Pacific Journal of Mathematics

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Vol. 147, No. 2 February 1991

PACIFIC JOURNAL OF MATHEMATICS
Vol. 147, No. 2, 1991

RINGS OF DIFFERENTIAL OPERATORS ON ONE DIMENSIONAL ALGEBRAS

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Let k be an algebraically closed field of characteristic zero, and A a finitely generated k-algebra of Krull dimension at most one. In this paper we study the ring of differential operators $\mathscr{D}(A)$. For example we obtain necessary and sufficient conditions for $\mathscr{D}(A)$ to be a direct sum of Simple rings, or to be left or right Noetherian.

0.1. Let k be an algebraically closed field of characteristic zero and A a finitely generated (commutative) k-algebra. The primary purpose of this paper is to study the ring $\mathscr{D}(A)$ of differential operators on A when $\dim(A)$, the Krull dimension of A, is at most one. If A is also reduced or is a domain $\mathscr{D}(A)$ has been studied extensively in **[10]** and **[15]** and we prove analogues of the main results of these papers. For example

THEOREM A. *Let A be a finitely generated k-algebra with Krull dimension at most one. Then*

(a) *$\mathscr{D}(A)$ is right Noetherian and finitely generated as a k-algebra.*

(b) *$\mathscr{D}(A)$ is left Noetherian if and only if A has an artinian quotient ring.*

0.2. One of the main ideas in **[15]** is to compare $\mathscr{D}(A)$, for A a domain, to $\mathscr{D}(\widetilde{A})$ where $\widetilde{A}$ is the integral closure of A. In particular, **[15**, Theorem B**]** gives necessary and sufficient conditions for $\mathscr{D}(A)$ and $\mathscr{D}(\widetilde{A})$ to be Morita equivalent. We prove a similar result here. We denote the nilradical of A by $N(A)$, and say that A has *injective normalisation* if every maximal ideal of $A/N(A)$ is contained in a unique maximal ideal of its integral closure.

THEOREM B. *Let A be a finitely generated algebra with* $\dim(A) \leq 1$ *and let $\widetilde{A}$ be the integral closure of $A/N(A)$. Then the following are equivalent:*

(1) *$\mathscr{D}(A)$ is Morita equivalent to $\mathscr{D}(\widetilde{A})$.*

(2) *$\mathscr{D}(A)$ is a direct sum of simple rings.*

(3) *$A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$ where each A_i is a primary ring with injective normalisation.*

0.3. As geometric motivation for the study of differential operators on non-reduced algebras we mention a result of S. P. Smith [**14**]. Let $R = k[x, y]$ and $f \in R$ a polynomial defining an irreducible curve X. If $\mathscr{O} = \mathscr{O}(X)$ it is known that the $\mathscr{D}(R)$-module $\mathscr{O}_f/\mathscr{O}$ has finite length and has a unique minimal submodule. There is some interest in describing this submodule. The main result of [**14**] is that when the normalisation map $\widetilde{X} \to X$ is injective $\mathscr{O}_f/\mathscr{O}$ is a simple $\mathscr{D}(R)$-module. This follows easily from the fact that $\mathscr{D}(R/f^nR)$ is a simple ring. In turn simplicity of $\mathscr{D}(R/f^nR)$ follows from a corollary to Theorem B in this case, see 2.6.

0.4. We outline the proof of Theorem A. The case where A has an artinian quotient ring is handled in §2. In general, there is a factor algebra $\overline{A}$ of A which has an artinian quotient ring and a surjective homomorphism $\phi: \mathscr{D}(A) \to \mathscr{D}(\overline{A})$, see 4.2. Since $\mathscr{D}(\overline{A})$ is Noetherian by the results of §2, it will suffice to prove

THEOREM C. *With the notation of* 4.2, *$J = \operatorname{Ker}\phi$ has finite length as a right $\mathscr{D}(A)$-module.*

For an arbitrary k-algebra A we define a *standard $\mathscr{D}(A)$-module* to be a right $\mathscr{D}(A)$-module of the form $\mathscr{D}(A, A/M)$ for some maximal ideal M of A. Standard modules have been studied in [**13**]. In §3 we show that if V is any A-module of finite length then $\mathscr{D}(A, V)$ is a direct sum of standard modules (Corollary 3.4). We also explain how Matlis duality may be used to study standard modules. Finally in Proposition 4.3 we show that if $\dim A \leq 1$ then any standard module has finite length. Theorem C follows easily from these results.

0.5. Let M be a maximal ideal of A and denote by A_M, $\widehat{A}_M$ respectively the localization and completion of A at M. Several results, including Theorem B, remain true when A is replaced by A_M or $\widehat{A}_M$. In 1.1 we abstract the properties of A, A_M and $\widehat{A}_M$ which are necessary for the proofs. In some situations, for example when using Matlis duality, the results for $\mathscr{D}(\widehat{A}_M)$ are more pertinent than those for $\mathscr{D}(A)$.

0.6. In §5 we describe a method which can be used to compute $\mathscr{D}(A)$ explicitly in a number of cases.

Any unexplained notation in this paper will be as in **[11]**. We have tried to merely sketch proofs which are routine adaptations of those in **[15]**. However the proof of the implication (2) $\Rightarrow$ (3) of Theorem B is new even in the case where A is a domain.

1. Preliminary results.

1.1. Let M be a maximal ideal of the finitely generated algebra A. Since several results on $\mathscr{D}(A)$, $\mathscr{D}(A_M)$ and $\mathscr{D}(\widehat{A}_M)$ can be proved simultaneously we abstract the required properties of the commutative algebras here. We assume

(a) $\dim(A) \leq 1$ and A has an artinian quotient ring F.

Next we prove

(b) There exists a subring B of F containing A such that B is a finitely generated A-module and $(B + N(F))/N(F)$ is the integral closure of $\overline{A} = (A + N(F))/N(F)$ in $F/N(F)$.

If $x \in F$ and $x + N(F)$ is integral over $\overline{A}$, then x is integral over A, so $A[x]$ is a finitely generated A-module. Since the integral closure of $\overline{A}$ is a finitely generated $\overline{A}$-module the result follows.

(c) There is a subalgebra $\overline{B}$ of B such that $B = \overline{B} \oplus N(B)$.

Since $B/N(B)$ is a direct sum of Dedekind domains and idempotents may be lifted over a nilpotent ideal, we have $B = B_1 \oplus \cdots \oplus B_r$ where each $B_i/N(B_i)$ is the coordinate ring of a nonsingular curve. By the infinitesimal lifting property, [**6**, Exercise II.8.6] there exists a subalgebra $\overline{B}_i$ of B_i with $B_i = \overline{B}_i \oplus N(B_i)$. Then (c) follows with $\overline{B} = \overline{B}_1 \oplus \cdots \oplus \overline{B}_r$.

We next show that the analogues of (a)–(c) remain valid when A is replaced by A_M or $\widehat{A}_M$.

For a multiplicatively closed set S in a commutative ring A we set $S(0) = \{a \in A \mid sa = 0 \text{ for some } s \in S\}$. Let $S = A \setminus M$, and let $0 = K_1 \cap \cdots \cap K_n$ be a minimal primary decomposition where K_i is P_i-primary. Assume the K_i are numbered so that $P_i \subseteq M$ if and only if $1 \leq i \leq m$. Then by [**1**, Proposition 4.9] $S(0) = K_1 \cap \cdots \cap K_m$. In particular $S(0)$ has no embedded primes, so passing to $A/S(0)$ we may assume S consists of regular elements without changing assumption (a). Since $S = k^*(1 + M)$ we have $\bigcap_n M^n = 0$ by Krull's intersection theorem. Hence $A \subseteq A_M \subseteq \widehat{A}_M$.

Let J be the Jacobson radical of the semilocal ring B_S. We prove

(a)$'$ $\dim(A_M) \leq 1$, $\dim(\widehat{A}_M) \leq 1$, F is the full quotient ring of A_M and $\widehat{A}_M$ has an artinian quotient ring Q.

(b)′ B_S (resp. $(\widehat{B}_S)_J$) is finitely generated as an A_M (resp. $\widehat{A}_M$) module and $B_S/N(B_S)$ (resp. $(\widehat{B}_S)_J)/N((\widehat{B}_S)_J)$ is the integral closure of $A_M/N(A_M)$ (resp. $\widehat{A}_M/N(\widehat{A}_M)$).

(c)′ Let $\overline{S}$ (resp. $\overline{J}$) be the image of S (resp. J) under the natural map $B \to \overline{B}$ (resp. $B_S \to \overline{B}_{\overline{S}}$). Then $B_S = \overline{B}_{\overline{S}} \oplus N(B_S)$ and $(\widehat{B}_S)_J = (\widehat{\overline{B}}_{\overline{S}})_{\overline{J}} \oplus N(\widehat{B}_S)_J$.

Applying the exact functor $- \otimes_B B_S$ to the sequence $0 \to N(B) \to B \xrightarrow{\pi} \overline{B} \to 0$ we obtain exactness of $0 \to N(B)_S \to B_S \xrightarrow{\pi_S} \overline{B}_{\overline{S}} \to 0$ where $\overline{S} = \pi(S)$. If $i\colon \overline{B} \to B$ is the inclusion map, then elements of $i(\overline{S})$ are units in B_S so there is a map $i_S\colon \overline{B}_{\overline{S}} \to B_S$ such that $\pi_S i_S$ is the identity on $\overline{B}_{\overline{S}}$. Thus $B_S = \overline{B}_{\overline{S}} \oplus N(B_S)$. To simplify notation set $\widehat{D} = (\widehat{B}_S)_J$ and $\widehat{E} = (\widehat{\overline{B}_{\overline{S}}})_{\overline{J}}$. A similar proof shows that $\widehat{D} = \widehat{E} \oplus N(\widehat{D})$.

We claim that $\widehat{D}$ has an artinian quotient ring. It suffices to show that $N(\widehat{D})$ is torsion free as an $\widehat{E}$-module. If x is any regular element of $\widehat{E}$, then since $\widehat{E}$ is the completion of a direct sum of Dedekind domains at a semimaximal ideal, there is a multiple y of x which is a regular element of $\overline{B}_{\overline{S}}$. Since B_S has an artinian quotient ring, $0 \to N(B_S) \to N(B_S)$ is exact where the map is multiplication by y. Applying exactness of the completion shows that y and hence x are nonzero divisors on $N(\widehat{D})$ as required.

Finally we claim that $\widehat{D}/\widehat{A}_M$ has finite length as an $\widehat{A}_M$-module. This is clear if $A_M = B_S$ so assume not and let $I = \operatorname{ann}_{A_M}(B_S/A_M)$ be the conductor. Since I is a proper ideal of A_M we have $I \subseteq M \subseteq J$. However B_S/I has finite length so $J^n \subseteq I$ for some n. It follows that the J-adic and I-adic topologies on B_S coincide and $(\widehat{B}_S)_J = (\widehat{B}_S)_I$. Similarly $\widehat{A}_M = \widehat{A}_I$. Now take the I-adic completion of the sequence of A_M-modules $0 \to A_M \to B_S \to B_S/A_M \to 0$. The last term is unchanged, since it has finite length and this proves the claim. Hence the conductor of $\widehat{D}$ in $\widehat{A}_M$ contains a regular element of $\widehat{D}$ and so the artinian quotient ring of $\widehat{D}$ is also the quotient ring of $\widehat{A}_M$. This proves the analogues of properties (a)–(c) for $\widehat{A}_M$ and their analogues for A_M are similar.

1.2. Let J be an ideal in a ring A such that $\bigcap J^n = 0$ and denote by $\widehat{A}$ the completion of A in the J-adic topology. If $d \in \mathscr{D}^q(A)$, an easy induction shows that $d(J^{n+q}) \subseteq J^n$. Therefore if $a = (a_0, a_1, \dots)$ is a Cauchy sequence in the J-adic topology, so also is $d(a) = (d(a_0), d(a_1), \ldots)$ and d extends to an element of $\mathscr{D}^q(\widehat{A})$. Furthermore if $\partial \in \mathscr{D}^q(\widehat{A})$ and ∂ vanishes on A then $\partial = 0$. To see this suppose $\alpha \in \widehat{A}$ and write $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in A$ and $\alpha_2 \in \widehat{J}^{n+q}$. We obtain $\partial(\alpha) = \partial(\alpha_2) \in \widehat{J}^n$ for all n so $\partial(\alpha) = 0$. Thus any element $d \in \mathscr{D}^q(A)$ extends uniquely to an element of $\mathscr{D}^q(\widehat{A})$ and we obtain maps

$$\widehat{A} \otimes_A \mathscr{D}^q(A) \to \mathscr{D}^q(\widehat{A}) \quad \text{and} \quad \widehat{A} \otimes_A \mathscr{D}(A) \to \mathscr{D}(\widehat{A}).$$

THEOREM. *If A is a finitely generated k-algebra then $\widehat{A} \otimes_A \mathscr{D}^q(A) = \mathscr{D}^q(\widehat{A})$ and $\widehat{A} \otimes_A \mathscr{D}(A) = \mathscr{D}(\widehat{A})$.*

Proof. If A is the local ring of a point on an algebraic variety, a corresponding result is proved by Ishibashi in [7, page 13, Corollary] and his proof extends to the present case. However we include an alternative proof for the convenience of the reader.

First suppose $A = k[x_1, \ldots, x_n]$ a polynomial algebra. If $\partial \in \mathscr{D}(\widehat{A})$ we show by induction on the order of ∂ that $\partial \in \widehat{A} \otimes_A \mathscr{D}(A) = \widehat{A}[\partial_1, \ldots, \partial_n]$ where $\partial_i = \partial/\partial x_i$. This is clear for $\partial \in \widehat{A}$. By induction we have $d_i = [\partial, x_i] \in \widehat{A}[\partial_1, \ldots, \partial_n]$. Now $[d_i, x_j]$ is the partial derivative of d_i with respect to ∂_j and $[d_i, x_j] = [d_j, x_i]$. Thus by a familiar argument involving exact differential equations, there exists $d \in \widehat{A}[\partial_1, \ldots, \partial_n]$ such that $[d, x_i] = d_i$ for all i and $d(1) = \partial(1)$. It is now easy to see that d and ∂ agree as operators on A, and as we observed above this implies $\partial = d$.

In general write $A = B/I$ where I is an ideal in $B = k[x_1, \ldots, x_n]$. Let J' be the inverse image of J in B. Then $\widehat{B}_{J'}/\widehat{I}_{J'} \cong \widehat{A}_J$. Let $S = \{\partial \in \mathscr{D}(B) | \partial(I) \subseteq I\}$ and $T = \{\partial \in \mathscr{D}(\widehat{B}) | \partial(\widehat{I}) \subseteq \widehat{I}\}$. Then by [**10**, Lemma 1.4] we can identify $\mathscr{D}(A)$ with $S/I\mathscr{D}(B)$ and in a similar way $\mathscr{D}(\widehat{A})$ may be identified with $T/\widehat{I}\mathscr{D}(\widehat{B})$. Write $S^q = S \cap \mathscr{D}^q(B)$ and $T^q = T \cap \mathscr{D}^q(\widehat{B})$. We first show that $T^q = \widehat{B} \otimes_B S^q$. By [**10**, Lemma 1.5] there is a finite set of elements $\{v_1, \ldots, v_r\}$ of I such that if $\partial \in \mathscr{D}^q(B)$ then $\partial \in S^q$ if and only if $\partial(v_i) \in I$ for all i. It is also easy to see that for $\partial \in \mathscr{D}^q(\widehat{B})$, $\partial \in T^q$ if and only if $\partial(v_i) \in \widehat{I}$ for all i. Let $\bigoplus^r A$ be the free A-module of rank r and $\phi\colon \mathscr{D}^q(B) \to \bigoplus^r A$ the homomorphism of left B-modules such that for $\partial \in \mathscr{D}^q(B)$, the jth component of $\phi(\partial)$ is

$\partial(v_j) + I$. Then $S^q = \operatorname{Ker}\phi$ and $T^q = \operatorname{Ker} 1 \otimes \phi \colon \widehat{B} \otimes \mathscr{D}^q(B) = \mathscr{D}^q(\widehat{B}) \to \bigoplus^r \widehat{A}$. However since $\widehat{B}$ is a flat B-module we also have $\operatorname{Ker} 1 \otimes \phi = \widehat{B} \otimes_B \operatorname{Ker}\phi$. Thus $T^q = \widehat{B} \otimes_B S^q$ as claimed. Applying $\widehat{B} \otimes_B -$ to the exact sequence $0 \to I\mathscr{D}^q(B) \to S^q \to \mathscr{D}^q(A) \to 0$ we obtain $\widehat{A} \otimes_A \mathscr{D}^q(A) = T^q/\widehat{I}\mathscr{D}^q(\widehat{B}) = \mathscr{D}^q(\widehat{A})$. Taking direct limits we have $\mathscr{D}(\widehat{A}) = \varinjlim \mathscr{D}^q(\widehat{A}) = \widehat{A} \otimes_A \varinjlim \mathscr{D}^q(A) = \widehat{A} \otimes_A \mathscr{D}(A)$ as required.

COROLLARY. *If A is a finitely generated k-algebra then for any integer n, $\mathscr{D}(\widehat{A}) = \mathscr{D}(A) + \widehat{J}^n \mathscr{D}(\widehat{A})$.*

Proof. This follows since $\widehat{A} = A + \widehat{J}^n$.

2. Algebras with an artinian quotient ring.

2.1. LEMMA. *Let A be any k-algebra and P a finitely generated projective A-module. Then $\mathscr{D}(A, P) \cong P \otimes_A \mathscr{D}(A)$ and $\mathscr{D}(P) \cong \operatorname{End}_{\mathscr{D}(A)}(\mathscr{D}(A, P))$. Furthermore, if P is a progenerator then $\mathscr{D}(P)$ and $\mathscr{D}(A)$ are Morita equivalent.*

Proof. By the dual basis lemma, there exists $x_i \in P$ and $\alpha_i \in P^* = \operatorname{Hom}_A(P, A)$, $1 \leq i \leq n$, such that $1_P = \sum x_i \alpha_i$, so the map $P \otimes_A \mathscr{D}(A) \to \mathscr{D}(A, P)$ sending $x \otimes d$ to xd has an inverse given by $d \to \sum x_i \otimes \alpha_i d$.

Define $\phi \colon \mathscr{D}(P) \to \operatorname{End}_{\mathscr{D}(A)}(\mathscr{D}(A, P))$ by letting $\phi(d)$ send d' to dd' for $d \in \mathscr{D}(P)$ and $d' \in \mathscr{D}(A, P)$. Then ϕ is onto since if $\sigma \in \operatorname{End}_{\mathscr{D}(A)}(\mathscr{D}(A, P))$ then $\phi(\sum \sigma(x_i)\alpha_i) = \sigma$. Clearly ϕ is one-one.

Finally if P_A is a progenerator then so is $(P \otimes \mathscr{D}(A))_{\mathscr{D}(A)}$ so the result follows.

THEOREM. *Let A be a finitely generated algebra which has an artinian quotient ring F and suppose* $\dim(A) \leq 1$. *If $A/N(A)$ is integrally closed in $F/N(F)$ then $\mathscr{D}(A)$ and $\mathscr{D}(A/N(A))$ are Morita equivalent.*

Proof. In this case we can take $B = A$ in statement (b) of 1.1. Hence by (c) in 1.1 we have $A = \overline{A} \oplus N$ where $\overline{A}$ is a subalgebra of A and

$N = N(A)$. Since A has an artinian quotient ring, N is torsionfree and hence projective as an $\overline{A}$-module. Also A is an $\overline{A}$ progenerator, so by the lemma $\mathscr{D}_{\overline{A}}(A)$ and $\mathscr{D}(\overline{A})$ are Morita equivalent. However by [**11**, Lemma 2.1], $\mathscr{D}(A) = \mathscr{D}_{\overline{A}}(A)$ so the result follows.

2.2. THEOREM. *Let A be a finitely generated k-algebra of Krull dimension at most one which has an artinian quotient ring F. Then*

(a) *$\mathscr{D}(A)$ is (left and right) Noetherian.*
(b) *$\mathscr{D}(A)$ is a finitely generated k-algebra.*
(c) *$\mathscr{D}(A)$ has left and right Krull dimension equal to that of A.*
(d) *If M is a simple $\mathscr{D}(A)$-module then $\mathrm{End}_{\mathscr{D}(A)} M = k$.*

Proof. It is known that the required properties of $\mathscr{D}(A)$ are Morita invariant. Let B be a subalgebra of F containing A chosen as in 1.1. Then by Theorem 2.1 $\mathscr{D}(B)$ and $\mathscr{D}(B/N(B))$ are Morita equivalent. Since $B/N(B)$ is a direct sum of coordinate rings of non-singular curves $\mathscr{D}(B/N(B))$ has all the required properties and thus so does $\mathscr{D}(B)$. Also $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are orders in the same semisimple artinian ring by [**11**, Theorem A] and B/A is a finitely generated torsion A-module. It is now straightforward to complete the proof as in [**15**, §2] or [**10**, §5].

2.3. LEMMA. *Suppose* $\dim(A) \leq 1$, *and A has an artinian quotient ring and injective normalisation. Then A is a direct sum of primary rings.*

Proof. We first handle the special case where A is reduced. Let $\overline{A}$ be the integral closure of A, M a maximal ideal of A and $S = A - M$. By hypothesis $\overline{A}_S$ is regular local and hence a domain by [**8**, Theorem 164]. Therefore A_M is also a domain, so by [**8**, Theorem 168] $A = A_1 \oplus \cdots \oplus A_n$ is a direct sum of domains A_i.

In general, by applying the above argument to $A/N(A)$ and then lifting idempotents over $N(A)$ we have $A = A_1 \oplus \cdots \oplus A_n$ where each A_i has a unique minimal prime. Since A has an artinian quotient ring, it follows that A_i is primary.

2.4. LEMMA. *Suppose* $\dim(A) \leq 1$ *and A is a simple left $\mathscr{D}(A)$-module. Then A is a primary ring with injective normalisation.*

Proof. Let S be the set of elements of A which are regular mod $N(A)$. Since $S(0)$ is a $\mathscr{D}(A)$-submodule of A by [**11**, Lemma 1.4], we have $S(0) = 0$. Hence A has an artinian quotient ring. If $A = A_1 \oplus A_2$ a direct sum of algebras then each A_i is a $\mathscr{D}(A)$-submodule by [**10**, Proposition 1.14]. Hence in view of Lemma 2.3, it is enough to show that A has injective normalisation. Also we can assume $\dim(A) = 1$. Let M be a maximal ideal of A. We claim that $\widehat{A} = \widehat{A}_M$ has a unique minimal prime. Let $K_1 \cap K_2 \cap \cdots \cap K_n = 0$ be a minimal primary decomposition in $\widehat{A}$ where K_i is P_i-primary, and assume for a contradiction that $n > 1$. Since $\widehat{A}$ has an artinian quotient ring, the ideals P_i, $1 \leq i \leq n$, are the minimal primes of $\widehat{A}$. Set $I = K_1 + K_2$. Since the set of zero divisors of $\widehat{A}$ equals $\bigcup_1^n P_i$ by [**1**, Proposition 4.7] and $I \not\subseteq \bigcup_1^n P_i$ by [**8**, Theorem 81], I contains a regular element. Thus $\widehat{A}_M/I$ is artinian and hence $I \cap A \neq 0$. Also $I \neq \widehat{A}$ since the K_i are proper ideals in the local ring $\widehat{A}$. Now by [**11**, Lemma 1.4] and [**1**, Proposition 4.9], each K_i is a $\mathscr{D}(\widehat{A})$-submodule of $\widehat{A}$. Therefore I is a $\mathscr{D}(\widehat{A})$-submodule. Since every element of $\mathscr{D}(A)$ extends uniquely to an element of $\mathscr{D}(\widehat{A})$, $I \cap A$ is a proper $\mathscr{D}(A)$-submodule of A. This is a contradiction, so the nilradical $N(\widehat{A}_M)$ is the unique minimal prime of $\widehat{A}_M$. Now $\widehat{A}_M/N(\widehat{A}_M)$ is the completion of $A/N(A)$ in the $M/N(A)$-adic topology. Since this is a domain, it follows by [**5**, Theorem 6.5] that $M/N(A)$ is contained in a unique maximal ideal of the integral closure of $A/N(A)$. Thus A has injective normalisation.

2.5. *Proof of Theorem* B (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Assume $\mathscr{D}(A)$ is a direct sum of simple rings, and let $1 = e_1 + \cdots + e_r$ be a decomposition of the identity into centrally primitive idempotents. Since any central element of $\mathscr{D}(A)$ belongs to A, we have $\mathscr{D}(A) = \mathscr{D}(Ae_1) \oplus \cdots \oplus \mathscr{D}(Ae_r)$ and each $\mathscr{D}(Ae_i)$ is simple. Therefore we can assume that $\mathscr{D}(A)$ is simple. This implies that A is a simple $\mathscr{D}(A)$-module because any proper factor module would have non-zero annihilator in A and hence in $\mathscr{D}(A)$. Hence by Lemma 2.4 A is a primary ring with injective normalisation.

(3) $\Rightarrow$ (1). We may assume that A is a primary ring with injective normalisation. Let F be the artinian quotient ring of A and let B be a subalgebra of F containing A and satisfying properties (a)–(c) of 1.1. We claim $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are Morita equivalent. Let M be a maximal ideal of A and Q the unique maximal ideal of B containing

M. Since B_Q/A_M is a finitely generated torsion A_M-module we have $Q^r \subseteq \operatorname{ann}_{A_M}(B_Q/A_M)$ for some r. Let $P = \mathscr{D}(B_Q, A_M) = \{d \in \mathscr{D}(B_Q) | d(B_Q) \subseteq A_M\}$. As in the proof of [15, 3.3 and 3.4], it is enough to show there exists a $d \in P$ such that $d(1) \in k \setminus \{0\}$. Now $B_Q = \overline{B}_{\overline{Q}} \oplus N(B_Q)$ where $\overline{B}_{\overline{Q}}$ is a subalgebra of B_Q with unique maximal ideal $\overline{Q}$. Since $\overline{B}_{\overline{Q}}$ is a discrete valuation ring, there exists $t \in \overline{B}_{\overline{Q}}$ and $\partial \in \operatorname{Der} \overline{B}_{\overline{Q}}$ such that $\overline{Q} = t\overline{B}_{\overline{Q}}$ and $\partial(t) = 1$. Set $d = \prod_{j=1}^{r-1}(t\partial - j) \in \mathscr{D}(\overline{B}_{\overline{Q}})$. By [11, Lemma 2.1] we can extend d to a differential operator on B_Q by setting $d(N(B_Q)) = 0$. Note that $d(t^n) = \lambda_n t^n$ for some $\lambda_n \in k$ and $\lambda_n = 0$ if and only if $1 \leq n \leq r-1$. As in [15] this implies $d(\overline{Q}^r) \subseteq \overline{Q}^r$. Since $B_Q = (k + kt + \cdots + kt^{r-1}) \oplus \overline{Q}^r \oplus N(B_Q)$, we have $d(B_Q) \subseteq k + \overline{Q}^r \subseteq A_M$ so $d \in P$ and $d(1) \in k \setminus \{0\}$ as required.

This shows that $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are Morita equivalent. Finally $\mathscr{D}(B)$ and $\mathscr{D}(B/N(B))$ are Morita equivalent by Theorem 2.1, and $B/N(B)$ is the integral closure of $A/N(A)$, so the theorem is proved.

2.6. Corollary. *Let A be a finitely generated k-algebra with* $\dim(A) \leq 1$. *The following conditions are equivalent.*

(1) *$\mathscr{D}(A)$ is a simple ring.*

(2) *A is a simple left $\mathscr{D}(A)$-module.*

(3) *A is a primary ring with injective normalisation.*

Proof. (1) $\Rightarrow$ (2) was noted in the proof of Theorem B.

(2) $\Rightarrow$ (3) is the statement of Lemma 2.4.

(3) $\Rightarrow$ (1) We can assume $\dim(A) = 1$. By Theorem B, $\mathscr{D}(A)$ is Morita equivalent to $\mathscr{D}(\tilde{A})$ where $\tilde{A}$ is the integral closure of $A/N(A)$. Since $A/N(A)$ is a domain, $\tilde{A}$ is the coordinate ring of a nonsingular, irreducible curve. Thus $\mathscr{D}(\tilde{A})$ is a simple ring and hence $\mathscr{D}(A)$ is also simple.

Remark. If $R = k[x, y]$ and $f = f(x, y) \in R$ is irreducible, it is easily seen that $A = R/f^nR$ is a primary ring for all $n \geq 1$. Thus if X is the planar curve defined by f and the normalisation map $\tilde{X} \to X$ is injective we have that $\mathscr{D}(A)$ is a simple ring and A is a simple $\mathscr{D}(A)$-module. This gives Proposition 2.8 and Corollary 2.9 of [14].

2.7. PROPOSITION. *Let A be a finitely generated k-algebra with $\dim(A) \leq 1$, and assume that A has an artinian quotient ring F. Then A has finite length as a left $\mathscr{D}(A)$-module.*

Proof. Let B be a subalgebra of F containing A and satisfying conditions (a)–(c) of 1.1. Set $I = \operatorname{ann}_A(B/A)$. Since B/A is a finitely generated torsion A-module, A/I has finite length. Also $B = B_1 \oplus \cdots \oplus B_r$ where each B_i is primary and $B_i/N(B_i)$ is the coordinate ring of a nonsingular curve. Thus B_i is a simple $\mathscr{D}(B_i)$-module. Let $\pi_i \colon B \to B_i$ be the projection map. If M is a $\mathscr{D}(A)$-submodule of A, we define $\lambda(M) = \{i | \pi_i(M) \neq 0\}$. We claim that $I(\bigoplus_{i\in\lambda(M)} B_i) \subseteq M$. Since also $M \subseteq \bigoplus_{i\in\lambda(M)} B_i$, and there are only finitely many choices for $\lambda(M)$ this will show that A has finite length as a $\mathscr{D}(A)$-module.

Assume $i \in \lambda(M)$. Let $e_i = \pi_i(1) \in B_i$; then $\pi_i(M) = e_i M$ so $\mathscr{D}(B)M \supseteq \mathscr{D}(B)e_i M = B_i$, since B_i is a simple $\mathscr{D}(B)$-module. Since $I\mathscr{D}(B) \subseteq \mathscr{D}(A)$ we obtain $M = \mathscr{D}(A)M \supset I\mathscr{D}(B)M \supset IB_i$ as required.

REMARK. The above remains true if A is not assumed to have an artinian quotient ring, see Proposition 4.3.

2.8. All of the results in this section hold if A is replaced by A_M or $\widehat{A}_M$. For the analogue of Theorem 2.2 with A replaced by $\widehat{A}_M$, we should observe that if B is an overring of A constructed in 1.1, then the endomorphism ring of any simple $\mathscr{D}(B/N(B))$-module equals k. This easily reduces to the following special case.

LEMMA. *If $B = k[[x]]$, $\mathscr{D} = \mathscr{D}(B)$ and M is any simple $\mathscr{D}$-module, then* $\operatorname{End}_{\mathscr{D}} M = k$.

Proof. Since M is holonomic, [**3**, Ch. 3, Prop. 3.11] applies.

3. Standard modules.

3.1. Let A be an arbitrary (commutative) k-algebra. If V is an A-module and M an ideal of A we denote by $\operatorname{ann}_V(M)$, $\operatorname{soc}_A(V)$ and $E_A(V)$ the annihilator of M in V, the socle of V and the injective hull of V respectively.

LEMMA. *Let Q be an ideal of A, and V an A-module with $QV=0$.*
(a) *If $d \in \mathscr{D}^n(A, V)$ then $d(Q^{n+1}) = 0$ (equivalently $dQ^{n+1} = 0$ in $\mathscr{D}(A, V)$).*
(b) *If Q is M-primary where M is a maximal ideal of A then $\mathscr{D}(A, V) = \{d \in \operatorname{Hom}_k(A, V) | d(M^n) = 0$ for some $n\}$. In particular, when $\mathscr{D}(A, V)$ is regarded as a right A-module via the action of $\mathscr{D}(A)$, then $\operatorname{soc}_A(D(A, V)) = \operatorname{ann}_{\mathscr{D}(A, V)} M$ is an essential submodule.*

Proof. (a) If $x \in Q$, then by induction on n, $0 = [d, x](Q^n) = d(xQ^n) - xd(Q^n)$. Hence $d(xQ^n) = 0$ and the result follows.
(b) Suppose $M^s \subseteq Q$ for some s. By (a) $d \in \mathscr{D}^n(A, V)$ implies $d(M^{s(n+1)}) = 0$. Conversely if $d \in \operatorname{Hom}_k(A, V)$ and $d(M^n) = 0$, then as in the proof of [**11**, Lemma 2.1] we have $[d, A]_i = [d, M]_i \subseteq \sum_{j+k=i} M^j d M^k$ which is zero for $i \geq s+n$. Thus $d \in \mathscr{D}^{s+n}(A, V)$. For the final statement suppose $d \in \mathscr{D}(A, V)$, $d \neq 0$ and choose n minimal with $d(M^n) = 0$. there exists $x \in M^{n-1}$ with $d(x) \neq 0$. Therefore $dx \in \operatorname{ann}_{\mathscr{D}(A, V)} M$ and the result follows.

COROLLARY. (*Compare* [**13**, *Corollary* 4.5].) *If M is a maximal ideal of A then as a right A-module*

$$\mathscr{D}(A, A/M) \cong E_A(A/M).$$

Proof. Set $E = E_A(M)$. It is well known that $\operatorname{Hom}_k(A, A/M)$ is an injective A-module, so E may be identified with a submodule of $\operatorname{Hom}_k(A, A/M)$. Also $E = \bigcup_n \operatorname{ann}_E M^n$ by [**12**, Proposition 4.23] so $E \subseteq \{d \in \operatorname{Hom}_k(A, A/M) | d(M^n) = 0$ for some $n\} = \mathscr{D}(A, A/M)$. By the lemma $\mathscr{D}(A, A/M)$ is an essential extension of $\operatorname{Hom}_k(A/M, A/M) \cong A/M$ so the result follows.

3.2. LEMMA. *Let K be an ideal of B, and $A = B/K$. If V is any A-module then there is an isomorphism of right $\mathscr{D}(A)$-modules*

$$\mathscr{D}_A(A, V) \cong \{d \in \mathscr{D}_B(B, V) | d(K) = 0\}.$$

Proof. Apply the functor $\mathscr{D}_B(-, V)$, which is left exact by [**15**, 1.3c)], to the sequence $0 \to K \to B \to A \to 0$ to obtain

$0 \to \mathscr{D}_B(A, V) \to \mathscr{D}_B(B, V) \to \mathscr{D}_B(K, V)$. The result follows since $\mathscr{D}_B(A, V) = \mathscr{D}_A(A, V)$.

3.3. Let $B = k[x_1, \ldots, x_n]$ be a polynomial algebra. For $1 \le i \le n$ set $\partial_i = \partial/\partial x_i$. Let $\underline{m} = (x_1, \ldots, x_n)$ and $D = k[\partial_1, \ldots, \partial_n]$. Then D is a subalgebra of $\mathscr{D}(B)$ with $\mathscr{D}(B) = D \oplus \underline{m}\mathscr{D}(B)$. Let $\pi \colon B \to B/\underline{m}$ be the natural map. The map $\mathscr{D}(B) \to \mathscr{D}(B, B/\underline{m})$ sending d to πd is surjective with kernel $\underline{m}\mathscr{D}(B)$. Thus as right $\mathscr{D}(B)$-modules, $D \cong \mathscr{D}(B)/\underline{m}\mathscr{D}(B) \cong \mathscr{D}(B, B/\underline{m})$. To avoid any confusion we shall write $D_{\mathscr{D}(B)}$ for D considered as a right $\mathscr{D}(B)$-module rather than as a subalgebra of $\mathscr{D}(B)$. Clearly $D_{\mathscr{D}(B)}$ is a holonomic $\mathscr{D}(B)$-module supported by the origin. (We refer to [**4**] for definitions.)

LEMMA. (a) *Let X be a right $\mathscr{D}(B)$-module such that* $\operatorname{soc}_B(X) = \operatorname{ann}_X(\underline{m})$ *is an essential B-submodule of X. Then $X = \operatorname{soc}_B(X) \otimes_k D$.*

(b) *The map $Y \to Y \otimes_k D$ defines an equivalence of categories between the category of finite dimension vector spaces over k and the category of holonomic $\mathscr{D}(B)$-modules supported by the origin.*

Proof. This follows from [**4**, V.3.1.2 and V.3.1.6] and induction on n.

3.4. THEOREM. *Let A be a finitely generated k-algebra and $0 \to U \to V \to W \to 0$ an exact sequence of A-modules where V has finite length. Then the sequence of right $\mathscr{D}(A)$-modules*

$$(*) \qquad 0 \to \mathscr{D}(A, U) \to \mathscr{D}(A, V) \to \mathscr{D}(A, W) \to 0$$

is exact and splits.

Proof. For each maximal ideal M let $V(M) = \{v \in V \mid vM^r = 0$ some $r\}$. Then $V = \bigoplus V(M)$, the sum ranging over maximal ideals M of A, so $\mathscr{D}(A, V) = \bigoplus \mathscr{D}(A, V(M))$. In proving the theorem we can assume $V = V(M)$, so $VM^r = 0$ for some r.

By Lemma 3.1 $\mathscr{D}(A, V) = \bigcup_s \mathscr{D}_s$ where $\mathscr{D}_s = \{d \in \operatorname{Hom}_k(A, V) \mid d(M^s) = 0\}$, which we identify with $\operatorname{Hom}_k(A/M^s, V)$. The exactness of the sequence $(*)$ follows easily from this.

Now write $A = B/K$ where $B = k[x_1, \dots, x_n]$ a polynomial algebra. We can assume $M = \underline{m}/K$ where $K \subseteq \underline{m} = (x_1, \dots, x_n)$. Let $D = k[\partial_1, \dots, \partial_n]$ as in 3.3. Since $\mathscr{D}(B, V)$ has a finite composition series with factors isomorphic to $D_{\mathscr{D}(B)}$, it is a holonomic $\mathscr{D}(B)$-module supported by the origin. Hence by Lemma 3.3 $\mathscr{D}(B, V) \cong \mathscr{D}(B, U) \oplus \mathscr{D}(B, W)$. Considering the submodules annihilated by K and using Lemma 3.2 we obtain $\mathscr{D}(A, V) \cong \mathscr{D}(A, U) \oplus \mathscr{D}(A, W)$ as required.

Let $l(V)$ denote the composition length of the A-module V.

COROLLARY. *Let V be an A-module of finite length. Then $\mathscr{D}(A, V)$ is a direct sum of $l(V)$ standard modules. Furthermore $l(V) = l(\operatorname{soc}_A \mathscr{D}(A, V))$.*

Proof. The first statement follows by applying Theorem 3.4 to a composition series for V, the second from the fact that any standard module has simple socle as an A-module by Corollary 3.1.

3.5. The following result will be useful in computing examples.

COROLLARY. *Let K be an ideal in the polynomial algebra $B = k[x_1, \dots, x_n]$ with $K \subseteq \underline{m} = (x_1, \dots, x_n)$. Let $A = B/K, D = k[\partial_1, \dots, \partial_n]$ as in 3.3 and $M = \underline{m}/K$. Suppose that V is an A-module of finite length such that $VM^s = 0$ for some s. Then $\mathscr{D}(A, V) = (\operatorname{soc}_A \mathscr{D}(A, V)) \operatorname{ann}_D K$.*

Proof. By Lemma 3.3 $\mathscr{D}(B, V) = (\operatorname{ann}_{\mathscr{D}(B, V)} \underline{m})D$. By Lemma 3.2 since $K \subseteq \underline{m}, \mathscr{D}(A, V) = \operatorname{ann}_{\mathscr{D}(B, V)} K \supseteq \operatorname{ann}_{\mathscr{D}(B, V)} \underline{m}$ so $\operatorname{ann}_{\mathscr{D}(B, V)} \underline{m} = \operatorname{ann}_{\mathscr{D}(A, V)} M = \operatorname{soc}_A \mathscr{D}(A, V)$. As a right $\mathscr{D}(B)$-module $\mathscr{D}(B, V)$ is isomorphic to a direct sum of copies of $D_{\mathscr{D}(B)}$. Hence $\mathscr{D}(A, V) = \operatorname{ann}_{\mathscr{D}(B, V)} K = (\operatorname{soc}_A \mathscr{D}(A, V)) \operatorname{ann}_D K$ as required.

3.6. We explain how Matlis duality can be used in the study of standard modules. Let M be a maximal ideal of A, $\widehat{A} = \widehat{A}_M$ and $\mathscr{D} = \mathscr{D}(A, A/M)$. Observe that $\mathscr{D}$ may be regarded in a natural way as a right $\mathscr{D}(\widehat{A})$-module. Indeed $A/M^n \cong \widehat{A}/\widehat{M}^n$ for all n, so by Lemma 3.1, $\mathscr{D}$ may be identified with $\mathscr{D}(\widehat{A}, \widehat{A}/M)$. Clearly any $\mathscr{D}(\widehat{A})$-submodule of $\mathscr{D}$ is a $\mathscr{D}(A)$-submodule. Conversely let X be a $\mathscr{D}(A)$-submodule of $\mathscr{D}$ and $f \in X$. Then $fM^n = 0$ for some n. If $d \in \mathscr{D}(\widehat{A})$ we can use Corollary 1.2 to write $d = d_1 + d_2$ where $d_1 \in \mathscr{D}(A)$ and $d_2 \in \widehat{M}^n\mathscr{D}(\widehat{A})$. Then $fd = fd_1 \in X$. Thus X is a $\mathscr{D}(\widehat{A})$ submodule of $\mathscr{D}$. If V is an $\widehat{A}$-submodule of $\mathscr{D}$, set $V^* = \{a \in \widehat{A} \mid v(a) = 0 \text{ for all } v \in V\}$ and if V is an ideal of $\widehat{A}$, set $V^* = \{d \in \mathscr{D} | d(v) = 0 \text{ for all } v \in V\}$. Then by [12, Theorem 5.21] $V \to V^*$ sets up a one-one order reversing correspondence between $\widehat{A}$-submodules of $\mathscr{D}$ and of $\widehat{A}$. It is easily verified that right $\mathscr{D}(\widehat{A})$-submodules of $\mathscr{D}$ and left $\mathscr{D}(A)$-submodules of $\widehat{A}$ correspond under this duality. Hence we have

PROPOSITION. *The maps $V \to V^*$ give a one-one order reversing correspondence between the submodules of the right $\mathscr{D}(A)$-module $\mathscr{D}(A, A/M)$ and submodules of the left $\mathscr{D}(\widehat{A})$-module $\widehat{A}$.*

4. Algebras with Krull dimension at most one.

4.1. If N is a $\mathscr{D}(A)$-submodule of A, there is a natural map $\mathscr{D}(A) \to \mathscr{D}(A/N)$ with kernel $\Delta(A, N)$ [11, 1.2]. It is of interest to know when this map is surjective. We prove one result in this direction. It is convenient to express A in the form $A = B/L$ where L is an ideal in a polynomial algebra B.

LEMMA. *Let L, N, Q be ideals of the polynomial algebra B such that $L = N \cap Q$, Q is M-primary where M is a maximal ideal of B, and N/L is a $\mathscr{D}(B/L)$-submodule of B/L. Then the natural map $\mathscr{D}(B/L) \to \mathscr{D}(B/N)$ is surjective.*

Proof. Given $d \in \mathscr{D}(B/N)$ we can lift d to an element $\partial \in \mathscr{D}(B)$ such that $\partial(N) \subseteq N$ and $d(b + N) = \partial(b) + N$ for all $b \in B$ [10, Lemma 1.4.]. Let $N = (f_1, \dots, f_p)$. We show there are elements $d_1, \dots, d_p \in \mathscr{D}(B)$ such that $\partial' = \partial + f_1 d_1 + \cdots + f_p d_p$ satisfies $\partial'(L) \subseteq L$. Then ∂' will induce an element of $\mathscr{D}(B/L)$ which maps onto d. Since $L = N \cap Q$, it is enough to ensure that $\partial'(L) \subseteq Q$. If ∂ has order t, then $\partial(M^s) \subseteq M^{s-t}$ for every s. Choose s large enough so that $M^{s-t} \subseteq Q$. Since M is a maximal ideal of B, $L/L \cap M^s$ is a finite dimensional vector space over k and we can choose elements

$v_1, \ldots, v_q \in L$ whose images form a basis. For each j write $\partial(v_j) = \sum_l f_l a_{jl}$ with $a_{jl} \in B$. Since B/M^s is a local artinian ring, there exists $\partial_l \in \mathscr{D}(B/M^s)$ such that $\partial_l(v_j + M^s) = -a_{jl} + M^s$ for all j, by [**11**, Lemma 2.1]. Using [**10**, Lemma 1.4] we can lift ∂_l to an element $d_l \in \mathscr{D}(B)$ such that $d_l(M^s) \subseteq M^s$ and $d_l(v_j) + M^s = -a_{jl} + M^s$ for all j. If $\partial' = \partial + \sum_l f_l d_l$ we have $\partial'(v_j) \in M^s \subseteq Q$ for all j and $\partial'(L \cap M^s) \subseteq \partial(M^s) + \sum f_l d_l(M^s) \subseteq M^{s-t} \subseteq Q$. Hence $\partial'(L) \subseteq Q$ as required.

4.2. We establish some notation which we will use from now on in dealing with algebras of dimension at most one. Let K be an ideal in a polynomial algebra B such that $A = B/K$ has dimension at most one. Let $K = \bigcap_{\lambda \in \Lambda} K_\lambda$ be a minimal primary decomposition where K_λ is P_λ-primary and set $\Omega = \{\lambda \in \Lambda | P_\lambda$ is minimal over $K\}$ and $I = \bigcap_{\lambda \in \Omega} K_\lambda$. By [**11**, Lemma 1.4], I/K is a $\mathscr{D}(A)$-submodule of A. Also $\overline{A} = B/I$ has an artinian quotient ring since it has no embedded primes. If $Q = \bigcap_{\lambda \in \Lambda \setminus \Omega} K_\lambda$, then $\overline{I} = I/K$ is isomorphic to an ideal of A/Q and so has finite length as an A-module. We let $J = \Delta(A, \overline{I})$ the kernel of the map $\phi\colon \mathscr{D}(A) \to \mathscr{D}(\overline{A})$.

Corollary. *The map ϕ is surjective.*

Proof. Let $\{K_\lambda | \lambda \in \Lambda \setminus \Omega\} = \{K_1, \ldots, K_m\}$ and suppose K_i is P_i-primary. Let $L_0 = I$ and for $1 \le i \le m$, $L_i = I \cap K_1 \cap \cdots \cap K_i$, so that $L_m = K$. By [**11**, Lemma 1.4] L_{i-1}/L_i is a $\mathscr{D}(A/L_i)$-submodule of A/L_i, and since P_i is a maximal ideal of A, it follows from Lemma 4.1, that the map $\mathscr{D}(A/L_i) \to \mathscr{D}(A/L_{i-1})$ is surjective. Since ϕ is obtained as the composite $\mathscr{D}(A/K) \to \mathscr{D}(A/L_{m-1}) \to \cdots \to \mathscr{D}(A/L_1) \to \mathscr{D}(A/I)$, the result follows.

4.3. Proposition. *Let A be a finitely generated algebra with* $\dim(A) \le 1$ *and M a maximal ideal of A. Then*

(a) *$\mathscr{D}(A, A/M)$ has finite length as a right $\mathscr{D}(A)$-module.*

(b) *$\widehat{A}_M$ has finite length as a left $\mathscr{D}(\widehat{A}_M)$-module.*

Proof. By Proposition 3.6 the two assertions are equivalent. Apply the left exact functor $\mathscr{D}_A(-, A/M)$ to the exact sequence $0 \to \overline{I} \to A \to \overline{A} \to 0$ to obtain $0 \to \mathscr{D}_A(\overline{A}, A/M) \to \mathscr{D}_A(A, A/M) \to \mathscr{D}_A(\overline{I}, A/M)$. Since $\overline{I}$ and A/M are both finite dimensional over

k, it is enough to show that $\mathscr{D}_A(\overline{A}, A/M) = \mathscr{D}_{\overline{A}}(\overline{A}, A/M)$ has finite length as a right $\mathscr{D}(\overline{A})$-module. Thus in proving the theorem we can assume that A has an artinian quotient ring. However we have already obtained statement (b) in this case, see 2.8.

We remark that the above result fails for domains of dimension two. For example let $A = k[x, y, z]/(x^3+y^3+z^3)$ and M the ideal of A generated by x, y, z. By [2, Proposition 1], M^n is a $\mathscr{D}(A)$-submodule of A for every n. Thus $\widehat{M^n}$ is a $\mathscr{D}(\widehat{A})$-submodule of $\widehat{A}$ for all n.

4.4. THEOREM C. *With the notation of* 4.2, J *has finite length as a right* $\mathscr{D}(A)$*-module.*

Proof. As noted in [11, 1.2] $J = \Delta(A, \overline{I}) = \mathscr{D}(A, \overline{I})$. Since $\overline{I}$ has finite length, this is a finite direct sum of standard modules by Corollary 3.4. By Proposition 4.3, each standard module has finite length so the result follows.

4.5. *Proof of Theorem* A. Let A be a finitely generated k-algebra with dimension at most one. If $\mathscr{D}(A)$ is left Noetherian then A has an artinian quotient ring by [11, Theorem A].

Conversely, suppose that A has an artinian quotient ring. Then by Theorem 2.2 $\mathscr{D}(A)$ is left and right Noetherian and finitely generated as a k-algebra. In particular this proves statement (b) of the theorem.

In general with the notation of 4.2, we have $\overline{A} = A/\overline{I}$ and $\mathscr{D}(\overline{A}) \cong \mathscr{D}(A)/J$. Since J and $\mathscr{D}(\overline{A})$ are Noetherian as right $\mathscr{D}(A)$-modules by Theorem 2.2 and Theorem C, $\mathscr{D}(A)$ is right Noetherian.

Let $J_1 = \Delta_A(\overline{I}, 0)$ and $N = J \cap J_1$. We know $\mathscr{D}(A)/J \cong \mathscr{D}(\overline{A})$ is a finitely generated k-algebra. Also J/N embeds in $\mathscr{D}(A)/J_1$ and hence in $\mathscr{D}_A(\overline{I})$ which is finite dimensional. It follows that $\mathscr{D}(A)/N$ is a finitely generated algebra. Now N is a $\mathscr{D}(A)$-submodule of J and so of finite length and since $J_1 J = 0$, N is a right $\mathscr{D}(A)/N$-module. Let X be a finite set of elements of $\mathscr{D}(A)$ whose images generate $\mathscr{D}(A)/N$ as a k-algebra and Y a finite set of generators for N as a right $\mathscr{D}(A)/N$ module. If S is the subalgebra of $\mathscr{D}(A)$ generated by $X \cup Y$, then $(S+N)/N = \mathscr{D}(A)/N$. Thus given $d \in \mathscr{D}(A)$, there exists $d' \in S$ such that $d - d' \in N = Y(\mathscr{D}(A)/N) \subseteq S$, so $S = \mathscr{D}(A)$.

REMARK. A similar proof shows that if A is a finitely generated k-algebra with Krull dimension at most one, and M is a maximal ideal of A, then the rings $\mathscr{D}(A_M)$ and $\mathscr{D}(\widehat{A}_M)$ are right Noetherian.

5. Examples.

5.1. In this section we indicate how $\mathscr{D}(A)$ may be calculated explicitly in certain cases. We write A in the form $A = B/K$ where K is an ideal in the polynomial algebra B, and use the fact that $\mathscr{D}(A) \cong \mathbb{I}_{\mathscr{D}(B)}(K\mathscr{D}(B))/K\mathscr{D}(B)$ where $\mathbb{I}_{\mathscr{D}(B)}(K\mathscr{D}(B)) = \Delta_B(K, K)$ is the idealiser of $K\mathscr{D}(B)$, see [**11**, 1.3.]. As a first illustration let $B = k[x, y]$, $M = (x, y)$, $P = (x)$, $K = PM = P \cap M^2 = (x^2, xy)$, and $A = B/K$. In the notation of 4.2 we have $\overline{I} = P/K$ and $\overline{A} = B/P \cong k[y]$. We note that $y\mathscr{D}(k[y]) \subseteq \mathscr{D}(A)$, $(x\partial/\partial x - 1)(\partial/\partial y)^i(K) \subseteq K$ and this last operator induces the operator $-(\partial/\partial y)^i$ on $k[y]$. Hence $S = y\mathscr{D}(k[y]) + (x\partial/\partial x - 1)k[\partial/\partial y]$ maps onto $\mathscr{D}(\overline{A})$ under the natural map $\mathscr{D}(A) \to \mathscr{D}(\overline{A})$. The kernel of this map is $J = \Delta(A, I) = \mathscr{D}(A, I)$. Since $x \in I$ and $xM \subseteq K$ we have $x \in \operatorname{soc}_A \mathscr{D}(A, \overline{I})$, and by Corollary 3.4, $\operatorname{soc}_A \mathscr{D}(A, \overline{I})$ is simple. Thus by Corollary 3.5, $J = x \operatorname{ann}_D K$ where $D = k[\partial/\partial x, \partial/\partial y]$. Now $\operatorname{ann}_D K = \operatorname{ann}_D P + \operatorname{ann}_D M^2 = k[\partial/\partial y] + k\partial/\partial x$. The justification for this statement will be given later. Putting the pieces together we obtain

$$\begin{aligned}\mathscr{D}(A) &\cong (S + J + K\mathscr{D}(B))/K\mathscr{D}(B) \\ &= (y\mathscr{D}(k[y]) + (x\partial/\partial x - 1)k[\partial/\partial y] \\ &\qquad + xk[\partial/\partial y] + kx\partial/\partial x + K\mathscr{D}(B))/K\mathscr{D}(B).\end{aligned}$$

We also note that $\Delta_A(\overline{I}, 0) = J_1$ is equal to the image of $y\mathscr{D}(k[y]) + (x\partial/\partial x - 1)k[\partial/\partial y] + xk[\partial/\partial y]$ in $\mathscr{D}(A)$, and the prime radical $N = J \cap J_1$ of $\mathscr{D}(A)$ is the image of $xk[\partial/\partial y]$, see [**11**, Theorem B].

A calculation of the algebra $\mathscr{D}(A)$ in this example is carried out by Muhasky in [**10**, Example 7.2], and Muhasky uses his calculation to show that $\mathscr{D}(A)$ is right but not left Noetherian.

5.2. We explain how the above calculation may be extended to cover other examples. Let $A, \overline{I}$ and $\overline{A} = A/\overline{I}$ be as in 4.2 and write $A = B/K$ where K is an ideal in $B = k[x_1, \dots, x_n]$ contained in $\underline{m} = (x_1, \dots, x_n)$. Let $\partial_i = \partial/\partial x_i$ for $1 \leq i \leq n$ and $D = k[\partial_1, \dots, \partial_n]$. There seems to be no description known in general for $\mathscr{D}(\overline{A})$, even when $\overline{A}$ is a domain, and we say nothing further about this problem here. Instead we assume $\mathscr{D}(\overline{A})$ is known, as for example in the case $\overline{A}$ is the coordinate ring of a non-singular curve. Since the map $\mathscr{D}(A) \to \mathscr{D}(\overline{A})$ is surjective we can find a subset S of $\mathscr{D}(A)$ which maps onto $\mathscr{D}(\overline{A})$. (The proof of Lemma 4.1 gives an algorithm for doing this, but as in Example 5.1, it is often easier to find S directly.) By [**15**, 1.3d)] we have $\mathscr{D}(B, I) = I\mathscr{D}(B)$. Thus identifying $\mathscr{D}(A)$ with $\amalg(K\mathscr{D}(B))/K\mathscr{D}(B)$, we have $J = \Delta(A, \overline{I}) = \{\partial \in I\mathscr{D}(B) | \partial(K) \subseteq K\}/K\mathscr{D}(B) = \operatorname{ann}_{I\mathscr{D}(B)/K\mathscr{D}(B)} K$. Let $\widetilde{J} = \{\partial \in I\mathscr{D}(B) | \partial(K) \subseteq K\}$; then $\mathscr{D}(A) \cong (S + \widetilde{J} + K\mathscr{D}(B))/K\mathscr{D}(B)$. By Corollary 3.5, $J = (\operatorname{soc}_A J) \operatorname{ann}_D K$ so the problem of describing J falls into two parts; the calculation of $\cos_A(J)$ and that of $\operatorname{ann}_D K$. We discuss these two parts in 5.3 and 5.4.

5.3. As in the proof of Theorem 3.4 we have $\overline{I} = \bigoplus \overline{I}(M)$ where the sum ranges over maximal ideals M of A and this yields a corresponding decomposition of J. Thus we may assume that $\operatorname{ann}_A \overline{I}$ is M-primary, where $M = \underline{m}/K$ and $\underline{m} = (x_1, \dots, x_n)$. Hence $\operatorname{soc}_A J = \operatorname{ann}_J M = \operatorname{ann}_{I\mathscr{D}(B)/K\mathscr{D}(B)} \underline{m} = \operatorname{soc}_B(I\mathscr{D}(B)/K\mathscr{D}(B))$. For arbitrary ideals $K \subseteq I$ of the polynomial algebra B such that $\operatorname{ann}_B I/K$ is $\underline{m}$-primary, we give a description of the module $I\mathscr{D}(B)/K\mathscr{D}(B)$ and its socle. We remark that $I\mathscr{D}(B)/K\mathscr{D}(B)$ is isomorphic to $\mathscr{D}(B, I/K)$ as a right $\mathscr{D}(B)$-module by [**15**, 1.3e)] so we also obtain a description of $\mathscr{D}(B, I/K)$.

Assume $I\underline{m}^{s+1} \subseteq K$ for some $s \geq 0$. For $t \in \mathbb{N}^n$ write $x^t = x_1^{t_1} \cdots x_n^{t_n}$ and $|t| = \sum t_i$. For $i = 1, \dots, n$, define

$$d_i = \prod_{j=1}^{s} (x_i \partial_i - j) \quad \text{if } s \geq 1$$

and $d_i = 1$ if $s = 0$, and $d = \prod_{i=1}^{n} d_i$. Let $I = (f_1, \dots, f_p)$. Each factor in the series

$$I \supseteq Im + K \supseteq \cdots \supseteq I\underline{m}^s + K \supseteq K$$

is spanned by elements of the form $f_i x^t$ for $1 \leq i \leq p$ and $t \in \mathbb{N}^n$. Hence we can choose subsets $T_i \subset \mathbb{N}^n$, $1 \leq i \leq p$, such that $\{f_i x^t + K | 1 \leq i \leq p, t \in T_i\}$ forms a basis for I/K and the images of those $f_i x^t$ which lie in $I\underline{m}^j + K$ form a basis of $(I\underline{m}^j + K)/(I\underline{m}^{j+1} + K)$. For each i and $t \in T_i$ set $\delta_{i,t} = f_i x^t d$ and let $\sigma \colon I\mathscr{D}(B) \to I\mathscr{D}(B)/K\mathscr{D}(B)$ be the natural map.

Lemma. (a) *$I\mathscr{D}(B)/K\mathscr{D}(B) = \bigoplus_{i=1}^{p} \bigoplus_{t\in T_i} \sigma\delta_{i,t}D$ and each $\sigma\delta_{i,t}D$ is a simple right $\mathscr{D}(B)$-module isomorphic to $D_{\mathscr{D}(B)}$.*

(b) $\operatorname{soc}(I\mathscr{D}(B)/K\mathscr{D}(B)) = \bigoplus_i \bigoplus_t k\sigma\delta_{i,t}$.

Proof. Clearly $I\mathscr{D}(B) = K\mathscr{D}(B) + \sum f_i x^t \mathscr{D}(B)$. Also $d_i x_i = x_i^{s+1}\partial_i^s$ by induction on s and hence $f_i x^t d x_i \in f_i x^t x_i^{s+1}\mathscr{D}(B) \subseteq K\mathscr{D}(B)$ for all i. Therefore $\sigma\delta_{i,t} \in \operatorname{soc}(I\mathscr{D}(B)/K\mathscr{D}(B))$. Since $d_i \equiv (-1)^s s! \mod M\mathscr{D}(B)$ we have $f_i x^t = \lambda\delta_{i,t} \mod f_i x^t M\mathscr{D}(B)$ where $\lambda \in k\setminus\{0\}$. Using this, we can show by reverse induction on $|t|$ that $\sigma f_i x^t \in \sum_{i,t}\sigma\delta_{i,t}D$ for all i, t. Hence $I\mathscr{D}(B)/K\mathscr{D}(B) = \sum_{i,t}\sigma\delta_{i,t}D$. Therefore $\operatorname{soc}(I\mathscr{D}(B)/K\mathscr{D}(B))$ is spanned by $\{\sigma\delta_{i,t} | 1 \le i \le p, t \in T_i\}$ and so $l(\operatorname{soc}(I\mathscr{D}(B)/K\mathscr{D}(B))) \le \sum_i |T_i|$. By Corollary 3.4 we have equality, so the elements $\{\sigma\delta_{i,t}\}$ are linearly independent. This proves (b) and (a) follows from Lemma 3.3.

5.4. We indicate a method which may be used to calculate $\operatorname{ann}_D K$ in some cases. By Corollary 3.1 we have $D \cong E_B(B/M)$ as a B-module. Also D can be regarded as a $\widehat{B}_M = k[[x_1, \dots, x_n]]$-module. It is easily seen that $\operatorname{ann}_D K = \operatorname{ann}_D \widehat{K}$, where $\widehat{K} = K\widehat{B}_M$. Also if $I = I_1 \cap I_2$ are ideals of $\widehat{B}_M$ then $\operatorname{ann}_D I = \operatorname{ann}_D I_1 + \operatorname{ann}_D I_2$ by [**12**, Theorem 5.21]. Let $\widehat{K} = \bigcap K_\lambda$ be a primary decomposition in $\widehat{B}_M$. It suffices to calculate $\operatorname{ann}_D K_\lambda$ for the primary ideals K_λ of $\widehat{B}_M$. We give several examples.

(a) For any $s \ge 1$ $\operatorname{ann}_D M^{s+1} = V_s$, the space of all polynomials in $\partial_1, \dots, \partial_n$ of degree at most s. Let $\pi\colon \widehat{B} \to \widehat{B}/\widehat{M} = k$ be the natural map. For any M-primary ideal Q, choose s with $M^{s+1} \subseteq Q$. Define $Q^\perp = \{\partial \in V_s | \pi(\partial(f)) = 0 \text{ for } f \in Q\}$. Then $(M^{s+1})^\perp = V_s$ and $\operatorname{ann}_D Q = Q^\perp$. This can be calculated simply using vector space duality.

(b) If $P = (x_1, \dots, x_m)\widehat{B}_M$, with $m < n$ then as is easily verified, $\operatorname{ann}_D P = k[\partial_{m+1}, \dots, \partial_n]$.

(c) For other primary ideals K, $\operatorname{ann}_D K$ may be harder to calculate. For example, let $n = 2$ and $K = (x_1^2 - x_2^3)\widehat{B}$. If $d = \sum a_{ij}\partial_1^i\partial_2^j$, then $d(x_1^2 - x_2^3)x_1^k x_2^l \equiv (k+2)!l!\,a_{k+2l} - k!\,(l+3)!\,a_{kl+3} \mod \widehat{M}$. Thus $\operatorname{ann}_D K = \{\sum a_{ij}\partial_1^i\partial_2^j | \binom{k+2}{2}a_{k+2l} = 3\binom{l+3}{3}a_{kl+3} \text{ for all } l, k \ge 0\}$.

5.5. **Example.** Let $B = k[x_1, x_2, x_3]$, $P = (x_1, x_2)$, $M = (x_1, x_2, x_3)$, $V = kx_2x_3 + k(x_1x_3 + x_1^2) + k(x_1^2 + x_2^2)$, $Q = M^3 + V$, $K = P \cap Q$ and $A = B/K$. We compute $\mathscr{D}(A)$ in this case. In the notation of 4.2, $I = P$ and $\overline{I} = P/K$. We have chosen this example because $\overline{I}$ is a noncyclic indecomposable A-module. We have $P = kx_1 + kx_2 + kx_1^2 + kx_1x_2 + K$. Let $d_1 = (x_1\partial_1 - 1)(x_2\partial_2 - 1)$, $d_2 = d_1(x_1\partial_1 - 2)$ and $S = x_3^2\mathscr{D}(k[x_3]) + d_2(x_3k[\partial_3] + k[\partial_3]) \subseteq \mathscr{D}(B)$. Then since $\partial_3(P) \subseteq P$ and $d_2(P) \subseteq K$ it is easily checked that $S(K) \subseteq K$; thus S induces differential operators on A. Clearly S (or rather its image in $\mathscr{D}(A)$) maps onto $\mathscr{D}(\overline{A}) = \mathscr{D}(k[x_3])$ under the map $\phi: \mathscr{D}(A) \to \mathscr{D}(\overline{A})$. The kernel of ϕ is $\Delta_A(A, \overline{I})$ whose socle is spanned by the images in $\mathscr{D}(A)$ of the elements x_1^2, x_1x_2, x_1d_1 and x_2d_1. Let $D = k[\partial_1, \partial_2, \partial_3]$ and $D' = \operatorname{ann}_D \widehat{K} = \operatorname{ann}_D \widehat{P} + \operatorname{ann}_D \widehat{Q} = k[\partial_3] + k\partial_1 + k\partial_2 + k\partial_1\partial_2 + k(2\partial_1\partial_3 + \partial_2^2 - \partial_1^2)$. Let $T = x_1^2D' + x_2^2D' + x_1d_1D' + x_2d_1D'$. Then

$$\Delta_A(A, \overline{I}) = (T + K\mathscr{D}(B))/K\mathscr{D}(B) \quad \text{and}$$
$$\mathscr{D}(A) = (S + T + K\mathscr{D}(B))/K\mathscr{D}(B).$$

6. A generalisation of Nakai's conjecture. It seems reasonable to conjecture that if A is a finitely generated algebra such that $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$ then A is a regular ring. If $A = \mathscr{O}(X)$ is the coordinate ring of an irreducible variety X this is a well-known conjecture of Nakai. It is known that Nakai's conjecture holds when X is a curve [**9**]. Also if $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$ then A is reduced [**11**, Theorem 4.2]. Here we prove

THEOREM. *Let A be a finitely generated k-algebra with* $\dim A \leq 1$. *If $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$, then A is regular.*

To prove this we need the following result which may be of independent interest.

PROPOSITION. *Let A be a finitely generated k-algebra with* $\dim(A) \leq 1$. *If* $\operatorname{gr}(\mathscr{D}(A))$ *is Noetherian, then $\mathscr{D}(A)$ is a direct sum of simple rings.*

Proof. The proof in [**15**] for domains works with some small changes. We sketch the argument. If $\operatorname{gr}(\mathscr{D}(A))$ is Noetherian, then so is $\mathscr{D}(A)$ and hence A has an artinian quotient ring F by [**11**, Theorem A]. Let B be a subalgebra of F such that B is a finitely generated A-module, and $B/N(B)$ is the integral closure of $A/N(A)$. We can assume that $\dim(A/Q) = 1$ for all minimal primes Q of A and similarly for B. Therefore since $\mathscr{D}(B)$ is Morita equivalent to $\mathscr{D}(B/N(B))$, $\mathscr{D}(B)$

is a direct sum of simple Noetherian hereditary rings of infinite dimension over k. Now, consider $P = \{\partial \in \mathscr{D}(F) | \partial(B) \subset A\}$. Then P contains a regular element c of A, $P_{\mathscr{D}(B)}$ is a progenerator and $E = \mathrm{End}(P)$ is Morita equivalent to $\mathscr{D}(B)$. We have

$$P \subset \mathscr{D}(A) \subset E \subset \mathscr{D}(F)$$

so

$$\mathrm{gr}(P) \subset \mathrm{gr}(\mathscr{D}(A)) \subset \mathrm{gr}(E) \subset \mathrm{gr}(\mathscr{D}(F)).$$

If $\mathscr{D}(A) = E$, then $\mathscr{D}(A)$ and $\mathscr{D}(B/N(B))$ are Morita equivalent and the result follows. If $\mathscr{D}(A) \neq E$, then $J = P\mathscr{D}(A)$ is an ideal of $\mathscr{D}(A)$, which is a proper left ideal of E. As in **[15]** we have $\dim_k(\mathrm{gr}(E)/\mathrm{gr}(J)) = \infty$ and $\dim_k(\mathrm{gr}(\mathscr{D}(A))/\mathrm{gr}(J)) < \infty$, so $\mathrm{gr}(E)$ cannot be finitely generated as a $\mathrm{gr}(\mathscr{D}(A))$-module. However $\mathrm{gr}(E)$ is isomorphic to $c\mathrm{gr}(E)$ which is an ideal of $\mathrm{gr}(\mathscr{D}(A))$. So $\mathrm{gr}(\mathscr{D}(A))$ is not Noetherian.

6.2. *Proof of Theorem* 6.1. If $\dim(A) \leq 1$ and $\mathscr{D}(A)$ is generated by $\mathscr{D}^1(A)$, then A is reduced by **[11]**. Also $\mathrm{gr}(\mathscr{D}(A))$ is finitely generated, so by the Proposition, and Theorem B, A is a direct sum of domains, so the result follows from the case where A is a domain, **[9]**.

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Received April 4, 1989 and in revised form November 13, 1989.

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