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The Segre class of a singular projective variety X is that of the normal cone of the diagonal in the product $X \times X$. This class was introduced by K. W. Johnson and W. Fulton to study immersions and embeddings. In our previous work we related the Segre classes and the Chern-Mather classes for hypersurfaces with codimension one singularities and $X^n \subset \mathbb{P}^{2n}$ with isolated singularities. In this paper we generalize these results to the case of $X^n \subset \mathbb{P}^N$ with singularities of codimension $N - n$ $(N \leq 2n)$.

The notion of Segre classes (of cones) has become of increasing importance as a key ingredient for constructing or analyzing various invariants, e.g., in intersection theory and group representation theory, etc.

The Segre class treated in this note is the Segre class of a singular projective variety, which was introduced by K. W. Johnson (and W. Fulton) to study immersions and embeddings of singular projective varieties [4]. This is the Segre class of the normal cone $C_\Delta(X \times X)$ of the diagonal Δ in the product $X \times X$. We call this class Johnson's Segre class, denoted by $S_*(X)$.

Another well-studied characteristic class of a singular variety is MacPherson's Chern class, the existence of which was conjectured by Deligne and Grothendieck. R. MacPherson [7] constructed this Chern class, using Chern-Mather classes and introducing the notion of local Euler obstruction. A. Dubson [2] gave a more concrete description for MacPherson's Chern class $C_*(X)$: Let $\mathcal{S}_X$ be a (in fact, any) Whitney stratification of X with the smooth part of X as the top-dimensional stratum and let $C_*^M(X)$ denote the Chern-Mather class of X. Then

$$C_*(X) = C_*^M(X) + \sum_{\substack{S \in \mathcal{S}_X \\ \dim S < \dim X}} m_S \cdot C_*^M(\overline{S}),$$

where m_S is a certain integer attached to each stratum S.

Motivated by Dubson's formula relating MacPherson's Chern class and the Chern-Mather class, the author [9] introduced the Segre-

Mather class, defined in a similar manner to the definition of the Chern-Mather class, trying to relate Johnson's Segre class and the Segre-Mather class. In order to relate these two classes, it turns out that we need to know (or identify) the irreducible components of the (projectivized) normal cone and their multiplicities. (This kind of situation occurs in obtaining the local index formula for a holonomic system [**1**, **2**, **5**]; i.e., one needs to identify the irreducible components of the characteristic variety (which is a cone) of the holonomic system and their multiplicities.) In [**9**] the author gave formulas $S_*(X)$ and $S_*^M(X)$ for the hypersurface case and for the case when $X^n \subset \mathbb{P}^{2n}$ with isolated singularities. In this note we generalize these two results to a more general case. This generalization was hinted by a question posed by S. Kleiman (private communication): what is the difference $S_*(X) - S_*^M(X)$ if $X^n \subset \mathbb{P}^{2n-1}$ with curves as singularities? An interesting feature of our result is that the difference between Johnson's Segre class and the Segre-Mather class of X is controlled by both the codimension of X in $\mathbb{P}^N$ (or more strongly, the local embedding dimension of X) and the codimension of the singular locus in X. Throughout this paper our ground field is the complex numbers $\mathbb{C}$.

I would like to thank the referee for some useful suggestions and comments.

1. Tangent star cone and Segre classes. Let X^n be a reduced (singular) projective variety of equidimension $= n$ in $\mathbb{P}^N$. Consider X to be embedded as the diagonal of $X \times X$. Then the *tangent star cone* of X, denoted by $\mathrm{TS}(X)$, is defined to be the *normal cone* of the diagonal X in $X \times X$. (Johnson [**4**] dubbed a fiber of the normal cone *tangent star*.) Namely, if the diagonal is defined by the ideal sheaf I, then

$$\mathrm{TS}(X) := \mathrm{Spec}\left(\bigoplus_{j\geq 0} I^j/I^{j+1}\right).$$

The *projectivized tangent star cone*, denoted by $P(X)$ (following Johnson's notation [**4**]), is defined by:

$$P(X) := \mathrm{Proj}\left(\bigoplus_{j\geq 0} I^j/I^{j+1}\right).$$

This is nothing but the exceptional divisor of the blow-up of $X \times X$ along the diagonal. It should be noted that $P(X)$ is of equidimension $= 2n - 1$. $P(X)$ has the canonical line bundle (i.e., the Serre line

bundle) $O_{P(X)}(1)$. Then *Johnson's ith Segre class* $S_i(X)$ of X is defined by:

$$S_i(X) := p_*(c_1(O_{P(X)}(1))^{n-1+i} \cap [P(X)]),$$

where $p: P(X) \to X$ is the projection. If X is non-singular, then $P(X)$ is isomorphic to the projectivization of the tangent bundle TX and it is known that $S_i(X)$ is the Poincaré dual of the usual ith Segre (cohomology) class $s_i(X) := s_i(TX)$, i.e., $S_i(X) = s_i(X) \cap [X]$. In fact, this is a special case of the following general fact:

Fact (e.g., see [3]). Let V be a vector bundle of rank r over a variety X. Let $\mathbb{P}(V)$ be the projectivization of the bundle V, $\pi: \mathbb{P}(V) \to X$ the projection and $O_{P(V)}(1)$ the canonical line bundle over $\mathbb{P}(V)$. Then we have

$$\pi_*(c_1(O_{\mathbb{P}(V)}(1))^{r-1+i} \cap [P(V)]) = s_i(V) \cap [X].$$

On the other hand, the *ith Segre-Mather class* $S_i^M(X)$ is defined as follows. Let $\nu: \hat{X} \to X$ be the Nash blow-up of X and let $\widehat{TX}$ be the (tautological) Nash tangent bundle over $\hat{X}$. Then

$$S_i^M(X) := \nu_*(s_i(\widehat{TX}) \cap [\hat{X}]),$$

where $s_i(\widehat{TX})$ is the ith Segre (cohomology) class of the bundle $\widehat{TX}$ over $\hat{X}$. $S_i^M(X)$ can, however, be described in a similar way to the definition of $S_i(X)$: Let $\mathbb{P}(\widehat{TX})$ be the projectivization of $\widehat{TX}$, $O_{\mathbb{P}(\widehat{TX})}(1)$ the canonical line bundle over $\mathbb{P}(\widehat{TX})$, $t: \mathbb{P}(\widehat{TX}) \to \hat{X}$ the projection map and $p_1 := \nu \cdot t$. Then by the above fact, we get

$$S_i^M(X) = p_{1*}(c_1(O_{\mathbb{P}(\widehat{TX})}(1))^{n-1+i} \cap [\mathbb{P}(\widehat{TX})]).$$

Since $\mathbb{P}(\widehat{TX})$ and $P(X)$ are isomorphic over the smooth part of X, one could expect that the difference between $S_i(X)$ and $S_i^M(X)$ is "supported" on the singular locus of X. (Note that they are classes rather than cycles.)

2. Projectivized tangent star cone $P(X)$. The definition of $P(X)$ is algebraically quite simple, but it is not so easy to capture it geometrically. Let X be irreducible of equidimension $= n$. Then $\mathbb{P}(\widehat{TX})$ is clearly irreducible and of equidimension $= 2n - 1$, but $P(X)$ is not necessarily irreducible and it may have many components lying over the singular locus of X.

EXAMPLE 2.1. Let X^1 be an irreducible singular plane curve. Then, set-theoretically $P(X)$ has as many extra components $\mathbb{P}^1$'s as the number of singularities and each extra component $\mathbb{P}^1$ projects onto only one singularity.

$P(X)$ can have many extra components projected onto one component of the singular locus.

EXAMPLE 2.2. Let $X^1 \subset P^3$ be the union of three lines $\mathbb{P}^1$'s intersecting at one point such that they do not lie in a plane. Then $P(X)$ has three extra components $\mathbb{P}^1$'s which are all projected to the singularity (= the intersection point of the three lines).

It is easy to make the following

Observation 2.3. Let X be an equidimensional projective variety and $X = X_1 \cup X_2 \cup \cdots \cup X_r$ be the decomposition of X into irreducible components. Then $\widehat{X} = \widehat{X}_1 \cup \widehat{X}_2 \cup \cdots \cup \widehat{X}_r$ is the (irreducible) decomposition of the Nash blow-up $\widehat{X}$ and the projectivization of the Nash tangent bundle $\mathbb{P}(\widehat{TX}) = \mathbb{P}(\widehat{TX}_1) \cup \cdots \cup P(\widehat{TX}_r)$ is the (irreducible) decomposition of $\mathbb{P}(\widehat{TX})$.

Now, as proved in **[9]**, we can get the following result, which is a key ingredient to relate Johnson's Segre class and the Segre-Mather class of X.

PROPOSITION 2.4 ([**9**, §3]). *Let X be a reduced projective equidimensional variety and $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. Then there exists a canonical morphism $q\colon \mathbb{P}(\widehat{TX}) \to P(X)$ such that* (i) $p_1 = p \cdot q$, (ii) *the image $q(\mathbb{P}(\widehat{TX}_i))$ is an irreducible component of $P(X)$ and* (iii) $q(\mathbb{P}(\widehat{TX})) = \overline{P(X_{\mathrm{sm}})}$, *where X_{sm} denotes the smooth part of X and $P(X_{\mathrm{sm}}) := P(X)|_{X_{\mathrm{sm}}} \cong \mathbb{P}(TX_{\mathrm{sm}})$*

Noticing that the multiplicity of each (typical) component $q(\mathbb{P}(\widehat{TX}_i))$ in $P(X)$ is equal to one, we can obtain the following

COROLLARY 2.5. *Let the situation be as before.*

$$[P(X)] = \sum_{i=1}^{r} q_*[\mathbb{P}(\widehat{TX}_i)] + \sum_j m_j \cdot e_j{}^*[C_j],$$

where C_j's are the other extra components of $P(X)$, with multiplicity $m_j := \mathrm{length}(\mathcal{O}_{P(X),C_j})$, lying above the singular locus of X and $e_j : C_j \hookrightarrow P(X)$ is the inclusion map.

Questions 2.6. (1) Is there any way, if possible, to identify irreducible components C_j's of $P(X)$ lying above the singular locus of X? (2) Is there any way (or algorithm), if possible, to compute or express the multiplicity $m_j = \text{lenght}(O_{P(X),C_j})$ in terms of some other well-developed invariants of singularities?

REMARK 2.7. For the hypersurface case $X^n \subset \mathbb{P}^{n+1}$, there is one and only one extra component C_j lying above each irreducible component Y_j^{n-1}, of exactly dimension $n-1$, of the singular locus Y of X, and $C_j = \mathbb{P}(T\mathbb{P}^{n+1}|_{Y_j^{n-1}})$ [**9**, §4]. By a result due to G. Kennedy [**6**, Theorem 3] we can show that its multiplicity m_j is equal to the multiplicity of the Jacobian ideal in the local ring of X at the generic point of Y_j^{n-1} (cf. [**8**]). For the case when $X^n \subset \mathbb{P}^{2n}$ with isolated singularities x_j's, there is at most one extra component C_j lying above each singularity x_j and if C_j appears above x_j then $C_j = \mathbb{P}(T\mathbb{P}^{2n}|_{x_j})$ and its multiplicity can be described as the shift multiplicity [**9**, §5]. In the following section we generalize these two results to a more general situation, except for the description of multiplicities, which seems to be hard to give.

3. A formula for Segre classes. Let the situation be as above. Let $\xi = c_1(O_{P(X)}(1))$, $\theta = c_1(O_{\mathbb{P}(\widehat{TX})}(1))$, $\theta_i = c_1(O_{\mathbb{P}(\widehat{TX}_i)}(1))$, $\xi_j = c_1(O_{C_j}(1))$ be the first Chern classes. Let $\pi_j\colon C_j \to X$ be the projection map, which is the restriction of $p\colon P(X) \to X$ to each C_j. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{P}(\widehat{TX}) & \xrightarrow{q} & P(X) & \xleftarrow{e_j} & C_j \\ \downarrow t & \overset{p_1}{\searrow} & \downarrow p & \overset{\pi_j}{\swarrow} & \\ \widehat{X} & \xrightarrow[\nu]{} & X & & \end{array}$$

Then we get the following formula, which was stated in [**9**] without proof. Here we give a proof for the sake of completeness.

PROPOSITION 3.1. *Let the situation and notation be as above.*

$$S_i(X) = S_i^M(X) + \sum_j m_j \cdot \pi_{j^*}(\xi_j^{n-1+i} \cap [C_j]).$$

Proof. By the definition of Segre classes and Corollary 2.5,

$$
\begin{aligned}
S_i(X) &= p_*(\xi^{n-1+i} \cap [P(X)]) \\
&= \sum_{s=1}^{r} p_*(\xi^{n-1+i} \cap q_*[\mathbb{P}(\widehat{TX}_s)]) + \sum_j m_j \cdot p_*(\xi^{n-1+i} \cap e_{j^*}[C_j]) \\
&\qquad\qquad \text{(by the projection formula)} \\
&= \sum_{s=1}^{r} p_* q_*(q^* \xi^{n-1+i} \cap [\mathbb{P}(\widehat{TX}_s)]) + \sum_j m_j \cdot p_* e_{j^*}(e_j^* \xi^{n-1+1} \cap [C_j]) \\
&\quad \text{(by the naturality of Chern classes and } q^*\xi = \theta \text{ and } e_j^*\xi = \xi_j) \\
&= \sum_{s=1}^{r} p_{1^*}(\theta_s^{n-1+i} \cap [\mathbb{P}(\widehat{TX}_s)]) + \sum_j m_j \cdot \pi_{j^*}(\xi_j^{n-1+i} \cap [C_j]) \\
&\qquad\qquad \text{(by the definition of Segre-Mather classes)} \\
&= \sum_{s=1}^{r} S_i^M(X_s) + \sum_j m_j \cdot \pi_{j^*}(\xi_j^{n-1+i} \cap [C_j]).
\end{aligned}
$$

Thus we get the equality in the proposition, because $S_i^M(X) = \sum_{s=1}^{r} S_i^M(X_s)$.

Now it remains to be more precise about the correction term

$$\sum_j m_j \cdot \pi_{j^*}(\xi_j^{n-1+i} \cap [C_j]),$$

which is relevant to Questions 2.6 in §2. Apart from being precise about the multiplicity m_j, we can at least claim the following, which is a generalization of our previous results [**9**, §§4 and 5].

THEOREM 3.2 (*see Remarks below*). *Let* X^n *be a reduced singular projective variety of equidimension* $= n$ *in* $\mathbb{P}^N$, *and* Y^k *denote the singular locus of* X, *which is of dimension* k. *Then*

(3.2.1) *if* $\operatorname{codim}(X^n, \mathbb{P}^N) < \operatorname{codim}(Y^k, X^n)$, *i.e.,* $N < 2n - k$, *then there is no extra component* C_j *lying above the singular locus of* X; *hence we have*

$$S_*(X) = S_*^M(X).$$

(3.2.2) *If* $\operatorname{codim}(X^n, \mathbb{P}^N) = \operatorname{codim}(Y^k, X^n)$, *i.e.,* $N = 2n-k$, *then there is at most one extra component* C_j *lying above each irreducible component* Y_j^k, *of exactly dimension* k, *of the singular locus* Y *of* X, *and if* C_j *appears over* Y_j^k, *then* $C_j = \mathbb{P}(T\mathbb{P}^N|_{Y_j^k})$. *And we have,*

for each i,

$$S_i(X) = S_i^M(X) + s_{i-n+k}(\mathbb{P}^N) \cap \left(\sum m_j \cdot [Y_j^k]\right),$$

where $s_{i-n+k}(\mathbb{P}^N)$ *is the usual Segre (cohomology) class of the ambient space* $\mathbb{P}^N$, $m_j = 0$ *(only) when* C_j *does not appear over* y_j^k *and* $m_j = \text{length}(O_{P(X),C_j})$ *when* C_j *appears over* y_j^k.

REMARK 3.3. It may be worthwhile to note a relevance to local embedding dimension, which was pointed out to me by the referee. Let m.l.e.dim. X denote the maximum of local embedding dimensions of X (in the sense of Zariski tangent space). Since the tangent star $\text{TS}(X)_x$ sits in the Zariski tangent space $\theta_x X$, (3.2.1) can be strengthened, without referring to the dimension N of the ambient space $\mathbb{P}^N$, as follows:

(3.2.1)′ if m.l.e.dim. $X < 2n-k$, then there is no extra component over the singular locus; hence $S_*(X) = S_*^M(X)$.

As to (3.2.2), we can say only the following (but not the Segre class formula):

(3.2.2)′ if m.l.e.dim. $X = 2n-k$, then there is at most one extra component C_j lying above each irreducible component Y_j^k, of exactly dimension k, of the singular locus Y of X.

Note that a key to obtaining the formula in (3.2.2) is the concrete description of C_j; i.e., $C_j = \mathbb{P}(T\mathbb{P}^N|_{Y^k_j})$. But in (3.2.2)′ one may not be able to get a concrete description of C_j.

REMARK 3.4. We emphasize that for the hypersurface case *all* $(n-1)$-dimensional components of the singular locus appear in the formula, whereas in the general setting (3.2.2) all k-dimensional components of the singular locus do not necessarily appear in the formula. Simple examples for this can be easily constructed.

Proof of Theorem 3.2. (3.2.1) is already observed in [**9**, Prop. 3.4], so we prove only (3.2.2), by "dimension count" argument. Since X is of equidimension n, $P(X)$ is of equidimension $2n-1$, as noted in §1. So each extra component C_j is of dimension $2n-1 = N+k-1$ and $\pi_j(C_j)$ is at most of dimension k. Thus the fiber of C_j over $y \in \pi_j(C_j) \subset Y^k$ must be of dimension $N-1$ and the dimension of $\pi_j(C_j)$ must be of k, which is the only possibility. Since $\pi_j(C_j)$ is irreducible because so is C_j, it follows that $\pi_j(C_j)$ must be an irreducible component of exactly dimension k of the singular locus

y^k of X. Since $P(X)$ is the projectivization of the tangent star cone $\mathrm{TS}(X)$, $\dim(\mathrm{TS}(X)_y) = N$, hence $\mathrm{TS}(X)_y = T_y\mathbb{P}^N$, the fiber of the tangent bundle $T\mathbb{P}^N$ at the point y. Therefore $C_j = \mathbb{P}(T\mathbb{P}^N|_{Y_j^k})$, where $Y_j^k = \pi_j(C_j)$. Therefore the formula in Proposition 3.1. becomes

$$S_i(X) = S_i^M(X) + \sum_j m_j \pi_{j^*}(\xi_j^{n-1+i} \cap [\mathbb{P}(T\mathbb{P}^N|_{Y_j^k})]).$$

By the dimension reason, it is clear that for $0 \leq i < n-k$,

$$(*) \qquad S_i(X) = S_i^M(X).$$

Observing that $n-1+i = N-(n-k)-1+i = N-1+(i-n+k)$ (using $N-n = n-k$), we have for $n-k \leq i \leq n$,

$$(**) \qquad S_i(X) = S_i^M(X) + \sum_j m_j \pi_{j^*}(\xi_j^{N-1+(i-n+k)} \cap [\mathbb{P}(T\mathbb{P}^N|_{Y_j^k})]).$$

Now by the general fact (in §1), we have

$$\begin{aligned}
\pi_{j^*}(\xi_j^{N-1+(i-n+k)} \cap [\mathbb{P}(T\mathbb{P}^N|_{Y_j^k})]) &= i_*(s_{i-n+k}(T\mathbb{P}^N|_{Y_j^k}) \cap [Y_j^k]), \\
&\qquad \text{where } i : Y_k^k \hookrightarrow X \text{ is the inclusion}, \\
&= i_*(s_{i-n+k}(i^*e^*T\mathbb{P}^N) \cap [Y_j^k]), \\
&\qquad \text{where } e : X \to \mathbb{P}^N \text{ is the inclusion}, \\
&\qquad \text{(by the naturality of Segre classes of vector bundles)} \\
&= i_*(i^*e^*s_{i-n+k}(T\mathbb{P}^N) \cap [Y_j^k]) \quad \text{(by the projection formula)} \\
&= e^*s_{i-n+k}(\mathbb{P}^N) \cap i_*[Y_j^k] \quad \text{(by abuse of notation)} \\
&= s_{i-n+k}(\mathbb{P}^N) \cap [Y_j^k].
\end{aligned}$$

Thus we get that for $n-k \leq i \leq n$,

$$S_i(X) = S_i^M(X) + s_{i-n+k}(\mathbb{P}^N) \cap \left(\sum_j m_j [Y_j^k] \right).$$

Since $s_{i-n+k}(\mathbb{P}^N) = 0$ for $i-n+k < 0$, combining $(*)$ and $(**)$, we get the formula in Statement (3.2.2). □

Problem 3.5. If X is a generic projection of a smooth n-fold in $\mathbb{P}^{2n+1}$ to $\mathbb{P}^{2n-k}$, then X satisfies the hypotheses of (3.2.2) of Theorem 3.2. Compute the multiplicities of the extra components (Even in this interesting and seemingly tractable situation computing multiplicities

seems to be hard. It may require some other geometric arguments, e.g., ramification, etc.)

REMARK 3.6. The case when $\operatorname{codim}(X^n, \mathbb{P}^N) > \operatorname{codim}(Y^k, X^n)$ (or, m.l.e.dim. $X > 2n - k$) is very subtle or complicated because it is possible that extra components lie over not only the top-dimensional singular locus but also lower-dimensional subvarieties of the singular locus. A simple example for this is the following: Let $X^2 \subset \mathbb{P}^4$ be the union of the three planes $\mathbb{P}_1^2$, $\mathbb{P}_2^2$, $\mathbb{P}_3^2$ defined by:

$$\mathbb{P}_1^2 := \{[z_0 : z_1 : 0 : 0 : z_4] \in \mathbb{P}^4\}, \quad \mathbb{P}_2^2 := \{[0 : z_1 : z_2 : 0 : z_4] \in \mathbb{P}^4\},$$

and

$$\mathbb{P}_3^2 := \{[0 : 0 : z_2 : z_3 : z_4] \in \mathbb{P}^4\}.$$

Then

$$\begin{aligned}
&\mathbb{P}_1^2 \cup \mathbb{P}_2^2 \subset \mathbb{P}_{12}^3 := \{[z_0 : z_1 : z_2 : 0 : z_4] \in \mathbb{P}^4\} \quad \text{and} \\
&\mathbb{P}_1^2 \cap \mathbb{P}_2^2 = \text{line } L_{12} := \{[0 : z_1 : 0 : 0 : z_4] \in \mathbb{P}^4\}. \\
&\mathbb{P}_2^2 \cup \mathbb{P}_3^2 \subset \mathbb{P}_{23}^3 := \{[0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^4\} \quad \text{and} \\
&\mathbb{P}_2^2 \cap \mathbb{P}_3^2 = \text{line } L_{23} = \{[0 : 0 : z_2 : 0 : z_4] \in \mathbb{P}^4\}.
\end{aligned}$$

$\mathbb{P}_1^2$ and $\mathbb{P}_3^2$ are transverse to each other, and

$$\mathbb{P}_1^2 \cap \mathbb{P}_2^2 \cap \mathbb{P}_3^2 = \{[0 : 0 : 0 : 0 : z_4]\} =: \{x_0\}.$$

The singular locus of our variety X^2 is $L_{12} \cup L_{23}$. Then there are three extra components C_0, C_{12}, C_{23} in $P(X)$. C_0 is supported on the single point $\{x_0\}$ because $\mathbb{P}_1^2$ and $\mathbb{P}_3^2$ are transverse to each other and so the tangent star $\mathrm{TS}(X)_{x_0}$ is nothing but the whole $T_{x_0}\mathbb{P}^4$. C_{12} and C_{23} are supported on the lines L_{12} and L_{23}, respectively, because $\mathbb{P}_1^2 \cup \mathbb{P}_2^2$ and $\mathbb{P}_2^2 \cup \mathbb{P}_3^2$ are already in the three dimensional projective spaces $\mathbb{P}_{12}^3$ and $\mathbb{P}_{23}^3$, respectively, and so we have $C_{12} = \mathbb{P}(T\mathbb{P}_{12}^3|_{L_{12}})$ and $C_{23} = \mathbb{P}(T\mathbb{P}_{23}^3|_{L_{23}})$. Let m_0, m_1, m_2 be the multiplicity of C_0, C_{12}, C_{23}, respectively, in $P(X)$. Then we have

$$\begin{cases}
S_0(X) = S_0^M(X), \\
S_1(X) = S_1^M(X) + m_1[L_{12}] + m_2[L_{23}], \\
S_2(X) = X_2^M(X) = m_1 s_1(\mathbb{P}_{12}^3) \cap [L_{12}] + m_2 s_1(\mathbb{P}_{23}^3) \cap [L_{23}] + m_0 \cdot [x_0].
\end{cases}$$

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