Pacific Journal of Mathematics

ORDERED GROUPS AND CROSSED PRODUCTS OF C^*-ALGEBRAS

GERARD J. MURPHY

Vol. 148, No. 2 April 1991

PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 2, 1991

ORDERED GROUPS AND CROSSED PRODUCTS OF C^*-ALGEBRAS

G. J. MURPHY

We define and analyse the concept of a crossed product of a C^*-algebra A by a semigroup. For a large class of semigroups we show that the crossed product is primitive if A is, and our constructions also give rise to simple C^*-algebras. Conditions are given for when the crossed product is type I or nuclear, and when covariant representations of a C^*-dynamical system give rise to faithful and/or irreducible representations of the crossed product.

Introduction. The theory of crossed products of C^*-algebras by automorphism groups is a deep and interesting area of the modern theory of operator algebras, as well as being a rich source of examples. It is natural to try to extend the ideas of this area to a more general setting. One way to do this is to consider crossed products by semigroups, and this paper develops some aspects of the theory. Surprisingly (or perhaps not) if the semigroup does not look much like a group the results turn out to be radically different in many respects from the classical case. For instance, the group C^*-algebra of an abelian group is of course itself abelian, and so, from the point of view of C^*-theory, not very interesting. But for a large and natural class of semigroups (namely the positive cones of abelian ordered groups) their C^*-algebras are not only non-abelian but actually primitive. This is useful because the primitive (and the simple) C^*-algebras are in a sense the building blocks of C^*-theory.

If we come down to very concrete detail, and look at the additive semigroup $\mathbf{N}$ of natural numbers, we find that its C^*-algebra is the Toeplitz algebra, i.e. the C^*-algebra generated by all Toeplitz operators with continuous symbol on the unit circle. Indeed, for any cone as above, its C^*-algebra can be faithfully represented as a C^*-algebra of Toeplitz operators in a generalized sense (see **[12]**). For the related situation of C^*-algebras generated by multivariable Wiener-Hopf operators see **[11]** and **[15]**. The papers **[4]**, **[10]** and **[21]** are also relevant. Indeed a very interesting theory of crossed products by semigroups is developed in **[10]**. This theory is quite different from ours however as the crossed product in **[10]** is in general a non-self-adjoint algebra. If (A, α, G) is a separable C^*-dynamical system with

G a discrete abelian group, then a special case of the Olesen-Pedersen spectral theory ([**16**], [**17**]) asserts that the cross product $A \times_\alpha G$ is simple (respectively prime) if and only if A is G-simple (respectively G-prime) and the Connes spectrum $\Gamma(\alpha)$ is equal to the dual group $\widehat{G}$. If we now suppose that G is totally ordered by some positive cone G^+ then the results for the crossed product $A \times_\alpha G^+$ are very different. Firstly, $A \times_\alpha G^+$ can never be simple if G is not trivial, and secondly $A \times_\alpha G^+$ is primitive provided only that A itself is primitive. In this case therefore one is spared the often quite difficult task of having to compute the Connes spectrum. In fact, however, we give necessary and sufficient conditions that $A \times_\alpha G^+$ be prime which are similar to the Olesen-Pedersen conditions above, but which require one to compute the Connes spectrum not of α but of a certain action γ of G on a C^*-algebra $O(G) \otimes A$. Here $O(G)$ is a certain C^*-algebra reflecting the order structure of G.

Although $A \times_\alpha G^+$ is usually not simple (except where the order on G is trivial), we do get new simple C^*-algebras arising from these constructions. There is a canonical map ε from $A \times_\alpha G^+$ to $A \times_\alpha G$, and in the case where G is an ordered subgroup of $\mathbf{R}$ and A is simple the kernel of ε is simple. In the special case where $A = \mathbf{C}$ the class of simple C^*-algebras that one gets was first investigated by Douglas in [**4**]. Recently the K-theory of these algebras has been computed (see [**7**], and for a simple special case [**13**]).

It is of interest to observe that a certain class of the algebras we study in this paper have already been used in K-theory. If α is an automorphism on a unital C^*-algebra A then one can show that $A \times_\alpha \mathbf{N}$ is isomorphic to the generalized Toeplitz algebra of α (in the sense of Pimsner-Voiculescu). An important step in deriving the six-term exact sequence for the K-theory of $A \times_\alpha \mathbf{Z}$ is showing that $K_j(A \times_\alpha \mathbf{N}) = K_j(A)$. The computation of the K-theory of the algebras $A \times_\alpha G^+$ in general would seem to be an interesting question.

We now give a brief section-by-section guide to this paper.

In §1 we construct the crossed product and induced covariant representations. The results here are basic to the rest of the paper, but this section is most like the classical theory. In §2 we introduce pre-ordered groups, and associate with each such group a certain C^*-algebra which reflects its order and group structure. In §3 we use this algebra, and a dilation theorem of McAsey and Muhly, to represent $A \times_\alpha G^+$ as a full hereditary subalgebra of a certain crossed product by G, and from this in turn we derive conditions on when $A \times_\alpha G^+$ is type I or nuclear and some other of the results already mentioned in this introduction. In §4 we analyse the covariant representations for ordered groups in detail, and from this deduce that $A \times_\alpha G^+$ is primitive when A is, and the results on simple algebras stated earlier.

1. Construction of the crossed product. Let M denote a monoid, with unit e, and let B be a unital C^*-algebra. We call a map $W\colon M \to B$, $x \mapsto W_x$, an *isometric homomorphism* if each W_x is an isometry and $W_{xy} = W_x W_y$ for all $x, y \in M$ (necessarily then $W_e = 1$). If $B = B(H)$ for a Hilbert space H we call (H, W) an *isometric representation* of M.

If M is left-cancellative, then isometric representations exist. To be specific, let H be a non-zero Hilbert space and let $l^2(M, H)$ denote the Hilbert space of all norm-square-summable maps f from M to H (i.e. $\sum_{x\in M} \|f(x)\|^2 < +\infty$) with the norm and scalar product given by $\|f\| = (\sum_{x\in M} \|f(x)\|^2)^{1/2}$, and $\langle f, g\rangle = \sum_{x\in M}\langle f(x), g(x)\rangle$. For each $x \in M$ we define an isometry W_x on $l^2(M, H)$ by the equation:

$$(W_x f)(z) = \begin{cases} f(y), & \text{if } z = xy, \\ 0, & \text{if } z \notin xM. \end{cases}$$

The map $W\colon M \to B(l^2(M, H))$, $x \mapsto W_x$, is an isometric homomorphism. We call $(l^2(M, H), W)$ the *canonical* isometric representation of M on $l^2(M, H)$. It is clear that W is injective.

If a monoid M admits an injective homomorphism into a C^*-algebra, then obviously M is left-cancellative.

A *C^*-dynamical system* will refer in this paper to a triple (A, α, M) where A is a C^*-algebra, M is a left-cancellative monoid, and α is a homomorphism from M to $\operatorname{Aut} A$ (so $\alpha_\varepsilon = \mathrm{id}$). We shall say (A, α, M) is *nontrivial* if A is non-zero and M is not a singleton, *separable* if A is separable and M countable, and *classical* if M is a group. If B is a C^*-algebra with multiplier algebra $M(B)$, a *covariant homomorphism* from (A, α, M) to B is a pair (φ, W) where $\varphi\colon A \to B$ is a $*$-homomorphism and $W\colon M \to M(B)$ is an isometric homomorphism, and φ, W interact via the equation

$$(*) \qquad \varphi\alpha_x(a) W_x = W_x \varphi(a) \qquad (x \in M, a \in A).$$

If $B = B(H)$ for H a Hilbert space we call (H, φ, W) a *covariant representation* of (A, α, M). If (A, α, M) is classical then $(*)$ is equivalent to the equation

$$(**) \qquad \varphi\alpha_x(a) = W_x \varphi(a) W_x^*$$

(as all W_x are then necessarily unitary), but for monoids which are not groups $(*)$ and $(**)$ may be inequivalent. This will be apparent in examples we shall be considering later.

As in the classical case each representation of A induces a covariant representation of (A, α, M). Its construction is similar to the classical case, but its theory is radically different for monoids which are not groups. Let (H, φ) be a representation of A, and suppose the Hilbert space H is nonzero. Let $(l^2(M, H), W)$ be the canonical isometric representation of M on $l^2(M, H)$. For $a \in A$ define $\overline{\varphi}(a) \in B(l^2(M, H))$ by the formula:

$$(\overline{\varphi}(a)f)(x) = \varphi\alpha_x^{-1}(a)f(x)$$

for all $f \in l^2(M, H)$ and all $x \in M$. The map

$$\overline{\varphi}\colon A \to B(l^2(M, H)), \qquad a \mapsto \overline{\varphi}(a),$$

is a $*$-homomorphism, and it is readily verified that $(l^2(M, H), \overline{\varphi}, W)$ is a covariant representation of (A, α, M), said to be *induced* by (H, φ). Note that if (H, φ) is a faithful (respectively non-degenerate) representation of A then $(l^2(M, H), \overline{\varphi})$ is also a faithful (respectively non-degenerate) representation of A.

Let (A, α, M) be a C^*-dynamical system where A is non-zero to avoid trivialities, an assumption we shall make tacitly henceforth. If F is the free $*$-algebra on the set $A \cup M$, let I be the smallest self-adjoint ideal of F for which F/I is unital, the map $\tilde{\rho}\colon A \to F/I$, $a \mapsto a+I$, is a $*$-homomorphism, the map $\widetilde{V}\colon M \to F/I$, $x \mapsto x+I$, is an isometry-valued homomorphism, and $\tilde{\rho}\alpha_x(a)\widetilde{V}_x = \widetilde{V}_x\tilde{\rho}(a)$ $(a \in A, x \in M)$. That such an ideal I exists follows from the fact that (A, α, M) admits a covariant representation (H, φ, W) where φ is non-zero (use the induced covariant representation corresponding to a faithful representation of A).

Note that $\tilde{\rho}(A) \cup \widetilde{V}_M$ generates F/I, where $\widetilde{V}_M = \{\widetilde{V}_x \mid x \in M\}$. If γ is any C^*-seminorm on F/I, then $a \mapsto \gamma(\tilde{\rho}(a))$ is a C^*-seminorm on A, so $\gamma(\tilde{\rho}(a)) \le \|a\|$. Also, $\gamma(\widetilde{V}_x)^2 = \gamma(\widetilde{V}_x^*\widetilde{V}_x) = \gamma(1) \le 1$. It follows that F/I admits a greatest C^*-seminorm, γ_0 say. Hence $J = \{c \in F/I \mid \gamma_0(c) = 0\}$ is a self-adjoint ideal of F/I, and the quotient $*$-algebra D_0 is a normed $*$-algebra with C^*-norm given by $\|c + J\| = \gamma_0(c)$ $(c \in F/I)$. Let D be the C^*-completion of D_0 and π the canonical $*$-homomorphism from F/I to D given by $\pi(c) = c + J$. Any $*$-homomorphism from F/I to a C^*-algebra can be factored uniquely through D via π.

For $a \in A$ let $\rho(a) = \pi\tilde{\rho}(a)$, and for $x \in M$ let $W'_x = \pi\tilde{V}_x$. We denote by $A \times_\alpha M$ the C^*-subalgebra of D generated by all $\rho(a)W'_x$ $(a \in A, x \in M)$. Clearly the map $\rho\colon A \to A \times_\alpha M$, $a \mapsto \rho(a)$, is a $*$-homomorphism. One easily verifies that if (u_λ) is an approximate unit for A, then $(\rho(u_\lambda))$ is one for $A \times_\alpha M$. Hence if $b \in A \times_\alpha M$, so is bW'_x $(= \lim_\lambda b\rho(u_\lambda)W'_x)$. Similarly $W'_x b \in A \times_\alpha M$. Thus we can define a multiplier $V_x \in M(A \times_\alpha M)$ by $V_x b = W'_x b$, $bV_x = bW'_x$. The map $V\colon M \to M(A \times_\alpha M)$, $x \mapsto V_x$, is an isometric homomorphism, and (ρ, V) is a covariant homomorphism from (A, α, M) to $A \times_\alpha M$.

We call $A \times_\alpha M$ the *crossed product* of A by M under the action α, or the *covariance algebra* of the C^*-dynamical system (A, α, M), and we call ρ and V the *canonical* maps. We summarize the important universal property of $A \times_\alpha M$ in the following result:

PROPOSITION 1.1. *Let (A, α, M) be a C^*-dynamical system. The canonical maps ρ and V are injective, and the C^*-algebra $A \times_\alpha M$ is generated by all $\rho(a)V_x$ $(a \in A, x \in M)$. If (φ, W) is any covariant homomorphism from (A, α, M) to a C^*-algebra B there exists a unique $*$-homomorphism $\varphi \times W\colon A \times_\alpha M \to B$ such that*

$$(\varphi \times W)(\rho(a)V_x) = \varphi(a)W_x \qquad (a \in A, x \in M).$$

Proof. Let $F, I, D, \tilde{\rho}, \tilde{V}, \pi, W'_x$ be as above. Let (φ, W) be a covariant homomorphism from (A, α, M) to B. There is a unique $*$-homomorphism $\psi\colon F \to B$ such that $\psi(a) = \varphi(a)$ $(a \in A)$ and $\psi(x) = W_x$ $(x \in M)$. Since $\psi(I) = 0$ we get an induced $*$-homomorphism $\tilde{\psi}\colon F/I \to B$, and hence a $*$-homomorphism $\check{\psi}\colon D \to B$ such that $\check{\psi}\pi = \tilde{\psi}$. Observe that

$$\check{\psi}\rho(a) = \check{\psi}\pi\tilde{\rho}(a) = \tilde{\psi}\tilde{\rho}(a) = \psi(a) = \varphi(a) \qquad (a \in A),$$

and

$$\check{\psi}(W'_x) = \check{\psi}\pi(\tilde{V}_x) = \tilde{\psi}(\tilde{V}_x) = \psi(x) = W_x \qquad (x \in M).$$

Hence $\check{\psi}(\rho(a)V_x) = \check{\psi}(\rho(a)W'_x) = \varphi(a)W_x$, so the map

$$\varphi \times W\colon A \times_\alpha M \to B, \qquad b \mapsto \check{\psi}(b),$$

is the unique $*$-homomorphism such that $(\varphi \times W)(\rho(a)V_x) = \varphi(a)W_x$ for all $a \in A$ and $x \in M$.

Now let (H, φ) be a faithful non-degenerate representation of A and $(l^2(M, H), \overline{\varphi}, W)$ the induced covariant representation of

(A, α, M). Since φ is injective so is $\overline{\varphi}$. If $0 = \rho(a)$ then $0 = (\overline{\varphi} \times W)\rho(a) = \overline{\varphi}(a)$, so $0 = a$. Thus ρ is injective. If $V_x = V_y$ then $\overline{\varphi}(a)W_x = (\overline{\varphi} \times W)(\rho(a)V_x) = (\overline{\varphi} \times W)(\rho(a)V_y) = \overline{\varphi}(a)W_y$, for all $a \in A$, so by non-degeneracy of the representation $(l^2(M, H), \overline{\varphi})$ of A we have $W_x = W_y$, and therefore $x = y$. Thus V is injective. □

If (A, α, M) is a C^*-dynamical system we can, and do henceforth, regard ρ as an embedding of A in $A \times_\alpha M$. Thus we identify a and $\rho(a)$, and view A as a C^*-subalgebra of $A \times_\alpha M$. Any approximate unit for A is one for $A \times_\alpha M$ also. In particular if A is unital so is $A \times_\alpha M$.

In the classical situation where M is a group, $A \times_\alpha M$ is the usual crossed product (as defined in **[18]** for example). In this case $A \times_\alpha M$ is the closed linear span of all aV_x $(a \in A, x \in M)$, as the linear span is a $*$-subalgebra. This is not true for our more general crossed products. We shall give a counter-example presently.

If M is any left-cancellative monoid and $\alpha: M \to \operatorname{Aut} \mathbf{C}$ the trivial homomorphism we set $C^*(M) = \mathbf{C} \times_\alpha M$. This algebra is unital and the canonical map $V: M \mapsto C^*(M)$ is the universal isometric homomorphism: if $W: M \to B$ is an isometric homomorphism into a unital C^*-algebra B then there exists a unique $*$-homomorphism $\varphi: C^*(M) \to B$ such that $\varphi V = W$. (Set $\varphi = \psi \times W$ where ψ is the unital homomorphism from $\mathbf{C}$ to B.)

Note that V_M generates $C^*(M)$.

If M is the additive monoid $\mathbf{N}$ then $C^*(\mathbf{N})$ is generated by the nonunitary isometry V_1, so we can identify $C^*(\mathbf{N})$ with the Toeplitz algebra, the C^*-algebra on the Hardy space H^2 generated by all Toeplitz operators with continuous symbol. This is in fact the motivating example for our more general theory. The element V_1^* is not in the closed linear span of all $V_n = V_1^n$ $(n \geq 0)$, as $V_1^* V_1 \neq V_1 V_1^*$, so $C^*(\mathbf{N})$ is not this closed linear span. Thus $C^*(\mathbf{N})$ is the counter-example we promised a moment ago.

We close this section with some trivial but useful remarks. Suppose that (A, α, M) is a C^*-dynamical system and that (φ, W) is a covariant homomorphism from (A, α, M) to a C^*-algebra B. Then

$$(\varphi \times W)(bV_x) = (\varphi \times W)(b)W_x$$

and

$$(\varphi \times W)(V_x b) = W_x(\varphi \times W)(b)$$

for all $b \in A \times_\alpha M$ and all $x \in M$. These equations hold because if (u_λ) is an approximate unit for A then

$$\begin{aligned}(\varphi \times W)(bV_x) &= \lim_\lambda (\varphi \times W)(bu_\lambda V_x)\\ &= \lim_\lambda (\varphi \times W)(b)\varphi(u_\lambda)W_x\\ &= \lim_\lambda (\varphi \times W)(bu_\lambda)W_x\\ &= (\varphi \times W)(b)W_x,\end{aligned}$$

and similarly

$$\begin{aligned}(\varphi \times W)(V_x b) &= \lim_\lambda (\varphi \times W)(V_x u_\lambda b)\\ &= \lim_\lambda (\varphi \times W)(\alpha_x(u_\lambda)V_x b)\\ &= \lim_\lambda \varphi\alpha_x(u_\lambda)W_x(\varphi \times W)(b)\\ &= \lim_\lambda W_x\varphi(u_\lambda)(\varphi \times W)(b)\\ &= \lim_\lambda W_x(\varphi \times W)(u_\lambda b)\\ &= W_x(\varphi \times W)(b).\end{aligned}$$

If (A, α, M) is a C^*-dynamical system and $\psi\colon A \times_\alpha M \to B$ is a surjective $*$-homomorphism onto a C^*-algebra B then there exists a unique covariant homomorphism (φ, W) from (A, α, M) to B such that $\varphi \times W = \psi$. Uniqueness is obvious by the preceding remark. To see existence, set $\phi(a) = \psi(a)$ $(a \in A)$, and set $W_x\psi(b) = \psi(V_x b)$, $\psi(b)W_x = \psi(bV_x)$ if $b \in A \times_\alpha M$ and $x \in M$.

2. Ordered groups. In this paper a *pre-ordered group* is a pair $(G, \leq)$ consisting of a discrete group G and a pre-order $\leq$ on G (that is, a reflexive transitive relation) such that if e is the unit of G and $G^+ = \{x \in G \mid e \leq x\}$ then

(a) The inequality $x \leq y$ implies $zxt \leq zyt$ for all $x, y, z, t \in G$;

(b) The cone G^+ generates G.

Note that Condition (b) is equivalent to the assertion that every element x of G can be written in the form $x = u^{-1}v$ for some $u, v \in G^+$.

If $\leq$ is a partial order (respectively a total order) we call $(G, \leq)$ a *partially ordered group* (respectively an *ordered group*). We shall be principally interested in the latter, as it is the case in which the strongest results can be obtained.

Pre-ordered groups exist in great abundance. We list a few examples here.

The additive group $\mathbf{R}$ is of course an ordered group with its usual order, as are all its subgroups. We shall always assume subgroups of $\mathbf{R}$ are endowed with the usual order.

All free groups can be made into ordered groups [**1**].

An abelian group can be made into an ordered group if and only if it is torsion-free [**9**]. It is well-known that a discrete abelian group is torsion-free if and only if its Pontryagin dual group $\widehat{G}$ is connected [**20**, p. 47]. In general a group can admit many translation-invariant total orderings for which the corresponding ordered groups are not isomorphic (as ordered groups).

If P is a cone in $\mathbf{R}^n$ such that $\mathbf{R}^n = P - P$ and we define $x \leq_P y$ to mean $y - x \in P$ then $(\mathbf{R}^n, \leq_P)$ is a partially ordered group with positive cone $\mathbf{R}^{n+} = P$.

If $(G_\lambda)_{\lambda \in \Lambda}$ is a family of pre-ordered groups the product group G is a pre-ordered group where for (x_λ) and (y_λ) in G we define $(x_\lambda) \leq (y_\lambda)$ to mean $x_\lambda \leq y_\lambda$ for all $\lambda \in \Lambda$.

In particular $\mathbf{Z}^n$ is a partially ordered group with positive cone $\mathbf{N}^n$.

Remark 2.1. If $x_1, \ldots, x_n$ are arbitrary elements of a pre-ordered group G then there is a positive element u in G such that $x_i \leq u$ for all j. To see this write $x_j = u_j^{-1} v_j$ where u_j, v_j belong to G^+, set $u = v_1 \cdots v_n$, and observe that $x_j \leq v_j \leq u$.

We shall need some results from a paper of McAsey and Muhly [**10**], so we introduce some of their terminology. If W is a map from a discrete group G to $B(H)$ where H is a Hilbert space we say W is *positive definite* if $W_e = 1$ and for every finite set $x_1, \ldots, x_n$ in G the matrix $(W_{x_i^{-1}x_j})_{ij}$ is positive in $M_n(B(H)) = B(H^{(n)})$. If moreover (A, α, G) is a C^*-dynamical system and (H, φ) is a representation of A then (H, φ, W) is a *positive definite covariant triple* if (W is positive definite and)

$$\varphi\alpha_x(a) W_x = W_x \varphi(a) \qquad (a \in A, x \in G).$$

The key fact concerning (H, φ, W) is that it can be dilated to get a covariant representation of (A, α, G):

Theorem 2.1 (*McAsey-Muhly* [**10**]). *Let (A, α, G) be a C^*-dynamical system where G is a discrete group, and let (H, φ, W) be a positive definite covariant triple for (A, α, G). Then there is a covariant representation (H', φ', W') of (A, α, G) and an isometry $V\colon H \to H'$ such that $\varphi(a) = V^*\varphi'(a)V$ for all $a \in A$ and $W_x = V^* W'_x V$ for all $x \in G$.*

The result is asserted and proved in **[10]** in considerably more generality than we have stated it here. (The blanket second-countability assumption in **[10]** is irrelevant to Theorem 2.1.) For related material on dilations see **[2]**, **[6]** and **[8]**.

PROPOSITION 2.2. *Let G be a pre-ordered group, and $W\colon G^+ \to B$ an isometric homomorphism into a unital C^*-algebra B. Then there is a unique extension $W\colon G \to B$ such that $W_{u^{-1}x} = W_u^* W_x$ for all $u \in G^+$ and $x \in G$. Moreover if $x_1, \dots, x_n \in G$ then the matrix $(W_{x_i^{-1}x_j})_{ij}$ is positive in $M_n(B)$.*

Proof. Uniqueness is clear from Condition (b) of the definition of a pre-ordered group.

Suppose that u, v and uv^{-1} belong to G^+. Then $W_u = W_{uv^{-1}} W_v$, so $W_v^* = W_u^* W_{uv^{-1}}$. Now suppose that an element x of G has two expressions of the form $x = u_1^{-1}v_1 = u_2^{-1}v_2$ where $u_j, v_j \in G^+$. Then $W_{u_1}^* W_{v_1} = W_{u_2}^* W_{v_2}$. This follows from our preceding observation and Remark 2.1, since there exists $u \in G^+$ such that uu_1^{-1} and uu_2^{-1} belong to G^+ and then $W_{u_1}^* W_{v_1} = W_u^* W_{uu_1^{-1}} W_{v_1} = W_u^* W_{ux} = W_u^* W_{uu_2^{-1}} W_{v_2} = W_{u_2}^* W_{v_2}$. It is therefore clear that we can extend W to G in such a way that $W_{u^{-1}x} = W_u^* W_x$ for all $u \in G^+$ and $x \in G$.

If $x_1, \dots, x_n$ is an arbitrary finite set in G use Remark 2.1 to choose $u \in G^+$ such that $y_j = ux_j \in G^+$ for $1 \leq j \leq n$. Then $(W_{x_i^{-1}x_j})_{ij} = (W_{y_i^{-1}y_j})_{ij} = (W_{y_i}^* W_{y_j})_{ij}$, so $(W_{x_i^{-1}x_j})_{ij}$ is positive. □

Suppose that (A, α, G) is a C^*-dynamical system where G is a preordered group, that (H, φ, W) is a covariant representation of the C^*-dynamical system (A, α, G^+), and that (to avoid ambiguity here) $\widetilde{W}$ denotes the canonical extension of W to G. Then as we have just seen $\widetilde{W}$ is positive definite, and it is easy to check that

$$(*) \qquad \varphi\alpha_x(a)\widetilde{W}_x = \widetilde{W}_x\varphi(a) \qquad (a \in A, x \in G).$$

Thus $(H, \varphi, \widetilde{W})$ is a positive definite covariant triple for (A, α, G) and therefore by Theorem 2.1 has a dilation to a covariant representation (H', φ', W') of (A, α, G). This will be of crucial importance for the sequel.

We shall use V to identify H as a closed vector subspace of H'. If $T \in B(H')$ we denote its compression to H by T_H. It is easy to verify that H is invariant for $\varphi'(A)$ from the fact that $\varphi'(a)_H = \varphi(a)$ and therefore $\varphi'(a)_H\varphi'(b)_H = \varphi'(ab)_H$ for all $a, b \in A$. Similarly H is invariant for all the unitaries W_x' $(x \in G^+)$ since the compression of a unitary to H is an isometry implies that H is invariant for the unitary.

We make a few further observations on the extension $\widetilde{W}$ of W to G: Firstly, $\widetilde{W}_{x^{-1}} = \widetilde{W}_x^*$ for all $x \in G$, and secondly, it is easy to see using equation $(*)$ that the linear span of all the elements $\varphi(a)W_{x_1}W_{x_2}\cdots W_{x_n}$ $(a \in A, x_1, \dots, x_n \in G)$ is a $*$-subalgebra of $\operatorname{im}(\varphi \times W)$, so its closure is $\operatorname{im}(\varphi \times W)$, since the elements $\varphi(a)W_x$ $(a \in A,\ x \in G^+)$ generate $\operatorname{im}(\varphi \times W)$.

In particular, $A \times_\alpha G^+$ is the closed linear span of all $aV_{x_1}V_{x_2}\cdots V_{x_2}$ $(a \in A,\ x_1, \dots, x_n \in G)$.

We shall be using these elementary observations frequently and tacitly.

To avoid ambiguity, having denoted the canonical map from G^+ to $M(A \times_\alpha G^+)$ by V, let us denote the canonical map from G to $M(A \times_\alpha G)$ by U. If U^+ is the restriction of U to G^+ and ρ: $A \to M(A \times_\alpha G)$ the inclusion map, then (ρ, U^+) is a covariant homomorphism from (A, α, G^+) to $A \times_\alpha G$. We set $\varepsilon = \rho \times U^+$, so ε: $A \times_\alpha G^+ \to A \times_\alpha G$ is the unique $*$-homomorphism such that $\varepsilon(aV_x) = aU_x$ $(a \in A, x \in G^+)$. Since ε is surjective we call it the *quotient* map. It gives us a means of relating the representations of $A \times_\alpha G$ with some of those of $A \times_\alpha G^+$. Far more important for our purposes is to relate all of the covariant representations of (A, α, G^+) with covariant representations of (A, α, G), as we shall do using Theorem 2.1.

We need to introduce a certain "universal" C^*-algebra $O(G)$ which reflects the order structure of G, and indirectly, the group structure also.

Suppose that $(G, \leq)$ is a pair consisting of a non-empty set G and a pre-order $\leq$ on G. Let F be the free $*$-algebra on G and let I be the smallest self-adjoint ideal containing the elements $x - x^*$ and $y - xy$ for all $x, y \in G$ such that $x \leq y$. Set $p'_x = x + I$, and denote by S' the $*$-subalgebra of F/I generated by the projections p'_x $(x \in G)$. If γ is a C^*-seminorm on S' then $\gamma(p'_x)^2 = \gamma(p'_x)$, so $\gamma(p'_x) \leq 1$. It follows from this observation that S' admits a greatest C^*-seminorm γ_0. The set J of all $b \in S'$ such that $\gamma_0(b) = 0$ is a self-adjoint ideal of S', and we can define a C^*-norm on the $*$-algebra S'/J by setting $\|b + J\| = \gamma_0(b)$ for all $b \in S'$. We denote the C^*-completion of S'/J by $O(G, \leq)$ or $O(G)$. Let $P_x = p'_x + J$. It is clear that P_x is a projection and that the C^*-algebra $O(G)$ is generated by the elements P_x $(x \in G)$. Note also that the map $x \mapsto P_x$ is decreasing.

The universal property of $O(G)$ is given by the following:

PROPOSITION 2.3. *Let* G *be a non-empty set and* $\leq$ *a pre-order on* G. *If* θ: $G \to B$ *is a decreasing map from* G *into the projections on a* C^**-algebra* B *then there is a unique* $*$*-homomorphism* φ: $O(G) \to B$ *such that* $\varphi(P_x) = \theta(x)$ *for all* $x \in G$.

Proof. Uniqueness is clear since the projections generate $O(G)$. To see existence let F, I, S', γ_0, J be as above. There exists a $*$-homomorphism ψ: $F \to B$ such that $\psi(x) = \theta(x)$ for all $x \in G$, and since $\theta(x) = \theta(x)^*$ and $\theta(x)\theta(y) = \theta(y)$ if $x \leq y$ we have $\psi(x - x^*)$ and $\psi(xy - y) = 0$. Hence $I \subseteq \ker(\psi)$, so there is an induced $*$-homomorphism $\tilde{\psi}$: $S' \to B$. The map

$$S' \to \mathbf{R}^+, \qquad b \mapsto \|\tilde{\psi}(b)\|,$$

is a C^*-seminorm on S', so it is dominated by γ_0, that is, $\|\tilde{\psi}(b)\| \leq \gamma_0(b)$. Hence $\tilde{\psi}(J) = 0$, and we obtain a norm-decreasing $*$-homomorphism φ: $S'/J \to B$ such that $\varphi(P_x) = \theta(x)$ for all $x \in G$. By density of S'/J in $O(G)$ and continuity of φ we can extend φ to obtain a $*$-homomorphism on $O(G)$. □

Observe that $O(G)$ can be badly behaved in general. For example, let H be an infinite-dimensional Hilbert space and let G denote the set of projections on H. Let $\leq$ be the reverse of the usual partial order on $B(H)_{sa}$ restricted to G. Then the inclusion map $G \to B(H)$ is decreasing, and therefore induces a $*$-homomorphism φ: $O(G) \to B(H)$ such that $\varphi(P_x) = x$ for all $x \in G$. Since the closed linear span of the projections is $B(H)$, φ is surjective. Hence $O(G)$ is not nuclear, since $B(H)$ is not.

Now suppose that G is an arbitrary pre-ordered group. For each $x \in G$ the map

$$G \to O(G), \qquad y \mapsto P_{xy},$$

is decreasing, and therefore by the universal property of $O(G)$ there is a unique $*$-homomorphism β_x: $O(G) \to O(G)$ such that $\beta_x(P_y) = P_{xy}$ for all $y \in G$. It is easily checked that $\beta_x \in \operatorname{Aut} O(G)$ and that the map

$$\beta\colon G \to \operatorname{Aut} O(G), \qquad x \mapsto \beta_x,$$

is a homomorphism. We shall call β the *canonical action* of G on $O(G)$.

In the next section we shall represent $A \times_\alpha G^+$ in terms of the algebras $O(G)$ and A and the actions β and α of G.

The algebra $O(G)$ is abelian if G is a totally ordered set. This implies that $O(G)$ is nuclear in this case, and this will be important for some results in the sequel. We can realize $O(G)$ in a more "concrete" fashion in this situation. Let $\Omega(G)^\sim$ denote the set of decreasing functions from G to $\{0, 1\}$. We define a linear order on $\Omega(G)^\sim$ by setting $\omega \leq \omega'$ if $\omega(x) \leq \omega'(x)$ $(x \in G)$. Denote by $+\infty$, $-\infty$ the functions on G that are constantly $1, 0$ respectively, so $\pm\infty \in \Omega(G)^\sim$ and $-\infty \leq \omega \leq +\infty$ for all $\omega \in \Omega(G)^\sim$. For $x \in G$ define $\overline{x} \in \Omega(G)^\sim$ by

$$\overline{x}(y) = \begin{cases} 1, & \text{if } y \leq x, \\ 0, & \text{if } y > x. \end{cases}$$

The map $G \to \Omega(G)^\sim$, $x \mapsto \overline{x}$, is strictly increasing.

We endow $\Omega(G)^\sim$ with the relative topology from the product space $\{0, 1\}^G$, and as the product is compact Hausdorff, it follows that $\Omega(G)^\sim$ is also a compact Hausdorff space (as it is a closed subset). Hence $\Omega(G) = \Omega(G)^\sim \backslash \{-\infty\}$ is a locally compact Hausdorff space.

For $x \in G$ let $\widetilde{P}_x \in C_0(\Omega(G))$ be the projection defined by $\widetilde{P}_x(\omega) = \omega(x)$. Clearly $x \leq y$ if and only if $\widetilde{P}_x \geq \widetilde{P}_y$. If $x \vee y = \max\{x, y\}$ we therefore have $\widetilde{P}_x \widetilde{P}_y = \widetilde{P}_{x \vee y}$ for all $x, y \in G$. This implies the linear span of all $\widetilde{P}_x$ $(x \in G)$ is a (separating) $*$-subalgebra of $C_0(\Omega(G))$, and therefore by the Stone-Weierstrass theorem, it is dense in $C_0(\Omega(G))$.

Proposition 2.4. *Suppose that $\leq$ is a total order on a non-empty set G. Then there is a unique $*$-isomorphism φ from $O(G)$ to $C_0(\Omega(G))$ such that $\varphi(P_x) = \widetilde{P}_x$ for all $x \in G$.*

Proof. Since the map $G \to C_0(\Omega(G))$, $x \mapsto \widetilde{P}_x$, is decreasing there is a unique $*$-homomorphism from $O(G)$ to $C_0(\Omega(G))$ such that $\varphi(P_x) = \widetilde{P}_x$ for all $x \in G$. We shall construct an inverse for φ. First observe that $O(G)$ is abelian since $P_x P_y = P_{x \vee y} = P_y P_x$, for all $x, y \in G$ (this uses the fact that G is totally ordered). If τ is a character on $O(G)$ define $\tau' \in \Omega(G)^\sim$ by setting $\tau'(x) = \tau(P_x)$. Thus for $x_1, \dots, x_n \in G$ and $\lambda_1, \dots, \lambda_n \in \mathbf{C}$ we have

$$\left|\tau\left(\sum_{i=1}^n \lambda_i P_{x_i}\right)\right| = \left|\sum_{i=1}^n \lambda_i \tau'(x_i)\right| \leq \left\|\sum_{i=1}^n \lambda_i \widetilde{P}_{x_i}\right\|.$$

Hence

$$\left\|\sum_{i=1}^{n}\lambda_i P_{x_i}\right\| \leq \left\|\sum_{i=1}^{n}\lambda_i \widetilde{P}_{x_i}\right\|.$$

We therefore have a well-defined linear map ψ from the linear span of all $\widetilde{P}_x$ $(x \in G)$ to $O(G)$ given by $\psi(\sum_{i=1}^{n}\lambda_i\widetilde{P}_{x_i}) = \sum_{i=1}^{n}\lambda_i P_{x_i}$. Clearly ψ is a $*$-homomorphism, and norm-decreasing by the inequalities above. Therefore we can extend it to a $*$-homomorphism from $C_0(\Omega(G))$ to $O(G)$. Since $\psi(\widetilde{P}_x) = P_x$ for all $x \in G$, the maps ψ and φ are inverse to each other, and so the result is proved. □

If G is not a singleton then $\Omega(G)\backslash\{+\infty\}$ is non-empty. For $x, y \in G$ set $[\overline{x}, \overline{y}) = \{\omega \in \Omega(G)^{\sim} \mid \overline{x} \leq \omega < \overline{y}\}$. These sets $[\overline{x}, \overline{y})$ $(x, y \in G)$ form a base of compact open sets for the topology of $\Omega(G)\backslash\{+\infty\}$, so $\Omega(G)^{\sim}$ is totally disconnected, as can also be seen by noting that $\{0, 1\}^G$ is totally disconnected. If G admits no greatest or least element then $\{\overline{x} \mid x \in G\}$ is dense in $\Omega(G)^{\sim}$.

Observe that even if G is only a pre-ordered group (that is, $\leq$ may not be a partial order) the algebra $O(G)$ is still abelian if every pair of elements of G can be compared $(x \leq y$ or $y \leq x)$. For example if $\mathbf{Z}^2$ is endowed with the pre-order defined by $(m, n) \leq (m', n')$ if $n \leq n'$, then $O(\mathbf{Z}^2)$ is abelian.

The algebra $O(G)$ is not abelian for all partially ordered groups. In particular, if $G = \mathbf{Z}^2$ is endowed with the product partial order, so $\mathbf{Z}^{2+} = \mathbf{N}^2$, then $O(G)$ is non-abelian. To see this let u, v be a pair of commuting isometries on a Hilbert space H whose range projections uu^* and vv^* do not commute. (For instance, take $H = H^2$ the Hardy space on the circle, and let u, v be the Toeplitz operators on H with symbols z and $(\lambda - z)/(1 - \overline{\lambda}z)$ where λ is a non-zero number of modulus less than 1, and z is the inclusion map of the circle in the plane.) If $(m, n) \in \mathbf{Z}^2$ define the projection $P_{(m,n)}$ to be 1 if $(m, n) \notin \mathbf{N}^2$ and to be $u^m v^n u^{m*} v^{n*}$ if $(m, n) \in \mathbf{N}^2$. Then the decreasing map $G \to B(H)$, $x \mapsto P_x$, induces a $*$-homomorphism φ: $O(G) \to B(H)$ whose range is not abelian since it contains uu^* and vv^*. Hence $O(G)$ is non-abelian as claimed.

3. The corner crossed product representation. If A and B are C^*-algebras we denote the maximal C^*-tensor product by $A \otimes B$. We shall need to use the universal property this enjoys, namely, if C is a C^*-algebra and φ: $A \to C$ and ψ: $B \to C$ are $*$-homomorphisms whose ranges commute then there exists a unique $*$-homomorphism

$\pi\colon A \otimes B \to C$ such that $\pi(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in A$ and $b \in B$.

Suppose (A, α, G) is a C^*-dynamical system where G is a preordered group. If $Z_0 = O(G) \otimes A$ we have a C^*-dynamical system (Z_0, γ, G) where $\gamma_* = \beta_x \otimes \alpha_x$ $(x \in G)$. Let Z denote the crossed product $Z_0 \times_\gamma G$ and let $U\colon G \to M(Z)$ be the canonical homomorphism.

Choose an approximate unit for Z_0 (and hence for Z) of the form $(f_\lambda \otimes u_\lambda)$ where (f_λ) is an approximate unit for $O(G)$ and (u_λ) is one for A. If $x \in G$ and $b \in Z$ one readily verifies that the nets $((P_x f_\lambda \otimes u_\lambda)b)$ and $(b(f_\lambda P_x \otimes u_\lambda))$ are convergent in Z. One way to see this is to show that the set B of all $b \in Z$ for which these nets converge is a C^*-subalgebra of Z containing all $(f \otimes a)U_y$ $(f \in O(G)$, $a \in A$, $y \in G)$, and so $B = Z$, as the elements $(f \otimes a)U_y$ generate Z. We can thus define $\overline{P}_x \in M(Z)$ by the equations

$$\begin{aligned} \overline{P}_x b &= \lim_\lambda (P_x f_\lambda \otimes u_\lambda) b, \\ b\overline{P}_x &= \lim_\lambda b(f_\lambda P_x \otimes u_\lambda). \end{aligned}$$

It is easily checked that $\overline{P}_x$ is a projection, that the map $x \mapsto \overline{P}_x$ is decreasing, and that $\overline{P}_x(f \otimes a) = P_x f \otimes a$, and $(f \otimes a)\overline{P}_x = fP_x \otimes a$ for all $f \in O(G)$ and $a \in A$. We have $U_y \overline{P}_x U_y^* = \overline{P}_{yx}$ $(x, y \in G)$. To see this it suffices to show that if $f \in O(G)$ and $a \in A$ then $U_y \overline{P}_x U_{y^{-1}}(f \otimes a) = \overline{P}_{yx}(f \otimes a)$, and this follows from the equations

$$\begin{aligned} U_y \overline{P}_x U_{y^{-1}}(f \otimes a) &= U_y \overline{P}_x(\beta_{y^{-1}}(f) \otimes \alpha_{y^{-1}}(a))U_{y^{-1}} \\ &= U_y(P_x \beta_{y^{-1}}(f) \otimes \alpha_{y^{-1}}(a))U_{y^{-1}} \\ &= \beta_y(P_x) f \otimes a \\ &= P_{yx} f \otimes a \\ &= \overline{P}_{yx}(f \otimes a). \end{aligned}$$

For $x \in G^+$ set $\widetilde{W}_x = pU_x p$ where $p = \overline{P}_e$, and observe that $pU_x p U_x^* = \overline{P}_e \overline{P}_x = \overline{P}_x = U_x p U_x^*$, so $pU_x p = U_x p$. Hence for all $x, y \in G^+$ we have $\widetilde{W}_x^* \widetilde{W}_x = p$, $\widetilde{W}_e = p$, and $\widetilde{W}_x \widetilde{W}_y = \widetilde{W}_{xy}$. We define $W_x \in M(pZp)$ for $x \in G^+$ by setting $W_x b = \widetilde{W}_x b$ and $bW_x = b\widetilde{W}_x$, if $b \in pZp$. The map $W\colon G^+ \to M(pZp)$, $x \mapsto W_x$, is an isometric homomorphism.

If φ denotes the $*$-homomorphism $A \to pZp$, $a \mapsto P_e \otimes a$, it is easily checked that (φ, W) is a covariant homomorphism from

(A, α, G^+) to pZp. We call the $*$-homomorphism $\varphi \times W$ the *canonical* map from $A \times_\alpha G^+$ to pZp. It is useful also to give p a name: it is the *distinguished* projection of $M(Z)$.

THEOREM 3.1. *Let* (A, α, G) *be a* C^**-dynamical system where* G *is a preordered group, let* $Z = (O(G) \otimes A) \times_\gamma G$, *and let* p *be the distinguished projection of* $M(Z)$. *Then the canonical map from* $A \times_\alpha G^+$ *to* pZp *is a* $*$*-isomorphism.*

Proof. We retain our previous notation.

We show first that $\varphi \times W$ is surjective. The algebra Z is the closed linear span of the elements bU_x $(b \in Z_0,\ x \in G)$, and therefore the closed linear span of the elements of the form

$$(*) \qquad (P_{x_1} \otimes a_1) \cdots (P_{x_n} \otimes a_n)U_x \qquad (x, x_j \in G,\ a_j \in A),$$

since the products $P_{x_1} \cdots P_{x_n}$ have closed linear span $O(G)$. If b is an element of the form in $(*)$ we claim that $pbp \in \operatorname{im}(\varphi \times W)$, and this will show that $\varphi \times W$ is surjective. To prove the claim observe that we can write b in the form

$$b = U_{y_1}^* U_{z_1} \varphi(a_1') U_{y_2}^* U_{z_2} \varphi(a_2') \cdots U_{y_n}^* U_{z_n} \varphi(a_n') U_{y_{n+1}}^* U_{z_{n+1}}$$

for some y_j, $z_j \in G^+$ and $a_j' \in A$, where we use the facts that $P_x \otimes a = U_x(P_e \otimes \alpha_{x^{-1}}(a))U_{x^{-1}}$ and that every element of G can be written in the form $y^{-1}z$ for some $y, z \in G^+$. Hence

$$pbp = W_{y_1}^* W_{z_1} \varphi(a_1') W_{y_2}^* W_{z_2} \varphi(a_2') \cdots W_{y_n}^* W_{z_n} \varphi(a_n') W_{y_{n+1}}^* W_{z_{n+1}},$$

since $p\varphi(a)p = \varphi(a)$ for all $a \in A$, and $U_x p = W_x$ if $x \in G^+$. It follows that $pbp \in \operatorname{im}(\varphi \times W)$ and the claim is proved.

Now we show that $\varphi \times W$ is injective. Represent $M(A \times_\alpha G^+)$ as a C^*-subalgebra of $B(H)$ for some Hilbert space H with $\mathrm{id}_H \in M(A \times_\alpha G^+)$. Let $\rho\colon A \to B(H)$ be the inclusion map. The triple (H, ρ, V) is a covariant representation of (A, α, G^+), where V denotes the canonical map from G^+ to $M(A \times_\alpha G^+)$, so by the dilation theorem of §2 there exists a covariant representation (H', ρ', V') of (A, α, G) dilating (H, ρ, V), where H is a closed vector subspace of H' invariant for $\rho'(a)$ $(a \in A)$ and V_x' $(x \in G^+)$.

Let $Q \in B(H')$ be the projection onto H. Then of course the invariance properties of H mean that $Q\rho'(a) = \rho'(a)Q$ for all $a \in A$ and $QV_x'Q = V_x'Q$ for all $x \in G^+$, and the dilation property means that $\rho'(a)_H = \rho(a)$ for all $a \in A$ and $(V_x')_H = V_x$ for all $x \in G^+$.

For an arbitrary element x of G set $Q_x = V'_xQV'^*_x$. Then Q_x is a projection, $Q_e = Q$, and the map $x \mapsto Q_x$ is decreasing since if $x \leq y$ then $Q_xQ_y = V'_xQV'_{x^{-1}}V'_yQV'^*_y = V'_xQV'_{x^{-1}y}QV'^*_y = V'_xV'_{x^{-1}y}QV'^*_y$ (as $x^{-1}y \in G^+$ implies that $QV'_{x^{-1}y}Q = V'_{x^{-1}y}Q$). Hence $Q_xQ_y = Q_y$. It follows from Proposition 2.3 that there exists a $*$-homomorphism $\psi_0\colon O(G) \to B(H')$ such that $\psi_0(P_x) = Q_x$ $(x \in G)$.

If $x \in G$ and $a \in A$ then $\rho'(a)$ commutes with $\psi_0(P_x)$, since

$$\begin{aligned}
\psi_0(P_x)\rho'(a) &= V'_xQV'_{x^{-1}}\rho'(a)\\
&= V'_xQ\rho'(\alpha_{x^{-1}}(a))V'_{x^{-1}}\\
&= V'_x\rho'(\alpha_{x^{-1}}(a))QV'_{x^{-1}}\\
&= \rho'(a)V'_xQV'_{x^{-1}}\\
&= \rho'(a)\psi_0(P_x).
\end{aligned}$$

This implies that $\rho'(a)$ commutes with all $\psi_0(f)$ $(f \in O(G))$. Hence there is a unique $*$-homomorphism $\psi_1\colon Z_0 \to B(H')$ such that $\psi_1(f \otimes a) = \psi_0(f)\rho'(a)$ for all $f \in O(G)$ and $a \in A$.

If $x, y \in G$ and $a \in A$ then

$$\begin{aligned}
\psi_1(\gamma_x(P_y \otimes a)) &= \psi_1(P_{xy} \otimes \alpha_x(a))\\
&= Q_{xy}\rho'(\alpha_x(a))\\
&= V'_xV'_yQV'_{y^{-1}}V'_{x^{-1}}\rho'(\alpha_x(a))\\
&= V'_x\psi_0(P_y)\rho'(a)V'_{x^{-1}}\\
&= V'_x\psi_1(P_y \otimes a)V'^*_x.
\end{aligned}$$

Hence for all $b \in Z_0$, $\psi_1(\gamma_x(b)) = V'_x\psi_1(b)V'^*_x$, so (H', ψ_1, V') is a covariant representation of (Z_0, γ, G). Observe that if $f \in O(G)$ and $a \in A$ then we have $(\psi_1 \times V')((f \otimes a)p) = \psi_0(fP_e)\rho'(a) = \psi_0(f)Q\rho'(a) = \psi_0(f)\rho'(a)Q = ((\psi_1 \times V')(f \otimes a))Q$. Hence

$$(\psi_1 \times V')(bp) = (\psi_1 \times V')(b)Q \quad \text{for all } b \in Z.$$

It follows that

$$\psi_2\colon pZp \to B(H), \qquad b \mapsto ((\psi_1 \times V')(b))_H,$$

is a $*$-homomorphism.

The composition $\psi_2(\varphi \times W)\colon A \times_\alpha G^+ \to B(H)$ is just the inclusion. To see this we need only show this map leaves aV_x fixed for each

$a \in A$ and $x \in G^+$, and this follows from the equations

$$\begin{aligned}
\psi_2(\varphi \times W)(aV_x) &= \psi_2(\varphi(a)W_x) \\
&= (\psi_1 \times V')(\varphi(a)W_x)_H \\
&= (\psi_1 \times V')((P_e \otimes a)U_x\overline{P}_e)_H \\
&= (\psi_1 \times V')((P_e \otimes a)U_x)Q_H \\
&= (\psi_1(P_e \otimes a)V'_x)_H \\
&= (Q\rho'(a)V'_x)_H \\
&= aV_x.
\end{aligned}$$

Since $\psi_2(\varphi \times W)$ is injective, so is $\varphi \times W$, and this means we have shown $\varphi \times W$ is a $*$-isomorphism. □

It is well known that if (A, α, G) is a classical C^*-dynamical system where A is nuclear and G is amenable, then $A \times_\alpha G$ is nuclear also.

THEOREM 3.2. *Let (A, α, G) be a C^*-dynamical system where G is an amenable ordered group and A a nuclear C^*-algebra. Then $A \times_\alpha G^+$ is nuclear.*

Proof. Since $O(G)$ is abelian, and A is nuclear, the algebra $O(G) \otimes A$ is nuclear. Hence $Z = (O(G) \otimes A) \times_\gamma G$ is nuclear, as G is amenable. It follows that the hereditary C^*-subalgebra pZp is nuclear, and therefore so is $A \times_\alpha G^+$. □

Of course, using the same proof, Theorem 3.2 is true if G is only assumed to be a pre-ordered amenable group for which $O(G)$ is nuclear.

Some of the deepest results of the theory of C^*-algebras are concerned with giving conditions on a C^*-dynamical system which ensure the crossed product is simple or prime. This is important as the simple and the prime C^*-algebras play a role in the C^*-theory analogous to that played by factors in the theory of von Neumann algebras. Incidentally there are some indications which suggest that prime C^*-algebras (i.e. those in which every pair of non-zero closed ideals have a non-zero intersection) are the more appropriate analogue of factors, rather than simple C^*-algebras. It turns out that while it is "hard" for $A \times_\alpha G$ to be simple it is impossible for $A \times_\alpha G^+$ to be so if G is non-trivial and partially ordered. However, while it is still "hard" for $A \times_\alpha G$ to be prime, it seems to be "easier" for $A \times_\alpha G^+$ to be prime (compare $C^*(\mathbf{Z})$ which is not prime, with $C^*(\mathbf{N})$ which is). More evidence for this claim will be given in §4.

PROPOSITION 3.3. *Let (A, α, G) be a non-trivial C^*-dynamical system where G is a partially ordered group. Then $A \times_\alpha G^+$ is not simple.*

Proof. Suppose that $A \times_\alpha G^+$ is simple, and suppose that the maps $V\colon G^+ \to M(A \times_\alpha G^+)$, $U\colon G \to M(A \times_\alpha G)$, and $\varepsilon\colon A \times_\alpha G^+ \to A \times_\alpha G$ are canonical. If $x \in G^+$ and $b_1, b_2 \in A \times_\alpha G^+$ with $b_1 V_x = b_2 V_x$ then $\varepsilon(b_1) U_x = \varepsilon(b_2) U_x$, so $\varepsilon(b_1) = \varepsilon(b_2)$ (as U_x is unitary). Hence $b_1 = b_2$, as ε is injective (its kernel must be zero by simplicity of $A \times_\alpha G^+$).

Suppose that (H, φ) is a faithful non-degenerate representation of A and $(l^2(G^+, H), \overline{\varphi}, W)$ is the induced covariant representation of (A, α, G^+). Then $(l^2(G^+, H), \overline{\varphi})$ is also faithful and non-degenerate. If $b \in A \times_\alpha G^+$ then $(b V_x V_x^*) V_x = b V_x$, so $vB_x V_x^* = b$, and therefore $\overline{\varphi}(b) W_x W_x^* = \overline{\varphi}(b)$. By non-degeneracy $W_x W_x^* = 1$, that is, W_x is a unitary for all $x \in G^+$.

Now choose a non-zero element η of H, and let f be the element of $l^2(G^+, H)$ such that $f(e) = \eta$ and $f(y) = 0$, $y > e$. Choose $x > e$. Then there exists $g \in l^2(G^+, H)$ such that $W_x g = f$, so $(W_x g)(e) = \eta \neq 0$, implying $e \in xG^+$, a contradiction since G is partially ordered. This proves the projection.

REMARK 3.1. The partial order assumption cannot be dropped in the preceding proposition. For example let G be a group endowed with the trivial pre-order such that $G^+ = G$. Then of course $A \times_\alpha G^+ = A \times_\alpha G$ is just a classical crossed product, and therefore it may be simple.

Let (A, α, G) be a C^*-dynamical system where G is a pre-ordered group, and let I be a G-invariant closed ideal of A. The closed linear span J of all $aV_{x_1} V_{x_2} \cdots V_{x_n}$ $(a \in I, x_1, \dots, x_n \in G)$ is an ideal of $A \times_\alpha G^+$, and any approximate unit for I is one for J also. Hence $A \cap J = I$. In fact J is the closed ideal of $A \times_\alpha G^+$ generated by I.

Recall that a classical C^*-dynamical system (A, α, G) is G-prime if for every pair of non-zero G-invariant closed ideals of A their intersection is non-zero.

PROPOSITION 3.4. *Let (A, α, G) be a C^*-dynamical system where G is a pre-ordered group and the crossed product $A \times_\alpha G^+$ is prime. Then (A, α, G) is G-prime.*

Proof. Let I_1, I_2 be non-zero G-invariant closed ideals of A generating the closed ideals J_1, J_2 respectively in $A \times_\alpha G^+$. As J_1, J_2 are non-zero, $J_1 \cap J_2$ contains a non-zero element, b say. Let (u_λ) and (v_μ) be approximate units in I_1 and I_2 respectively. Then $b = \lim_{\lambda,\mu} bu_\lambda v_\mu$, so for some indices λ and μ the product $u_\lambda v_\mu$ is non-zero, and since $u_\lambda v_\mu \in I_1 \cap I_2$ this shows that $I_1 \cap I_2$ is non-zero. □

We recall some definitions and results of the classical theory. Suppose that (A, α, G) is a non-trivial separable C^*-dynamical system where G is an abelian group. The *Arveson spectrum* $\mathrm{Sp}(\alpha)$ of α is the set of all $\gamma \in \widehat{G}$ (where $\widehat{G}$ is the dual group of the discrete group G) such that there exist unit vectors $a_n \in A$ for which

$$\lim_{n\to\infty} \|\alpha_x(a_n) - \gamma(x)a_n\| = 0 \qquad (x \in G).$$

The set $\mathrm{Sp}(\alpha)$ is closed in $\widehat{G}$ and its annihilator $\mathrm{Sp}(\alpha)^\perp$ is the set of all elements x of G for which $\alpha_x = \mathrm{id}$. If B is a G-invariant C^*-subalgebra of A we get a new C^*-dynamical system $(B, \alpha \mid B, G)$ by restricting α to B. The *Connes spectrum* of α is a closed subgroup of $\widehat{G}$ defined by the equation

$$\Gamma(\alpha) = \bigcap_B \mathrm{Sp}(\alpha \mid B)$$

where B runs over all non-zero G-invariant hereditary C^*-subalgebras of A. The following conditions are equivalent.

(a) The crossed product $A \times_\alpha G$ is prime (respectively simple);

(b) The algebra A is G-prime (respectively G-simple) and $\Gamma(\alpha) = \widehat{G}$.

These results can be found in **[16]** and **[17]**.

If G is a pre-ordered group, the corresponding equivalences for the C^*-dynamical system (A, α, G^+) do not hold. This is not surprising, as Condition (b) makes no reference to the order structure of G. For example, consider the C^*-dynamical system $(\mathbf{C}, \alpha, \mathbf{Z})$ (α trivial, of course). The algebra $\mathbf{C} \times_\alpha \mathbf{Z}^+ = C^*(\mathbf{N})$ is prime (it is the Toeplitz algebra, as we saw already), but $\Gamma(\alpha) \neq \widehat{\mathbf{Z}}$. If instead α is the usual action by an irrational rotation of angle θ on the circle group $\mathbf{T}$, then $(C(\mathbf{T}), \alpha, \mathbf{Z})$ is G-simple and $\Gamma(\alpha) = \widehat{\mathbf{Z}}$, as is well known, but $C(\mathbf{T}) \times_\alpha \mathbf{Z}^+$ is not simple (Proposition 3.3).

THEOREM 3.5. *Let (A, α, G) be a non-trivial separable C^*-dynamical system where G is an abelian pre-ordered group. The following are* *equivalent conditions*:

(a) *The crossed product* $A \times_\alpha G^+$ *is prime*;

(b) *The tensor product* $O(G) \otimes A$ *is* G*-prime for the action* $\gamma = \beta \otimes \alpha$, *and* $\Gamma(\gamma) = \widehat{G}$.

Proof. Let $Z_0 = O(G) \otimes A$ and $Z = Z_0 \times_\gamma G$, and let p be the distinguished projection of $M(Z)$.

Suppose J is a closed ideal of Z containing pZp. Then J contains $p(P_e \otimes a)p = P_e \otimes a$ for all $a \in A$. Hence if $U: G \to M(Z)$ is the canonical map, J contains $U_x(P_e \otimes a)U_x^* = \beta_x(P_E) \otimes \alpha_x(a) = P_x \otimes \alpha_x(a)$. It follows that $Z_0 \subseteq J$, and so $Z = J$. Thus pZp is a full hereditary C^*-subalgebra of Z.

If C is a non-zero C^*-algebra let $\mathrm{Prim}(C)$ denote its primitive ideal space. Then C is prime iff every two non-empty open sets of $\mathrm{Prim}(C)$ have non-empty intersection.

As pZp is full and hereditary in Z, the map

$$\mathrm{Prim}(Z) \to \mathrm{Prim}(pZp), \qquad J \mapsto J \cap pZp,$$

is a homeomorphism, and therefore Z is prime iff pZp is prime. The theorem now follows using the Pedersen-Olesen results applied to (Z_0, γ, G), and the $*$-isomorphism of $A \times_\alpha G^+$ with pZp. □

If G is an abelian partially ordered group the Toeplitz algebra $T(G)$ of G as defined in [**12**] is just the algebra $C^*(G^+)$. It was shown in [**12**] that $C^*(G^+)$ is primitive if G is totally ordered. The central idea of the proof is essentially a use of the special case of Theorem 3.5 when $A = \mathbf{C}$.

If (A, α, G) is a non-trivial C^*-dynamical system where G is a preordered group, then as we saw in the proof of Theorem 3.5, pZp is a full hereditary C^*-subalgebra of Z. Hence Z is type I iff pZp is type I. Otherwise put, Z is type I iff $A \times_\alpha G^+$ is type I.

Incidentally, if (A, α, G) is separable then $A \times_\alpha G^+$ $(\cong pZp)$ is stably isomorphic to Z by a well-known result of Brown [3] on full hereditary subalgebras.

We are now going to need the following result:

THEOREM 3.6 (*Zeller-Meier* [**22**]). *Let* (A, α, G) *be a classical separable* C^**-dynamical system, where* G *acts freely on* $\widehat{A}$ (*the spectrum of* A). *The following conditions are equivalent*:

(a) $A \times_\alpha G$ *is type* I;

(b) A *is type* I *and every* G*-orbit in* $\widehat{A}$ *is discrete.*

If a group G acts on sets Ω_1, Ω_2 we get an action of G on $\Omega_1 \times \Omega_2$ by setting $x(\omega_1, \omega_2) = (x\omega_1, x\omega_2)$.

THEOREM 3.7. *Let (A, α, G) be a non-trivial separable C^*-dynamical system where G is an ordered group acting freely on $\Omega(G) \times \widehat{A}$. Then the following conditions are equivalent:*

(a) $A \times_\alpha G^+$ *of type* I;

(b) A *is type* I *and the G-orbits in $\Omega(G) \times \widehat{A}$ are discrete.*

Proof. By [5] there is a canonical homeomorphism $\theta\colon \widehat{Z}_0 \to \Omega(G) \times \widehat{A}$ where as usual $Z_0 = O(G) \otimes A$. One easily checks that $\theta(x\omega, xt) = x\theta(\omega, t)$ $(\omega \in \Omega(G),\ t \in \widehat{A},\ x \in G)$. Also A is type I iff $O(G) \otimes A$ is type I. The result is now immediate from Theorem 3.6. □

4. Covariant representations. The theory that we develop in this section is concerned only with the totally ordered case. Although some fragments can probably be done in greater generality, we shall give counter-examples to show that the principal results do not extend to the partially ordered case. Thus for ease of exposition we shall confine our attention to totally ordered groups throughout.

We shall be principally concerned with the question of what conditions on a covariant representation (H, φ, W) ensure the corresponding representation $(H, \varphi \times W)$ is faithful. However we begin with a result on irreducible representations.

THEOREM 4.1. *Let (A, α, G) be a C^*-dynamical system where G is an ordered group. Let (H, φ) be a non-zero irreducible representation of A and suppose that $(l^2(G^+, H), \overline{\varphi}, W)$ is the induced covariant representation of (A, α, G^+). Then $(l^2(G^+, H), \overline{\varphi} \times W)$ is an irreducible representation of (A, α, G^+).*

Proof. Let P be a projection in the commutant of $\operatorname{im}(\overline{\varphi} \times W)$, so that $P\overline{\varphi}(a)W_x = \overline{\varphi}(a)W_xP$ $(a \in A,\ x \in G^+)$. Since $(l^2(G^+, H), \overline{\varphi})$ is nondegenerate we have $PW_x = W_xP$. (To see this choose an approximate unit (u_λ) for A and note that $(\overline{\varphi}(u_\lambda))$ converges strongly to 1 on $l^2(G^+, H)$.)

For $x \in G^+$ and $\eta \in H$ define $\eta_x \in l^2(G^+, H)$ by

$$\eta_x(y) = \begin{cases} \eta, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

If $x, z \in G^+$ we have $W_z\eta_x = \eta_{zx}$, and if $x < z$, then $W_z^*\eta_x = 0$. It follows that if $\eta, \eta' \in H$ and $y, z \in G^+$ with $y \neq z$ then $\langle P\eta_y, \eta'_z\rangle = 0$ (for example, if $y < z$, then $\langle P\eta_y, \eta'_z\rangle = \langle W_z^*P\eta_y, \eta'_e\rangle = \langle PW_z^*\eta_y, \eta'_e\rangle = 0$). In particular, if $z > e$ we have $0 = \langle P\eta_e, \eta'_z\rangle = \langle (P\eta_e)(z), \eta'\rangle$, for all $\eta' \in H$. Hence $(P\eta_e)(z) = 0$. Thus there is a unique element $Q\eta \in H$ such that $P\eta_e = (Q\eta)_e$. Clearly the map $Q\colon H \to H$, $\eta \mapsto Q\eta$, is continuous and linear.

Let $S \in B(l^2(G^+, H))$ be the diagonal operator given by $(Sf)(y) = Qf(y)$ $(f \in l^2(G^+, H), y \in G^+)$. If $\eta \in H$ we have $P\eta_y = PW_y\eta_e = W_y(Q\eta)_e = (Q\eta)_y = S\eta_y$. Hence $P = S$. It follows that Q is a projection. Now if $a \in A$ then $P\overline{\varphi}(a) = \overline{\varphi}(a)P$, so if $\eta \in H$ we have $P\overline{\varphi}(a)\eta_e = \overline{\varphi}(a)P\eta_e$ implies $Q\varphi(a)\eta = \varphi(a)Q\eta$. Hence $Q \in (\operatorname{im}\varphi)'$, and therefore $Q = 0$ or 1 by irreducibility of (H, φ). Thus $P = 0$ or 1, and hence $(l^2(G^+, H), \overline{\varphi} \times W)$ is irreducible. □

Let (A, α, G) be a non-trivial C^*-dynamical system where G is an ordered group. We say a covariant representation (H, φ, W) of (A, α, G^+) is *skew* if for $a \in A$ and $x \in G^+$, the equality $\varphi(\alpha_x(a)) = W_x\varphi(a)W_x^*$ implies that $a = 0$ or $x = e$. If (H, φ, W) is skew then φ is injective, and W_x is non-unitary for $x > e$. If (H, φ) is a faithful representation of A then $(l^2(G^+, H), \overline{\varphi}, W)$, the induced covariant representation of (A, α, G^+), is skew. For if $x > e$ and $a \in A$ are such that $\overline{\varphi}\alpha_x(a) = W_x\overline{\varphi}(a)W_x^*$, then given any $f \in l^2(G^+, H)$ we have $(\overline{\varphi}\alpha_x(a)f)(e) = (W_x\overline{\varphi}(a)W_x^*f)(e) = 0$ (as $e \notin xG^+$), so $\varphi\alpha_x(a)f(e) = 0$. Hence $\varphi\alpha_x(a) = 0$, so $\alpha_x(a) = 0$, and therefore $a = 0$.

It follows that $\alpha_x(a) = V_xaV_x^* \Rightarrow a = 0$ or $x = e$, for $a \in A$ and $x \in G^+$. (Take a faithful representation of A and apply the induced covariant representation to the above equation.) Hence if (H, φ, W) is any covariant representation of (A, α, G^+) where $\varphi \times W$ is injective we must have (H, φ, W) skew, for if $\varphi\alpha_x(a) = W_x\varphi(a)W_x^*$ we have $(\varphi \times W)(\alpha_x(a)) = (\varphi \times W)(V_xaV_x^*)$, so $\alpha_x(a) = V_xaV_x^*$, implying that $a = 0$ or $x = e$.

Now suppose that (A, α, G) is a C^*-dynamical system where G is an abelian ordered group. Let $\widehat{G}$ denote the Pontryagin dual group of G. Of course, as G is discrete, $\widehat{G}$ is compact. If $\gamma \in \widehat{G}$ then the map $V^\gamma\colon G^+ \to M(A \times_\alpha G^+)$, $x \mapsto \gamma(x)V_x$, is an isometric homomorphism. Letting $\rho\colon A \to A \times_\alpha G^+$ be the inclusion map, (ρ, V^γ) is a covariant homomorphism from (A, α, G^+) to $A \times_\alpha G^+$, so $\delta_\gamma = \rho \times V^\gamma$ is a $*$-homomorphism from $A \times_\alpha G^+$ to itself. Since δ_γ is the unique

$*$-homomorphism such that $\delta_\gamma(aV_x) = \gamma(x)aV_x$ $(a \in A, x \in G^+)$, it is clear that $\delta_\gamma\delta_{\gamma'} = \delta_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \widehat{G}$. Thus $\delta_\gamma \in \operatorname{Aut}(A \times_\alpha G^+)$, and $\delta\colon \widehat{G} \to \operatorname{Aut}(A \times_\alpha G^+)$, $\gamma \mapsto \delta_\gamma$, is a homomorphism. We call δ the *(dual) action* of $\widehat{G}$ on $A \times_\alpha G^+$, and we say a subset S of $A \times_\alpha G^+$ is *$\widehat{G}$-invariant* if $\delta_\gamma(S) = S$ $(\gamma \in \widehat{G})$.

Let us say that a covariant representation (H, φ, W) of (A, α, G^+) is *amenable* if there is a homomorphism $\delta\colon \widehat{G} \to \operatorname{Aut}(\operatorname{im}(\varphi \times W))$, $\gamma \mapsto \delta_\gamma$, such that $\delta_\gamma(\varphi(a)W_x) = \gamma(x)\varphi(a)W_x$ $(a \in A, x \in G^+, \gamma \in \widehat{G})$. Clearly δ is unique. We call it the *action* of $\widehat{G}$ on $\operatorname{im}(\varphi \times W)$. We shall use the same symbol δ for this action, and for the action on $A \times_\alpha G^+$—there should no risk of confusion. The reason for the terminology *amenable* will be apparent shortly. We shall see that δ plays a crucial role in analysing the covariant representation (H, φ, W).

A routine argument shows that a covariant representation (H, φ, W) of (A, α, G^+) is amenable if and only if $\ker(\varphi \times W)$ is $\widehat{G}$-invariant. If there exist unitaries $U_\gamma \in \varphi(A)'$ such that the Weyl commutation relations

$$U_\gamma W_x = \gamma(x) W_x U_\gamma \qquad (x \in G^+, \gamma \in \widehat{G})$$

hold, then it is clear that $\operatorname{im}(\varphi \times W)$ is invariant under $\operatorname{Ad} U_\gamma$. Letting δ_γ be the restriction of $\operatorname{Ad} U_\gamma$ to $\operatorname{im}(\varphi \times W)$ we get an action δ of $\widehat{G}$ on $\operatorname{im}(\varphi \times W)$, so (H, φ, W) is amenable.

If (H, φ) is a representation of A then the induced covariant representation $(l^2(G^+, H), \overline{\varphi}, W)$ of (A, α, G^+) is amenable. (Define unitaries $U_\gamma \in \overline{\varphi}(A)'$ by setting $(U_\gamma f)(x) = \gamma(x)f(x)$ $(f \in l^2(G^+, H), x \in G^+)$. Then $U_\gamma W_x = \gamma(x)W_xU_\gamma$ $(x \in G^+, \gamma \in \widehat{G})$. Hence $(l^2(G^+, H), \overline{\varphi}, W)$ is amenable by the remarks above.)

Not all covariant representations are amenable. We present an easy counter-example. Let G be non-trivial. If $\varphi\colon \mathbf{C} \to B(\mathbf{C})$, $\lambda \mapsto \lambda 1$, and $W\colon \widehat{G} \to B(\mathbf{C})$, $x \mapsto 1$, then $(\mathbf{C}, \varphi, W)$ is a non-amenable covariant representation of $(\mathbf{C}, \alpha, G^+)$ (of course α is the trivial action on $\mathbf{C}$).

Again suppose that (A, α, G) is a C^*-dynamical system where G is an abelian ordered group. Let (H, φ, W) be an amenable covariant representation of (A, α, G^+). Then for each $b \in B = \operatorname{im}(\varphi \times W)$ the map $\widehat{G} \to B$, $\gamma \mapsto \delta_\gamma(b)$, is continuous. This is so because the set of all $b \in B$ for which the above map is continuous is a C^*-algebra containing the generators $\varphi(a)W_x$ $(a \in A, x \in G^+)$ of B, and hence this algebra is B itself.

Let $d\gamma$ denote normalized Haar measure on $\widehat{G}$. For $b \in B$ we set

$$\mu(b) = \int_{\widehat{G}} \delta_\gamma(b)\, d\gamma.$$

We call the map $\mu\colon B \to B$, $b \mapsto \mu(b)$, the *mean* associated to the covariant representation (H, φ, W). Clearly μ is linear and norm-decreasing. We define

$$B^\delta = \{b \in B \mid \delta_\gamma(b) = b (\gamma \in \widehat{G})\}.$$

This is a C^*-algebra of B, which we call the *fixed-point algebra* of B. Clearly $\delta_\gamma(\mu(b)) = \mu(b)$ $(\gamma \in \widehat{G})$, so $\mu\mu(b) = \mu(b)$ $(b \in B)$. Hence $\mu^2 = \mu$ and $\mu(B) = B^\delta$. It is clear that if $b \in B^+$, then $\mu(b) \geq 0$, and if additionally $\mu(b) = 0$ then $b = 0$. This strict positivity of μ will be a key point in our result on skew covariant representations. We now need to identify the algebra B^δ more closely. Set $Q_x = W_x W_x^*$ for $x \in G$. Then Q_x is a projection, $Q_x = 1$ if $x \leq e$, and $Q_y Q_x = Q_{y \vee z}$ for $y, z \in G$.

If $b \in B$ and $x \in G$ it is easily checked that

$$\delta_\gamma(bW_x) = \gamma(x)\delta_\gamma(b)W_x.$$

Hence if $b = \varphi(a)W_{x_1} \cdots W_{x_n}$ with $a \in A$ and $x_1, \ldots, x_n \in G$ we have

$$\begin{aligned}
\mu(b) &= \int_{\widehat{G}} \gamma(x_1 \cdots x_n)\varphi(a)W_{x_1} \cdots W_{x_n}\, d\gamma \\
&= \left(\int_{\widehat{G}} \gamma(x_1 \cdots x_n)\, d\gamma\right)\varphi(a)W_{x_1} \cdots W_{x_n} \\
&= \begin{cases} \varphi(a)W_{x_1} \cdots W_{x_n}, & \text{if } x_1 \cdots x_n = e, \\ 0, & \text{if } x_1 \cdots x_n \neq e. \end{cases}
\end{aligned}$$

A simple induction argument on n shows that $W_{x_1} \cdots W_{x_n}$ is of the form Q_x for some $x \in G$ if $x_1 \cdots x_n = e$. Hence $\mu(b) = \varphi(a)Q_x$ or $\mu(b) = 0$. In either case $\mu(b)$ is in the closed linear span C of all the elements $\varphi(a)Q_x$ $(a \in A, x \in G)$. As B is the closed linear span of all elements $\varphi(a)W_{x_1} \cdots W_{x_n}$ $(a \in A, x_1, \ldots, x_n \in G)$, so $\mu(B) \subseteq C$, and obviously $C \subseteq \mu(B)$, so $C = \mu(B)$.

Explicitly, we have just shown that B^δ is the closure of the linear span C_0 of all $\varphi(a)Q_x$ $(a \in A, x \in G^+)$. Note also that C_0 is obviously a $*$-subalgebra of B^δ, as $\varphi(a)Q_x = Q_x\varphi(a)$.

If we regard $M(A, \times_\alpha, G^+)$ as a C^*-algebra on some Hilbert space K with $\mathrm{id}_K \in M(A \times_\alpha G^+)$ and let $\rho\colon A \to B(K)$ be the inclusion map then (K, ρ, V) is an amenable covariant representation of

(A, α, G^+) and $\rho\times V\colon A\times_\alpha G^+ \to B(K)$ is the inclusion map. We call (K, ρ, V) the *identity* covariant representation of (A, α, G^+). We therefore have a mean $\mu\colon A\times_\alpha G^+ \to A\times_\alpha G^+$ and fixed-point algebra $(A\times_\alpha G^+)^\delta$. Also, (K, ρ, V) is skew if (A, α, G) is non-trivial.

A few general remarks are needed before the next lemma. Let C be a C^*-algebra. If $p_1, \ldots, p_n$ are pairwise orthogonal projections in C then

$$\left\|\sum_{i=1}^n p_i c p_i\right\| = \max_{1\le i\le n} \|p_i c p_i\| \qquad (c\in C).$$

If $q_1, \ldots, q_n$ are projections in C such that $q_1 \ge q_2 \ge \cdots \ge q_n$ then $q_1 - q_2, \ldots, q_{n-1} - q_n, q_n$ are pairwise orthogonal projections. Moreover, if $c_1, \ldots, c_n \in C$ and we set $b_i = c_1 + \cdots + c_i$ $(1 \le i \le n)$ then

$$\sum_{i=1}^n c_i q_i = \sum_{i=1}^{n-1} b_i(q_i - q_{i+1}) + b_n q_n.$$

Lemma 4.2. *Let (A, α, G) be a non-trivial C^*-dynamical system where G is an abelian ordered group, and suppose that (H, φ, W) is an amenable skew covariant representation of (A, α, G^+). Then there exists a unique $*$-isomorphism $\theta\colon (A\times_\alpha G^+)^\delta \to (\mathrm{im}(\varphi\times W))^\delta$ such that $\theta(aV_xV_x^*) = \varphi(a)W_xW_x^*$ $(a\in A, x\in G)$.*

Proof. Uniqueness of θ is obvious. Put $P_x = V_xV_x^*$ and $Q_x = W_xW_x^*$. To see existence of θ it suffices to show

$$\left\|\sum_{i=1}^n a_i P_{x_i}\right\| = \left\|\sum_{i=1}^n \varphi(a_i) Q_{x_i}\right\|$$

for $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in G$. We may even suppose that $e \le x_1 < \cdots < x_n$, so that $P_{x_1} > \cdots > P_{x_n}$ and $Q_{x_1} > \cdots > Q_{x_n}$.

Claim. $\|\varphi(a)Q_x\| = \|a\| = \|\varphi(a)(Q_x - Q_y)\|$ if $e \le x < y$ and $a\in A$.

The result follows easily from the claim, because

$$\left\|\sum_{i=1}^n \varphi(a_i)Q_{x_i}\right\| = \left\|\sum_{i=1}^{n-1}\left(\sum_{j=1}^i \varphi(a_j)\right)(Q_{x_i} - Q_{x_{i+1}}) + \sum_{j=1}^n \varphi(a_j)Q_{x_j}\right\|$$

$$= \max_{1\le i\le n}\left\|\sum_{j=1}^i a_j\right\| = \left\|\sum_{i=1}^n a_i P_{x_i}\right\|,$$

by the remarks preceding this lemma. To prove the claim, let us first note that for $e \leq x < y$ the maps from A to $B(H)$ given by $a \mapsto \varphi(a)Q_x$ and by $a \mapsto \varphi(a)(Q_x - Q_y)$ are $*$-homomorphisms, so the claim is proved if we show they are injective. Now if $\varphi(a)Q_x = 0$ then $W_x \varphi\alpha_{x^{-1}}(a)W_x^* = 0$, so $\varphi\alpha_{x^{-1}}(a) = 0$, implying that $a = 0$, by injectivity of φ. On the other hand if $\varphi(a)(Q_x - Q_y) = 0$, set $z = x^{-1}y$ (so $z > e$), and observe that $W_x \varphi\alpha_{x^{-1}}(a)W_x^* = W_x W_z \varphi\alpha_{zx}^{-1}(a)W_z^* W_x^*$, so for $b = \alpha_{zx}^{-1}(a)$ we have $\varphi\alpha_z(b)W_z\varphi(b)W_z^*$, implying that $b = 0$ by skewness of (H, φ, W). Hence $a = 0$. □

THEOREM 4.3. *Let* (A, α, G) *be a non-trivial* C^**-dynamical system where* G *is an abelian ordered group, and let* (H, φ, W) *be a covariant representation of* (A, α, G^+). *The following statements are equivalent:*

(a) $\varphi \times W$ *is injective,*

(b) (H, φ, W) *is amenable and skew.*

Proof. We have already seen that (a) $\Rightarrow$ (b). Assume therefore that (b) holds. Let μ and ν be the means associated to the identity covariant representation of (A, α, G^+) and to the covariant representation (H, φ, W) respectively. Let $P_x = V_x V_x^*$ and $Q_x = W_x W_x^*$ $(x \in G)$. By Lemma 4.2 there is a $*$-isomorphism

$$\theta\colon (A \times_\alpha G^+)^\delta \to (\mathrm{im}(\varphi \times W))^\delta$$

such that $\theta(aP_x) = \varphi(a)Q_x$ $(a \in A, x \in G)$. We claim that $\nu(\varphi \times W) = \theta\mu$. To see this it suffices to show that

$$(*) \qquad \nu(\varphi \times W)(aV_{x_1} \cdots V_{x_n}) = \theta\mu(aV_{x_1} \cdots V_{x_n})$$

for all $x_1, \ldots, x_n \in G$ and $a \in A$. But if $x_1 \cdots x_n \neq e$ then both sides of $(*)$ are obviously zero. So we may suppose that $x_1 \cdots x_n = e$ in which case $aV_{x_1} \cdots V_{x_n}$ is of the form aP_x for some $x \in G$. Then $\nu(\varphi \times W)(aP_x) = \nu(\varphi(a)Q_x) = \varphi(a)Q_x = \theta(aP_x) = \theta\mu(aP_x)$. Thus $(*)$ holds and the claim that $\nu(\varphi \times W) = \theta\mu$ is proved.

Now suppose that $b \in \ker(\varphi \times W)$. Then $\nu(\varphi \times W)(b^*b) = 0$, so $\theta\mu(b^*b) = 0$. Hence $\mu(b^*b) = 0$ (as θ is a $*$-isomorphism), from which $b^*b = 0$ (by strict positivity of μ), and so $b = 0$. Thus $\ker(\varphi \times W) = 0$ and we have shown that (b) $\Rightarrow$ (a). □

REMARK 4.1. If G is an abelian partially ordered group recall that the algebra $C^*(B^+) = \mathbf{C} \times_\alpha G^+$, where the action α is (necessarily) trivial. Let $H^2(G)$ be the closed linear span in the Hilbert space

$L^2(\widehat{G})$ of the elements $(\varepsilon_x)_{x\in G^+}$ where for $x \in G$ the map $\varepsilon_x\colon \widehat{G} \to \mathbf{T}$ is defined by setting $\varepsilon_x(\gamma) = \gamma(x)$. For $x \in G^+$ let W_x be the isometry in $B(H^2(G))$ defined by setting $W_x(f) = \varepsilon_x f$ $(f \in H^2(G))$. The map $W\colon G^+ \to B(H^2(G))$, $x \mapsto W_x$, is an isometric homomorphism, and it is easy to check that (ψ, W) is an amenable skew covariant homomorphism of $(\mathbf{C}, \alpha, G^+)$, where $\psi\colon \mathbf{C} \to B(H^2(G))$ is the unital homomorphism. However if G is not totally ordered then $\psi \times W$ is not necessarily injective. For example, take $G = \mathbf{Z}$, with the positive cone $G^+ = \mathbf{N}\backslash\{1\}$. Then G is a partially ordered group and it is shown in **[12]** that in this case $\psi \times W$ is not injective. Thus the totally ordered assumption in Theorem 4.3 cannot be weakened to a partially ordered condition.

THEOREM 4.4. *Let (A, α, G) be a non-trivial C^*-dynamical system where G is an abelian ordered group. If (H, φ) is a faithful representation of A and $(l^2(G^+, H), \overline{\varphi}, W)$ is the induced covariant representation of (A, α, G^+) then $\overline{\varphi} \times W$ is injective.*

Proof. The triple $(l^2(G^+, H), \overline{\varphi}, W)$ is skew and amenable, so $\overline{\varphi} \times W$ is injective, by Theorem 4.3. □

THEOREM 4.5. *Let (A, α, G) be a non-trivial C^*-dynamical system where A is primitive and G is an abelian ordered group. Then $A \times_\alpha G^+$ is primitive.*

Proof. Let (H, φ) be a faithful irreducible representation for A. Note that $\varphi \neq 0$ as $A \neq 0$. If $(l^2(G^+, H), \overline{\varphi}, W)$ is the induced covariant representation of (A, α, G^+) then by Theorems 4.1 and 4.4 $(l^2(G^+, H), \overline{\varphi} \times W)$ is a faithful irreducible representation of $A \times_\alpha G^+$, and therefore $A \times_\alpha G^+$ is primitive. □

If G is any abelian ordered group it follows from Theorem 4.5 that $C^*(G^+)$ is primitive. This was shown also in **[12]** by quite different means, using the results of **[16]** and **[17]** on Connes spectra that were already mentioned in §3.

We can strengthen some of the results of this section in the case of subgroups of $\mathbf{R}$. First a definition: If (A, α, G) is a C^*-dynamical system where G is an ordered group, we call a covariant representation (H, φ, W) of (A, α, G^+) *pure* if $\bigcap_{x\in G^+} W_x(H) = 0$. If (H, φ, W) is arbitrary we can split it up into a pure and a "unitary" part. To see this, set $H_0 = \bigcap_{x\in G^+} W_x(H)$ and $H_1 = H \ominus H_0$. Clearly,

H_0, H_1 are closed vector subspaces of H and $H_0 \oplus H_1 = H$, and it is a routine exercise to show they are reducing spaces for all W_x $(x \in G^+)$ and all $\varphi(a)$ $(a \in A)$. If both H_0, H_1 are non-zero we can define the maps $\varphi^{(j)}\colon A \to B(H_j)$, $a \mapsto \varphi(a)_{H_j}$, and $W^{(j)}\colon G^+ \to B(H_j)$, $x \mapsto (W_x)_{H_j}$, and get covariant representations $(H_j, \varphi^{(j)}, W^{(j)})$ $(j = 0, 1)$ of (A, α, G^+). The triple $(H_1, \varphi^{(1)}, W^{(1)})$ is pure sine $\bigcap_{x \in G^+} W_x^{(1)}(H_1) = 0$. Clearly each $W_x^{(0)}$ is unitary $(x \in G^+)$. We thus have an analogue of the Wold-von Neumann decomposition of an isometlry into its pure and unitary parts. Observe that $\varphi \times W = (\varphi^{(0)} \times W^{(0)}) \oplus (\varphi^{(1)} \times W^{(1)})$. Thus $\varphi \times W$ is injective if one of these summands is.

THEOREM 4.6. *Let (A, α, G^+) be a C^*-dynamical system where G is a subgroup of* $\mathbf{R}$. *Then any pure covariant representation (H, φ, W) of (A, α, G^+) is amenable.*

Proof. In Douglas' terminology the map $x \mapsto W_x$ is a pure one-parameter semigroup of isometries, so by his results in [4] there exists for each $t \in \mathbf{R}$ a unitary $U_t \in B(H)$ such that $U_t W_x = e^{ixt} W_x U_t$ $(x \in G^+)$, and $U_t \in \{W_x W_x^* \mid x \in G^+\}''$. Thus $\operatorname{Ad} U_t(W_x) = e^{ixt} W_x$ $(x \in G^+)$ and $\operatorname{Ad} U_t(\varphi(a)) = \varphi(a)$ $(a \in A)$, so $\operatorname{im}(\varphi \times W)$ is invariant under $\operatorname{Ad} U_t$. Denote by $\tilde{\delta}_t$ the $*$-isomorphism of $\operatorname{im}(\varphi \times W)$ got by restricting $\operatorname{Ad} U_t$. Now define $\gamma_t \in \widehat{G}$ by $\gamma_t(x) = e^{ixt}$. For δ the action of $\widehat{G}$ on $A \times_\alpha G^+$ we have therefore $\tilde{\delta}_t(\varphi \times W) = (\varphi \times W)\delta_{\gamma_t}$. Hence $J = \ker(\varphi \times W)$ satisfies $\delta_{\gamma_t}(J) \subset J$ for all $t \in \mathbf{R}$. But $\Gamma = \{\gamma_t \mid t \in \mathbf{R}\}$ is a subgroup of $\widehat{G}$ with annihilator $\Gamma^\perp = 0$, so Γ is dense in $\widehat{G}$. By the continuity of the map $\widehat{G} \to A \times_\alpha G^+$, $\gamma \mapsto \delta_\gamma(b)$, for each $b \in A \times_\alpha G^+$, we conclude that $b \in J \Rightarrow \delta_\gamma(b) \in J$ $(\gamma \in \widehat{G})$. Thus J is G-invariant and so (H, φ, W) is amenable. □

THEOREM 4.7. *Let (A, α, G) be a non-trivial C^*-dynamical system where G is a subgroup of* $\mathbf{R}$. *If (H, φ, W) is a skew covariant representation of (A, α, G^+) then $\varphi \times W$ is injective.*

Proof. If (H, φ, W) is pure the result follows immediately from Theorems 4.3 and 4.6. If (H, φ, W) is not pure then for $H_0 = \bigcap_{x \in G^+} W_x(H)$ and $H_1 = H \ominus H_0$ we have H_0 and H_1 are non-zero, so (H, φ, W) splits into its "unitary" and pure parts $(H_0, \varphi^{(0)}, W^{(0)})$ and $(H_1, \varphi^{(1)}, W^{(1)})$ respectively. Now $(H_1, \varphi^{(1)}, W^{(1)})$ is easily seen to be skew as (H, φ, W) is, so again by Theorems 4.3 and 4.6, $(H_1, \varphi^{(1)}, W^{(1)})$ is injective, and therefore $\varphi \times W$ is injective. □

If (A, α, G) is a C^*-dynamical system with G an ordered group and if $\varepsilon\colon A \times_\alpha G^+ \to A \times_\alpha G$ is the quotient map, set $K(A, \alpha, G) = \ker(\varepsilon)$. We therefore have a short exact sequence

$$0 \to K(A, \alpha, G) \to A \times_\alpha G^+ \to A \times_\alpha G \to 0.$$

LEMMA 4.8. *Let (A, α, G) be a C^*-dynamical system where G is an ordered group. Then $K(A, \alpha, G)$ is the closed ideal in $A \times_\alpha G^+$ generated by all $a - aV_xV_x^*$ $(a \in A, x \in G^+)$.*

Proof. Let the elements $a - aV_xV_x^*$ $(a \in A, x \in G^+)$ generate the closed ideal J. If $U\colon G \to M(A \times_\alpha G)$ is canonical and $\varepsilon\colon A \times_\alpha G^+ \to A \times_\alpha G$ is the quotient map then $\varepsilon(a - aV_xV_x^*) = a - aU_xU_x^* = 0$, so $J \subseteq K(A, \alpha, G)$. Thus if $B = (A \times_\alpha G^+)/J$ we get an induced $*$-homomorphism $\tilde{\varepsilon}\colon B \to A \times_\alpha G$ given by $\tilde{\varepsilon}(b + J) = \varepsilon(b)$.

Let φ be the $*$-homomorphism from A to B given by $\varphi(a) = a + J$, and let $W\colon G^+ \to M(B)$, $x \mapsto W_x$, be the homomorphism into the unitary group given by defining $W_x(b + J) = V_xb + J$, $(b + J)W_x = bV_x + J$ for $b \in A \times_\alpha G^+$, $x \in G^+$. (That W_x are isometries is obvious. To see they are unitaries it suffices to show that $b - V_xV_x^*b \in J$ if $b \in A \times_\alpha G^+$. But this is clear, for if (u_λ) is an approximate unit for A then we have $b - V_xV_x^*b = \lim_\lambda(u_\lambda - u_\lambda V_xV_x^*)b$.) We can obviously extend W to a unitary-valued homomorphism $W\colon G \to M(B)$, and it is easy to check (φ, W) is a covariant homomorphism from (A, α, G) to B. The $*$-homomorphism $\varphi \times W\colon A \times_\alpha G \to B$ satisfies $(\varphi \times W)\tilde{\varepsilon}(aV_x + J) = (\varphi \times W)(aU_x) = \varphi(a)W_x = aV_x + J$ for $a \in A$ and $x \in G^+$. Hence $(\varphi \times W)\tilde{\varepsilon} = \mathrm{id}$, so $\tilde{\varepsilon}$ is injective. It follows that $K(A, \alpha, G) = J$. □

THEOREM 4.9. *Let (A, α, G^+) be a non-trivial C^*-dynamical system where A is simple and G is a subgroup of* **R**. *Then (A, α, G) is simple.*

Proof. Let J be a closed ideal in $K(A, \alpha, G)$, $J \neq K(A, \alpha, G)$, and let $\psi\colon A \times_\alpha G^+ \to (A \times_\alpha G^+)/J$ be the quotient map. As we saw in §1, there exists a unique covariant homomorphism (φ, W) from (A, α, G^+) to $(A \times_\alpha G^+)/J$ such that $\varphi \times W = \psi$.

For $x \in G^+$ define

$$I_x = \{a \in A \mid a - aV_xV_x^* \in J\}.$$

Then I_x is a closed ideal in A, so if it contains a non-zero element it is equal to A (by simplicity of A). Let $R_x = 1 - V_xV_x^*$. This is a projection and $R_xa = aR_x$ $(a \in A)$. Also $aR_{x+y} = aR_x + V_x\alpha_x^{-1}(a)R_yV_x^*$. Using this equation one easily checks that the set

$$L = \{x \in GT^+ \mid I_x = A\}$$

is closed under addition, and it is even easier to see that $0 \leq y \leq x \in L \Rightarrow y \in L$ $(y, x \in G^+)$. By the archimedean property of G we therefore have $L = \{0\}$ or $L = G^+$.

Suppose that $a \in A$ and $x \in G^+$ are such that $\varphi\alpha_x(a) = W_x\varphi(a)W_x^*$. Then $\psi(\alpha_x(a) - V_xaV_x^*) = 0$, so $\alpha_x(a) - \alpha_x(a)V_xV_x^* \in J$. If $x > 0$ and $a \neq 0$ then $I_x = A$ and $L = G^+$. Hence $b - bV_yV_y^* \in J$ $(b \in A, y \in G^+)$ so by Lemma 4.8, $J = K(A, \alpha, G)$, a contradiction. Thus either $x = 0$ or $a = 0$, and so (φ, W) is skew. By Theorem 4.7, $\psi = \varphi \times W$ is injective, so $J = 0$. Thus $K(A, \alpha, G)$ is simple. □

REMARK 4.2. The above result does not hold for arbitrary ordered groups. If one takes $A = \mathbf{C}$ and takes G to be the lexicographic product of $\mathbf{Z}$ with itself then it follows from Theorems 2.2 and 2.3 of [**14**] that $K(A, \alpha, G)$ contains the C^*-algebra K of compact operators on a separable infinite-dimensional Hilbert space as a closed ideal such that the quotient algebra $C^*(G^+)/K$ is isomorphic to $C^*(\mathbf{N}) \otimes C(\mathbf{T})$, and therefore $K \neq K(A, \alpha, G)$, since $C^*(G^+)/K(A, \alpha, G) = C(\mathbf{T}^2)$.

REFERENCES

[1] G. Birkhoff, *Review of Everett and Ulam, On ordered groups*, Math. Reviews, **7** (1946).

[2] S. Brehmer, *Über vertauschbar Kontraktionen des Hilbertschen Raumes*, Acta Sci. Math. (Szeged), **22** (1961), 106–111.

[3] L. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math., **71** (1977), 335–348.

[4] R. G. Douglas, *On the C^*-algebra of a one-parameter semigroup of isometries*, Acta Math., **128** (1972), 143–152.

[5] J. M. G. Fell, *The structure of the algebras of operator fields*, Acta Math., **106** (1961), 233–280.

[6] C. Foiaş and B. Sz.-Nagy, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.

[7] R. Ji and J. Xia, *On the classification of commutator ideals*, J. Funct. Anal., (to appear).

[8] T. Itô, *On the commutative family of subnormal operators*, J. Fac. Sci. Hokkaido Univ., (1) **14** (1958), 1–15.

[9] F. W. Levi, *Ordered groups*, Proc. Indian Acad. Sci., **16** (1942), 256–263.

[10] M. McAsey and P. Muhly, *Representations of non-self-adjoint crossed products*, Proc. London Math. Soc., (3) **47** (1983), 128–144.

[11] P. Muhly and J. Renault, *C^*-algebras of multivariable Wiener-Hopf operators*, Trans. Amer. Math. Soc., **274** (1982), 1–44.

[12] G. J. Murphy, *Ordered groups and Toeplitz algebras*, J. Operator Theory **18** (1987), 303–326.

[13] ——, *Simple C^*-algebras and subgroups of* $\mathbf{Q}$, Proc. Amer. Math. Soc., **107** (1989), 97–100.

[14] ——, *Toeplitz operators and algebras*, (submitted).

[15] A. Nica, *Some remarks on the groupoid approach to the Wiener-Hopf operators*, J. Operator Theory, **18** (1987), 163–198.

[16] D. Olesen and G. K. Pedersen, *Applications of the Connes spectrum to C^*-dynamical systems*, J. Funct. Anal., **30** (1978), 179–197.

[17] ——, *Application of the Connes spectrum to C^*-dynamical systems* II, J. Funct. Anal., **36** (1980), 18–32.

[18] G. K. Pedersen, *C^*-algebras and their automorphism groups*, LMS monographs **14**, Academic Press, London and New York, 1979.

[19] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain crossed-products of C^*-algebras*, J. Operator Theory, **4** (1980), 93–118.

[20] W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, New York and London, 1962.

[21] H. Salas, *Semi-groups of isometries with commutating range projections*, J. Operator Theory, **14** (1985), 311–346.

[22] G. Zeller-Meier, *Produits croisés d'une C^*-algebre par un groupe d'automorphismes*, J. Math. Pures et Appl. **47** (1968), 101–239.

Received May 1, 1989 and in revised form October 1, 1989.

UNIVERSITY COLLEGE CORK
CORK, IRELAND

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

ASSOCIATE EDITORS

SUPPORTING INSTITUTIONS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555-05.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $190.00 a year (10 issues). Special rate: $95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics
Vol. 148, No. 2 April, 1991