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In this paper we determine the rational formal groups defined over a field of characteristic zero. This answers a question originally posed by Robert MacPherson.

While one can answer this question using Weil's theorem which asserts that every birational group is birationally isomorphic to an actual algebraic group **[W1]**, below we give an elementary argument using methods similar to those used in **[C]**.

THEOREM. *Every rational formal group law over an algebraically closed field K of characteristic zero is of the form*

$$L^{-1}G(L(x), L(y))$$

where $G(x, y)$ is either $x+y$ or $x+y+xy$ and L is a linear functional transformation over K such that $L(0)=0$.

One deduces easily from this that

COROLLARY. *The rational formal group laws over a field K of characteristic zero are the rational functions*

$$(x+y+cxy)/(1-dxy)$$

where c and d are elements of K. Moreover, this formal group is rationally isomorphic to $x+y$ over K if $c^2-4d=0$ and to $x+y+xy$ over $K(\sqrt{(c^2-4d)})$ otherwise.

Proof of theorem. Recall that now K is algebraically closed. Suppose $F(x, y)$ is a rational formal group.

Let $\omega = dx/F_2(x, 0)$ and $g(x) = F(x, x)$ (the rational function giving multiplication by 2 on F). Then ω and g satisfy the hypothesis of the following proposition:

PROPOSITION. *Suppose $\omega \in K(x)\,dx$ and $g \in K(x)$, $\omega \neq 0$, $\mathrm{ord}_0\,\omega = 0$, $g(0)=0$ and $g^*\omega = 2\omega$. Then $\omega = L^*(dx)$ or $L^*(c\,dx/(x+1))$*

where L is a linear fractional transformation defined over K such that $L(0) = 0$ and $c \in K^*$.

Proof. Let Y denote the set of poles and Z the set of zeros of ω. It follows from the hypothesis that $g^{-1}Y = Y$ and $g^{-1}Z = Z$.

The equation $g^*\omega = 2\omega$ implies that

$$\sum \operatorname{ord}_Q g^*\omega = \sum \operatorname{ord}_Q \omega$$

where the sums run over $Q \in Y = g^{-1}Y$. Suppose $Q \in \mathbb{P}^1(K)$. Then we also have the formula

$$\sum \operatorname{ord}_P g^*\omega = \deg(g) \cdot \operatorname{ord}_Q \omega + (\deg(g) - \#g^{-1}(Q))$$

where the sum runs over $P \in g^{-1}(Q)$. Suppose now Q is a pole of ω. The right-hand side of this formula is less than or equal to $\operatorname{ord}_Q \omega$. Hence the last two formulas imply that

$$\deg(g) \cdot \operatorname{ord}_Q \omega + (\deg(g) - \#g^{-1}(Q)) = \operatorname{ord}_Q \omega$$

for alll $Q \in Y$. This occurs for a given $Q \in Y$ iff $\deg(g) = 1$ or $\operatorname{ord}_Q \omega = -1$ (in which case $\#g^{-1}(Q) = 1$).

Suppose first that $\deg(g) = 1$ and ω has a pole of order greater than one. Since $g^*\omega = 2\omega$, no iterate of g is the identity. As $g(0) = 0$ it follows that there exists exactly one non-zero point fixed by some iterate of g. Since $g^{-1}Y = Y$, $g^{-1}Z = Z$ and $\operatorname{ord}_0 \omega = 0$, we see that ω has only one pole and no zeros. It follows that $\omega = L^*(dx)$ for some linear fractional transformation L which we may assume vanishes at the origin.

Suppose now that ω has only simple poles. If Q is a pole of ω we know that $g^{-1}(Q)$ consists of exactly one point, P say, and we have the formula

$$\operatorname{Res}_P g^*\omega = \deg(g) \operatorname{Res}_Q \omega$$

by a local computation. Since $g^*(\omega) = 2\omega$, this becomes

$$\operatorname{Res}_P \omega = (\deg(g)/2) \operatorname{Res}_Q \omega.$$

Now we know that $g^{-1}Y = Y$. Hence, there exists a Q in Y and a positive integer n such that $\{Q\} = g^{-n}(Q)$. By iterating the previous equation we deduce that

$$(\deg(g)/2)^n \operatorname{Res}_Q \omega = \operatorname{Res}_Q \omega.$$

Hence, as $\operatorname{Res}_Q \omega \neq 0$ and $\deg(g) \in \mathbb{Z}_{>0}$, $\deg(g) = 2$.

The facts that $g^{-1}Y = Y$ and $g^{-1}Z = Z$ imply that the zeros and poles of ω lie among the branch points of $g\colon \mathbb{P}^1 \to \mathbb{P}^1$. Since g has degree 2 it has only two branch points. Since ω is not equal to zero, has only simple poles and its residues sum to zero it must have exactly two poles and no zeros. Hence $\omega = L^*(c\,dx/(x+1))$ for some linear fractional transformation L and some constant $c \in K^*$. Since $\mathrm{ord}_0\,\omega = 0$, we may assume $L(0) = 0$. This proves the proposition. □

The theorem follows from the proposition noting that $F(x, y) = L^{-1}G(L(x), L(y))$ where $G(x, y) = x + y$ if $\omega = L^*(dx)$ and $G(x, y) = x + y + xy$ if $\omega = L^*(c\,dx/(x+1))$. □

REMARK. The only place in the above argument where the algebraic closedness of K was used in a serious manner was in the last step which required finding a linear fractional transformation which moved one pole of ω to 0 and the other to ∞.

REFERENCES

[C] R. Coleman, *One dimensional algebraic formal groups*, Pacific J. Math., **122** (1986), 35–41.

[W] A. Weil, *On algebraic groups of transformations*, Amer. J. Math., **77** (1955), 355–391.

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