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As is proved by Kerzman-Stein, over a compact strongly pseudo convex boundary in C^n, Szegö projection S is the operator defined by Henkin-Ramirez *modulo compact operators*. While, over *a special ball*, U_ε, in the strongly pseudo convex boundary, in order to obtain a local embedding theorem of CR-structures, Kuranishi constructed the Neumann type operator N_b for $\bar{\partial}_b$ and so we have *a local Szegö operator* by

$$S_{U_\varepsilon} = \mathrm{id} - \bar{\partial}_b^* N_b \bar{\partial}_b \quad \text{on } U_\varepsilon,$$

where $\bar{\partial}_b^*$ means the adjoint operator of $\bar{\partial}_b$. There might be a relation between S_{U_ε} and the Romanov kernel like the case of the Szegö operator and the Henkin-Ramirez kernel. We study this problem and show some estimates for the Romanov kernel.

0. Introduction. Let $(M, {}^\circ T'')$ be an abstract strongly pseudo convex CR-manifold. Then as is well known, if $\dim_R M = 2n-1 \geq 7$, $(M, {}^\circ T'')$ is locally embeddable in a complex euclidean space C^n((Ak3), (K)). In the proof of this local embedding theorem, it is shown that: over *a special ball* in the strongly pseudo convex boundary, an L^2-estimate for $\bar{\partial}_b$, which is stronger than the standard L^2-estimate, is established and so the L^2-solution operator for $\bar{\partial}_b$ is obtained. This operator plays an essential role in our local embedding theorem. Therefore it must be important to study this solution operator for $\bar{\partial}_b$ precisely.

In order to get a solution operator, there exists another method. By using an integral formula, a local solution operator for $\bar{\partial}_b$ is constructed explicitly by Henkin and Harvey-Polking. Obviously, these solution operators are different. And it seems quite interesting to study the relation between the L^2-solution for $\bar{\partial}_b$ and the explicit solution, obtained by using an integral formula. We recall the $\bar{\partial}$-case over a strongly pseudo convex domain in C^n. In this case, the explicit solution, constructed by Lieb and Range, is a certain kind of the essential part of the Kohn's L^2-solution. Therefore we could hope for a similar result in the $\bar{\partial}_b$ case over *a special ball* in the strongly pseudo convex boundary. As mentioned already, our L^2-a priori estimate is different from the standard L^2-estimate. Therefore in the above sense, it seems to be natural to consider that the explicit solution operator would satisfy the similar L^2-estimate. In this paper, we discuss this point over rigid hypersurfaces in C^n (for the definition, see §3 in this paper). And we prove our a priori estimate (Main Theorem in §5 in this paper) for the explicit solution operator.

1. CR-structure and $\overline{\partial}_b$-operator. Let M be a real hypersurface in C^n. Let p be a reference point of M. We assume that p is a smooth point, namely let ρ be a defining function of M in a neighborhood of p in C^n, i.e., there is a neighborhood $V(p)$ of p satisfying:

$$M \cap V(p) = \{q : q \in V(p), \rho(q) = 0\}$$

and

$$d\rho \neq 0 \quad \text{over } M \cap V(p).$$

Then over $M \cap V(p)$, we can introduce an CR-structure induced from C^n. Namely, let

$$^\circ T'' = T''C^n \cap C \otimes TM \quad \text{over } M \cap V(p).$$

Then this $^\circ T''$ satisfies

$$\text{(1-1)} \quad {}^\circ T'' \cap {}^\circ \overline{T}'' = 0, \quad f\text{-}\dim_C(C \otimes TM/({}^\circ T'' + {}^\circ \overline{T}'')) = 1,$$

$$\text{(1-2)} \quad [\Gamma(M \cap V(p), {}^\circ T''), \Gamma(M \cap V(p), {}^\circ T''] \subset \Gamma(M \cap V(p), {}^\circ T'').$$

This pair $(M \cap V(p), {}^\circ T'')$ is called a CR-structure, or a CR-manifold.

Let $(M \cap V(p), {}^\circ T'')$ be a CR-manifold. We introduce a C^∞ vector bundle decomposition

$$\text{(1-3)} \qquad C \otimes TM = {}^\circ T'' + {}^\circ \overline{T}'' + C\zeta,$$

where

$$\text{(1-3-1)} \qquad \zeta \text{ is a real vector field},$$

$$\text{(1-3-2)} \qquad \zeta_q \notin {}^\circ T''_q + {}^\circ \overline{T}''_q \quad \text{for } q \text{ in } M \cap V(p).$$

By using this decomposition, we have a Levi form

$$L(X, Y) = \sqrt{-1}[X, \overline{Y}]_\zeta \quad \text{for } X, Y \text{ in } \Gamma(M \cap V(p), {}^\circ T''),$$

where $[X, \overline{Y}]_\zeta$ means the ζ-part of $[X, \overline{Y}]$ according to (1-3). As is well known, this map L makes sense for elements X, Y in ${}^\circ T''$. And if this Levi form is positive or negative definite, $(M \cap V(p), {}^\circ T'')$

is called a strongly pseudo convex real hypersurface. Next we briefly explain $\overline{\partial}_b$-complex. For u in $\Gamma(M \cap V(p), C)$, we set

$$\overline{\partial}_b u(x) = Xu \quad \text{for } X \text{ in } {}^\circ T'',$$

where $\Gamma(M \cap V(p), c)$ means the spacing consisting of C^∞ functions over $M \cap V(p)$. Namely we have a first order differential operator

$$\overline{\partial}_b : \Gamma(M \cap V(p), c) \to \Gamma(M \cap V(p), ({}^\circ T'')^*).$$

By the same way as for usual differential forms, we have

$$\overline{\partial}_b^{(p)} : \Gamma(M \cap V(p), \Lambda^p({}^\circ T'')^*) \to \Gamma(M \cap V(p), \Lambda^{p+1}({}^\circ T'')^*)$$

and so

$$\overline{\partial}_b^{(p+1)} \circ \overline{\partial}_b^{(p)} = 0.$$

2. Kuranishi's L^2-estimate. Let $(M, {}^\circ T'')$ be a strongly pseudo convex CR manifold, embedded as a real hypersurface in C^n. Let p be a reference point of M. Then by a change of coordinates, we can assume that there is a neighborhood $W(p)$ of p in C^n, satisfying:

$$M \cap W(p) = \{(z_1, \dots, z_n) : (z_1, \dots, z_n) \in W(p),\ \operatorname{Im} z_n = h(z_1, \dots, z_{n-1}, \operatorname{Re} z_n)\},$$

where $z_i(p) = 0$, $1 \leq i \leq n-1$, and h is a real valued C^∞ function, and

$$(\partial^2 h / \partial z_i \partial \overline{z}_j)(0) = \delta_{ij}, \qquad 1 \leq i, j \leq n-1,$$
$$(\partial^2 h / \partial z_i \partial z_j)(0) = \delta_{ij}, \qquad 1 \leq i, j \leq n-1,$$
$$dh(0, \dots, 0) = 0.$$

In this set up, we introduce a neighborhood $M \cap U_\varepsilon(p)$ of p as follows:

$$M \cap U_\varepsilon(p) = \{(z_1, \dots, z_n) : (z_1, \dots, z_n) \in W(p),\ \operatorname{Im} z_n = h(z_1, \dots, z_{n-1}, \operatorname{Re} z_n),\ 2\operatorname{Re}\{(1/2\sqrt{-1}) z_n + z_n^2\} < \varepsilon\}.$$

Now we briefly sketch Kuranishi's L^2-estimate over $M \cap U_\varepsilon(p)$. Obviously by the above assumption, our $M \cap U_\varepsilon(p)$ is diffeomorphic to the real $2n-1$ dimensional ball. We denote this diffeomorphism map by h and we fix this. If ε is chosen sufficiently small, there is a system of bases $Y_1', Y_2', \dots, Y_{n-1}'$ of ${}^\circ T''$ over $M \cap U_\varepsilon(p)$, where ${}^\circ T''$ means the CR structure over $M \cap U_\varepsilon(p)$ induced from C^n. In our case, we can define a real vector field ζ, dual to

$$\sqrt{-1}\partial \rho,$$

where $\rho = \operatorname{Im} z_n - h(z_1, \ldots, z_{n-1}, \operatorname{Re} z_n)$. And by using this ζ, we have a C^∞ vector bundle decomposition and so we have the Levi form. By the Schmidt orthogonal process, form $Y_1', Y_2', \ldots, Y_{n-1}'$, we have a system of bases $Y_1, Y_2, \ldots, Y_{n-1}$ of ${}^\circ T''$ satisfying

$$-\sqrt{-1}[Y_i, \overline{Y}_j]_\zeta = \delta_{ij},$$

where $-\sqrt{-1}[Y_i, \overline{Y}_j]_\zeta$ means the coefficient of the ζ part of $[Y_i, \overline{Y}_j]$ according to the above C^∞ vector bundle decomposition. By using this $Y_1, Y_2, \ldots, Y_{n-1}$, we put an L^2-norm on

$$\Gamma(M \cap U_\varepsilon(p), \Lambda^p({}^\circ T'')^*).$$

Namely for u in $\Gamma(M \cap U_\varepsilon(p), \Lambda^p({}^\circ T'')^*)$, we have C^∞ functions u_I by

$$u_I = u(Y_{i_1}, \ldots, Y_{i_p}), \qquad I = (i_1, \ldots, i_p).$$

By using these u_I, we set

$$\|u\|^2_{M\cap U_\varepsilon(p)} = \sum_I \int_{B_1(0)} |u_I \circ h|^2 \, dx_1 \cdots dx_{2n-1},$$

where I runs through all ordered indices of length p and h is a diffeomorphism map from $M \cap U_\varepsilon(p)$ to $B_1(0)$ defined as above. Furthermore we must introduce several notations. Namely $\overline{\partial}_1^*$ denotes the adjoint operator of $\overline{\partial}_1$ with respect to the above L^2-norm. And we set

$$b = \sqrt{\sum_{i=1}^{n-1} |Y_i t|^2},$$

where $t = 2\operatorname{Re}\{1/2\sqrt{-1})z_n + z_n^2\}$. And we set the characteristic curve C by

$$C = \{(z_1, \ldots, z_n), (z_1, \ldots, z_n) \in M \cap U_\varepsilon(p), \\ Y_i t = 0, \ 1 \le i \le n-1\}.$$

Then in **[K]**, Kuranishi obtained

$$\|(1/b)v\|^2_{M\cap U_\varepsilon(p)} \le c\{\|\overline{\partial}_b v\|^2_{M\cap U_\varepsilon(p)} + \|\overline{\partial}_b^* v\|^2_{M\cap U_\varepsilon(p)}\}$$

for v in $\Gamma(M \cap U_\varepsilon(p) - C, ({}^\circ T'')^*)$ satisfying:

$$v(Y^0) = 0 \quad \text{on } \{(z_1, \ldots, z_n) : (z_1, \ldots, z_n) \in M \cap U_\varepsilon(p) - C, \\ t = \varepsilon\},$$

where

$$Y^0 = \sum_{i=1}^{n-1} (\overline{Y}_i t/b) Y_i,$$

if $\dim_R M = 2n-1 \geq 7$. Actually, Kuranishi obtained the estimate more precisely. However, in this paper, we discuss this estimate. Then, the L^2-solution operator $\overline{\partial}_b^* N_b$ satisfies

$$\|(1/b)(\overline{\partial}_b^* N_b v)\|_{M\cap U_\varepsilon(p)} \leq c\|v\|_{M\cap U_\varepsilon(p)}$$

for v in $\Gamma(M \cap U_\varepsilon(p) - C, ({}^\circ T'')^*)$, which is of L^2. We show that an explicit solution obtained by Henkin and Harvey-Polking satisfies the similar estimate.

3. Rigid hypersurfaces in C^n. In this paper, we study the $\overline{\partial}_b$-operator over a special kind of real hypersurfaces in C^n. Namely let

$$M = \{(z_1, \dots, z_n): \operatorname{Im} z_n = k(z_i, \overline{z}_j),\ 1 \leq i, j \leq n-1\},$$

where k is a real valued C^∞ function which depends only on z_i, $\overline{z}_j$, and not on z_n, $\overline{z}_n$ satisfying:

$$k(0, 0) = 0 \quad \text{and} \quad dk(0, 0) = 0.$$

We call M satisfying these relations a rigid hypersurface. Let M be a rigid hypersurface. And let M be strongly pseudo convex near the origin. Then by a change of coordinates, the defining equation of M becomes

$$\operatorname{Im} z_n'' = \sum_{i=1}^{n-1} |z_i''|^2 + \text{ terms of higher order in } z_j'', \overline{z}_j'',$$

where $1 \leq j \leq n-1$.

4. Integral formula for $\overline{\partial}_b$ and the Romanov kernel. Let u, v be C^∞ functions from $C^n \times C^n$ to C^n,

$$u(\zeta, z) = (u_1(\zeta, z), \dots, u_n(\zeta, z)),$$
$$v(\zeta, z) = (v_1(\zeta, z), \dots, v_n(\zeta, z)).$$

We use the following notations:

$$u(\zeta, z)(\zeta - z) = \sum_{j=1}^{n} u_j(\zeta, z)(\zeta_j - z_j),$$

$$u(\zeta, z)\, d(\zeta - z) = \sum_{j=1}^{n} u_j(\zeta, z)\, d(\zeta_j - z_j),$$

$$\bar{\partial} u(\zeta, z)\, d(\zeta - z) = \sum_{j=1}^{n} \bar{\partial} u_j(\zeta, z) \wedge d(\zeta_j - z_j),$$

and we define the following kernels:

$$\text{(4-1-1)} \quad \Omega^u(\zeta, z) = (2\pi i)^{-n}((u(\zeta, z)\, d(\zeta - z))/(u(\zeta, z)(\zeta - z))) \wedge ((\bar{\partial} u(\zeta, z)\, d(\zeta - z))/(u(\zeta, z)(\zeta - z)))^{n-1},$$

$$\text{(4-1-2)} \quad \Omega^v(\zeta, z) = (2\pi i)^{-n}((v(\zeta, z)\, d(\zeta - z))/(v(\zeta, z)(\zeta - z))) \wedge ((\bar{\partial} v(\zeta, z)\, d)(\zeta - z))/(v(\zeta, z)(\zeta - z)))^{n-1},$$

$$\begin{aligned} \text{(4-1-3)} \quad & \Omega^{u,v}(\zeta, z) \\ &= (2\pi i)^{-n}((u(\zeta, z)\, d(\zeta, z))/(u(\zeta, z)(\zeta - z))) \\ &\quad \wedge ((\bar{\partial} v(\zeta, z)\, d(\zeta - z))/(v(\zeta, z)(\zeta - z))) \\ &\quad \wedge \sum_{j+k=n-2} ((\bar{\partial} u(\zeta, z)\, d(\zeta - z))/(u(\zeta, z)(\zeta - z)))^j \\ &\quad \wedge ((\bar{\partial} v(\zeta, z)\, d(\zeta - z))/(v(\zeta, z)(\zeta - z)))^k. \end{aligned}$$

Then as is well known, in **[B]** and **[BS]**, we have

$$\bar{\partial}\Omega^{u,v}(\zeta, z) = \Omega^v(\zeta, z) - \Omega^u(\zeta, z),$$
$$\bar{\partial}\Omega^v(\zeta, z) = 0.$$

Let M be as in §1 in this paper. Then we can define *formally*

$$R_M(u, v)(\phi)(z) := \left\{ \int_{\zeta \in M} \Omega^{u,v}(\zeta, z) \wedge \phi(\zeta) \right\}_{T_M},$$

$$L(u)(\phi)(z) := \int_{\zeta \in M} \Omega^u(\zeta, z) \wedge \phi(\zeta),$$

for $\phi \in \mathscr{D}^{0,1}(M \cap U)$, where $\{\ \}_{T_M}$ means the tangential part of $\{\ \}$. Of course without any assumption for u, v and M, the operators R_M, L do not make sense. However if we assume that u is a local support function for (M, D) at a point p (for the definition, see 2.4 Definition in **[BS]**), then $R_M(u, v)(\phi)$, $L(u)(\phi)$ make sense. And furthermore, the boundary value of $L(u)(\phi)$ from D^- and D^+ exists respectively, where D means U and

$$D^+ = \{z : z \in C^n, \rho(z) > 0\},$$
$$D^- = \{z : z \in C^n, \rho(z) > 0\}.$$

And for $\phi \in \mathscr{D}^{0,1}(M \cap U)$,

$$\phi = -(\bar\partial_b R_M(u, v)(\phi) + R_M(u, v)\bar\partial_b \phi) + L_M^+(v)(\phi) - L_M^-(u)(\phi) \quad \text{on } M \cap U.$$

Note from this equality, the terms $L_M^+(v)(\phi)$ and $L_M^-(u)(\phi)$ are obstructions to solving the equations $\bar\partial_b g = \phi$. If we set

$$u_j(\zeta, z) = \partial\rho/\partial\zeta_j(\zeta), \quad v_j(\zeta, z) = -\partial\rho/\partial z_j(z), \qquad 1 \le j \le n,$$

then $u(\zeta, z) = (u_1(\zeta, z), \dots, u_n(\zeta, z))$ and $v(\zeta, z) = (v_1(\zeta, z), \dots, v_n(\zeta, z))$ are local support functions for (M, D^-) and (M, D^+) respectively. And in the case,

$$L_M^-(u)(\phi) = 0 \quad \text{unless } \phi \in \mathscr{D}^{p,0}(M \cap U),$$
$$L_M^+(v)(\phi) = 0 \quad \text{unless } \phi \in \mathscr{D}^{p,n-1}(M \cap U).$$

And so we have: for $\phi \in \mathscr{D}^{p,1}(M \cap U)$,

$$\phi = -\{\bar\partial_b R_M(u, v)(\phi) + R_M(u, v)(\bar\partial_b \phi)\},$$

if $n \ge 3$.

Henceforth, we abbreviate R for $R_M(u, v)$, where u and v are defined as above, and $R\phi$ stands for $R_M(u, v)(\phi)(z)$.

5. Kuranishi's L^2-estimate for the Romanov kernel. In §4, we see that the Romanov kernel R is a certain kind of the solution operator for $\bar\partial_b$. Concerning this R kernel, in this section, we show an L^2-estimate which the L^2 solution satisfies. Namely, we show

MAIN THEOREM. *For any ϕ in $\Gamma(M \cap U_\varepsilon(p) - C, ({}^\circ T'')^*)$, which is of L^2, and for any $\delta < 1$, we have:*

$$\|(1/b^\delta)R\phi\|_{M\cap U_\varepsilon(p)} \le C_\delta \|\phi\|_{M\cap U_\varepsilon(p)},$$

where C_δ depends only on δ.

In order to prove the main theorem, we first show

LEMMA 5.1.

$$C_1 \sqrt{\sum_{i=1}^{n-1} |z_i''|^2} \le b \le C_2 \sqrt{\sum_{i=1}^{n-1} |z_i''|^2},$$

where C_1, C_2 *are positive constants, and* b *is defined by*

$$b = \sqrt{\sum_{i=1}^{n-1} |Y_i''t|^2},$$

where $\{Y_i''\}_{1\le i\le n-1}$ *is obtained from* $\{Y_i\}_{1\le i\le n-1}$, *by the Schmidt orthogonal process, and*

$$Y_i = \partial/\partial \overline{z}_i'' - (\rho_{\bar{i}}/\rho_{\bar{n}})\partial/\partial \overline{z}_n'', \qquad 1 \le i \le n-1,$$

$$\rho = \operatorname{Im} z_n'' - \sum_{i=1}^{n-1} |z_i''|^2 - Q(z_j'', \overline{z}_j''),$$

where $\{z_i''\}_{1\le i\le n}$ *means the coordinate obtained in* §*3 in this paper.*

Proof of Lemma 5.1. By the construction of Y_i'', Y_i'' is a linear combination of Y_j, $1 \le j \le n$, satisfying:

$$Y_i'' := \sum_{j=1}^{n-1} a_{ji} Y_j,$$

where a_{ji} is a C^∞ function over $M \cap U_\varepsilon(p)$ and $a_{ji}(p) = 0$. So

$$Y_i''t = Y_i t + \sum_{j=1}^{n-1} a_{ji} Y_j t.$$

While

$$\begin{aligned} Y_j t &= (\partial/\partial \overline{z}_j'' - (\rho_{\bar{i}}/\rho_{\bar{n}})\partial/\partial \overline{z}_n'')2\operatorname{Re}\{(1/2\sqrt{-1})z_n'' + z_n''^2\} \\ &= z_i''(1 + 4\sqrt{-1}z_n''). \end{aligned}$$

Therefore we have our lemma. □

And we have

LEMMA 5.2. *There is a constant* c *satisfying:*

$$\int_{\zeta\in M\cap U_\varepsilon(p)} (1/b^\delta)|\Omega^{u,v}(\zeta, z)|dV_\zeta \le c \quad \textit{for } z \textit{ in } U_\varepsilon(p).$$

This lemma is proved in [HP]. So we briefly sketch the proof. For a system of coordinates of $M \cap U_\varepsilon(p)$, we can adopt $(z_1'', \dots, z_{n-1}'', t)$, which we constructed in §3 in this paper, where $t = \operatorname{Re} z_n''$. Then over $M \cap U_\varepsilon(p)$,

$$c_1\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) \le |z_n''| \le c_2\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right),$$

where c_1, c_2 are positive constants. So over $M \cap U_\varepsilon(p)$,

$$c_3\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) \leq |u(\zeta - z'')| \leq c_4\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right),$$

where c_3, c_4 are positive constants. And

$$c_5\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right) \leq |v(\zeta - z'')| \leq c_6\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right),$$

where c_5, c_6 are positive constants. And

$$u\,d(\zeta - z) \wedge v\,d(\zeta - z) = Q(|\zeta - z|).$$

So each coefficient of $(1/b^\delta)R$ is dominated by

$$\left(\sum_{i=1}^{n-1} |z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2}\right)\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right)^{-n}.$$

And this is locally integrable on $C^{n-1} \times R$ if $\delta < 1$. In fact, by using polar coordinates, we compute the following integral. We set

$$\begin{aligned}
x_1 &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{2n-3}\cos\theta_{2n-2},\\
y_1 &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{2n-3}\sin\theta_{2n-2},\\
x_2 &= r\cos\theta_1\cdots\cos\theta_{2n-4}\sin\theta_{2n-3},\\
y_2 &= r\cos\theta_1\cdots\sin\theta_{2n-4},\\
&\cdots\\
x_{n-1} &= r\cos\theta_1\sin\theta_2,\\
y_{n-1} &= r\sin\theta_1,
\end{aligned}$$

where $z_j'' = x_j + \sqrt{-1}y_j$, $1 \leq j \leq n-1$. Then

$$\left(\sum_{i=1}^{n-1} |z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1} |z_i''|^2}\right)\left(|t| + \sum_{i=1}^{n-1} |z_i''|^2\right)^{-n}$$
$$= r^{-\delta}(t + r)(t + r^2)^{-n}.$$

So

$$\int_{M\cap U_\varepsilon(p)} \left(\sum_{i=1}^{n-1}|z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1}|z_i''|^2}\right) \times \left(|t| + \sum_{i=1}^{n-1}|z_i''|^2\right) dV_{z,t}$$
$$\leq \int_0^\varepsilon \int_0^\infty r^{-\delta}(t+r)(t+r^2)^{-n} r^{2n-3}\,dt\,dr$$
$$= \int_0^\varepsilon \int_0^\infty \{(1/(t+r^2)^{n-1})r^{2n-3-\delta} + (1/(t+r^2)^n)((r-r^2)/r^\delta)r^{2n-3}\}\,dt\,dr.$$

While

$$\int_0^\infty (1/(t+r^2)^{n-1})r^{2n-3-\delta}\,dt$$
$$= -(1/(n-2))[(1/(t+r^2)^{n-2})r^{2n-3-\delta}]_0^\infty$$
$$= (1/(n-2))r^{1-\delta},$$

$$\int_0^\infty (1/(t+r^2)^n)((r-r^2)r^{2n-3}\,dt$$
$$= -(1/(n-1))[(1/(t+r^2)^{n-1}(1-r))r^{2n-2-\delta}]_0^\infty$$
$$= (1/(n-1))(1-r)r^{-\delta}.$$

Therefore

$$\int_{M\cap U_\varepsilon(p)} \left(\sum_{i=1}^{n-1}|z_i''|^2\right)^{-(\delta/2)} \left(|t| + \sqrt{\sum_{i=1}^{n-1}|z_i''|^2}\right) \left(|t| + \sum_{i=1}^{n-1}|z_i''|^2\right) dV_{z,t}$$
$$\leq \int_0^\varepsilon ((1/(n-2))r^{1-\delta} + (1/(n-1))r^{-\delta} - (1/(n-1))r^{1-\delta})\,dr$$
$$= (1/((n-2)(2-\delta)))\varepsilon^{1-(\delta/2)} + (1/((n-1)(1-\delta)))\varepsilon^{(1/2)-(\delta/2)} - (1/((n-1)(2-\delta)))\varepsilon^{1-(\delta/2)}.$$

Therefore we have our lemma. □

Now we prove our main theorem.

$$
\begin{aligned}
&\int_{M\cap U_\varepsilon(p)} (1/b^{2\delta})|R_M(u, v)(\phi)|^2\, dV \\
&\le \int_{M\cap U_\varepsilon(p)} \left\{ (1/b^{2\delta}) \left(\int_{M\cap U_\varepsilon(p)} \Omega^{u,v}(\zeta, z)\phi(\zeta)\, dV_\zeta \right)^2 \right\} dV_z \\
&\le \int_{M\cap U_\varepsilon(p)} \left(\int_{M\cap U_\varepsilon(p)} |(1/b^{\delta})\Omega^{u,v}(\zeta, z)\phi(\zeta)|\, dV_\zeta \right)^2 dV_z \\
&\le \int_{M\cap U_\varepsilon(p)} \left\{ \left(\int_{M\cap U_\varepsilon(p)} |(1/b^{\delta})\Omega^{u,v}(\zeta, z)|\, dV_\zeta \right) \right. \\
&\qquad \left. \times \left(\int_{M\cap U_\varepsilon(p)} |(1/b^{\delta})\Omega^{u,v}(\zeta, z)|\, |\phi(\zeta)|^2\, dV_\zeta \right) \right\} dV_z \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the Schwarz lemma)} \\
&\le c^2 \int_{M\cap U_\varepsilon(p)} |\phi(\zeta)|^2\, dV_\zeta \qquad \text{(by Lemma 5.2 in this paper)} \\
&\le c^2 \|\phi\|^2_{M\cap U_\varepsilon(p)}.
\end{aligned}
$$

So we have our theorem. □

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