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EXPLICIT $\bar{\partial}$ -PRIMITIVES OF HENKIN-LEITERER KERNELS ON STEIN MANIFOLDS

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In this paper we construct explicitly $\bar{\partial}$ -primitives and use them to obtain a representation formula for holomorphic functions and a theorem on extendability of CR-functions.

1. Introduction. Let X be a Stein manifold of dimension n , $h : X \rightarrow \mathbb{C}^p$ ($p \leq n - 1$) a holomorphic map and let $Z(h) = \{\zeta \in X : h(\zeta) = 0\}$. If $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$ is a Henkin-Leiterer type kernel on X (see §2 for notation) then $K(\zeta, z)$ is a $\bar{\partial}$ -closed $(n, n - 1)$ -form in ζ , for a fixed z , i.e., $\bar{\partial}_\zeta K(\zeta, z) = 0$, whose singularity occurs at $\zeta = z$. On the other hand, since $X - Z(h)$ is $(n - 2)$ -complete (see Sorani and Villani [8, p. 435]), it follows that the cohomology group

$$H^{n-1}(X - Z(h), \mathcal{O}^n) \cong H_{\bar{\partial}}^{(n, n-1)}(X - Z(h))$$

vanishes (see Andreotti and Grauert [1, p. 250]). Therefore, for a fixed $z \in Z(h)$, there exists an $(n, n - 2)$ -form $\eta(\zeta, z)$, in $X - Z(h)$, so that

$$\bar{\partial}_\zeta \eta(\zeta, z) = K(\zeta, z).$$

For some problems, however, it is important to have explicit formulas for such $\bar{\partial}$ -primitives, η , of K ; the problems we have in mind are related to integral representations (see for example Stout [9] and Hatziafratis [2]) and extendability of CR-functions (see for example Lupacchiolu [6], Tomassini [11] and Stout [10]). Since such forms $\eta(\zeta, z)$ are not unique, their dependence on z , for example, may be difficult to control with cohomological arguments.

In this paper we construct explicitly such $\bar{\partial}$ -primitives and use them to obtain a representation formula for holomorphic functions and a theorem on extendability of CR-functions.

The arrangement of the paper is as follows. First in §2 we review the main points of the Henkin-Leiterer construction; with X and h as above we consider a domain $D \subset X$, a Stein neighborhood W of \bar{D} and we briefly discuss what a Leray section $s^* = s^*(\zeta, z)$ and the associated Henkin-Leiterer kernel $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$ are.

Then in §3 we carry out the construction of the $\bar{\partial}$ -primitives $\eta_h(\zeta, z)$ and in Theorem 3.1 we prove that indeed $\bar{\partial}_\zeta \eta_h(\zeta, z) = K(\zeta, z)$ for $\zeta \in W - Z(h - h(z))$, ζ, z being always so that $s^*(\zeta, z)$ is defined. (At this point we would like to point out that we were led to consider this construction by the paper of Laurent-Thiebaut [5] in which the case $p = 1$ is studied.)

Our main application of this construction is a Cauchy type integral representation formula for holomorphic functions. Fix a $z \in D$, we consider an open set $\Gamma \subset \partial D$ (open in ∂D) with $\partial\Gamma$ smooth so that $\Gamma \supset (\partial D) \cap Z(h - h(z))$ and we prove (Theorem 3.2) that for $f \in C(\bar{\Gamma} \cup D) \cap \mathcal{O}(D)$ we have

$$f(z) = \int_{\zeta \in \Gamma} f(\zeta) K(\zeta, z) - \int_{\zeta \in \partial\Gamma} f(\zeta) \eta_h(\zeta, z).$$

This integral formula expresses the value of f at z in terms of its values on a part of the boundary of D namely $\bar{\Gamma}$. In particular it provides a formula for extending CR-functions from parts of the boundary (if such extensions exist); this is the point of Theorem 4.1 in §4. This theorem gives a necessary and sufficient condition for the extendability of a CR-function f from a part of the boundary of D to a holomorphic function in D ; roughly speaking the condition says that certain integrals involving the CR-function and taken over certain cycles which lie in the domain (on ∂D) of f should agree.

Finally with regards to the Theorem 3.2 we mention the work of Patil [7] where a different method was devised for recovering, in some cases, an H^2 -function from its boundary values on a set of positive measure.

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2. Henkin-Leiterer type kernels. In this section we will establish notation and recall the main points of the Henkin-Leiterer construction on Stein manifolds.

Let X be a Stein manifold of dimension n and let $T(X)$ denote its holomorphic tangent bundle with the fiber above z ($z \in X$) denoted by $T_z(X)$. Then, following Henkin and Leiterer [4, Ch. 4], there exists a holomorphic map $s : X \times X \rightarrow T(X)$ and a holomorphic function $\varphi : X \times X \rightarrow \mathbb{C}$ so that

- (i) $s(\zeta, z) \in T_z(X)$ for $(\zeta, z) \in X \times X$,
- (ii) $s(z, z) = 0$ and $s(\cdot, z)$ is a biholomorphic map from a neighborhood of $z \in X$ to a neighborhood of $0 \in T_z(X) \cong \mathbb{C}^n$,

(iii) $\varphi(z, z) = 1$ and there exists a positive integer ν_0 so that $\varphi^{\nu_0}(\zeta, z) \|s(\zeta, z)\|^{-2}$ is a C^2 -function on $X \times X - \Delta = X \times X - \{(z, z) : z \in X\}$, for any norm $\|\cdot\|$ on $T(X)$; in particular $\varphi^\nu \|s\|^{-2}$ is of class C^r on $X \times X - \Delta$ provided that $\nu \geq \nu_1(r)$ for some integer $\nu_1(r)$.

Now fix $D \subset X$, a relatively compact domain in X with smooth boundary. Recall that a Leray section for (D, s, φ) is a C^1 -map $s^* = s^*(\zeta, z)$ defined for $z \in D$ and for ζ in a neighborhood of ∂D , denoted by $\text{Dom}(s^*(\cdot, z))$ and depending on z , with values in $T^*(X)$, the holomorphic cotangent bundle of X , so that:

- (i) $s^*(\zeta, z) \in T_z^*(X)$ ($T_z^*(X)$ denotes the fiber of $T^*(X)$ above z),
- (ii) $\langle s^*(\zeta, z), s(\zeta, z) \rangle \neq 0$ whenever $\varphi(\zeta, z) \neq 0$ and
- (iii) there is an integer ν^* so that the function

$$\varphi^{\nu^*}(\zeta, z) (\langle s^*(\zeta, z), s(\zeta, z) \rangle)^{-1}$$

is of class C^1 for $(\zeta, z) \in V \times L$, for each compact subset L of D and where V is a neighborhood of ∂D , depending on L . Here $\langle \cdot, \cdot \rangle$ denotes the pairing of cotangent vectors with tangent vectors.

For examples of Leray sections, which always exist in the above setting, see [4, p. 165].

To a Leray section s^* , Henkin and Leiterer associate an $(n, n-1)$ -form in the following way:

$$K^{(s^*, \nu)}(\zeta, z) = \varphi^\nu(\zeta, z) \frac{\omega'_\zeta(s^*(\zeta, z)) \wedge \omega_\zeta(s(\zeta, z))}{\langle s^*(\zeta, z), s(\zeta, z) \rangle^n}$$

where ν is assumed to be large enough so that $K^{(s^*, \nu)}(\zeta, z)$ is continuous in each $V \times L$ ($\nu \geq n\nu^*$ is enough); the differential forms $\omega'_\zeta(s^*(\zeta, z))$ are defined in terms of local coordinates (U, χ) at z ; let (s_1, \dots, s_n) and (s_1^*, \dots, s_n^*) be the expressions of s and s^* in terms of the local coordinate system (U, χ) , i.e.,

$$s(\zeta, z) = \sum_{j=1}^n s_j(\zeta, z) \left(\frac{\partial}{\partial \chi_j} \right)_z \quad \text{and} \quad s^*(\zeta, z) = \sum_{j=1}^n s_j^*(\zeta, z) (d\chi_j)_z;$$

here $\{(\partial/\partial \chi_j)_z\}_{j=1}^n$ is the usual basis of $T_z(X)$ with respect to (U, χ) and $\{(d\chi_j)_z\}_{j=1}^n$ is the corresponding basis for $T_z^*(X)$.

Then

$$\omega_\zeta(s(\zeta, z)) = d_\zeta s_1(\zeta, z) \wedge \dots \wedge d_\zeta s_n(\zeta, z)$$

and

$$\omega'_\zeta(s(\zeta, z)) = c_n \sum_{j=1}^n (-1)^{j-1} s_j^*(\zeta, z) \bigwedge_{k \neq j} d_\zeta s_k^*(\zeta, z)$$

where $c_n = (-1)^{n(n-1)/2} (n-1)! / (2\pi i)^n$.

Of course by the way $\omega_\zeta(s(\zeta, z))$ and $\omega'_\zeta(s^*(\zeta, z))$ are defined, they depend on the choice of the local coordinates (U, χ) . It turns out, however, that their wedge product and therefore $K^{(s^*, \nu)}(\zeta, z)$ are independent of the choice of local coordinates, i.e., $K^{(s^*, \nu)}(\zeta, z)$ is a globally defined $(n, n-1)$ -form, see [4, p. 166].

REMARK. The discussion, given in §1, in which we justify by a cohomological argument the existence of $\bar{\partial}$ -primitives, $\eta(\zeta, z)$, of $K(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$, applies for a particular class of Leray sections, the ones which are defined for $(\zeta, z) \in X \times X$, i.e., $D = X$ and $\text{Dom}(s^*(\cdot, z)) = X$; the point here is that, in the general case, $\text{Dom}(s^*(\cdot, z)) - Z(h)$ is not $(n-2)$ -complete; however it is possible to give a cohomological argument to prove existence of the $\bar{\partial}$ -primitives in the general case too; this argument amounts to modifying, in a way, $s^*(\zeta, z)$ so that the argument given in §1 applies (see also the remark following the proof of Theorem 3.1 below).

3. Construction of the $\bar{\partial}$ -primitives. With the notation of §2, let us consider a holomorphic map $h : W \rightarrow \mathbb{C}^p$, $p \leq n-1$, where W is a Stein neighborhood of \bar{D} ; let $Z(h - h(z))$ denote the zero-set of $h - h(z)$, i.e.,

$$Z(h - h(z)) = \{\zeta \in W : h(\zeta) = h(z)\}.$$

In this section we will construct a $\bar{\partial}$ -primitive of $K^{(s^*, \nu)}(\zeta, z)$ in $W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$; in this construction, z is a fixed point of D ; the dependence of the construction on z , however, will be immediately clear, because of the explicit way the construction is carried out.

According to [4, Lemma 4.7.2] there exist holomorphic maps $h_i^* : W \times W \rightarrow T^*(X)$, $i = 1, \dots, p$, so that $h_i^*(\zeta, z) \in T_z^*(X)$ and

$$\langle h_i^*(\zeta, z), s(\zeta, z) \rangle = \varphi(\zeta, z) \cdot (h_i(\zeta) - h_i(z))$$

for $(\zeta, z) \in W \times W$ and $i = 1, \dots, p$. Using such holomorphic maps h_i^* we now define a C^∞ -map $t^* : W \times W \rightarrow T^*(X)$ in the following way:

$$t^*(\zeta, z) = \sum_{i=1}^p (\bar{h}_i(\zeta) - \bar{h}_i(z)) h_i^*(\zeta, z);$$

then it is clear that t^* is a well-defined C^∞ -map with $t^*(\zeta, z) \in T_z^*(X)$.

Also notice that

$$(I) \quad \langle t^*(\zeta, z), s(\zeta, z) \rangle = \varphi(\zeta, z) \sum_{i=1}^p |h_i(\zeta) - h_i(z)|^2.$$

Let

$$\begin{aligned} & \eta_h^{(s^*, \nu)}(\zeta, z) \\ &= -c'_n \varphi^{\nu-n+1} \sum_{l=0}^{n-2} \varphi^l \frac{\det[s_j^*, t_j^*, \overbrace{\partial_\zeta s_j^*}^l, \overbrace{\partial_\zeta t_j^*}^{n-l-2}]}{(\langle s^*, s \rangle^{l+1} (\sum_{i=1}^p |h_i(\zeta) - h_i(z)|^2)^{n-l-1}} \wedge \omega_\zeta(s(\zeta, z)) \end{aligned}$$

where $c'_n = (-1)^{n(n-1)/2} (2\pi i)^{-n}$; (s_1^*, \dots, s_n^*) and (t_1^*, \dots, t_n^*) are the expressions of $s^*(\zeta, z)$ and $t^*(\zeta, z)$, respectively, with respect to the local coordinates (U, χ) considered in §2; let us point out that $\omega_\zeta(s(\zeta, z))$, in the definition of $\eta_h^{(s^*, \nu)}(\zeta, z)$ above, is computed with respect to the same coordinates (U, χ) ; thus if (s_1, \dots, s_n) are the expressions of $s(\zeta, z)$ with respect to (U, χ) then $\omega_\zeta(s(\zeta, z)) = \partial_\zeta s_1 \wedge \dots \wedge \partial_\zeta s_n$. In the determinants which appear in the definition of $\eta_h^{(s^*, \nu)}$, j runs from $j = 1$ to $j = n$ forming the n rows of them.

Although the differential form $\eta_h^{(s^*, \nu)}(\zeta, z)$ is introduced locally, it turns out that it is invariantly defined since we have

LEMMA 3.1. $\eta_h^{(s^*, \nu)}(\zeta, z)$ is a globally defined $(n, n-1)$ -form, i.e., it is independent of the choice of local coordinates, with $\zeta \in W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ and a fixed $z \in D$.

Proof. Let $(\tilde{U}, \tilde{\chi})$ be another coordinate system at z ; let $(\tilde{s}_1^*, \dots, \tilde{s}_n^*)$, $(\tilde{t}_1^*, \dots, \tilde{t}_n^*)$ and $(\tilde{s}_1, \dots, \tilde{s}_n)$ be the expressions of s^* , t^* and s , respectively, with respect to $(\tilde{U}, \tilde{\chi})$. Then

$$\begin{aligned} (\tilde{s}_j) &= G \cdot (s_j), \\ (\tilde{s}_j^*) &= (G')^{-1} \cdot (s_j^*), \\ (\tilde{t}_j^*) &= (G')^{-1} \cdot (t_j^*), \end{aligned}$$

where $G = G(z)$ is the transition matrix from (U, χ) to $(\tilde{U}, \tilde{\chi})$ for the holomorphic vector bundle $T(X)$, in which case $(G')^{-1}$, the inverse of the transpose of G , is the transition matrix from (U, χ) to $(\tilde{U}, \tilde{\chi})$ for the bundle $T^*(X)$; of course $G = G(z)$ depends only on z ; here (s_j) denotes the transpose of (s_1, \dots, s_n) and similarly

for the others; the dot denotes matrix multiplication. Therefore,

$$\begin{aligned}(\partial_{\zeta} \tilde{s}_j) &= G \cdot (\partial_{\zeta} s_j), \\ (\overline{\partial}_{\zeta} \tilde{s}_j^*) &= (G')^{-1} \cdot (\overline{\partial}_{\zeta} s_j^*), \\ (\overline{\partial}_{\zeta} \tilde{t}_j^*) &= (G')^{-1} \cdot (\overline{\partial}_{\zeta} t_j^*).\end{aligned}$$

It follows from the above relations and properties of determinants with entries differential forms (see [3, p. 94]) that

$$\det[\tilde{s}_j^*, \tilde{t}_j^*, \overbrace{\partial_{\zeta} \tilde{s}_j^*}^l, \overbrace{\partial_{\zeta} \tilde{t}_j^*}^{n-l-2}] = \det[(G')^{-1}] \det[s_j^*, t_j^*, \overbrace{\partial_{\zeta} s_j^*}^l, \overbrace{\partial_{\zeta} t_j^*}^{n-l-2}]$$

and

$$\partial_{\zeta} \tilde{s}_1 \wedge \cdots \wedge \partial_{\zeta} \tilde{s}_n = \det(G) \partial_{\zeta} s_1 \wedge \cdots \wedge \partial_{\zeta} s_n.$$

Since $\det[(G')^{-1}] = [\det(G)]^{-1}$, it follows that $\eta_h^{(s^*, \nu)}(\zeta, z)$ is, indeed, independent of local coordinates. This completes the proof of the lemma.

REMARK. The holomorphic maps h_i^* ($i = 1, \dots, p$) are by no means unique; thus the differential form $\eta_h^{(s^*, \nu)}$ depends on the choice of h_i^* . We will come back to this point later.

LEMMA 3.2. *Let σ^* and τ^* be defined, for (ζ, z) with $\varphi(\zeta, z) \neq 0$ and $\zeta \in W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$, as follows:*

$$\begin{aligned}\sigma^*(\zeta, z) &= (\langle s^*(\zeta, z), s(\zeta, z) \rangle)^{-1} \cdot s^*(\zeta, z) \quad \text{and} \\ \tau^*(\zeta, z) &= (\langle t^*(\zeta, z), s(\zeta, z) \rangle)^{-1} \cdot t^*(\zeta, z).\end{aligned}$$

Then

$$\eta_h^{(s^*, \nu)}(\zeta, z) = -c_n' \cdot \varphi^{\nu} \cdot \sum_{l=0}^{n-2} \det[\sigma_j^*, \tau_j^*, \overbrace{\partial_{\zeta} \sigma_j^*}^l, \overbrace{\partial_{\zeta} \tau_j^*}^{n-l-2}] \wedge \omega_{\zeta}(s)$$

where σ_j^* and τ_j^* are the expressions of σ^* and τ^* with respect to the local coordinates (U, χ) and $\omega_{\zeta}(s) = \omega_{\zeta}(s(\zeta, z))$ is the differential form as in the definition of $\eta_h^{(s^*, \nu)}$ with respect to the same coordinates (U, χ) .

Proof. First notice that σ^* and τ^* are well-defined since $\varphi(\zeta, z) \neq 0$ implies $\langle s^*(\zeta, z), s(\zeta, z) \rangle \neq 0$ and together with $\zeta \notin Z(h - h(z))$, they imply also that $\langle t^*(\zeta, z), s(\zeta, z) \rangle \neq 0$; this is because of (I). It follows from the definition of σ^* and τ^* that

$$\begin{aligned}\overline{\partial}_{\zeta} \sigma_j^* &= (\langle s^*, s \rangle)^{-1} \overline{\partial} s_j^* + s_j^* \overline{\partial}_{\zeta} [(\langle s^*, s \rangle)^{-1}] \quad \text{and} \\ \overline{\partial}_{\zeta} \tau_j^* &= (\langle t^*, s \rangle)^{-1} \overline{\partial} t_j^* + t_j^* \overline{\partial}_{\zeta} [(\langle t^*, s \rangle)^{-1}].\end{aligned}$$

Now the lemma follows from the above equations, from (I) and properties of determinants.

We are ready now to prove that $\eta_h^{(s^*, \nu)}$ is a $\bar{\partial}_\zeta$ -primitive of $K^{(s^*, \nu)}$. More precisely we have

THEOREM 3.1. *Let D be a domain on the Stein manifold X , $\dim_{\mathbb{C}} X = n$, and $h : W \rightarrow \mathbb{C}^p$ a holomorphic map, $p \leq n - 1$, where W is a Stein neighborhood of \bar{D} . Let $s^* = s^*(\zeta, z)$ and $K^{(s^*, \nu)}$ be as in § 2 and let $\eta_h^{(s^*, \nu)}$ be the above constructed differential form. Then, for a fixed $z \in D$, we have*

$$\bar{\partial}_\zeta \eta_h^{(s^*, \nu)}(\zeta, z) = d_\zeta \eta_h^{(s^*, \nu)}(\zeta, z) = K^{(s^*, \nu)}(\zeta, z)$$

for $\zeta \in W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$.

Proof. Let us consider first (ζ, z) with $\varphi(\zeta, z) \neq 0$. Then, by the definition of σ^* and τ^* ,

$$(1) \quad \langle \sigma^*, s \rangle = 1 \quad \text{and} \quad \langle \tau^*, s \rangle = 1.$$

Working always with a fixed coordinate system (U, χ) at z , (1) can be written as

$$(2) \quad \sum_{j=1}^n \sigma_j^* s_j = 1 \quad \text{and} \quad \sum_{j=1}^n \tau_j^* s_j = 1.$$

It follows from (2) that $s_j \neq 0$ for at least one $j \in \{1, \dots, n\}$. We may assume, without loss of generality, that $s_1 \neq 0$. Then, by Lemma 3.2,

$$(3) \quad \eta_h^{(s^*, \nu)} = -\frac{c'_n}{s_1} \varphi^\nu \sum_{l=0}^{n-2} \det \begin{bmatrix} \sigma_1^* s_1 & \tau_1^* s_1 & \overbrace{\bar{\partial}(\sigma_1^* s_1)}^l & \overbrace{\bar{\partial}(\tau_1^* s_1)}^{n-l-2} \\ \sigma_j^* & \tau_j^* & \bar{\partial} \sigma_j^* & \bar{\partial} \tau_j^* \end{bmatrix} \wedge \omega_\zeta(s);$$

in the determinants in (3) j runs from $j = 2$ to $j = n$ forming the 2nd up to the n th row of them. In obtaining (3) we also used the fact that $s_1 = s_1(\zeta, z)$ is holomorphic in ζ (throughout this proof $\bar{\partial} = \bar{\partial}_\zeta$). Next, multiplying the j th-rows of each determinant in (3) ($2 \leq j \leq n$) by s_j and adding them to the first row of it we obtain, in view of (2),

$$(4) \quad \eta_h^{(s^*, \nu)} = -\frac{c'_n}{s_1} \varphi^\nu \sum_{l=0}^{n-2} \det \begin{bmatrix} 1 & 1 & \overbrace{0}^l & \overbrace{0}^{n-l-2} \\ \sigma_j^* & \tau_j^* & \bar{\partial} \sigma_j^* & \bar{\partial} \tau_j^* \end{bmatrix} \wedge \omega_\zeta(s).$$

Applying $\bar{\partial} = \bar{\partial}_\zeta$ to both sides of (4) and using the fact that φ is holomorphic in ζ , we obtain

$$\begin{aligned} \bar{\partial}\eta_h^{(s^*, \nu)} = & -\frac{c'_n}{s_1}\varphi^\nu \sum_{l=0}^{n-2} \left(\det \begin{bmatrix} 0 & 1 & \overbrace{0}^l & \overbrace{0}^{n-l-2} \\ \bar{\partial}\sigma_j^* & \tau_j^* & \bar{\partial}\sigma_j^* & \bar{\partial}\tau_j^* \end{bmatrix} \wedge \omega_\zeta(s) \right. \\ & \left. + \det \begin{bmatrix} 1 & 0 & \overbrace{0}^l & \overbrace{0}^{n-l-2} \\ \sigma_j^* & \bar{\partial}\tau_j^* & \bar{\partial}\sigma_j^* & \bar{\partial}\tau_j^* \end{bmatrix} \wedge \omega_\zeta(s) \right) \end{aligned}$$

or, after a computation,

$$\begin{aligned} (5) \quad \bar{\partial}\eta_h^{(s^*, \nu)} = & -\frac{c'_n}{s_1}\varphi^\nu \sum_{l=0}^{n-2} (\det[\overbrace{\bar{\partial}\sigma_j^*}^l, \overbrace{\bar{\partial}\tau_j^*}^{n-l-1}] - \det[\overbrace{\bar{\partial}\sigma_j^*}^{l+1}, \overbrace{\bar{\partial}\tau_j^*}^{n-l-2}]) \wedge \omega_\zeta(s) \\ = & \frac{c'_n}{s_1}\varphi^\nu \det[\overbrace{\bar{\partial}\sigma_j^*}^{n-1}] \wedge \omega_\zeta(s) - \frac{c'_n}{s_1}\varphi^\nu \det[\overbrace{\bar{\partial}\tau_j^*}^{n-1}] \wedge \omega_\zeta(s); \end{aligned}$$

all the determinants in (5) are $(n-1) \times (n-1)$ and j runs from $j=2$ to $j=n$ forming their $(n-1)$ rows. Now we claim that

$$(6) \quad \frac{c'_n}{s_1}\varphi^\nu \det \left(\left[\overbrace{\bar{\partial}\sigma_j^*}^{n-1} \right]_{j=2}^n \right) \wedge \omega_\zeta(s) = K^{(s^*, \nu)}$$

and

$$(7) \quad \frac{1}{s_1} \det \left[\overbrace{\bar{\partial}\tau_j^*}^{n-1} \right]_{j=2}^n = 0.$$

First let us prove (6). It follows from the definition of $K^{(s^*, \nu)}$ and the relations between s_j^* and σ_j^* (exactly as in the proof of Lemma 2.2) that

$$K^{(s^*, \nu)} = \frac{c'_n\varphi^\nu}{s_1} \det \left(\left[\begin{array}{cc} \sigma_1^* s_1 & \overbrace{\bar{\partial}(\sigma_1^* s_1)}^{n-1} \\ \sigma_j^* & \bar{\partial}\sigma_j^* \end{array} \right]_{j=2}^n \right) \wedge \omega_\zeta(s).$$

Therefore, in view of (2),

$$K^{(s^*, \nu)} = \frac{c'_n\varphi^\nu}{s_1} \det \left(\left[\begin{array}{cc} 1 & \overbrace{0}^{n-1} \\ \sigma_j^* & \bar{\partial}\sigma_j^* \end{array} \right]_{j=2}^n \right) \wedge \omega_\zeta(s)$$

which immediately implies (6).

Similarly, to prove (7) we write its left-hand side (in view of the relation between τ^* and t^*) as follows:

$$(8) \quad \frac{1}{s_1} \det[\overbrace{\bar{\partial} \tau_j^*}^{n-1}] = (\langle t^*, s \rangle)^{-n} \det([t_j^*, \overbrace{\bar{\partial} t_j^*}^{n-1}]_{j=1}^n).$$

Let h_{ij}^* ($1 \leq i \leq p$, $1 \leq j \leq n$) be the expressions of h_i^* with respect to the local coordinates (U, χ) , i.e.,

$$h_i^*(\zeta, z) = \sum_{j=1}^n h_{ij}^*(\zeta, z)(d\chi_j)_z.$$

Recalling that $t^* = \sum_{i=1}^p (\bar{h}_i - \bar{h}_i(z))h_i^*$ we obtain

$$(9) \quad t_j^* = \sum_{i=1}^p (\bar{h}_i - \bar{h}_i(z))h_{ij}^* \quad \text{and} \quad \bar{\partial} t_j^* = \sum_{i=1}^p h_{ij}^* \bar{\partial} \bar{h}_i,$$

since h_{ij}^* are holomorphic in ζ . Now to prove (7) we distinguish two cases:

1st case: $p \leq n - 2$; in this case

$$(10) \quad \bar{\partial} t_{j_1} \wedge \cdots \wedge \bar{\partial} t_{j_{n-1}} = 0$$

for $1 \leq j_1 < \cdots < j_{n-1} \leq n$; this follows from (9); but (10) and (8) imply (7) in this case.

2nd case: $p = n - 1$; in this case, substituting (9) into the right-hand side of (8), we obtain

$$(11) \quad \det[t_j^*, \overbrace{\bar{\partial} t_j^*}^{n-1}]_{j=1}^n = p! \det \left(\left[\sum_{i=1}^p (h_i - h_i(z))h_{ij}^*, h_{1j}^*, \dots, h_{pj}^* \right]_{j=1}^n \right) \\ \times \overline{\partial \bar{h}_1} \wedge \cdots \wedge \overline{\partial \bar{h}_p} = 0;$$

since (11) and (8) imply (7), the proof of (7) is complete. Finally (7), (6) and (5) imply the formula of the theorem in the case $\varphi(\zeta, z) \neq 0$ and, since the set $\{\varphi(\zeta, z) \neq 0\}$ is dense, this completes the proof of the theorem.

REMARK. As we pointed out before, $\eta_h^{(s^*, \nu)}$ depends on the choice of $\{h_i^*\}_{i=1}^p$; in the case $p \leq n - 2$, however, this dependence is not essential in a sense which we will make precise now.

Let $[\eta_h^{(s^*, \nu)}]$ denote the cohomology class of $\eta_h^{(s^*, \nu)}$ in the Dolbault cohomology group $H_{\bar{\partial}}^{(n, n-2)}(V_z - Z(h - h(z)))$ where V_z is an open neighborhood of ∂D with $\bar{V}_z \subset W \cap \text{Dom}(s^*(\cdot, z))$ (here z is fixed, as usual, and ζ is the variable).

Let $(h_i^*)' : W \times W \rightarrow T^*(X)$, $i = 1, \dots, p$, be holomorphic maps, with $(h_i^*)'(\zeta, z) \in T_z^*(X)$ and $\langle (h_i^*)', s \rangle = \varphi \cdot (h_i - h_i(z))$, i.e., another choice for h_i^* and let $(\eta_h^{(s^*, \nu)})'$ denote the $\bar{\partial}$ -primitive of $K^{(s^*, \nu)}$ in $W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ associated to $(h_i^*)'$. We claim that

$$[\eta_h^{(s^*, \nu)}] = [(\eta_h^{(s^*, \nu)})'];$$

in other words, the cohomology class $[\eta_h^{(s^*, \nu)}]$ is independent of the choice of h_i^* . To prove this we argue as follows. Let $\psi(\zeta, z)$ be a C^∞ function with $0 \leq \psi(\zeta, z) \leq 1$, having compact support contained in $W \cap \text{Dom}(s^*(\cdot, z))$, which is identically one in a neighborhood of \bar{V}_z . Let $\bar{s}(\zeta, z)$ denote a Leray section for (D, s, φ) with $\text{Dom}(\bar{s}(\cdot, z)) = W$ and defined for $z \in W$; such a Leray section always exists (see [4, p. 164]; let us point out that $\bar{s}(\zeta, z)$ is not the complex conjugate of $s(\zeta, z)$). Define

$$\lambda^*(\zeta, z) = \varphi^{\nu_1}(\zeta, z) \left[\frac{\psi(\zeta, z)}{\langle s^*(\zeta, z), s(\zeta, z) \rangle} s^*(\zeta, z) + \frac{1 - \psi(\zeta, z)}{\langle \bar{s}(\zeta, z), s(\zeta, z) \rangle} \bar{s}(\zeta, z) \right]$$

where $\nu_1 = \max(\nu_0, \nu^*)$. Since

$$\langle \lambda^*(\zeta, z), s(\zeta, z) \rangle = \varphi^{\nu_1}(\zeta, z),$$

it follows that λ^* is a Leray section for (D, s, φ) ; thus we may associate to λ^* the $\bar{\partial}$ -primitives $\eta_h^{(\lambda^*, \nu)}$ and $(\eta_h^{(\lambda^*, \nu)})'$ of $K^{(\lambda^*, \nu)}$, in $W - Z(h - h(z))$, corresponding to h_i^* and $(h_i^*)'$. It follows from Theorem 3.1 that

$$\bar{\partial}(\eta_h^{(\lambda^*, \nu)} - (\eta_h^{(\lambda^*, \nu)})') = K^{(\lambda^*, \nu)} - K^{(\lambda^*, \nu)} = 0 \quad \text{in } W - Z(h - h(z));$$

but $W - Z(h - h(z))$ is $(n - 3)$ -complete (here $p \leq n - 2$; see [7, p. 435]) whence $H^{n-2}(W - Z(h - h(z)), \mathcal{O}^n) = 0$ (see [1, p. 250]); therefore, from Dolbault's theorem, there exists an $(n, n - 3)$ -form θ in $W - Z(h - h(z))$ with

$$\eta_h^{(\lambda^*, \nu)} - (\eta_h^{(\lambda^*, \nu)})' = \bar{\partial}\theta.$$

Since $\psi \equiv 1$ in a neighborhood of \bar{V}_z , it follows from the proof of Lemma 3.2 that

$$\begin{aligned}\eta_h^{(\lambda^*, \nu)} &= \eta_h^{(s^*, \nu)} \quad \text{and} \\ (\eta_h^{(\lambda^*, \nu)})' &= (\eta_h^{(s^*, \nu)})' \quad \text{in } V_z - Z(h - h(z))\end{aligned}$$

whence

$$\eta_h^{(s^*, \nu)} - (\eta_h^{(s^*, \nu)})' = \bar{\partial}\theta \quad \text{in } V_z - Z(h - h(z)).$$

This proves the claim that the cohomology class $[\eta_h^{(s^*, \nu)}]$ does not depend on the choice of h_i^* . Notice also that if Γ is an open subset of ∂D with $\bar{\Gamma} \subset \partial D - Z(h - h(z))$ and f is a smooth CR-function on Γ then

$$f\eta_h^{(s^*, \nu)} - f(\eta_h^{(s^*, \nu)})' = \bar{\partial}(f\theta) = d(f\theta)$$

whence we obtain

$$\int_c f\eta_h^{(s^*, \nu)}(\cdot, z) = \int_c f(\eta_h^{(s^*, \nu)})'(\cdot, z)$$

for every $(2n - 2)$ -dimensional cycle c in Γ .

The following theorem is a generalization of the Henkin-Leiterer version of the Cauchy-Fantappiè formula; its proof is similar to the proof of Proposition 2.4 in [6, p. 185].

THEOREM 3.2. *Let D be a domain on the Stein manifold X and let h , W , $K^{(s^*, \nu)}$ and $\eta_h^{(s^*, \nu)}$ be as in Theorem 3.1. Let $z \in D$ and let $\Gamma \subset \partial D$ be an open subset of ∂D with $\partial\Gamma$ smooth and so that $\Gamma \supset (\partial D) \cap Z(h - h(z))$. Then for $f \in C(\bar{\Gamma} \cup D) \cap \mathcal{O}(D)$, i.e., continuous on $\bar{\Gamma} \cup D$ and holomorphic in D , we have the following representation formula:*

$$f(z) = \int_{\zeta \in \Gamma} f(\zeta)K^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial\Gamma} f(\zeta)\eta_h^{(s^*, \nu)}(\zeta, z).$$

Proof. Let $G \subset D$ be an open subset of D so that $\partial G \cap \partial D = \Gamma$ and $D \cap Z(h - h(z)) \subset G$. We also assume that $\partial G = \Gamma \cup \Gamma_0$ where $\Gamma_0 = \partial G \cap \bar{D} \subset W$. Then, by [4, Theorem 4.3.4], we have

$$\begin{aligned}(1) \quad f(z) &= \int_{\partial G} fK^{(s^*, \nu)}(\cdot, z) \\ &= \int_{\Gamma} fK^{(s^*, \nu)}(\cdot, z) + \int_{\Gamma_0} fK^{(s^*, \nu)}(\cdot, z).\end{aligned}$$

Since $\Gamma_0 \subset W - Z(h - h(z))$ it follows from Theorem 3.1, Stokes's theorem and the fact that f is holomorphic in D that

$$(2) \quad \begin{aligned} \int_{\Gamma_0} f K^{(s^*, \nu)}(\cdot, z) &= \int_{\partial\Gamma_0} f \eta_h^{(s^*, \nu)}(\cdot, z) \\ &= - \int_{\partial\Gamma} f \eta_h^{(s^*, \nu)}(\cdot, z). \end{aligned}$$

Now the formula of the theorem follows from (1) and (2).

REMARK. If $\Gamma = \partial D$ then $\partial\Gamma = \emptyset$ and the formula of Theorem 3.2 reduces to that of [4, Theorem 4.3.4].

4. Extending CR-functions. Let (D, s, φ) , W and s^* be as in §3; we assume furthermore that $s^*(\zeta, z)$ is defined, as a Leray section, for all $(\zeta, z) \in W \times W$. Let E be a closed subset of ∂D so that each connected component of $\partial D - E$ contains a peak point for $\mathcal{O}(\overline{D})$, i.e., a point ζ_0 for which there exists a $g \in \mathcal{O}(\overline{D})$ with $|g(\zeta_0)| > |g(\zeta)|$ for $\zeta \in \overline{D} - \{\zeta_0\}$. For each $z \in W - E$ let

$$\mathcal{P}_z = \{h : W \rightarrow \mathbb{C}^{n-2} : h \text{ holomorphic, } z \in Z(h) \text{ and } Z(h) \cap E = \emptyset\}.$$

We can now state a criterion for extendability of CR-functions defined on $\partial D - E$; a version of it in \mathbb{C}^n , with the Bochner-Martinelli kernel in place of the Henkin-Leiterer type kernel, is in [2]; its proof is based on ideas from [6] and [5].

THEOREM 4.1. *With notation as above, suppose that $\mathcal{P}_z \neq \emptyset$ for each $z \in W - E$ and let f be a smooth CR-function on $\partial D - E$. Then a necessary and sufficient condition that f extends to a holomorphic function in D is*

$$(1) \quad \int_{\zeta \in \partial\Gamma} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z) = \int_{\zeta \in \partial\Gamma} f(\zeta) \eta_g^{(s^*, \nu)}(\zeta, z)$$

for $h, g \in \mathcal{P}_z$, $\Gamma \supset (\partial D) \cap (Z(h) \cup Z(g))$ open (in ∂D) with $\overline{\Gamma} \subset \partial D - E$ and $\partial\Gamma$ smooth and $z \in W - E$.

REMARKS. (i) Of course $\eta_h^{(s^*, \nu)}$ and $\eta_g^{(s^*, \nu)}$ are $\bar{\partial}$ -primitives of $K^{(s^*, \nu)}$ in $W \cap \text{Dom}(s^*(\cdot, z)) - Z(h - h(z))$ and $W \cap \text{Dom}(s^*(\cdot, z)) - Z(g - g(z))$ respectively. As we pointed out before, in the remark following the proof of Theorem 3.1, given Γ , the value of the integral

$$\int_{\zeta \in \partial\Gamma} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z)$$

is uniquely determined by h , i.e., it is independent of the choice of h_i^* .

(ii) Observe that if Γ' has the properties required for Γ then, by Theorem 3.1 and Stokes's theorem,

$$\begin{aligned} & \int_{\zeta \in \partial \Gamma'} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial \Gamma} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z) \\ &= \int_{\zeta \in (\Gamma' - \Gamma) \cup (\Gamma - \Gamma')} f(\zeta) K^{(s^*, \nu)}(\zeta, z) \end{aligned}$$

with the various parts of $(\Gamma' - \Gamma) \cup (\Gamma - \Gamma')$ appropriately oriented; therefore if (1) holds for Γ it will also hold for Γ' .

Proof of Theorem 4.1. First the necessity of (1) follows immediately from Theorem 3.2.

Now we prove sufficiency of (1), i.e., we assume that (1) holds and we prove that f extends to a holomorphic function in D . To this end let $z \in W - \partial D$ and let $h \in \mathcal{P}_z$; choose Γ and define

$$(2) \quad F(z) = \int_{\zeta \in \Gamma} f(\zeta) K^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial \Gamma} f(\zeta) \eta_h^{(s^*, \nu)}(\zeta, z).$$

Condition (1) now guarantees that $F(z)$ is well-defined, i.e., it is independent of the various choices (basically of the choice of h , in view of the previous remarks). Next we prove that F is holomorphic; for this we compute $\bar{\partial}_z F$.

$$(3) \quad \bar{\partial}_z F(z) = \int_{\zeta \in \Gamma} f(\zeta) \bar{\partial}_z K^{(s^*, \nu)}(\zeta, z) - \int_{\zeta \in \partial \Gamma} f(\zeta) \bar{\partial}_z \eta_h^{(s^*, \nu)}(\zeta, z).$$

This computation is justified, in part, by the explicit formula for $\eta_h^{(s^*, \nu)}$; notice that if $h \in \mathcal{P}_z$ then $h - h(z') \in \mathcal{P}_{z'}$ for z' close to z ; thus, in (3),

$$\bar{\partial}_z \eta_h^{(s^*, \nu)}(\zeta, z) = \bar{\partial}_{z'} \eta_h^{(s^*, \nu)}(\zeta, z')|_{z'=z};$$

the point here is that h , too, depends on z . But

$$(4) \quad \bar{\partial}_z K^{(s^*, \nu)}(\zeta, z) = \bar{\partial}_\zeta \tilde{K}(\zeta, z)$$

where

$$\tilde{K}(\zeta, z) = -(n-1)c'_n \varphi^\nu \frac{\det[s_j^*, \bar{\partial}_z s_j^*, \overbrace{\bar{\partial}_\zeta s_j^*}^{n-2}] \wedge \omega_\zeta(s)}{\langle s^*, s \rangle^n}$$

(with the notation of §3; in particular we make use of a local coordinate system (U, χ) as in §3; the independence of $\tilde{K}(\zeta, z)$ of the

choice of (U, χ) is proved exactly as Lemma 3.1); this is proved in [3, p. 107]; $\tilde{K}(\zeta, z)$ is defined for $z \in W$ and $\zeta \in W - \{z\}$.

But, by Theorem 3.1,

$$(5) \quad \bar{\partial}_\zeta(\bar{\partial}_z \eta_h^{(s^*, \nu)}) = \bar{\partial}_z K^{(s^*, \nu)}(\zeta, z).$$

It follows from (4) and (5) that

$$\bar{\partial}_\zeta[\bar{\partial}_z \eta_h^{(s^*, \nu)} - \tilde{K}] = 0 \quad \text{in } W - Z(h).$$

Since $W - Z(h)$ is $(n - 3)$ -complete, it follows that there exists an $(n, n - 3)$ -form $\mu(\zeta, z)$ in ζ , whose coefficients are $(0, 1)$ -forms in z (locally in (U, χ)), so that

$$(6) \quad \bar{\partial}_z \eta_h^{(s^*, \nu)} - \tilde{K} = \bar{\partial}_\zeta \mu \quad \text{in } W - Z(h)$$

(this argument is similar to the remark following the proof of Theorem 3.1).

But (3), in view of (4) and (6), becomes:

$$\bar{\partial}_z F = \int_\Gamma f \bar{\partial}_\zeta \tilde{K} - \int_{\partial\Gamma} f \tilde{K} - \int_{\partial\Gamma} f \bar{\partial}_\zeta \mu,$$

from which, by Stokes's theorem and the fact that f is a CR-function we obtain $\bar{\partial}_z F = 0$; thus F is holomorphic in $W - \partial D$. An argument similar to that in [6, pp. 188–190] proves that $F = 0$ in $W - \bar{D}$ and that $F|_D$ is indeed a holomorphic extension of f . For the Plemelj type formula in the setting of Stein manifolds, which is required here, see [5].

This completes the proof of Theorem 4.1.

Comments. (i) The point of using the differential form \tilde{K} in the proof of Theorem 4.1 is that, although $\eta_h^{(s^*, \nu)}$ is not defined on $Z(h)$, $\bar{\partial}_z \eta_h^{(s^*, \nu)}$ is $\bar{\partial}_\zeta$ -cohomologous to \tilde{K} in a neighborhood of $\partial\Gamma$, and \tilde{K} is defined in $W - \{z\}$.

(ii) A point which may be investigated further is to find geometric conditions under which equality (1) holds; for example, if $\dim_{\mathbb{C}}(Z(h) \cap Z(g)) \geq 1$, does it follow that (1) holds?

(iii) If $h, q \in \mathcal{P}_z$ and $h_1 = \cdots = h_{n-2}$ and $g_1 = \cdots = g_{n-2}$ ($n \geq 3$) then the difference $\eta_h^{(s^*, \nu)} - \eta_g^{(s^*, \nu)}$ is $\bar{\partial}$ -exact in $W - (Z(h) \cup Z(g))$ (this is proved in [5]), which implies that (1) holds in this case.

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