{\otimes} \mathcal{K}_\mathbb{Z} \overline{\otimes} \mathcal{K}_\mathbb{B}$$

and, by (11), formula (12) holds for  $K(\tau)$  of the form  $a(\tau) \otimes \tilde{K}$ ,  $a \in \mathbf{CO}(\mathbb{R})$  and  $\tilde{K} \in \mathcal{K}_\mathbb{B}$ . We can then find a sequence  $K_m(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_\mathbb{B})$  such that  $K_m(\tau) \rightarrow K(\tau)$ , uniformly in  $\tau \in \mathbb{R}$ , and (12) is valid for each  $K_m(\tau)$ . Then

$$Y_\varphi K_m(\varphi - M)Y_{-\varphi-j} \rightarrow Y_\varphi K(\varphi - M)Y_{-\varphi-j}$$

in  $\mathcal{K}_{\mathbb{Z} \times \mathbb{B}}$ , uniformly in  $e^{2\pi i\varphi} \in \mathbb{S}^1$ . Since the supremum-norm of a function on  $\mathbb{S}^1$  taking values in  $\mathcal{L}(L^2(\mathbb{Z}) \overline{\otimes} L^2(\mathbb{B}))$  equals the norm of the corresponding multiplication operator on  $L^2(\mathbb{S}^1) \overline{\otimes} L^2(\mathbb{Z}) \overline{\otimes} L^2(\mathbb{B})$ , the convergence above also holds in  $\mathcal{L}(L^2(\mathbb{S}^1) \overline{\otimes} L^2(\mathbb{Z}) \overline{\otimes} L^2(\mathbb{B}))$ .  $\square$

Let  $\mathbf{M}_{SL}$  denote the maximal-ideal space of  $\mathcal{L} / \mathcal{K}_{\mathbb{S}^1}$  and let

$$\sigma^{SL}: \mathcal{L} / \mathcal{K}_{\mathbb{S}^1} \rightarrow \mathbf{C}(\mathbf{M}_{SL})$$

denote the composition of the Gelfand map with the canonical projection. We then have (cf. [2], for example):  $\mathbf{M}_{SL} = \mathbb{S}^1 \times \{-1, +1\}$  and

$$\sigma_a^{SL}(\cdot, \pm 1) = a(\cdot), \quad \text{for } a \in \mathbf{C}^\infty(\mathbb{S}^1)$$

and

$$\sigma_{b(D_\theta)}^{SL}(\cdot, \pm 1) = b(\pm\infty) \quad \text{for } b \in \mathbf{CS}(\mathbb{Z}).$$

Let  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}})$  denote the  $\mathcal{K}_{\mathbb{Z} \times \mathbb{B}}$ -valued functions on  $\mathbf{M}_{SL}$ . Here  $\mathcal{K}_{\mathbb{Z} \times \mathbb{B}}$  denotes the compact ideal of  $L^2(\mathbb{Z}) \overline{\otimes} L^2(\mathbb{B})$ ,  $\mathcal{K}_{\mathbb{Z} \times \mathbb{B}} = \mathcal{K}_\mathbb{Z} \overline{\otimes} \mathcal{K}_\mathbb{B}$ .

**THEOREM 1.6.** *There exists an onto  $*$ -isomorphism*

$$\Psi: \frac{\mathcal{E}_\varphi}{\mathcal{K}(\mathcal{H})} \rightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}})$$

such that if  $\tilde{\gamma}$  denotes the composition of  $\Psi$  with the canonical projection  $\mathcal{E}_\varphi \rightarrow \mathcal{E}_\varphi/\mathcal{K}(\mathcal{H})$  and  $A \in \mathcal{E}_\varphi$  is such that  $B = F^{-1}AF$  is of the form  $B = b(D)K(\tau)T_j$ , where  $b \in \mathbf{CS}(\mathbb{R})$ ,  $K(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_\mathbb{B})$  and  $j \in \mathbb{Z}$ , we then have:

$$\tilde{\gamma}_A(e^{2\pi i\varphi}, \pm 1) = b(\pm\infty)Y_\varphi K(\varphi - M)Y_{-\varphi-j}.$$

*Proof.* Let  $\Psi$  be given by

$$\frac{\mathcal{E}_\varphi}{\mathcal{K}(\mathcal{H})} \rightarrow \frac{\widehat{\mathcal{E}}_\varphi}{\mathcal{K}(\mathcal{H})} \rightarrow \frac{\mathcal{S}\mathcal{L} \overline{\otimes} \mathcal{K}_\mathbb{Z} \overline{\otimes} \mathcal{K}_\mathbb{B}}{\mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbb{B}}} \rightarrow \mathbf{C}(M_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}}),$$

where in the first step we take  $A + \mathcal{K}(\mathcal{H}) \in \mathcal{E}_\varphi/\mathcal{K}(\mathcal{H})$  to  $F^{-1}AF + \mathcal{K}(\mathcal{H})$ , next to

$$WF^{-1}AFW^{-1} + \mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbb{B}},$$

and in the last step we use the onto \*-isomorphism (see [1]):

$$\frac{\mathcal{S}\mathcal{L} \overline{\otimes} \mathcal{K}_\mathbb{Z} \overline{\otimes} \mathcal{K}_\mathbb{B}}{\mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbb{B}}} \rightarrow \mathbf{C}(M_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}})$$

$$A \otimes K_1 \otimes K_2 + \mathcal{K}_{\mathbb{S}^1 \times \mathbb{Z} \times \mathbb{B}} \mapsto \sigma_A^{SL}(\varphi, \pm 1)K_1 \otimes K_2.$$

Defined this way,  $\Psi$  has the desired properties, by Proposition 1.5 and its proof. □

**2. Definition of two symbols on  $\mathcal{E}_\varphi$ .** Our first task in this section is to give a precise description of the symbol space of  $\mathcal{E}_\varphi$ , i.e., the maximal-ideal space of the commutative  $C^*$ -algebra  $\mathcal{E}_\varphi/\mathcal{E}_\varphi$ . The symbol space of  $\mathcal{E}$ , the  $C^*$ -algebra generated by the operators listed in (1) except the periodic functions  $e^{ij t}$ , was described in [3]:

**THEOREM 2.1** (*Theorem 2.3 of [3]*). *The symbol space  $\mathbf{M}$  of  $\mathcal{E}$  can be identified with the bundle of unit spheres of the cotangent bundle of the compact manifold with boundary  $[-\infty, +\infty] \times \mathbb{B}$ , where  $[-\infty, +\infty]$  denotes the compactification of  $\mathbb{R}$  obtained by adding the points  $-\infty$  and  $+\infty$ . The  $\sigma$ -symbols of the generators  $A_1, A_2, A_4, A_5$  and  $A_6$  are given below as functions of the local coordinates  $(t, x; \tau, \xi)$ , where  $(t, \tau) \in [-\infty, +\infty] \times \mathbb{R}^*$ ,  $(x, \xi) \in T^*\mathbb{B}$  and  $\tau^2 + h^{jk}\xi_j\xi_k = 1$ :*

$$\sigma_{A_1} = a(x), \quad \sigma_{A_2} = b(t), \quad \sigma_{A_4} = 0, \quad \sigma_{A_5} = \tau, \quad \sigma_{A_6} = b^j(x)\xi_j.$$

When periodic functions are adjoined to the algebra, the points over  $|t| = \infty$  become circles. More precisely, we have:

**THEOREM 2.2.** *The symbol space  $\mathbf{M}_P$  of  $\mathcal{E}_\varphi$  is homeomorphic to the closed subset of  $\mathbf{M} \times \mathbb{S}^1$  described in local coordinates by*

$$\{((t, x; \tau, \xi), e^{i\theta}); (t, x; \tau, \xi) \in \mathbf{M}, \theta \in \mathbb{R} \text{ and } \theta = t \text{ if } |t| < \infty\}.$$

Using this description of  $\mathbf{M}_P$ , the  $\sigma$ -symbols of the generators in (1) are respectively given by

$$a(x), \quad b(t), \quad e^{ij\theta}, \quad 0, \quad \tau \quad \text{and} \quad b^j(x)\xi_j.$$

*Proof.* Let  $\mathbf{P}_{2\pi}$  denote the closed algebra generated by  $\{e^{ij\theta}; j \in \mathbb{Z}\}$ , i.e., the  $2\pi$ -periodic continuous functions on  $\mathbb{R}$ . Its maximal-ideal space is  $\mathbb{S}^1$ , with  $e^{i\theta} \in \mathbb{S}^1$  defining the multiplicative linear functional  $f \rightarrow f(\theta)$ .

With  $\mathcal{E}$  denoting the commutator ideal of  $\mathcal{E}$ , the maximal-ideal space of  $\mathcal{E}/\mathcal{E}$  is  $\mathbf{M}$ , as described in Theorem 2.1. By definition of the Gelfand map, a point  $(t, x; \tau, \xi)$  defines the multiplicative linear functional

$$A + \mathcal{E} \rightarrow \sigma_A(t, x; \tau, \xi).$$

The following maps are canonically defined:

$$(13) \quad i_1: \frac{\mathcal{E}}{\mathcal{E}} \rightarrow \frac{\mathcal{E}_\varphi}{\mathcal{E}_\varphi}$$

and

$$(14) \quad i_2: \mathbf{P}_{2\pi} \rightarrow \frac{\mathcal{E}_\varphi}{\mathcal{E}_\varphi}.$$

(It is obvious that  $\mathcal{E} \subseteq \mathcal{E}_\varphi$ )

Let us denote by  $\iota$  the product of the dual maps of  $i_1$  and  $i_2$ , i.e.,

$$(15) \quad \begin{aligned} \iota: \mathbf{M}_P &\rightarrow \mathbf{M} \times \mathbb{S}^1, \\ w &\mapsto (w \circ i_1, w \circ i_2), \end{aligned}$$

where  $w$  denotes a multiplicative linear functional on  $\mathcal{E}_\varphi/\mathcal{E}_\varphi$ .

As the images of  $i_1$  and  $i_2$  generate  $\mathcal{E}_\varphi/\mathcal{E}_\varphi$ ,  $\iota$  is an injective map, clearly continuous, which proves that  $\mathbf{M}_P$  is homeomorphic to a compact subset of  $\mathbf{M} \times \mathbb{S}^1$ . Now we proceed to investigate which points of  $\mathbf{M} \times \mathbb{S}^1$  belong to the image of  $\iota$ . This dual-map argument is essentially ‘‘Herman’s Lemma’’ (cf. [4]).

As in the proof of Proposition 1.2, here again we use general results on comparison algebras. It follows from Theorem VII-1-5 of [2] that

for every point of the cosphere-bundle of  $\Omega$ ,  $(t, x; \tau, \xi) \in \mathcal{S}^*\Omega$ , there is a multiplicative linear functional on  $\mathcal{E}_{\mathcal{F}}/\mathcal{E}_{\mathcal{F}}$  that takes any function  $a$ , belonging to the closed algebra generated by  $A^{\sharp}$  in (6), to  $a(x, t)$  and  $D\Lambda$ ,

$$D = \frac{1}{i} \frac{\partial}{\partial t} + \frac{1}{i} b^j(x) \frac{\partial}{\partial x^j} + q(x) \in \mathcal{D}^{\sharp},$$

to  $\tau + b^j(x)\xi_j$ . This multiplicative linear functional must correspond to the point

$$((t, x; \tau, \xi), e^{it}) \in \mathbf{M} \times \mathbb{S}^1,$$

with  $|t| < \infty$ .

Suppose now that  $((t, x; \tau, \xi), e^{i\theta})$  is in the image of  $\iota$  and that  $|t| < \infty$ . Let  $\omega$  denote the corresponding multiplicative linear functional on  $\mathcal{E}_{\mathcal{F}}/\mathcal{E}_{\mathcal{F}}$  and  $\chi$  denote a function in  $\mathbf{C}_0^{\infty}(\Omega)$  with  $\chi(t) = 1$ . It is clear that  $\chi(\cdot)e^{i(\cdot)} + \mathcal{E}_{\mathcal{F}}$  belongs to the image of  $i_1$  and thus, by (15),

$$\omega(\chi(\cdot)e^{i(\cdot)} + \mathcal{E}_{\mathcal{F}}) = e^{i\theta}.$$

On the other hand, since  $e^{i(\cdot)} + \mathcal{E}_{\mathcal{F}}$  belongs to the image of  $i_2$ , we get:

$$\omega(\chi(\cdot)e^{i(\cdot)} + \mathcal{E}_{\mathcal{F}}) = \omega(\chi(\cdot) + \mathcal{E}_{\mathcal{F}})\omega(e^{i(\cdot)} + \mathcal{E}_{\mathcal{F}}) = e^{i\theta}.$$

We then obtain  $e^{i\theta} = e^{it}$ .

For  $t = \pm\infty$  and any  $e^{i\theta} \in \mathbb{S}^1$ , let us consider the sequence  $t_m = \theta \pm 2\pi m$ . Since  $\mathbf{M}_P$  is closed and

$$((t_m, x; \tau, \xi), e^{it_m}) \rightarrow ((t, x; \tau, \xi), e^{i\theta}) \quad \text{as } m \rightarrow \infty,$$

we conclude that  $((t, x; \tau, \xi), e^{i\theta}) \in \mathbf{M}_P$ . □

**REMARK 2.3.** We have just proved above that

$$\mathbf{W}_P := \{((t, x; \tau, \xi), e^{i\theta}) \in \mathbf{M}_P; |t| < \infty\}$$

is dense in  $\mathbf{M}_P$ .

Next we define the  $\gamma$ -symbol.

The  $\mathbf{C}^*$ -algebra  $\mathcal{E}_{\mathcal{F}}/\mathcal{H}(\mathcal{H})$  has the closed two-sided ideal  $\mathcal{E}_{\mathcal{F}}/\mathcal{H}(\mathcal{H})$ , which was proven to be  $*$ -isomorphic to  $\mathbf{C}(M_{SL}, \mathcal{H}_{\mathbb{Z} \times \mathbb{B}})$  in Theorem 1.6. Every  $A \in \mathcal{E}_{\mathcal{F}}$  determines a bounded operator of  $\mathcal{L}(\mathcal{E}_{\mathcal{F}}/\mathcal{H}(\mathcal{H}))$  by  $E + \mathcal{H}(\mathcal{H}) \rightarrow AE + \mathcal{H}(\mathcal{H})$ , thus defining

$$T: \mathcal{E}_{\mathcal{F}} \rightarrow \mathcal{L}(\mathcal{E}_{\mathcal{F}}/\mathcal{H}(\mathcal{H})).$$

Let us define

$$(16) \quad \begin{aligned} \gamma: \mathcal{E}_\varphi &\rightarrow \mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}})) \\ A &\mapsto \gamma_A = \Psi T_A \Psi^{-1}, \end{aligned}$$

for  $\Psi$  defined in Theorem 1.6.

For  $E \in \mathcal{E}_\varphi$ ,  $\gamma_E$  is the operator multiplication by  $\tilde{\gamma}_E \in \mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}})$  (see Theorem 1.6). Let  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$  denote the continuous functions on  $\mathbf{M}_{SL}$  taking values in  $\mathcal{L}_{\mathbb{Z} \times \mathbb{B}} := \mathcal{L}(L^2(\mathbb{Z} \times \mathbb{B}))$ . Identifying functions in  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$  with the corresponding multiplication operator of  $\mathcal{L}(\mathbf{C}(\mathcal{M}_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}}))$ , we can say then that  $\gamma$  is an extension of  $\tilde{\gamma}$ .

**PROPOSITION 2.4.** *There exists a \*-homomorphism*

$$\gamma: \mathcal{E}_\varphi \rightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}}),$$

where

$$\mathbf{M}_{SL} = \{e^{2\pi i\varphi}; \varphi \in \mathbb{R}\} \times \{+1, -1\},$$

given on the generators (1), according to notation established in §1 and in Theorem 1.6, by:

$$(17) \quad \begin{aligned} \gamma_{A_1} &= a(x); & \gamma_{A_2} &= b(\pm\infty); \\ \gamma_{A_3} &= Y_{-j}; & \gamma_{A_4} &= Y_\varphi \tilde{\Lambda}(\varphi - M) Y_{-\varphi}; \\ \gamma_{A_5} &= Y_\varphi K(\varphi - M) Y_{-\varphi}, & \text{where } K(\tau) &= -\tau \tilde{\Lambda}(\tau), \tau \in \mathbb{R} \text{ and} \\ \gamma_{A_6} &= Y_\varphi L(\varphi - M) Y_{-\varphi}, & \text{where } L(\tau) &= D_x \tilde{\Lambda}(\tau), \tau \in \mathbb{R}. \end{aligned}$$

Furthermore,  $\gamma$  restricted to the  $C^*$ -algebra  $C_\varphi^\diamond$ , generated by the operators in (1) except  $b \in \mathbf{CS}(\mathbb{R})$ , is an isometry.

*Proof.* Let us calculate  $\gamma$ , defined in (16), for the generators  $A_1, \dots, A_6$  of (1). By Proposition 1.2, it is enough to calculate the result of a left multiplication by  $A_p$ ,  $p = 1, \dots, 6$ , on operators  $E \in \mathcal{E}_\varphi$  such that  $F^{-1}EF$  are of the form  $c(D)K(\tau)T_l$ ,  $c \in \mathbf{CS}(\mathbb{R})$ ,  $K \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_{\mathbb{B}})$  and  $l \in \mathbb{Z}$ . For such an  $E$ , we get  $F^{-1}(A_p E)F$ ,  $p = 1, \dots, 6$ , equal to, modulo compact operators,

$$\begin{aligned} &c(D)a(x)K(\tau)T_l, & (cb)(D)K(\tau)T_l, & c(D)K(\tau + j)T_{j+l}, \\ &c(D)\tilde{\Lambda}(\tau)K(\tau)T_l, & -c(D)\tau\tilde{\Lambda}(\tau)K(\tau)T_l & \text{ and } c(D)D_x\tilde{\Lambda}(\tau)K(\tau)T_l, \end{aligned}$$

respectively. Here we have used (3) and

$$[c(D), B_k] \in \mathcal{K}(\mathcal{H}), \quad k = 4, 5, 6$$

(cf. [3], Proposition 1.2). By Theorem 1.6, we get:

$$\begin{aligned} \gamma_{A_1 E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi \tilde{K}(\varphi - M)Y_{-\varphi-1} \\ &= a(x)\gamma_E(\varphi, \pm 1) \quad (\tilde{K} = aK), \\ \gamma_{A_2 E}(\varphi, \pm 1) &= (cb)(\pm\infty)Y_\varphi K(\varphi - M)Y_{-\varphi-1} = b(\pm\infty)\gamma_E(\varphi, \pm\infty), \\ \gamma_{A_3 E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi K(\varphi + j - M)Y_{-\varphi-j-1} = Y_j\gamma_E(\varphi, \pm 1), \\ \gamma_{A_4 E}(\varphi, \pm 1) &= c(\pm\infty)Y_\varphi(\tilde{\Lambda}K)(\varphi - M)Y_{-\varphi-1} \\ &= Y_\varphi\tilde{\Lambda}(\varphi - M)Y_{-\varphi}\gamma_E(\varphi, \pm 1) \end{aligned}$$

and analogously for  $p = 5$  and  $6$ . This proves formulas (17).

For any  $A \in \mathcal{E}_\varphi$  such that  $F^{-1}AF = J(\tau) \in \mathbf{CO}(\mathbb{R}, \mathcal{K}_\mathbb{B})$ , it is also clear, using (5), that

$$\gamma_A(\varphi, \pm 1) = Y_\varphi J(\varphi - M)Y_{-\varphi}.$$

Hence, by (4),  $\gamma_{A_k^*}$  also belongs to  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$ .

The norm of the operator of  $\mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}}))$  given by multiplication by a function in  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$  is equal to the sup-norm of this function. In other words, the  $C^*$ -algebra  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$  is isometrically imbedded in  $\mathcal{L}(\mathbf{C}(\mathbf{M}_{SL}, \mathcal{K}_{\mathbb{Z} \times \mathbb{B}}))$ . As the image of a dense subalgebra of  $\mathcal{E}_\mathbb{P}$  is contained in  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$ , we conclude that  $\gamma$  maps  $\mathcal{E}_\varphi$  into  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$ .

Using the identification

$$L^2(\mathbb{S}^1) \otimes L^2(\mathbb{Z}) \otimes L^2(\mathbb{B}) = L^2(\mathbb{S}^1, L^2(\mathbb{Z} \times \mathbb{B})),$$

it can be straightforwardly verified that, for  $A(\tau) \in \mathbf{CB}(\mathbb{R}, \mathcal{L}_\mathbb{B})$ ,  $WA(\tau)W^{-1} \in \mathbf{C}(\mathbb{S}^1, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$  and it is given by  $Y_\varphi A(\varphi - M)Y_{-\varphi}$ . This means that for  $k = 1, 4, 5, 6$ , we have

$$\gamma_{A_k} = WF^{-1}A_kFW^{-1} \quad \text{and} \quad \gamma_{A_k^*} = WF^{-1}A_k^*FW^{-1}.$$

It is also clear that  $WT_jW^{-1} = Y_{-j}$  and, hence,

$$\gamma_A = WF^{-1}A(WF^{-1})^{-1}, \quad \text{for } A \in \mathcal{E}_\varphi^\circ,$$

proving that

$$\|\gamma_A\|_{\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})} = \|A\|_{\mathcal{L}(\mathcal{K})} \quad \text{and} \quad \gamma_{A^*} = (\gamma_A)^* \quad \text{for } A \in \mathcal{E}_\varphi^\circ.$$

This finishes the proof, since it is obvious that  $\gamma_{A_2^*} = (\gamma_{A_2})^*$ . □

The  $\sigma$ -symbol and the  $\gamma$ -symbol, defined in Theorem 2.2 and Proposition 2.4 respectively, are related by:

**PROPOSITION 2.5.** *For every  $A \in \mathcal{E}_\varphi$ ,  $\|\sigma_A|_{\mathbf{M}_p \setminus \mathbf{W}_p}\| \leq \|\gamma_A\|$ , i.e.,*

$$\sup\{|\sigma_A((t, x; \tau, \xi), e^{i\theta})|; |t| = \infty\} \leq \sup\{\|\gamma_A(m)\|_{\mathcal{L}_{\mathbb{Z} \times \mathbb{B}}}; m \in \mathbf{M}_{SL}\}.$$

*Proof.* Since the commutators of  $A_2$  with the other generators in (1) and their adjoints are compact (cf. [3], Proposition 1.2), the set of operators of the form

$$(18) \quad A = \sum_{j=1}^N b_j(t)A_j + K,$$

$$b_j \in \mathbf{CS}(\mathbb{R}), A_j \in \mathcal{E}_\varphi^\infty, K \in \mathcal{K}(\mathcal{H}), N \in \mathbb{N},$$

is dense in  $\mathcal{E}_\varphi$ . As  $\sigma_K = 0$  and  $\gamma_K = 0$  for  $K \in \mathcal{K}(\mathcal{H})$ , it suffices to assume  $A$  of the form (18) with  $K = 0$ .

For such an  $A$ , Theorem 2.2 implies:

$$\sigma_A((t, x; \tau, \xi), e^{i\theta}) = \sum_{j=1}^N b_j(t)\sigma_{A_j}((t, x; \tau, \xi), e^{i\theta}).$$

Letting  $A^\pm$  denote the operators  $\sum_{j=1}^N b_j(\pm\infty)A_j$ , it is clear then that

$$\begin{aligned} \sigma_A((+\infty, x; \tau, \xi), e^{i\theta}) &= \sigma_{A^+}((\pm\infty, x; \tau, \xi), e^{i\theta}) \quad \text{and} \\ \sigma_A((-\infty, x; \tau, \xi), e^{i\theta}) &= \sigma_{A^-}((\pm\infty, x; \tau, \xi), e^{i\theta}); \end{aligned}$$

hence:

$$(19) \quad \|\sigma_A|_{\mathbf{M}_p \setminus \mathbf{W}_p}\| \leq \max\{\|\sigma_{A^+}\|, \|\sigma_{A^-}\|\}.$$

The map  $\sigma: \mathcal{E}_\varphi \rightarrow \mathbf{C}(\mathbf{M}_p)$  was defined as the composition of the Gelfand map (an isometry) with the canonical projection  $\mathcal{E}_\varphi \rightarrow \mathcal{E}_\varphi/\mathcal{K}(\mathcal{H})$ . It then follows that

$$\|\sigma_{A^\pm}\| \leq \|A^\pm\|.$$

As  $A^\pm \in \mathcal{E}_\varphi^\infty$ , where  $\gamma$  is an isometry,

$$(20) \quad \|\sigma_{A^\pm}\|_{\mathbf{C}(\mathbf{M}_p)} \leq \|\gamma_{A^\pm}\|_{\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})}.$$

By Proposition 2.4,

$$\gamma_A(\varphi, +1) = \sum_{j=1}^N b_j(+\infty)\gamma_{A_j}(\varphi, +1) = \gamma_{A^+}(\varphi, +1)$$

and

$$\gamma_A(\varphi, -1) = \gamma_{A^-}(\varphi, -1).$$

Furthermore, for any  $A \in \mathcal{E}_\varphi^\infty$ , it is clear from (17) that  $\gamma_A(\varphi, +1) = \gamma_A(\varphi, -1)$  and, therefore,

$$(21) \quad \|\gamma_A\| = \max\{\|\gamma_{A^+}\|, \|\gamma_{A^-}\|\}$$

We are finished by (19), (20) and (21). □

If  $\gamma_A = 0$ , then,  $\sigma_A|_{\mathbf{M}_P \setminus \mathbf{W}_P} = 0$ . The converse is also true:

**PROPOSITION 2.6.** *An operator  $A \in \mathcal{E}_\varphi$  belongs to the kernel of  $\gamma$  if and only if  $\sigma_A$  vanishes on  $\mathbf{M}_P \setminus \mathbf{W}_P$ . Furthermore, we have:*

$$(22) \quad \ker \gamma \cap \ker \sigma = \mathcal{K}(\mathcal{H}).$$

*Proof.* Let  $\mathcal{T}_0$  denote the  $C^*$ -algebra generated by multiplications by functions in  $C_0^\infty(\Omega)$  and by the operators of the form  $D\Lambda$ , where  $D$  is a first order linear differential operator on  $\Omega$  with smooth coefficients of compact support. Given  $A_0$ , one of these generators just described, we can find  $\chi \in C_0^\infty(\mathbb{R})$  such that  $\chi A_0 = A_0$  and then  $\gamma_{A_0} = \gamma_\chi \gamma_{A_0} = 0$ , by Proposition 2.4. So, we have  $\mathcal{T}_0 \subseteq \ker \gamma$ .

Using the nomenclature of [2],  $\mathcal{T}_0$  is the minimal comparison algebra associated with the triple  $\{\Omega, dS, H\}$ . It can be easily concluded from [2], Lemma VII-1-2, that  $A \in \mathcal{E}_\varphi$  belongs to  $\mathcal{T}_0$  if and only if  $\sigma_A$  vanishes on  $\mathbf{M}_P \setminus \mathbf{W}_P$ , proving that  $\mathcal{T}_0 \subseteq \ker \gamma$ , by Proposition 2.5.

Since  $\ker \sigma = \mathcal{E}_\varphi$  and  $\ker \gamma = \mathcal{T}_0$ , the equality in (22) follows from [2], Theorem VII-1-3. □

**3. A Fredholm criterion and an application to differential operators.** We will now give a necessary and sufficient criterion for an  $N \times N$ -matrix whose entries are operators in  $\mathcal{E}_\varphi$ , regarded as a bounded operator on  $L^2(\Omega, \mathbb{C}^N)$ ,  $N \geq 1$ , to be Fredholm. Let us denote  $L^2(\Omega, \mathbb{C}^N)$  by  $\mathcal{H}^N$  and by  $\mathcal{E}_\varphi^N$  the  $C^*$ -subalgebra of  $\mathcal{L}(\mathcal{H}^N)$

$$\mathcal{E}_\varphi^N := \{((A_{jk})); A_{jk} \in \mathcal{E}_\varphi, 1 \leq j, k \leq N\}.$$

It is easy to see that the compact ideal of  $\mathcal{L}(\mathcal{H}^N)$  coincides with the matrices with entries in  $\mathcal{K}(\mathcal{H})$ , i.e.,

$$\mathcal{K}(\mathcal{H}^N) = \mathcal{K}^N := \{((K_{jk})); K_{jk} \in \mathcal{K}(\mathcal{H}), 1 \leq j, k \leq N\}.$$

Let us define two symbols on  $\mathcal{E}_\varphi^N$ :

$$\sigma_A^N = ((\sigma_{A_{jk}}))_{1 \leq j, k \leq N} \quad \text{and} \quad \gamma_A^N = ((\gamma_{A_{jk}}))_{1 \leq j, k \leq N},$$



where  $A = ((A_{jk}))_{1 \leq j, k \leq N} \in \mathcal{E}_{\mathcal{F}}^N$ . The following proposition follows immediately from the definitions above and Proposition 2.6.

**PROPOSITION 3.1.** *The  $\gamma^N$ -symbol of an operator  $A \in \mathcal{E}_{\mathcal{F}}^N$  is identically zero if and only if its  $\sigma^N$ -symbol vanishes on  $\mathbf{M}_P \setminus \mathbf{W}_P$ . Furthermore, we have:*

$$(23) \quad \ker \sigma^N \cap \ker \gamma^N = \mathcal{K}^N.$$

**THEOREM 3.2.** *For an operator  $A = ((A_{jk}))_{1 \leq j, k \leq N} \in \mathcal{E}_{\mathcal{F}}^N$  to be Fredholm, it is necessary and sufficient that:*

- (i)  $\sigma_A^N$  be invertible, i.e., the  $N \times N$ -matrix  $((\sigma_{A_{jk}}(m)))$  be invertible for all  $m \in \mathbf{M}_P$ , and
- (ii)  $\gamma_A^N$  be invertible, i.e., the  $N \times N$ -matrix, with entries in  $\mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{Z} \times \mathbb{B}})$ ,  $((\gamma_{A_{jk}}(m)))$  be invertible for all  $m \in \mathbf{M}_{SL}$ .

*Proof.* Suppose that  $A$  is Fredholm and let  $B$  be such that  $1 - AB$  and  $1 - BA$  are compact. We have  $B \in \mathcal{E}_{\mathcal{F}}^N$ , since  $\mathcal{E}_{\mathcal{F}}^N / \mathcal{K}^N$  is a  $C^*$ -subalgebra of  $\mathcal{L}(\mathcal{K}^N) / \mathcal{K}^N$ . We then get

$$\sigma_{1-AB}^N = \sigma_{1-BA}^N = 0 \quad \text{and} \quad \gamma_{1-AB}^N = \gamma_{1-BA}^N = 0$$

and, hence,

$$1 = \sigma_A^N \sigma_B^N = \sigma_B^N \sigma_A^N \quad \text{and} \quad 1 = \gamma_A^N \gamma_B^N = \gamma_B^N \gamma_A^N.$$

Conversely, suppose that (i) and (ii) above are satisfied. Since  $\gamma^N: \mathcal{E}_{\mathcal{F}}^N \rightarrow \mathbf{C}(\mathbf{M}_{SL}, N \times N\text{-matrices with entries in } \mathcal{L}(L^2(\mathbb{Z}) \otimes L^2(\mathbb{B})))$  is a  $*$ -homomorphism (by Proposition 2.4), its range is a  $C^*$ -algebra. There must be then a  $B \in \mathcal{E}_{\mathcal{F}}^N$  such that  $\gamma_B^N = (\gamma_A^N)^{-1}$ . Since  $1 - AB \in \ker \gamma^N$ ,  $1 - \sigma_A^N \sigma_B^N$  vanishes on  $\mathbf{M}_P \setminus \mathbf{W}_P$ , by Proposition 3.1. As the map  $\sigma$  is surjective, so is  $\sigma^N$ . An operator  $Q \in \mathcal{E}_{\mathcal{F}}^N$  can therefore be found such that its symbol  $\sigma_Q^N$  equals the continuous function vanishing on  $\mathbf{M}_P \setminus \mathbf{W}_P$

$$(\sigma_A^N)^{-1} - \sigma_B^N.$$

By Proposition 3.1 again,  $Q \in \ker \gamma^N$  and, then,

$$\gamma_{1-A(B+Q)}^N = \gamma_{1-(B+Q)A}^N = 0.$$

Since we also have

$$\sigma_{1-A(B+Q)}^N = 1 - \sigma_A^N \sigma_B^N - \sigma_A^N \sigma_Q^N = 0 = \sigma_{1-(B+Q)A}^N,$$

the operator  $B + Q$  is an inverse for  $A$ , modulo a compact operator, by equation (23). □

In order to apply this result to differential operators, it is convenient to conjugate the  $\gamma$ -symbol with the discrete Fourier transform. We define:

$$(24) \quad \Gamma: \mathcal{E}_\varphi \rightarrow \mathbf{C}(\mathbf{M}_{SL}, \mathcal{L}_{\mathbb{S}^1 \times \mathbb{B}})$$

$$A \mapsto \Gamma_A(m) = F_d^{-1} \gamma_A(m) F_d, \quad m \in \mathbf{M}_{SL},$$

where  $F_d: L^2(\mathbb{S}^1) \rightarrow L^2(\mathbb{Z})$ ,  $\mathbb{S}^1 = \{e^{i\theta}; \theta \in \mathbb{R}\}$ , was defined in (7), and, as usual,  $F_d$  also denotes  $F_d \otimes I_{\mathbb{B}}$ .

Next we calculate  $\Gamma_A$  for the generators of  $\mathcal{E}_\varphi$ . It is obvious that, for  $a \in \mathbf{C}^\infty(\mathbb{B})$ ,

$$(25) \quad \Gamma_a(\varphi, \pm 1) = a, \quad (e^{2\pi i \varphi}, \pm 1) \in \mathbf{M}_{SL},$$

and, for  $b \in \mathbf{CS}(\mathbb{R})$ ,

$$(26) \quad \Gamma_b(\varphi, \pm 1) = b(\pm\infty), \quad \text{independent of } \varphi.$$

For  $j \in \mathbb{Z}$ ,  $F_d^{-1} Y_{-j} F_d$  equals the operator multiplication by  $e^{ij\theta}$  on  $\mathbb{S}^1 = \{e^{i\theta}, \theta \in \mathbb{R}\}$ , and then, by (24) and (17),

$$(27) \quad \Gamma_{e^{+ij\theta}}(\varphi, \pm 1) = e^{ij\theta}, \quad \text{for all } (e^{2\pi i \varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Let  $a \in \mathbf{C}(\Omega)$  be of the form

$$(28) \quad a(t, x) = a_+(t, x)\chi_+(t, x) + a_-(t, x)\chi_-(t, x) + a_0(t, x),$$

where  $a_\pm$  are continuous and  $2\pi$ -periodic in  $t$ ,  $a_0 \in \mathbf{CO}(\Omega)$  and  $\chi_\pm \in \mathbf{CS}(\mathbb{R})$  satisfy  $\chi_\pm(\pm\infty) = 1$ ,  $\chi_+ + \chi_- = 1$ . By the continuity of  $\Gamma$ , (25), (26) and (27), it follows that

$$(29) \quad \Gamma_a(\varphi, \pm 1) = a_\pm(\theta, x), \quad \text{for } (e^{2\pi i \varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Note that (28) gives  $\Gamma_{A_1}$ ,  $\Gamma_{A_2}$  and  $\Gamma_{A_3}$ , for  $A_p$  as defined on page 283.

Now we calculate  $F_d^{-1} K(\varphi - M) F_d$ , for  $\varphi \in \mathbb{R}$  and  $K(\tau) = \tilde{\Lambda}(\tau)$ ,  $-\tau \tilde{\Lambda}(\tau)$  or  $D_x \tilde{\Lambda}(\tau)$ , which is needed for obtaining  $\Gamma_{A_p}$ ,  $p = 4, 5, 6$ . Let us use again that  $-\Delta_{\mathbb{B}}$  has an orthonormal basis of eigenfunctions  $w_m$ ,  $m \in \mathbb{N}$ , with eigenvalues  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots, \lambda_m \rightarrow \infty$  as  $m \rightarrow \infty$ , and define the unitary map

$$U: L^2(\mathbb{B}) \rightarrow L^2(\mathbb{N}),$$

$$u \mapsto (w_m, u)_{m \in \mathbb{N}}.$$

By the spectral theorem, the conjugate  $U(1 + (\varphi - j)^2 - \Delta_{\mathbb{B}})^{-1/2} U^{-1}$  equals the operator multiplication by  $(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}$  on  $L^2(\mathbb{N})$ ,

for each  $j \in \mathbb{Z}$ ,  $\varphi \in \mathbb{R}$ . The operator  $\tilde{\Lambda}(\varphi - M) \in \mathcal{L}_{\mathbb{Z} \times \mathbb{B}}$  acts on

$$\mathbf{u} = (u_j)_{j \in \mathbb{Z}} \in L^2(\mathbb{Z}; L^2(\mathbb{B}))$$

by

$$\tilde{\Lambda}(\varphi - M)\mathbf{u} = ((1 + (\varphi - j)^2 + \Delta_{\mathbb{B}})^{-1/2}u_j)_{j \in \mathbb{Z}}$$

and, thus,

$$(30) \quad (I_{\mathbb{Z}} \otimes U)\tilde{\Lambda}(\varphi - M)(I_{\mathbb{Z}} \otimes U)^{-1} = (1 + (\varphi - j)^2 + \lambda_m)^{-1/2},$$

where, by  $(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}$ , we now mean the corresponding multiplication operator on  $L^2(\mathbb{Z}) \otimes L^2(\mathbb{N})$ .

Let us adopt the notation:

$$(31) \quad 1 + (\varphi - D_{\theta})^2 - \Delta_{\mathbb{B}} := (F_d \otimes U)^{-1}(1 + (\varphi - j)^2 + \lambda_m)(F_d \otimes U).$$

It is easy to see that  $1 + (\varphi - D_{\theta})^2 - \Delta_{\mathbb{B}}$  is the unique self-adjoint realization of the differential expression  $1 + (\varphi + i\frac{\partial}{\partial \theta})^2 - \Delta_{\mathbb{B}}$  on  $\mathbb{S}^1 \times \mathbb{B}$  (see Lemma 3.3). By (30) and (31) then, we obtain:

$$(32) \quad (F_d \otimes I_{\mathbb{B}})^{-1}\tilde{\Lambda}(\varphi - M)(F_d \otimes I_{\mathbb{B}}) = (1 + (\varphi - D_{\theta})^2 - \Delta_{\mathbb{B}})^{-1/2},$$

for every  $\varphi \in \mathbb{R}$ . Using that  $Y_{\varphi} = F_d^{-1}e^{-i\varphi\theta}F_d$ ,  $\varphi \in \mathbb{R}$  and (17), it follows that:

$$(33) \quad \Gamma_{\Lambda}(\varphi, \pm 1) = e^{-i\varphi\theta}(1 + (D_{\theta} - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2}e^{i\varphi\theta},$$

$$(e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Since, for each  $j \in \mathbb{Z}$  and each  $\varphi \in \mathbb{R}$ ,

$$U(\varphi - j)(1 + (\varphi - j)^2 - \Delta_{\mathbb{B}})^{-1/2}U^{-1}$$

equals the operator multiplication by

$$(\varphi - j)(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}$$

on  $L^2(\mathbb{N})$ , we obtain, in a way analogous to how (33) was obtained:

$$(34) \quad \Gamma_{A_4}(\varphi, \pm 1) = e^{-i\varphi\theta}(D_{\theta} - \varphi)(1 + (D_{\theta} - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2}e^{i\varphi\theta},$$

$$(e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

Here we have assumed the notation:

$$(\varphi - D_{\theta})(1 + (\varphi - D_{\theta})^2 - \Delta_{\mathbb{B}})^{-1/2}$$

$$:= (F_d \otimes U)^{-1}(\varphi - j)(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}(F_d \otimes U).$$

For the last type of generator, we need the following lemma.

LEMMA 3.3. *The subspace*

$$\{u \in L^2(\mathbb{S}^1 \times \mathbb{B}); (1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}u \in C^\infty(\mathbb{S}^1 \times \mathbb{B})\}$$

is dense in  $L^2(\mathbb{S}^1 \times \mathbb{B})$ , for every  $\varphi \in \mathbb{R}$ .

*Proof.* The statement is true for  $\varphi = 0$ , since

$$1 + D_\theta^2 - \Delta_{\mathbb{B}} = 1 - \Delta_{\mathbb{S}^1 \times \mathbb{B}}$$

is essentially self-adjoint on  $C^\infty(\mathbb{S}^1 \times \mathbb{B})$ , by [2], Theorem IV-1-8, for example. For  $\varphi \in \mathbb{R}$ ,

$$(1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}(1 + D_\theta^2 - \Delta_{\mathbb{B}})^{1/2}$$

is a Banach-space isomorphism, since it is unitarily equivalent to the multiplication by the function on  $\mathbb{Z} \times \mathbb{N}$

$$(1 + (\varphi - j)^2 + \lambda_m)^{-1/2}(1 + j^2 + \lambda_m)^{-1/2},$$

which is bounded and bounded away from zero. □

For every  $v \in C^\infty(\mathbb{S}^1 \times \mathbb{B})$ , it is clear that

$$D_x F_d v = F_d D_x v,$$

where, on the right-hand side,  $D_x$  is regarded as a differential expression on  $\mathbb{S}^1 \times \mathbb{B}$  and, on the left-hand side,  $D_x$  acts, as a differential operator on  $\mathbb{B}$ , on each component  $w_j \in C^\infty(\mathbb{B})$  of

$$w = (w_j)_{j \in \mathbb{Z}} = F_d v \in L^2(\mathbb{Z}; L^2(\mathbb{B})).$$

By Lemma 3.3, it therefore follows that

$$(35) \quad \begin{aligned} F_d [D_x (1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}] F_d^{-1} \\ = D_x [F_d (1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2} F_d^{-1}]. \end{aligned}$$

The right-hand side of (35) equals  $D_z \tilde{\Lambda}(\varphi - M)$ , by (32). We have, hence:

$$(36) \quad \Gamma_{A_\varphi}(\varphi, \pm 1) = e^{-i\varphi\theta} [D_x (1 + (\varphi - D_\theta)^2 - \Delta_{\mathbb{B}})^{-1/2}] e^{i\varphi\theta}.$$

Equations (29), (31), (32), (33), (34) and (36) prove:

PROPOSITION 3.4. *The map  $\Gamma$  defined in (24) is given on the generators of  $\mathcal{E}_\varphi$  (with  $m = (e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}$  and  $\Gamma_A(\varphi, \pm 1) \in \mathcal{L}_{\mathbb{S}^1 \times \mathbb{B}}$ ,  $\mathbb{S}^1 = \{e^{i\theta}; \theta \in \mathbb{R}\}$ ) by:*

$$\Gamma_a(\varphi, \pm 1) = a_\pm(\theta, x), \quad \text{for } a \text{ as in (28)}$$

$$\Gamma_\Lambda(\varphi, \pm 1) = e^{-i\varphi\theta} (1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2} e^{i\varphi\theta}$$

$$\Gamma_{-i \frac{\partial}{\partial t} \Lambda}(\varphi, \pm 1) = e^{-i\varphi\theta} (D_\theta - \varphi) (1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2} e^{i\varphi\theta}$$

$$\Gamma_{D_x \Lambda}(\varphi, \pm 1) = e^{-i\varphi\theta} D_x (1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2} e^{i\varphi\theta}.$$

**REMARK 3.5.** Because of the way  $\Gamma$  was defined, it is obvious that condition (ii) of Theorem 3.2 can be replaced by

(ii') The matrix  $\Gamma_A^N(m) := ((\Gamma_{A_{jk}}(m)))_{1 \leq j, k \leq N}$  is invertible for all  $m \in \mathbf{M}_{SL}$ .

Our next and final objective is to find necessary and sufficient conditions for a differential operator with semi-periodic coefficients on  $\Omega$  to be Fredholm. Most of the ideas and proofs in what follows are borrowed from [2], §§VII.3 and IX.3, where the more general problem of finding differential expressions within reach of a Comparison Algebra is addressed.

**PROPOSITION 3.6.** *Let  $L$  be an  $M$ th order differential expression on  $\mathbb{B}$ , with smooth coefficients. The operator  $L\Lambda^M$ , defined initially on the dense subspace  $\Lambda^{-M}(C_0^\infty(\Omega))$ , can be extended to a bounded operator  $A$  in  $\mathcal{L}(\mathcal{H})$ . Moreover, we have that  $A \in \mathcal{C}_\varphi$ ,  $\sigma_A$  coincides with the principal symbol of  $L$  on  $\mathbf{W}_P$  (points of  $\mathbf{M}_P$  over  $|t| < \infty$ ) and*

$$\Gamma_A(\varphi, \pm 1) = e^{-i\varphi\theta} L(1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}})^{-1/2} e^{i\varphi\theta},$$

$$(e^{2\pi i\varphi}, \pm 1) \in \mathbf{M}_{SL}.$$

*Proof.* It is easy to see that any  $M$ th order differential expression on a compact manifold equals a sum of products of at most  $M$  first-order differential expressions. (See, for example, the proof of Proposition VI-3-1 of [2].) It is therefore enough to consider  $L$  of the form

$$L = D_1 D_2 \cdots D_M,$$

where  $D_j, j = 1, \dots, M$ , are first order expressions. For  $M = 1$ , the proposition is true by Theorem 2.2 and Proposition 3.4.

Using that  $\Lambda^2 = H^{-1}$ ,  $H = 1 - \Delta_{\mathbb{R}} - \Delta_{\mathbb{B}}$ , it is easy to see that, for  $u \in \Lambda^{-2}(C_0^\infty(\Omega))$ , and  $D_1$  and  $D_2$  first order expressions, we have:

$$(37) \quad D_1 D_2 \Lambda^2 u = D_1 \Lambda^2 D_2 u + D_1 \Lambda^2 [H, D_2] \Lambda^2 u.$$

The commutator  $[H, D_2]$  is a second order expression on  $\mathbb{B}$  and can therefore be expressed as a sum of products of at most two first order differential expressions:

$$[H, D_2] = \sum_{j=1}^p F_j G_j.$$

This shows that, on the dense subspace  $\Lambda^{-2}(C_0^\infty(\Omega))$ ,  $D_1 D_2 \Lambda^2$  equals the operator

$$(D_1 \Lambda)(D_2^* \Lambda)^* + (D_1 \Lambda) \sum_{j=1}^p (F_j^* \Lambda)^* (G_j \Lambda) \Lambda \in \mathcal{C}_\varphi,$$

where  $D^*$  denotes the formal adjoint of a differential expression  $D$ . Since  $\sigma_\Lambda = 0$ , we get:

$$\sigma_{D_1 D_2 \Lambda^2} = \sigma_{D_1 \Lambda} \sigma_{D_2^* \Lambda},$$

which, restricted to  $W_P$ , coincides with the principal symbol of  $D_1$ ,  $D_2$ , by Theorem 2.2. It also follows that:

$$\Gamma_{D_1 D_2 \Lambda^2} = \Gamma_{D_1 \Lambda} \Gamma_{D_2^* \Lambda}^* + \Gamma_{D_1 \Lambda} \sum_{j=1}^p \Gamma_{F_j^* \Lambda}^* \Gamma_{G_j \Lambda} \Gamma_\Lambda.$$

By Proposition 3.4, we get:

$$\begin{aligned} e^{i\varphi\theta} \Gamma_{D_1 D_2 \Lambda^2}(\varphi, \pm 1) e^{-i\varphi\theta} &= (D_1 \Lambda_\varphi)(D_2^* \Lambda_\varphi)^* + D_1 \Lambda_\varphi \sum_{j=1}^p (F_j^* \Lambda_\varphi)^* (G_j \Lambda_\varphi) \Lambda_\varphi \\ &= D_1 \Lambda_\varphi^2 D_2 + D_1 \Lambda_\varphi^2 \sum_{j=1}^p F_j G_j \Lambda_\varphi^2, \end{aligned}$$

where  $\Lambda_\varphi = H_\varphi^{-1/2}$ ,  $H_\varphi = 1 + (D_\theta - \varphi)^2 - \Delta_{\mathbb{B}}$ . Since  $[H, D_2]$  and  $[H_\varphi, D_2]$  are equal (as expressions on  $\mathbb{B}$ ), we get:

$$e^{i\varphi\theta} \Gamma_{D_1 D_2 \Lambda^2}(\varphi, \pm 1) e^{-i\varphi\theta} = D_1 \Lambda_\varphi^2 D_2 + D_1 \Lambda_\varphi^2 [H_\varphi, D_2] \Lambda_\varphi^2 = D_1 D_2 \Lambda_\varphi^2$$

proving the proposition for  $L = D_1 D_2$ .

Suppose now that the proposition is true for sums of products of at most  $M$  first order differential expressions and let  $L = D_1 D_2 \cdots D_{M+1}$  be a product of first order expressions. Define:  $F = D_1 D_2$  and  $G = D_3 \cdots D_{M+1}$ . Using the formula

$$\begin{aligned} L \Lambda^{M+1} u &= F \Lambda^2 G \Lambda^{M-1} u + F \Lambda^2 [H, G] \Lambda^{M+1} u, \\ &u \in \Lambda^{-M-1}(C_0^\infty(\Omega)), \end{aligned}$$

the proposition follows for this  $L$ , by the same argument as above. □

Let  $\{U_\beta\}$  be a finite atlas on  $\mathbb{B}$  and  $\{\varphi_\beta\}$  a subordinate partition of unity, i.e.  $\text{support } \varphi_\beta \subset U_\beta$ . Let  $L$  be a differential operator on  $\Omega$ , acting on  $\mathbb{C}^N$ -valued functions, locally given on  $U_\beta$  by

$$(38) \quad L = \sum_{j=0}^{\tilde{M}} \sum_{|\alpha| \leq M_j} A_{\beta,j,\alpha}(t, x) \left( \frac{1}{i} \frac{\partial}{\partial x} \right)^\alpha \left( \frac{1}{i} \frac{\partial}{\partial t} \right)^j,$$

where

$$\left( \frac{1}{i} \frac{\partial}{\partial x} \right)^\alpha := \left( -i \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( -i \frac{\partial}{\partial x_n} \right)^{\alpha_n}, \quad \text{for } \alpha \in \mathbb{N}^n$$

and  $|\alpha| = \alpha_1 + \cdots + \alpha_n$ . We will say that  $L$  has *semi-periodic coefficients* if the matrices

$$\tilde{A}_{\beta,j,\alpha}(t, x) := \varphi_\beta(x) A_{\beta,j,\alpha}(t, x),$$

regarded as functions on  $\Omega$ , have as entries functions of the type (28). It is easy to see that this definition is independent of the choice of atlas on  $\mathbb{B}$ . We want to decide when

$$L: H^M(\Omega, \mathbb{C}^N) \rightarrow L^2(\Omega, \mathbb{C}^N)$$

is a Fredholm operator, assuming that  $L$  has semi-periodic coefficients. Here  $M$  denotes the order of  $L$ ,  $M = \max\{M_j + j, j = 1, \dots, \tilde{M}\}$ .

We also denote by  $\Lambda$  the operator  $\Lambda \otimes I_N$  on  $\mathcal{L}(L^2(\Omega, \mathbb{C}^N))$ , where  $I_N$  denotes the  $N \times N$  identity matrix. Since  $\Lambda$  commutes with  $\frac{\partial}{\partial t}$  and  $L = \sum L_\beta$ , for  $L_\beta := \varphi_\beta L$ , we get:

$$L\Lambda^M = \sum_{\beta,j,\alpha} (t, x) \left( \frac{1}{i} \frac{\partial}{\partial x} \right)^\alpha \Lambda^{|\alpha|} \left( \frac{1}{i} \frac{\partial}{\partial t} \right)^j \Lambda^j \Lambda^{M-|\alpha|-j}.$$

After multiplying  $(\frac{1}{i} \frac{\partial}{\partial x})^\alpha$  above by  $\chi_{\beta,j,\alpha} \in \mathbf{C}_0^\infty(U_\beta)$ ,  $\chi_{\beta,j,\alpha}(x) = 1$  for  $x$  in the support of  $\tilde{A}_{\beta,j,\alpha}$ , we still get the same operator and  $\chi_{\beta,j,\alpha}(x)(\frac{1}{i} \frac{\partial}{\partial x})^\alpha$  is now a differential expression defined on  $\mathbb{B}$ . We can therefore apply Proposition 3.6 and conclude that  $L\Lambda^M \in \mathcal{E}_\varphi^N$ . Using, moreover, that  $\sigma_{\Lambda^{M-|\alpha|-j}} = 0$  for  $|\alpha| + j < M$ , we get:

$$\sigma_{L\Lambda^M}(t, x; \tau, \xi) = \sum_\beta \sum_{|\alpha|+j=M} \tilde{A}_{\beta,j,\alpha}(t, x) \xi^\alpha \tau^j, \quad |t| < \infty.$$

The right-hand side of the previous equation coincides with the principal symbol of  $L$  restricted to the co-sphere bundle of  $\Omega$ . Invertibility

of the  $\sigma$ -symbol is therefore equivalent to uniform ellipticity of  $L$ , by Remark 2.3.

The operator-valued symbol  $\Gamma_{L\Lambda^M}$  is also given by Proposition 3.6 (and Proposition 3.4):

$$e^{-i\varphi\theta}\Gamma_{L\Lambda^M}(\varphi, \pm 1)e^{i\varphi\theta} = \sum_{\beta, j, \alpha} \tilde{A}_{\beta, j, \alpha}^{\pm}(\theta, x) \left(\frac{1}{i} \frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{1}{i} \frac{\partial}{\partial \theta} - \varphi\right)^j \Lambda_{\varphi}^M,$$

where we have used that  $\Lambda_{\varphi}$  and  $\frac{\partial}{\partial \theta}$  commute. We have denoted by  $\tilde{A}_{\beta, j, \alpha}^{\pm}$  the  $2\pi$ -periodic continuous functions such that

$$\tilde{A}_{\beta, j, \alpha}(t, x) - \chi_+(t)\tilde{A}_{\beta, j, \alpha}^+(\theta, x) - \chi_-(t)\tilde{A}_{\beta, j, \alpha}^-(\theta, x) \in \mathbf{CO}(\Omega).$$

(See (28).)

Let  $L_{\beta}^{\pm}(\varphi)$  denote the differential expressions on  $S^1 \times \mathbb{B}$

$$L_{\beta}^{\pm}(\varphi) := \sum_{j=0}^{\tilde{M}} \sum_{|\alpha| \leq M_j} \tilde{A}_{\beta, j, \alpha}^{\pm}(\theta, x) \left(\frac{1}{i} \frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{1}{i} \frac{\partial}{\partial \theta} - \varphi\right)^j,$$

and define the operator

$$(39) \quad L^{\pm}(\varphi) := \sum_{\beta} L_{\beta}^{\pm}(\varphi): H^M(S^1 \times \mathbb{B}, \mathbb{C}^N) \rightarrow L^2(S^1 \times \mathbb{B}, \mathbb{C}^N).$$

Since  $\Lambda_{\varphi}$  is an isomorphism from

$$L^2(S^1 \times \mathbb{B}, \mathbb{C}^N) \quad \text{onto} \quad H^M(S^1 \times \mathbb{B}, \mathbb{C}^N),$$

the above considerations, together with Theorem 3.2 and Remark 3.5 prove the following theorem.

**THEOREM 3.7.** *Let  $L$  denote an  $M$ th order differential operator on  $\Omega$  of the form (38), with continuous semi-periodic coefficients, and let  $L^{\pm}(\varphi)$  denote the differential operators on  $S^1 \times \mathbb{B}$  defined in (39). Then*

$$L: H^M(\Omega, \mathbb{C}^N) \rightarrow L^2(\Omega, \mathbb{C}^N)$$

*is Fredholm if and only if  $L$  is uniformly elliptic and  $L^{\pm}(\varphi)$  are invertible for all  $\varphi \in [0, 1]$ .*



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