Pacific Journal of Mathematics

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Vol. 147, No. 1 January 1991

PACIFIC JOURNAL OF MATHEMATICS
Vol. 147, No. 1, 1991

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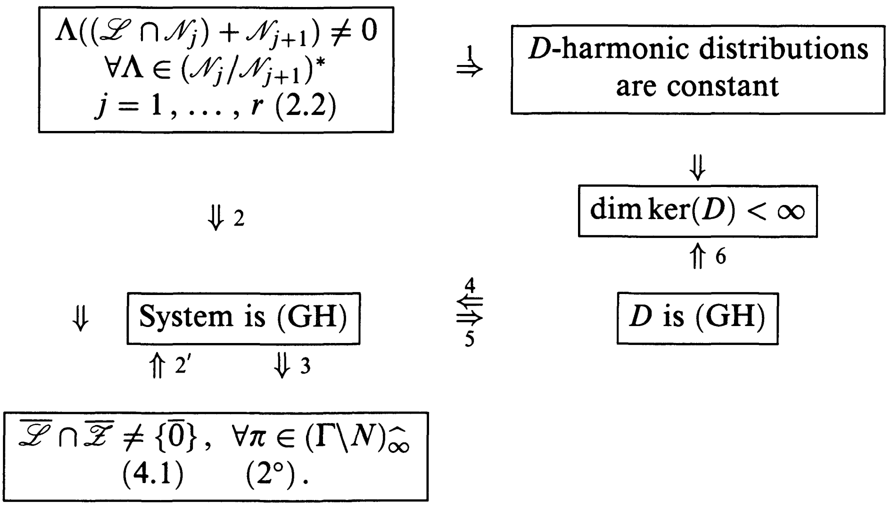
Let $M = \Gamma\backslash N$ be a compact nilmanifold. A system of differential operators $D_1, \dots, D_k$ on M is *globally hypoelliptic* (GH) if when $D_1 f = g_1, \dots, D_k f = g_k$ with $f \in \mathscr{D}'(M)$, $g_1, \dots, g_k \in C^\infty(M)$ then $f \in C^\infty(M)$. Let $X_1, \dots, X_k$ be real vector fields on M induced by the Lie algebra $\mathscr{N}$ of N. We study the relationships between (GH) of the system $X_1, \dots, X_k$ on M, (GH) of the operator $D = X_1^2 + \cdots + X_k^2$, the constancy of D-harmonic distributions on M, and related algebraic conditions on $X_1, \dots, X_k \in \mathscr{N}$.

0. Introduction. Let $M = \Gamma\backslash N$ be a compact nilmanifold, where N is a connected, simply connected real nilpotent Lie group with a discrete subgroup Γ. There is a unique probability measure μ defined on the Borel sets on M and invariant under the action of N on M by right translations. Every μ-integrable function f on M defines a distribution by the formula $(f, \phi) = \int_M f\phi\, d\mu$, $\phi \in C^\infty(M)$. Let $\mathscr{N}$ be the Lie algebra of N. If $X \in \mathscr{N}$ then X induces a vector field (which we will denote also by X) on $\Gamma\backslash N$ by $(Xf)(\Gamma n) = (d/dt)|_{t=0} f(\Gamma n \exp tX)$. Consider the left-invariant sum of squares of such vector fields $X_1, \dots, X_k \in \mathscr{N}$. This second order differential operator $D = X_1^2 + \cdots + X_k^2$ can be regarded as acting on the right on distributions on $\Gamma\backslash N$. A distribution $u \in \mathscr{D}'(M)$ is *D-harmonic* if $Du = 0$ on M. The operator D is *globally hypoelliptic* (GH) if when $Df = g$ with $f \in \mathscr{D}'(M)$, $g \in C^\infty(M)$, then $f \in C^\infty(M)$. The *system* of vector fields $X_1, \dots, X_k$ on M is (GH) if when $X_1 f = g_1, \dots, X_k f = g_k$ with $f \in \mathscr{D}'(M)$, $g_1, \dots, g_k \in C^\infty(M)$, then $f \in C^\infty(M)$. In this paper we investigate relationships between (GH) of D, (GH) of the corresponding system $X_1, \dots, X_k$ of vector fields, the constancy of D-harmonic distributions on M, and related algebraic conditions on $X_1, \dots, X_k \in \mathscr{N}$.

Our results are summarized in the figure below. In this figure, functionals $\Lambda \in \mathscr{N}_j^*$ are assumed to be *integral*, i.e. $\Lambda(\log\Gamma \cap \mathscr{N}_j) \subseteq \mathbb{Z}$; $\mathscr{N} = \mathscr{N}_1 \supset \mathscr{N}_2 \supset \cdots \supset \mathscr{N}_r \supset \mathscr{N}_{r+1} = \{0\}$ is the *lower central series* of $\mathscr{N}$ (we say $\mathscr{N}$ is of *step r*), and $\mathscr{L}$ is the subalgebra of $\mathscr{N}$ Lie-generated by $X_1, \dots, X_k$. Let $\mathscr{W}_\pi$ be an ideal in $\ker(d\pi)$ such that

$\mathcal{N}/\mathcal{W}_\pi$ has one dimensional center on which π is non-trivial. Then $\overline{\mathcal{L}} := \mathcal{L} + \mathcal{W}_\pi$ and $\overline{\mathcal{Z}} := \mathcal{Z} + \mathcal{W}_\pi$.



We explain below the labeled implications in the above figure referring the reader to indicated sections of the paper for details.

1. This is Theorem (2.1). Condition (2.2) with $j = 1$ provides constancy of the D-harmonic distributions on the associated torus.

2. This holds with the necessary assumption that the system $X_1, \ldots, X_k$ is (GH) on the associated torus (proved in **[C-R2]**, Theorem 1).

2′. This requires the assumption that the system $X_1, \ldots, X_k$ is (GH) on the associated torus (implicitly contained in **[C-R2]** and discussed here in §4).

3. This is proved in §4 for N with exclusively flat coadjoint orbits (which includes step 2 groups), and also for any nilpotent semidirect product $\mathbb{R} \ltimes \mathbb{R}^n$.

4. This is always true. (If $X_1 f, \ldots, X_k f$ are C^∞, then so is $Df = (X_1^2 + \cdots + X_k^2)f$ and by (GH) of D, $f \in C^\infty$.)

5. We prove this converse to implication 4 for N of step 2, if $D = X_1^2 + X_2^2$ with $X_1 \in \mathcal{N}$, $X_2 \in \mathcal{N}_2$ and with a necessary growth condition on X_2 in §1. A growth condition on X_1 follows from (GH) of the system X_1, X_2. Implication 5 is false for solvmanifolds, even if all the vector fields $X_1, \ldots, X_k$ are algebraic, and hence satisfy all growth conditions. Indeed, the example in §3 shows such a D with a non-L^2 distribution in its kernel.

6. See e.g. **[G-W3]**, Lemma 3, p. 161.

We would like to thank L. Baggett, A. Hulanicki, J. Jenkins, and P. Ohring for many helpful discussions during the preparation of this paper.

1. 2-step nilmanifolds. In this section we show that global hypoellipticity of the system $X_1, \dots, X_k$ is insufficient for (GH) of D, even if N is step 2, (Example (1.4)). Growth conditions on all the vector fields are needed. Under such conditions D can be proven to be (GH), at least on step 2 nilmanifolds (Theorem (1.1)). We'll see in §3 that this cannot happen in general solvmanifolds.

(1.1) THEOREM. *Let $\mathscr{N}$ be a step 2 rational nilpotent Lie algebra, N the corresponding connected, simply connected group, and Γ a co-compact discrete subgroup of N. Let $Y_1, \dots, Y_n$; $Z_1, \dots, Z_k$ be a linear basis for $\mathscr{N}$ selected from $\log\Gamma$ and such that $Y_l + [\mathscr{N}, \mathscr{N}]$, $l = 1, \dots, n$ is a basis of $\mathscr{N}/[\mathscr{N}, \mathscr{N}]$, and Z_p, $p = 1, \dots, k$ is a basis of $[\mathscr{N}, \mathscr{N}]$. Then the operator*

$$D = X_1^2 + X_2^2,$$

where $X_1 = \alpha_1 Y_1 + \cdots + \alpha_n Y_n$, $X_2 = \beta_1 Z_1 + \cdots + \beta_k Z_k$, is (GH) on the compact nilmanifold $\Gamma\backslash N$, provided both $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_k$ satisfy the following growth condition (which we state for the α's only):

$$(1.2) \qquad |\alpha_1 k_1 + \cdots + \alpha_n k_n| \geq C(k_1^2 + \cdots + k_n^2)^{-p},$$

for some p, $C > 0$ and all integers $k_1, \dots, k_n$ not all zero.

Proof. Let $Du = g \in C^\infty(\Gamma\backslash N)$ with $u \in \mathscr{D}'(\Gamma\backslash N)$. We use an irreducible (non-canonical) Fourier series decomposition of u, $u = u_0 + \sum_\pi \sum_{q=1}^{m(\pi)} u_{\pi,q}$, where $u_0 \in \mathscr{D}'(\Gamma[N, N]\backslash N)$. Thus u_0 lives on the associated torus. The sum is over all ∞-dimensional representations $\pi \in (\Gamma\backslash N)^\wedge$ (with multiplicities $m(\pi)$). Also, g has a Fourier series decomposition with $g_0 \in C^\infty(\Gamma[N, N]\backslash N)$. Condition (1.2) assures that the operator $\overline{D} = (\alpha_1 \partial/\partial x_1 + \cdots + \alpha_n \partial/\partial x_n)^2$ on the associated torus is (GH) by the Theorem in **[G-W1]**. We conclude that u_0 is in fact smooth. The proof that the sum over ∞-dimensional π is smooth is a modification of the proof of global regularity of a real vector field on a compact nilmanifold (Theorem 1, page 351 of **[C-R3]**). For each fixed ∞-dimensional π we construct a suitable Schrödinger model. Since π is ∞-dimensional, there exists i such that $[Y_i, X_1] \notin \ker(d\pi)$. Let $\mathscr{W}_\pi$ be an ideal in $\ker(d\pi)$ such that $\mathscr{N}/\mathscr{W}_\pi$ has 1-dimensional center. Passage to this quotient does not affect $d\pi(D)$. Introducing a Kirillov subalgebra generated by the images of $X := Y_i$ and $Y := X_1$ in $\mathscr{N}/\mathscr{W}_\pi$, we obtain a Schrödinger model for π. In that model $d\pi(D) = -\lambda^2\xi_1^2 - \Lambda(X_2)^2$, where $\Lambda \in \mathscr{N}^*$ corresponds via Kirillov theory **[K]** to π. Moreover, $\Lambda([\mathscr{N}, \mathscr{N}] \cap \log\Gamma) \subseteq \mathbb{Z}$ and $\lambda = \Lambda([Y_1, X_1])$. We use the formula (1.8) on page 353 of **[C-R1]** to write for any $U \in \mathscr{U}(\mathscr{N})$, the universal enveloping algebra of $\mathscr{N}$:

$$(1.3)\quad (Uf)_\pi = \pi\{[D[D\dots[D, U]\dots]]g + D[D\dots[D, U]\dots]g + \cdots + D^{m-2}[D, U]g + D^{m-1}Ug\}P_\pi^{-m} \equiv h_m P_\pi^{-m}.$$

Here $P_\pi(\xi_1, \dots, \xi_k) = -\lambda^2\xi_1^2 - \Lambda(X_2)^2$ and (instead of (1.9) on page 353 of **[C-R1]**) we use the estimate

$$|h_m P_\pi^{-m}| \le |h_m||\Lambda(X_2)|^{-2m} \le C^{-2}|\Lambda(Z_1)^2 + \cdots + \Lambda(Z_k)^2|^p |h_m| \equiv |\pi(V)h_m|$$

for some $V \in \mathscr{U}(\mathscr{N})$.

The second inequality is where we need the assumption (1.2) about the coefficients $\beta_1, \dots, \beta_k$ of X_2. Also, (1.3) works only if $(\operatorname{ad} D)^m U = 0$ for some m (depending of course on U). Since $D = X_1^2 + X_2^2$ with X_2 central in $\mathscr{N}$, this is the same as $(\operatorname{ad}(X_1^2))^m U = 0$. The latter condition is true for any nilpotent Lie algebra $\mathscr{N}$, any $X_1 \in \mathscr{N}$ and $U \in \mathscr{U}(\mathscr{N})$. To see this, wlog we assume that $U = U_1U_2\cdots U_p$ with $U_i \in \mathscr{N}$, $i = 1, \dots, p$. Note that $\operatorname{ad}(X_1^2)$ is a derivation of the associative algebra $\mathscr{U}(\mathscr{N})$. By Leibnitz's rule $\operatorname{ad}(X_1^2)^m U =$ *a linear combination of the terms of the form of* $\operatorname{ad}(X_1^2)^{l_1}U_1 \cdots \operatorname{ad}(X_1^2)^{l_p}U_p$, where $l_1 + \cdots + l_p = m$. Thus it suffices to show that there exists a number l such that $\operatorname{ad}(X_1^2)^l$ maps $\mathscr{N}$ into 0. This last statement is contained in Lemma 5.1 on page 230 of **[G]**.

(1.4) EXAMPLE. Let N be a direct product of the 3 dimensional Heisenberg group and $\mathbb{R}$. Let X, Y, Z, and Z_1 with $[X, Y] = Z$ be a rational basis of $\mathscr{N}$. Consider $D = (X + \alpha Y)^2 + (Z + \beta Z_1)^2$ with α irrational non-Liouville and β a Liouville number. As in the proof of Theorem (1.1), for $\pi \in (\Gamma\backslash N)^\wedge$, pick a Schrödinger model with Kirillov subalgebra generated by Y and $X + \alpha Y$. In that model $d\pi(D) = -\lambda^2\xi^2 - (\lambda + \beta\lambda_1^2)$ with $\lambda = \Lambda(Z)$, $\lambda_1 = \Lambda(Z_1)$, where $\Lambda \in \mathscr{N}^*$ corresponds to π. Computations similar to those of Example 1 on page 355 of **[C-R3]** show that D cannot be (GH) on $\Gamma\backslash N$.

2. *D*-harmonic distributions on nilmanifolds. The following Theorem (2.1) does not require any growth assumptions on $X_1, \dots, X_k$. (Whether D in the Theorem is (GH), even with $X_1, \dots, X_k$ and their commutators satisfying (1.2), is still an open problem. This problem is still open even if $X_1, \dots, X_k$ are algebraic.) Consider the case $k = 1$, with $\Gamma\backslash N$ being the torus, say two dimensional, and $D = (\alpha_1 Y_1 + \alpha_2 Y_2)^2 = X_1^2$. Then Theorem (1.1) corresponds to the statement that D is (GH) provided α_2/α_1 is an irrational non-Liouville number. On the other hand, the 2-torus version of Theorem (2.1) says that $\ker D = \mathbb{C} \cdot 1$ provided α_2/α_1 is irrational.

(2.1) THEOREM. *Let $\mathscr{N}$ be a rational nilpotent Lie algebra of step r, N the corresponding connected, simply connected group, and Γ a cocompact discrete subgroup of N. Let $X_1, \dots, X_k$ generate a Lie subalgebra $\mathscr{L}$ of $\mathscr{N}$. Suppose that $\mathscr{L}$ has the property*

(2.2) *For each non-zero integral linear functional $\Lambda \in (\mathscr{N}_j/\mathscr{N}_{j+1})^*$, $\Lambda((\mathscr{L} \cap \mathscr{N}_j) + \mathscr{N}_{j+1}) \neq 0$, $j = 1, \dots, r$.*
[$\Lambda \in \mathscr{N}_j^*$ is called integral if $\Lambda(\log \Gamma \cap \mathscr{N}_j) \subset \mathbb{Z}$.]

If $u \in \mathscr{D}'(\Gamma\backslash N)$ and $(X_1^2 + \cdots + X_k^2)u = 0$, then u can be identified with a constant function.

REMARK 1. $X_1, \dots, X_k$ satisfying condition (2.2) in general do not generate the whole tangent space of $\Gamma\backslash N$. Consequently, D-harmonic distributions a priori need not even be continuous functions. Therefore, compactness of $\Gamma\backslash N$ alone cannot guarantee such 'harmonic' distributions to be constants.

REMARK 2. If $r = 1$, then Theorem (2.2) is about a torus. (Recall that $\mathscr{N} = \mathscr{N}_1$ and $\mathscr{N}_2$ is the commutator of $\mathscr{N}$.)

We start the proof of Theorem (2.1) with the following proposition.

(2.3) PROPOSITION. *The condition* (2.2) *above and the following condition* (2.4) *are equivalent for every compact nilmanifold $\Gamma\backslash N$.*

(2.4) *For each $\pi \in (\Gamma\backslash N)^\wedge \sim \{1\}$, if $1 \leq j \leq r$ is such that $\pi(N_{j+1}) \equiv I$, but $\pi(N_j) \not\equiv I$, then $d\pi(\mathscr{L} \cap \mathscr{N}_j) \neq 0$, where $N_j = \exp \mathscr{N}_j$, and $(\Gamma\backslash N)^\wedge$ denotes the irreducible unitary representations of N contained in the quasi-regular representation of N on $L^2(\Gamma\backslash N)$.*

Proof (2.2) $\Leftrightarrow$ (2.4). Each $\pi \in (\Gamma\backslash N)^\wedge$ corresponds to some $\widetilde{\Lambda} \in \mathscr{N}^*$ integral on a rational maximal subordinate subalgebra $\mathscr{M}$ of $\mathscr{N}$

([H],[R1]). In particular, for j as in (2.4) $\mathscr{N}_j \subseteq \mathscr{M}$, and

$$(2.5) \qquad d\pi|_{\mathscr{N}_j} = i\widetilde{\Lambda}|_{\mathscr{N}_j} = i\Lambda \quad \text{with } \Lambda \in (\mathscr{N}_j/\mathscr{N}_{j+1})^* \text{ integral.}$$

Conversely, any integral $\Lambda \in (\mathscr{N}_j/\mathscr{N}_{j+1})^*$ can be extended by 0 on a rational basis of $\mathscr{N}$ to an integral $\widetilde{\Lambda} \in \mathscr{N}^*$. $\pi \in \widehat{N}$ corresponding via Kirillov theory to $\widetilde{\Lambda}$ is in the spectrum of $\Gamma\backslash N$ ([M]). Equation (2.5) holds as before. Thus (2.2) and (2.4) are equivalent.

In view of the Proposition (2.3), all we need to prove Theorem (2.1) is the following:

(2.6) LEMMA. *Let $X_1, \dots, X_k$ generate a Lie subalgebra $\mathscr{L}$ of a nilpotent Lie algebra $\mathscr{N}$. Let $\pi \in \widehat{N}$ be such that $d\pi(\mathscr{L} \cap \mathscr{N}_r) \neq 0$. Then for every $u_\pi \in (H_\pi^\infty)'$, $d\pi(X_1^2 + \cdots + X_k^2)u_\pi = 0$ implies $u_\pi = 0$. Here $N = \exp \mathscr{N}$ and $\mathscr{N}_r$ is the lowest non-zero term of the lower central series of $\mathscr{N}$.*

Proof of Theorem (2.1). We write an irreducible Fourier series expansion

$$u = \sum_{\pi \in (\Gamma\backslash N)^\wedge} \sum_{q=1}^{m_\pi} u_{\pi,q} = u_0 + \sum_{j=1}^{r} \sum_{\pi \in \Pi_j} \sum_{q=1}^{m_\pi} u_{\pi,q},$$

where

$$\Pi_j = \{\pi \in (\Gamma\backslash N)^\wedge : \pi(N_{j+1}) \equiv I,\ \pi(N_j) \not\equiv I\}, \qquad j = 1, \dots, r.$$

Note that Π_1 consists of all 1-dimensional non-trivial representations in $(\Gamma\backslash N)^\wedge$. We apply Lemma (2.6) to $u_{\pi,q}$ with $\pi \in \Pi_r$, then again apply Lemma (2.6) to $\mathscr{N}/\mathscr{N}_r$ which takes care of $u_{\pi,q}$ with $\pi \in \Pi_{r-1}$ in the above sum, etc. We are left with u_0 which corresponds to trivial π, i.e. $u = u_0$ is a constant function on M.

The proof of Lemma (2.6) will follow from Lemma (2.7) below, but first we need some definitions (cf. [F-S]).

A Lie algebra $\mathscr{L}$ is called *graded* if it has a direct sum decomposition $\mathscr{L} = \sum_{j=1}^r \bigoplus V^j$ with the property that $[V^j, V^k] \subset V^{k+j}$ if $k + j \leq r$ and $[V^k, V^j] = 0$ if $k + j > r$. A graded algebra is always nilpotent. A connected simply connected nilpotent Lie group L is called graded if its Lie algebra $\mathscr{L}$ is graded.

Any graded (nilpotent) Lie algebra $\mathscr{L}$ has a natural family of *dilations* $\{\alpha_\lambda\}_{\lambda>0}$ (one parameter group of automorphisms of $\mathscr{L}$) defined on each V^j by $\alpha_j(Y) = \lambda^j Y$, $Y \in V^j$, $\lambda > 0$. By the exponential map α_λ corresponds to a one-parameter group of automorphisms of L, the simply connected nilpotent Lie group corresponding to $\mathscr{L}$.

A linear differential operator P on a graded group L is *homogeneous* of degree d if $P(f \circ \alpha_\lambda) = \lambda^d (Pf) \circ \alpha_\lambda$ for any $f \in C^\infty(L)$.

We call a differential operator P on a graded group L a *Rockland operator* if (i) P is left-invariant and homogeneous, and (ii) $d\pi(P)$ is injective on H_π^∞ for every $\pi \in \widehat{L}$ except the trivial representation. By a theorem of Helffer and Nourrigat **[H-N]**, a Rockland operator (on a graded group L) is *hypoelliptic*: i.e. if u is a distribution on L such that Pu is C^∞ on an open $\Omega \subset L$, then u is C^∞ on Ω.

(2.7) LEMMA. *Let $\mathscr{L}$ be a graded Lie subalgebra of a nilpotent Lie algebra $\mathscr{N}$, and let $P \in \mathscr{U}(\mathscr{L})$, the universal enveloping algebra of $\mathscr{L}$, be a Rockland operator on the graded group L corresponding to $\mathscr{L}$. If $\pi \in \widehat{N}$ is such that $d\pi(\mathscr{L} \cap \mathscr{N}_r) \neq 0$, then $d\pi(P)u_\pi = 0$ for $u_\pi \in (H_\pi^\infty)'$ implies $u_\pi = 0$. ($\mathscr{N}_r$ is the lowest non-zero term of the lower central series of $\mathscr{N}$.)*

Proof of Lemma (2.7). Suppose $d\pi(P)u = 0$ for some $0 \neq u \in (H_\pi^\infty)'$. We are going to show then there is a non-smooth function $\tilde{u}$ on L such that $P\tilde{u} = 0$. That would contradict the hypoellipticity of P on L. We adapt the proof of Lemma (4.6) of Rothschild and Stein **[R-S]** to our situation. Let $\psi \in H_\pi^\infty$ be such that $(u, \psi) \neq 0$, and let $\{\alpha_\lambda\}_{\lambda > 0}$ be the one-parameter group of dilations of L. For each dilation $l \to \alpha_\lambda(l)$ $(\lambda \in \mathbb{R}^+)$ of L define the representation π_λ of L by $\pi_\lambda(l) := \pi(\alpha_\lambda(l))$. Observe that if $\pi(P)u = 0$, it follows from the homogeneity of P that $\pi_\lambda(P)u = 0$ too. Let

$$\tilde{u}(l) = \int_1^\infty (\pi_\lambda(l)u, \psi)\lambda^Q \, d\lambda, \qquad l \in L, \tag{2.8}$$

where the exponent Q is to be specified later. First we check that the integral in (2.8) converges for each fixed $l \in L$. Since $u \in (H_\pi^\infty)'$, we have

$$|(\pi_\lambda(l)u, \psi)| = |(u, \pi(\alpha_\lambda(l^{-1}))\psi)| \leq C|||\pi(\alpha_\lambda(l^{-1}))\psi|||. \tag{2.9}$$

By **[K]**, H_π can be identified with $L^2(\mathbb{R}^P)$, H_π^∞ with $\mathscr{S}(\mathbb{R}^P)$, the Schwartz space of rapidly decreasing functions, and we can think of $|||\cdot|||$ in (2.9) as being a combination of $\mathscr{S}(\mathbb{R}^P)$ seminorms of the form $|||\phi||| = \|x^\beta D_\alpha \phi(x)\|$, where $\|\ \|$ is the $L^2(\mathbb{R}^P)$ norm. In our case

$$\phi = \pi(\alpha_\lambda(l^{-1}))\psi = \pi\left(\exp\left(\sum_j \lambda^j Y_j\right)\right)\psi, \tag{2.10}$$

where

$$\log l = Y = \sum Y_j \in \bigoplus_{j=1}^{r} V^j = \mathscr{L}. \tag{2.11}$$

The representation $\pi \in \widehat{N}$ acting on $\phi \in L^2(\mathbb{R}^P)$ can be written (cf. **[H-N]**, page 904)

$$\pi(\exp Y)\phi(x) = \exp(i\langle\Lambda, v(Y, x)\rangle)\phi(\sigma(x, Y)), \tag{2.12}$$

where $\sigma \in \mathbb{R}^P$ $(= \mathscr{M}\backslash\mathscr{N})$ and $v \in \mathscr{N}$ are polynomials in $x \in \mathbb{R}^P$ and $Y \in \mathscr{L}$, and $\mathscr{M} \subset \mathscr{N}$ is a maximal subordinate subalgebra for $\Lambda \in \mathscr{N}^*$. Combining (2.10) and (2.12) we see that $|||\phi|||$ can be estimated by a combination of expressions of the form

$$\|v_1(\cdot, Y)(D_\alpha\psi)(\sigma(\cdot, Y))\|, \qquad Y = \log l \tag{2.13}$$

with σ as in (2.12) and some polynomial v_1 of $x \in \mathbb{R}^P$ and $Y \in \mathscr{L}$, the norm $\|\ \|$ being the $L^2(\mathbb{R}^P)$ norm with respect to the $x \in \mathbb{R}^P$ variable marked by a dot. Since π is unitary, we can rewrite (2.13) as

$$\|\pi(\exp(-Y))\{v_1(\cdot, Y)(D_\alpha\psi)(\sigma(\cdot, Y))\}\| \tag{2.13a}$$

which can be written

$$\|v_2(\cdot, Y)(D_\alpha\psi)(\cdot)\| \tag{2.13b}$$

with some polynomial v_2. This is because $\sigma(x, Y)$ (see (2.12)) comes from the multiplication of $\exp X$ (on the right) by $\exp Y$, and subsequent multiplication of $\exp\sigma(x, Y)$ by $\exp(-Y)$ makes it x again. Suppose now that Y depends on λ via $Y = Y_\lambda = \sum \lambda^j Y_j$ (cf. (2.11)). It follows from (2.13b) that

(2.14) $|||\phi||| \le$ *a polynomial in* λ *of some degree* Q_1 *with coefficients of the form* $\|v_3(\cdot; Y_1, \ldots, Y_r)(D_\alpha\psi)(\cdot)\|$, ($v_3$ *being a polynomial in* $x \in \mathbb{R}^P$ *and* $Y_1, \ldots, Y_r$) *and with* Q_1 *independent of* Y *or* x.

Thus $\tilde{u}(l)$ is well-defined and continuous for any $Q \ge Q_1 + 2$. As in **[R-S]** $P\tilde{u} = 0$, since the differentiation under the integral sign can be justified as follows. (Recall that P acts on the right and is left-invariant.)

$$(2.15)\quad \begin{aligned} |P(\pi_\lambda(l)u\,,\,\psi)| &= |(\pi_\lambda(l)\pi_\lambda(P)u\,,\,\psi)| \\ &= \lambda^d|(\pi_\lambda(l)\pi(P)u\,,\,\psi)| = \lambda^d|(u\,,\,\pi({}^tP)\pi(l^{-1})\psi)| \\ &\le \lambda^d C|||\pi({}^tP)\pi(\alpha_\lambda(l^{-1}))\psi||| \le \lambda^d C_1|||\pi(\alpha_\lambda(l^{-1})\psi)|||' \end{aligned}$$

where $d =$ degree of homogeneity of P and $|||\ |||'$ means that we've "absorbed" $\pi(P)$ into the Schwartz space seminorm $|||\ |||$. The last expression in (2.15) can now be estimated in exactly the same way as the one in (2.9), resulting in an estimate similar to (2.14), with a polynomial in λ of degree Q_2, say. Thus $\int_1^\infty P(\pi_\lambda(l)u\,,\,\psi)\lambda^{-Q}\,d\lambda$ converges absolutely whenever $Q \ge Q_2+2$, and $P\tilde{u} = \int_1^\infty P(\pi_\lambda(l)u\,,\,\psi)\lambda^{-Q}\,d\lambda = \int_1^\infty(\pi_\lambda(l)\pi_\lambda(P)u\,,\,\psi)\lambda^{-Q}\,d\lambda = 0$.

The key thing now is that by the assumption of the lemma there is a $Z \in \mathscr{N}_r \cap \mathscr{L}$ such that $d\pi(Z) = ic \neq 0$, $c \in \mathbb{R}$. For this Z we have $\pi_\lambda(\exp tZ) = e^{ict\lambda^r}$ and

$$\begin{aligned} \tilde{u}(\exp tZ) &= \int_1^\infty (\pi(\exp tZ)u\,,\,\psi)\lambda^{-Q}\,d\lambda \\ &= (u\,,\,\psi)\int_1^\infty e^{ict\lambda^r}\lambda^{-Q}\,d\lambda \\ &= (u\,,\,\psi)r^{-1}\int_1^\infty e^{ict\lambda'}\lambda'^{-Q_3}\,d\lambda' \end{aligned}$$

where $Q_3 = Q/r + (r-1)/r$ and $\lambda' = \lambda^r$. We pick now Q in (2.8) so that $Q \ge \max(Q_1\,,\,Q_2)+2$, and that Q_3 is an integer ≥ 2. With this choice of Q, as in **[R-S]** $\tilde{u}(l)$ defines a distribution on L, $P\tilde{u} \equiv 0$, yet $\tilde{u}$ restricted to $\exp(\mathbb{R}Z) \subset L$ is not smooth.

(2.16) Corollary. *In particular, Lemma* (2.7) *holds true for* $\mathscr{N} = \mathscr{F}$, *the free nilpotent Lie algebra of step* r *on* $k+m$ *generators* $\widetilde{X}_1, \dots, \widetilde{X}_k;\ \widetilde{Y}_1, \dots, \widetilde{Y}_m$, *and* $\widetilde{\mathscr{L}}$ *the subalgebra of* $\mathscr{F}$ *generated by* $\widetilde{X}_1, \dots, \widetilde{X}_k$. *Here* $P = \widetilde{X}_1^2 + \cdots + \widetilde{X}_k^2$ *is the Rockland operator* (*cf.* **[F-S]**, *p.* 130) *and* $\rho \in \widehat{F}$ *is such that* $d\rho(\widetilde{Z}) \neq 0$ *for some* $\widetilde{Z} \in \widetilde{L} \cap \mathscr{F}_r$.

Proof of Lemma (2.6). Suppose $d\pi(X_1^2 + \cdots + X_k^2)u = 0$ for $u \in (H_\pi^\infty)'$ and let $Z \in \mathscr{L} \cap \mathscr{N}_r$ be such that $d\pi(Z) \neq 0$. By the above corollary, $u = 0$. This can be seen as follows. We construct a chain of subgroups from L to N, each of codimension 1 in the next, so that $\mathscr{N} = \mathbb{R}Y_1 \ltimes \cdots \ltimes \mathbb{R}Y_m \ltimes \mathscr{L}$, for some $Y_1, \dots, Y_m \in \mathscr{N}$. Let Φ be a homomorphism of $\mathscr{F}$ onto $\mathscr{N}$ given on the generators of $\mathscr{F}$ by

$\Phi(\widetilde{X}_j) = X_j$, $j = 1, \ldots, k$; $\Phi(\widetilde{Y}_j) = Y_j$, $j = 1, \ldots, m$. Let $\widetilde{Z}$ be a preimage of Z in $\widetilde{\mathscr{L}} \cap \mathscr{F}_r$. We apply the corollary to $\rho = \pi \circ \Phi \in \widehat{F}$ noticing that $H_\pi = H_\rho$ and $d\rho(\widetilde{X}_1^2 + \cdots + \widetilde{X}_k^2) = d\pi(X_1^2 + \cdots + X_k^2)$.

Lemma (2.7) also implies the following version of Theorem (2.1) in case $\mathscr{L}$, the Lie algebra generated by $X_1, \ldots, X_k$, is graded.

(2.1′) THEOREM. *Let $M = \Gamma\backslash N$ be a compact nilmanifold and let $\mathscr{L}$ be a graded subalgebra of $\mathscr{N}$. Suppose that $\mathscr{L}$ has the property (2.2). Let $P \in \mathscr{U}(\mathscr{L})$ be a Rockland operator on L acting on $\Gamma\backslash N$. If $u \in \mathscr{D}'(\Gamma\backslash N)$ is in the kernel of P then $u =$ const.*

REMARK 1. Theorem (2.1′) states that if $\mathscr{L}$ is a large enough graded subalgebra of $\mathscr{N}$ (i.e. $\mathscr{L}$ satisfies (2.2)) and $P \in \mathscr{U}(\mathscr{L})$ is homogeneous, then injectivity of $d\rho(P)$ on H_ρ^∞ for all non-trivial ρ in $\widehat{L}$ implies injectivity of $d\pi(P)$ on $(H_\pi^\infty)'$ for all non-trivial π in $(\Gamma\backslash N)^\frown$.

REMARK 2. The existence of Z in $\mathscr{N}_r \cap \mathscr{L}$ at the end of the proof of Lemma (2.7) and the choice of $\widetilde{Z}$ in $\widetilde{\mathscr{L}} \cap \mathscr{F}_r$ in the proof of Lemma (2.6) use condition (2.2). We don't know whether the assumption (2.2) in Theorem (2.1) can be replaced by the weaker condition (2°) of Theorem (4.1).

3. A solvmanifold (counter)example. Here we produce an example of a (GH) system of two vector fields on a class of 3-dimensional ("hyperbolic") *solv*manifolds. We show that the kernel of the sum of squares of these vector fields contains a distribution which is not a C^∞-function. Also, Lemma (3.4) might be of independent interest.

Consider the following class of three-dimensional compact solvmanifolds, $M = \Gamma\backslash S$ (see **[A-G-H]** and **[Br1, 2]** for details). S is the semidirect product of $\mathbb{R}$ and $\mathbb{R}^2$ (with $\mathbb{R}^2$ normal in S), in which the group operation is

$$(x, t)(x', t') = (x + A^t x', t + t'), \qquad x, x' \in \mathbb{R}^2, t, t' \in \mathbb{R}.$$

Here A^t, $t \in \mathbb{R}$, is a 1-parameter subgroup of $\mathrm{SL}(2, \mathbb{R})$ through a fixed matrix $A \in \mathrm{SL}(2, \mathbb{Z})$. The discrete subgroup Γ can be taken to be the set of points in S with integer coordinates. (The fact that $A \in \mathrm{SL}(2, \mathbb{Z})$ is equivalent to A mapping the integer lattice $\mathbb{Z}^2$ into itself.) We'll consider the case in which A has unequal positive eigenvalues λ and λ^{-1}. Choosing the eigenvectors of A as a basis of $\mathbb{R}^2$ we can write the group operation in S in the new u, v coordinates

$$(u, v, t)(u', v', t') = (u + \lambda^t u', v + \lambda^{-t} v', t + t'), \quad u, u', v, v', t, t' \in \mathbb{R}. \tag{3.1}$$

In these new coordinates $\mathbb{R}^2 \cap \Gamma$ is no longer $\mathbb{Z}^2$. (For each $\lambda > 1$ such that λ, λ^{-1} are eigenvalues of a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ we get a distinct solvmanifold $\Gamma_\lambda \backslash S$, although S is not changed up to isomorphism by altering λ.) Letting T, U, and V be the infinitesimal generators of the one-parameter subgroups of S corresponding to t, u, and v we have $[T, U] = \ln \lambda U$, $[T, V] = -\ln \lambda V$. We consider the operator $P = T^2 + U^2$ and the system $\{T, U\}$ of vector fields induced on $\Gamma \backslash S$ by T and U.

(3.2) Proposition. *Let T, U and $\Gamma \backslash S$ be as described above. Then*

(a) *The system of vector fields $\{T, U\}$ on $M = \Gamma \backslash S$ is (GH);*

(b) *The operator $P = T^2 + U^2$ is not (GH) on $\Gamma \backslash S$. In fact, there is a distribution $u \in \mathscr{D}'(\Gamma \backslash S) \sim L^2(\Gamma \backslash S)$ such that $Pu = 0$.*

Proof of (a). Let $u \in \mathscr{D}'(\Gamma \backslash S)$ be such that $Tu = f$, $Uu = g$, and $Vu = h$, with $f, g, h \in C^\infty(\Gamma \backslash S)$. Let $u = u_0 + \sum_{\pi, j} u_{\pi, j}$ be an irreducible Fourier series of u, the summation being over infinite dimensional $\pi \in (\Gamma \backslash S)^\wedge$ with j counting the multiplicities and with $u_0 \in \mathscr{D}'(\Gamma[S, S] \backslash S)$, so u_0 lives on the associated torus. On that 1-dim torus, T acts as d/dt. Thus $Tu_0 = f_0 \in C^\infty(\Gamma[S, S] \backslash S)$ and u_0 is smooth. As for the $u_{\pi, j}$'s, each ∞-dimensional $\pi \in (\Gamma \backslash S)^\wedge$ acts on $L^2(\mathbb{R}, dt)$ and $d\pi(U) u_{\pi, j} = 2\pi i \alpha \lambda^t u_{\pi, j}$ for some $0 \neq \alpha \in \mathbb{R}$. Since $d\pi(V)$ acts by multiplication by $2\pi i \beta \lambda^{-t}$ with $0 \neq \beta \in \mathbb{R}$, $u_{\pi, j} = (-4\pi^2 \alpha\beta)^{-1} d\pi(V) d\pi(U) u_{\pi, j} = (-4\pi^2 \alpha\beta)^{-1} (Vg)_{\pi, j}$. In fact, for any $R \in \mathscr{U}(\mathscr{S})$ we have $(Ru)_{\pi, j} = (-4\pi^2 \alpha\beta)^{-1} (RVg)_{\pi, j}$, and

$$\begin{aligned} \|Ru\|^2_{L^2(\Gamma \backslash S)} &= (4\pi^2)^{-2} \sum_{\pi, j} \|RVg\|^2 (\alpha\beta)^{-2} + \|(Ru)_0\|^2 \\ &\leq C \sum_{\pi, j} \|RVg\|^2 + \|(Ru)_0\|^2 \\ &\leq C \|RVg\|^2_{L^2(\Gamma \backslash S)} < \infty. \end{aligned} \tag{3.3}$$

The last expression is finite since $g \in C^\infty(\Gamma \backslash S)$. The first inequality in (3.3) is a consequence of the following.

(3.4) Lemma. *Let $S = \mathbb{R}^2 \rtimes \mathbb{R}$ be a solvable Lie group with the group law* (3.1). *Let Γ be a cocompact discrete subgroup of S. Then*

$\Gamma \cap \mathbb{R}^2 \times \{0\}$ *is an abelian lattice of points* $(a, b) \in \mathbb{R}^2$ *having the property that the product* ab *is bounded away from zero, except of course for the group identity.*

(3.5) COROLLARY. *In the setting of the lemma above, the dual lattice* $\Gamma^* = \{\chi_{\alpha,\beta} : \Gamma \to 1\}$ *is also a lattice of points* (α, β) *such that the product* $\alpha\beta$ *is bounded away from* 0, *except for* $(\alpha, \beta) = (0, 0)$.

Proof of Lemma (3.4). Let $(0, 0, m)$ and $(a, b, 0) \in \Gamma \subset S$. Then $(0, 0, m)(a, b, 0)(0, 0, m)^{-1} = (\lambda^m a, \lambda^{-m} b, 0)$. Suppose $(a_n, b_n, 0)$, $n = 1, 2, \ldots$ were a sequence of points in $\Gamma \cap \mathbb{R}^2 \times \{0\}$ such that $a_n b_n \to 0$ as $n \to \infty$. Wlog we may suppose $a_n \geq b_n > 0$. Then for every n there would be an integer k_n such that

$$\lambda^{2(k_n - 1)} < b_n / a_n \leq \lambda^{2k_n}. \tag{3.6}$$

Define a new sequence of points of Γ by

$$(a'_m, b'_n, 0) := (\lambda^{k_n} a_n, \lambda^{-k_n} b_n, 0), \qquad n = 1, 2, \ldots.$$

We have $a'_n b'_n = a_n b_n \to 0$ and $b'_n / a'_n = \lambda^{-2k_n} b_n / a_n$. The inequalities (3.6) imply now

$$\lambda^{-2} < b'_n / a' \leq 1, \qquad n = 1, 2, \ldots.$$

Thus $\Gamma \ni (a'_n, b'_n, 0) \to (0, 0, 0)$ as $n \to \infty$, which violates the discreteness of Γ.

Proof of (b). For a fixed infinite dimensional $\pi \in (\Gamma \backslash S)^\frown$ acting on $L^2(\mathbb{R}, dt)$ we'll construct a non-zero function $u(t)$ on $\mathbb{R}$ such that $d\pi(P)u = 0$ and $d\pi(U)u \in L^2(\mathbb{R})$. Such a u defines a distribution $\tilde{u}$ on $\phi \in C^\infty(\Gamma \backslash S)$:

$$\begin{aligned}
|(\tilde{u}, \phi)| &:= |(u, Q\phi_\pi)| \\
&= |(u, (-4\pi^2 \alpha\beta)^{-1} \pi(U)\pi(V)Q\phi_\pi)| \\
&= |(\pi(U)u, (VQ\phi)_\pi)|\, |4\pi^2 \alpha\beta|^{-1} \\
&\leq C \|\pi(U)u\|\, \|(V\phi)_\pi\| \leq C_1 \|V\phi\|_{L^2(\Gamma\backslash S)} \\
&\leq C_2 \|V\phi\|_{L^\infty(\Gamma\backslash S)},
\end{aligned}$$

where ϕ_π denotes a projection onto a fixed irreducible subspace H_π of $L^2(\Gamma \backslash S)$ corresponding to π, and $Q : H_\pi \to L^2(\mathbb{R})$ is an intertwining operator onto a Schrödinger model for π.

To find such u we write

$$\begin{aligned}
d\pi(P)u &= (d^2/dt^2 - 4\pi^2 \alpha^2 \lambda^{2t})u \\
&= \{(\ln \lambda) r^2 (d/dr^2 + r^{-1} d/dr) - (2\pi\alpha r)^2\} u_1
\end{aligned} \tag{3.7}$$

where we have put $r = \lambda^t$, and we have defined $u_1(r) = u(t)$, $r > 0$, and we take advantage of the fact that $d^2/dr^2 + r^{-1}d/dr$, $r > 0$, is the radial part of the Laplacian Δ on the plane. Thus (3.7) becomes equivalent to

$$(3.8) \qquad (\Delta - a^2)u_2 = 0, \quad a = 2\pi\alpha/\ln\lambda$$

for a radial function u_2 on $\mathbb{R}^2 \sim 0$. A solution u_2 of

$$(3.9) \qquad (\Delta - a^2)u_2 = \delta_0 \quad \text{on } \mathbb{R}^2,$$

where δ_0 is the Dirac function supported at the origin 0 of $\mathbb{R}^2$, satisfies (3.8), and if it is radial, also (3.7). Applying the Fourier transform on $\mathbb{R}^2$ to (3.9) we obtain (cf. e.g. **[S-W]**, page 6)

$$\begin{aligned}\hat{u}_2(\xi, \eta) &= -(4\pi^2)^{-1}(\xi^2 + \eta^2 + a^2)^{-1} \\ &= -(4\pi^2)^{-1}\int_0^\infty \exp[-(\xi^2 + \eta^2 + a^2)s]\,ds \\ &= \int_{\mathbb{R}^2} \exp[-2\pi i(\xi x + \eta y)]u_2(x, y)\,dx\,dy\end{aligned}$$

where

$$(3.10) \quad u_2(x, y) = u_1(r) = -(16\pi^2)^{-1}\int_0^\infty s^{-1}e^{-bs}\exp(-\pi^2 r^2/s)\,ds$$

with $r^2 = x^2 + y^2$ and $b = (\alpha/\ln\lambda)^2$. Thus letting $r = \lambda^t$, $-\infty < t < \infty$, $u_1(r)$ given by (3.10) is a non-zero solution to (3.7) we were after. It remains to show that $d\pi(U)u \in L^2(\mathbb{R})$. For this we write

$$\begin{aligned}(3.11) \qquad \|d\pi(U)u\|^2 &= \int_{-\infty}^\infty |2\pi\alpha r u_1(r)|^2\,dt \\ &= c\int_0^\infty r\left(\int_0^\infty \ldots ds \int_0^\infty \ldots d\sigma\right) dr,\end{aligned}$$

where $c_1\int_0^\infty \ldots ds = u_1(r) = c_1\int_0^\infty \ldots d\sigma$ are given by (3.10). Changing the order of integration in (3.11) to $dr\,ds\,d\sigma$ and grouping the terms containing r only, the dr integral becomes

$$\begin{aligned}\int_0^\infty \exp[-\pi^2 r^2(s^{-1} + \sigma^{-1})]r\,dr &= (s^{-1} + \sigma^{-1})^{-1}\int_0^\infty \exp(-\pi^2 r^2)r\,dr \\ &= 2\pi^2(s^{-1} + \sigma^{-1})^{-1}.\end{aligned}$$

Substituting this back into (3.10) we obtain

$$\begin{aligned}(3.11) &= C' \int_0^\infty \int_0^\infty \exp[-b(s+\sigma)](s\sigma)^{-1}(s^{-1}+\sigma^{-1})^{-1}\,ds\,d\sigma \\ &= C' \int_0^\infty \int_0^\infty \exp(-bs)\exp(-b\sigma)(s+\sigma)^{-1}\,ds\,d\sigma \\ &\le (C'/2)\left(\int_0^\infty \exp(-bs)s^{-1/2}\,ds\right)^2 < \infty .\end{aligned}$$

REMARK. Similarly one can show that $\infty = \|u\|_{L^2(\mathbb{R})}$. We claim $\tilde{u} \in \mathscr{D}'(\Gamma\backslash S)$ is not given by any L^2-function on $\Gamma\backslash S$. Suppose the negation, i.e. $\tilde{u}$ is given by some $w \in L^2(\Gamma\backslash S)$. Since $\tilde{u} : H_{\pi'}^\infty \to 0$ for all $\pi' \not\cong \pi$, $\pi' \in (\Gamma\backslash S)^\frown$, we have $w \in H_\pi$. Then $Qw \in L^2(\mathbb{R})$ and $Qw = u$ a.e. because it gives the same distribution—a contradiction. In particular, $\tilde{u}$ cannot be continuous or smooth on $\Gamma\backslash S$.

4. Necessary and sufficient conditions for (GH) of systems. Theorem 1 on page 366 of **[C-R2]** states a necessary and sufficient condition for (GH) of a system $\mathbb{L}$ on $\Gamma\backslash N$. The proof of necessity, however, has a gap in the last paragraph of page 367. Namely, it is not clear whether or not there is a $\Lambda' \in \mathscr{O}_N(\Lambda)$ such that $\Lambda'(\mathscr{L}) = 0$. On the other hand, the proof of sufficiency establishes the (at least formally) stronger sufficiency theorem below.

(4.1) THEOREM. *If* (1°) $\mathbb{L} + [\mathscr{N}, \mathscr{N}]$ *is* (*GH*) *on* $\Gamma[N, N]\backslash N$, *and* (2°) *for each* $\pi \in (\Gamma\backslash N)^\frown_\infty$, $(\mathscr{L} + \mathscr{W}_\pi) \cap \mathscr{Z}(\mathscr{N}/\mathscr{W}_\pi) \neq \{\bar{0}\}$, *then* $\mathbb{L}$ *is* (*GH*) *on* $\Gamma\backslash N$. (*Here,* $\mathscr{W}_\pi$ *is an ideal in* $\ker(d\pi)$ *such that* $\dim \mathscr{Z}(\mathscr{N}/\mathscr{W})_\pi = 1$ *and* $\pi \mid Z(N/W_\pi) \neq I$. *Also,* $\pi \in (\Gamma\backslash N)^\frown_\infty$ *means* π *is infinite dimensional, and* $\mathscr{L}$ *is a Lie subalgebra of* $\mathscr{N}$ *generated by* $\mathbb{L}$.)

(4.2) CONJECTURE. Conditions (1°) and (2°) of Theorem (4.1) are necessary for (GH) of $\mathscr{L}$ on $\Gamma\backslash N$.

Although we do *not* have any counter-example to this conjecture, we have been able to prove it only under special conditions.

To prove the conjecture, we assume that $\mathscr{L}$ is (GH) on $\Gamma\backslash N$, so that (1°) is automatically satisfied. Then we suppose that $\overline{\mathscr{L}} \cap \mathscr{Z}(\mathscr{N}/\mathscr{W})_\pi = 0$, and we try to prove there exists $\Lambda' \in \mathscr{O}_N(\Lambda)$ such that $\Lambda'(\mathscr{L}) = 0$. By the lemma on page 368 of **[C-R2]**, this would contradict (GH) of $\mathscr{L}$ on $\Gamma\backslash N$.

(4.3) PROPOSITION. *Suppose that* $\pi \in (\Gamma\backslash N)^{\wedge}_{\infty}$ *implies either that the corresponding co-adjoint orbit* $\mathscr{O}(\pi)$ *in* $\mathscr{N}^*$ *is flat, or else that* π *is induceable from a polarization of codimension one in* $\mathscr{N}$. *If* $\mathbb{L} = \{X_1, \ldots, X_k\}$ *is a (GH) system of vector fields on* $\Gamma\backslash N$, *then* (1°) $\mathbb{L} + [\mathscr{N}, \mathscr{N}]$ *is (GH) on* $\Gamma[N, N]\backslash N$, *and* (2°) *for each* $\pi \in (\Gamma\backslash N)^{\wedge}_{\infty}$, $(\mathscr{L} + \mathscr{W}_\pi) \cap \mathscr{Z}(\mathscr{N}/\mathscr{W}_\pi) \neq \{\bar{0}\}$.

Before proving this theorem and giving examples, we state the following immediate consequences. If $\mathscr{N}$ is of step 2, then Proposition (4.3) shows that Conjecture (4.2) is true for N, since all orbits will be flat. Also, for each natural number $n \geq 2$, there exist nontrivial rational nilpotent Lie algebras of step n such that all orbits (i.e., of *all* dimensions) will be flat **[R3]**. Thus the conjecture will have been proved for a large class of nilmanifolds. Also, the conjecture will have been proved for the important class of nilpotent semi-direct products $\mathbb{R} \ltimes \mathbb{R}^n$, with arbitrarily long lower central series.

Proof. The case of flat orbits is easiest. Fix a $\pi \in (\Gamma\backslash N)^{\wedge}_{\infty}$. By repeatedly factoring out the part of the center in the kernel of $d\pi$, we may assume wlog that $\mathscr{Z}(\mathscr{N}) = \mathbb{R}Z$, that $\mathscr{L} \cap \mathscr{Z} = \{0\}$, and that $\pi = \pi_\Lambda$ where $\Lambda = Z^*$. Then $\mathscr{O}_N(\Lambda) = Z^* + (Z)^\perp$. $\mathscr{L}$ is spanned by a basis $L_1, \ldots, L_k$, not in $\mathscr{Z}$. Pick $\Lambda_1, \ldots, \Lambda_k$ in $Z^\perp$ such that

$$\Lambda_j(L_i) = \begin{cases} 0, & \text{if } i \neq j; \\ -\Lambda(L_j), & \text{if } i = j. \end{cases}$$

Then let $\Lambda' = Z^* + \sum_1^k \Lambda_j \in \mathscr{O}_N(\Lambda)$, and $\Lambda'(\mathscr{L}) = 0$. This proves the conjecture in the flat orbit case.

Now, suppose π is induced from a (rational) polarization $\mathscr{M}$ of codimension 1. (Hence $\mathscr{M}$ is an ideal.)

(4.4) LEMMA. *If* $\mathscr{Z}(\mathscr{N}) = \mathbb{R}Z$ *and* $\mathscr{M}$ *is a polarizing ideal for any* $\Lambda \in \mathscr{N}^*$ *with* $\Lambda(Z) \neq 0$, *then* $\mathscr{M}$ *is abelian.*

Proof of Lemma. Since $\mathscr{M}$ is subordinate, $Z \notin [\mathscr{M}, \mathscr{M}]$, and $\mathscr{M} \triangleleft \mathscr{N}$ implies $[\mathscr{M}, \mathscr{M}]$ is an ideal too, since $(\operatorname{ad} X)[M_1, M_2] = [(\operatorname{ad} X)M_1, (\operatorname{ad} X)M_2]$. If there exists $0 \neq [M_1, M_2] \in [\mathscr{M}, \mathscr{M}]$, then $[M_1, M_2] \notin \mathscr{Z}$, so there exists $U_1 \in \mathscr{N}$ such that $(\operatorname{ad} U_1)[M_1, M_2] = [(\operatorname{ad} U_1)M_1, (\operatorname{ad} U_1)M_2] \in [\mathscr{M}, \mathscr{M}] \sim \{0\}$ too. Hence $(\operatorname{ad} U_1)[M_1, M_2] \in [\mathscr{M}, \mathscr{M}] \sim \mathscr{Z}$. So there is a U_2 such that $[(\operatorname{ad} U_2)(\operatorname{ad} U_1)M_1, (\operatorname{ad} U_2)(\operatorname{ad} U_1)M_2] \neq 0$, and so on. Since there is no end to this process, the nilpotence of $\mathscr{N}$ has been violated. Thus $[\mathscr{M}, \mathscr{M}] = 0$.

This proves the lemma.

Now we have $\mathscr{N} = \mathbb{R} \ltimes \mathscr{M}$, where $\mathscr{M} \cong \mathbb{R}^n$ is a rational abelian ideal of codimension one in $\mathscr{N}$. Since $\mathscr{L}$ is (GH) on $\Gamma\backslash N$, and since $\mathscr{M}$ is rational, $\mathscr{L} \not\subset \mathscr{M}$. Thus there exists $X \in \mathscr{L} \sim \mathscr{M}$. We are supposing $\mathscr{L} \cap \mathscr{Z} = \{0\}$, and it will suffice to prove that there exists $\Lambda' \in \mathscr{O}_N(\Lambda)$ such that $\Lambda'(\mathscr{L}) = 0$.

If $\mathscr{L}$ *were not* abelian, there would exist a central element $C \neq 0$, $C \in [\mathscr{N}, \mathscr{N}] \subset \mathscr{M}$, so $[C, \mathscr{M}] = 0$. But $[C, X] = 0$, and $\mathscr{N} = \mathbb{R}X \oplus \mathscr{M}$. Thus $[C, \mathscr{N}] = 0$, so that $C \in \mathscr{Z}(\mathscr{N}) \sim \{0\}$. But $C \in \mathscr{L}$, and $\mathscr{L} \cap \mathscr{Z} \neq \{0\}$. This is a contradiction, and so $\mathscr{L}$ is abelian.

Now, suppose there existed $L \in \mathscr{L} \cap \mathscr{M} \sim \{0\}$. Then $[X, L] = 0 = [\mathscr{M}, L]$, so $L \in \mathscr{Z} \sim \{0\}$. This is a contradiction. So if $X \neq L \in \mathscr{L} \sim \{0\}$, then $L = \alpha X + M$, for some $\alpha \neq 0$ and $M \in \mathscr{M}$. Also, since $\mathscr{L}$ is abelian, $[X, M] = 0 = [\mathscr{M}, M]$. Hence $M \in \mathscr{Z} \sim \{0\}$, so that $\mathscr{L}$ contains the $\mathbb{R}$-span of X and $\alpha X + M$. But then $\mathscr{Z} \subset \mathscr{L}$. This is a contradiction.

Hence $\mathscr{L} = \mathbb{R}X$. Now, pick $Y \in \mathscr{M} \sim \mathscr{Z}$ such that $[X, Y] = Z$, and $(\mathrm{Ad}^* \exp \mathbb{R}Y)\Lambda = Z^* + \mathbb{R}X^*$. Hence there exists $\Lambda' \in \mathscr{O}_N(\Lambda)$ such that $\Lambda'(X) = 0 = \Lambda'(\mathscr{L})$.

This proves the proposition.

(4.5) Example. Let $\mathscr{N}$ be spanned by X_1, X_2, Y_1, Y_2, and Z, where all non-zero brackets are generated by $[X_1, Y_1] = Y_2$, $[X_1, Y_2] = Z$, and $[X_2, Y_1] = Z$. Thus $\mathscr{N}$ is 3-step, with $\mathscr{N}_2 = [\mathscr{N}, \mathscr{N}]$ spanned by Y_2 and Z, and $\mathscr{Z} = \mathbb{R}Z$.

Let $X = X_1 + \alpha Y_1$ and $Y = X_2 + \alpha Y_2 + \beta Z$, where α is an irrational, non-Liouville number. Then $\mathscr{L} = \mathbb{R}X \oplus \mathbb{R}Y$ is an abelian Lie subalgebra of $\mathscr{N}$, (GH) on $\Gamma[N, N]\backslash N$. Since $\mathscr{L} \cap \mathscr{Z} = \{0\}$, condition (2°) of Proposition (4.3) is not satisfied. However, *every* Kirillov orbit $\mathscr{O}_N \subset \mathscr{N}^*$ is flat (of all possible dimensions) [**R3**]. By Proposition (4.3), $\mathscr{L}$ is *not* (GH) on $\Gamma\backslash N$, regardless of the choice of β, and regardless of the choice of α. This example illustrates the *necessity* of conditions (1°) and (2°) of Proposition (4.3).

(4.6) Example. Let $\mathscr{N} = \mathbb{R} \ltimes \mathbb{R}^{n+1}$ be the n-step nilpotent "chain" Lie algebra spanned by X, $Y_1, \ldots, Y_n$, $Y_{n+1} = Z$, with all non-zero brackets generated by $[X, Y_i] = Y_{i+1}$, $i = 1, \ldots, n$. Then let $\mathscr{L} = \mathbb{R}L$, where $L = X + \alpha_1 Y_1 + \cdots + \alpha_n Y_n + \alpha_{n+1} Z$. Then $\mathscr{L}$ is (GH) on $\Gamma[N, N]\backslash N$, but $\mathscr{L}$ is *not* (GH) on $\Gamma\backslash N$, since condition (2°) of Proposition (4.3) is not satisfied. That is $\mathscr{L} \cap \mathscr{Z} = \{0\}$.

There are of course many variations and combinations of these two examples.

The following example supports Conjecture (4.2) by showing how $\mathscr{L} \cap \mathscr{Z} = \{0\}$ can lead to $\mathscr{L}$ not being (GH) even under circumstances not covered by Proposition (4.3). In particular, it will be a 3-step *non*-flat orbit example in which $\mathscr{L}$ is (GH) on $\Gamma[N, N]\backslash N$ and also on $\Gamma Z\backslash N$ and on $\Gamma M\backslash N$, and yet $\mathscr{L}$ is not (GH) on $\Gamma\backslash N$, apparently because $\mathscr{L} \cap \mathscr{Z} = \{0\}$. here, $\mathscr{N}/\mathscr{M}$ will be of dimension two.

(4.7) EXAMPLE. Let $\mathscr{N}$ be the $\mathbb{R}$-span of W_1, W_2, X_1, X_2, Y_1, Y_2, and Z. Let all non-zero brackets be generated by $[W_i, X_i] = Y_i$, and $[W_i, Y_i] = Z$, $i = 1, 2$. Let $\mathscr{L}$ be the $\mathbb{R}$-span of $\{W_1 + \alpha W_2 + \beta X_1 + \gamma X_2 + \xi Z, \ Y_1 - \alpha Y_2 + \eta Z\}$ where the real numbers, 1, α, β, γ are linearly independent over $\mathbb{Q}$ and satisfy the growth condition (1.2), and ξ, $\eta \in \mathbb{R}$ are arbitrary but fixed. The abelian Lie algebra $\mathscr{L}$ is (GH) on $\Gamma[N, N]\backslash N$ and on $\Gamma Z\backslash N$, but $\mathscr{L} \cap \mathscr{Z} = \{0\}$. Let $\Lambda = Z^*$, so the polarizing subalgebra is the ideal $\mathscr{M}$ spanned by $\{X_1, X_2, Y_1, Y_2, Z\}$. However, we can act on Λ by $\exp t(W_1 - W_2)$ to get $\Lambda' : Y_1 - \alpha Y_2 + \eta Z_2 \mapsto 0$. And we can act on Λ' by $\exp s(Y_1 + Y_2)$ to get $\Lambda'' : W_1 + \alpha W_2 + \beta X_1 + \gamma X_2 + \xi Z \mapsto 0$. Thus $\Lambda'' \in \mathscr{O}_N(\Lambda)$, and $\Lambda''(\mathscr{L}) = \{0\}$. By Lemma on page 368 of **[C-R2]** $\mathscr{L}$ is *not* (GH) on $\Gamma\backslash N$.

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Received February 14, 1989 and in revised form October 13, 1989. Research by the first author was supported in part by Louisiana Education Quality Support Fund contract number 86-LBR-016-04 and by the National Science Foundation. The second author's research was supported in part by the National Science Foundation.

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Pacific Journal of Mathematics
Vol. 147, No. 1 January, 1991