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The concept of the C_n-space by F. Williams is generalized to the one defined on the category of higher homotopy associative H-spaces. This generalized concept is used to obtain the mod p version of the torus theorem by J. Hubbuck.

1. Introduction. In 1969 J. Hubbuck proved the following theorem:

THE TORUS THEOREM ([7, *Theorem* 1.1]). *Let X be a connected finite* CW*-complex. If X admits a homotopy commutative multiplication, then X has the homotopy type of a torus.*

The above property depends essentially on the mod 2 structure of X. In fact, Hubbuck used the 2-localized K-theory to prove the above theorem. Later J. Lin reproved the above theorem by using another method. In the paper he gave the explicit mod 2 version of the above theorem which is stated as follows:

THE MOD 2 TORUS THEOREM ([**12**, *Theorem* 1]). *Let X be a simply connected* CW*-complex whose* mod 2 *cohomology $H^*(X;\mathbf{Z}/2)$ is finite. If X admits a homotopy commutative multiplication, then*

$$\widetilde{H}^*(X;\mathbf{Z}/2)=0.$$

Beside the above theorem, Iriye and Kono [**8**, Th. 1.3] also showed that the mod 2 structure is essential for the homotopy commutative H-spaces. They proved that if p is an odd prime, then any p-localized H-space admits a homotopy commutative multiplication.

In this paper we describe the odd prime version of The Torus Theorem. To do so we generalize the homotopy commutativity of H-spaces to the higher ones. The concept of the higher homotopy commutativity was first introduced by M. Sugawara [**21**]. He used it to give a criterion of a homotopy commutative H-space to be the loop space of an H-space. Later F. Williams [**25**] considered another type of higher homotopy commutativity which is weaker than Sugawara's one. Both concepts are defined on the category of associative H-spaces. We generalize the concept of Williams to the one which is defined on the higher homotopy associative H-spaces. We call these generalized spaces the quasi C_n-spaces. In this sense if a space is a homotopy commutative H-space, then it is a quasi C_2-space, and if a space is the loop space of an H-space, then it is a quasi C_∞-space. Then our main theorem is stated as follows:

THEOREM 1.1. *Let X be a simply connected* CW*-complex with the finite* mod p *cohomology $H^*(X;\mathbf{Z}/p)$ for a prime p. If X is a quasi C_p-space, then*

$$\widetilde{H}^*(X;\mathbf{Z}/p)=0.$$

In the above theorem, the condition C_p cannot be relaxed to C_{p-1}. In fact we show in §2 that the p-localized odd sphere $S^{2t-1}_{(p)}$ is a quasi C_{p-1}-space.

Now Theorem 1.1 implies The Mod 2 Torus Theorem since a homotopy commutative H-space is a quasi C_2-space (Proposition 2.3). Furthermore since the loop space of an H-space is a quasi C_n-space for all n (Theorem 2.2), Theorem 1.1 implies the following theorem which was originally proved by Aguadé and Smith.

THEOREM ([**2**]). *Let X be a simply connected* CW*-complex with the finite* mod p *cohomology $H^*(X;\mathbf{Z}/p)$ for an odd prime p. If X has a homotopy type of the loop space of an H-space, then*

$$\widetilde{H}^*(X;\mathbf{Z}/p)=0.$$

Recently McGibbon studied the higher homotopy commutativity of Sugawara type. Then he got the similar results to Theorem 1.1 under the assumption that X is a C_p-space in the sense of Sugawara ([**15**, Th. 3]). Since a C_p-space in the sense of Sugawara is also a quasi C_p-space (cf. [**15**, Prop. 6]), Theorem 1.1 generalizes his result.

Now the explicit definition of the quasi C_n-space is given in §2, and we state in Theorem 2.2 that our definition generalize Williams' one which is proved in §5. We also study the localized spheres as the examples in §2. Section 3 is for the preparation of the proof of our main theorem. We study the cohomology of the exterior A_n-spaces. Then we generalize Borel's result about the primitivity of the generators of the cohomology of homotopy associative H-spaces. We give the proof of our main theorem in §4.

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2. Quasi C_n-spaces. In this section we define a quasi C_n-form on an A_n-space. We follow the techniques of Iwase [9] on A_n-space.

Let X be an A_n-space $(n \geq 2)$ with the projective i-spaces $XP(i)$ $(i \leq n)$ (see §5). Then $XP(i)/XP(i-1)$ is naturally homeomorphic to $S^i \wedge X^{\Lambda(i)}$, where $Y^{\Lambda(t)}$ is the t-fold smash product $Y \wedge \cdots \wedge Y$ of a space Y. Since there is a natural homeomorphism $S^i \wedge X^{\Lambda(i)} \to (S^1)^{\Lambda(i)} \wedge X^{\Lambda(i)} \xrightarrow{\lambda} (S^1 \wedge X)^{\Lambda(i)} \to (\Sigma X)^{\Lambda(i)}$, we have the induced map $\rho_i \colon XP(i) \to (\Sigma X)^{\Lambda(i)}$, where

$$\lambda(s_1, \dots, s_i, x_1, \dots, x_i) = (s_1, x_1, \dots, s_i, x_i).$$

Let $\mathscr{S}(i)$ be the ith symmetric group. Then $\tau \in \mathscr{S}(i)$ acts on $Y^{\Lambda(i)}$ by $\tau(y_1, \dots, y_i) = (y_{\tau^{-1}(1)}, \dots, y_{\tau^{-1}(i)})$. Denote by $(Y)_i$ the ith James reduced product space of Y.

DEFINITION 2.1. Let X be an A_n-space $(2 \leq n \leq +\infty)$. Then a *quasi C_n-form* on X is a family of maps $\{\varphi_i \colon (\Sigma X)_i \to XP(i)\}_{1 \leq i \leq n}$ so that the following conditions are satisfied:

(1) $\varphi_1 = \mathrm{id}_{\Sigma X}$,

(2) $\varphi_i | (\Sigma X)_{i-1} = \iota_{i-1}\varphi_{i-1}$ $(2 \leq i \leq n)$,

where $\iota_{i-1} \colon XP(i-1) \to XP(i)$ is the inclusion,

(3) $\rho_i \varphi_i \simeq (\Sigma_{\tau \in \mathscr{S}(i)} \tau)\xi_i$,

where $\xi_i \colon (\Sigma X)_i \to (\Sigma X)^{\Lambda(i)}$ is the natural projection, and the summation on the right-hand side is defined by using the obvious co-H-structure of $(\Sigma X)^{\Lambda(i)}$.

An A_n-space with a given C_n-form is called a *quasi C_n-space.*

The above definition is a generalization of Williams' C_n-form defined on associative H-spaces ([25]). In fact it is noted in [25, Remark 19] without a proof that an associative H-space X is a C_n-space in the sense of Williams if and only if there is a map $\varphi \colon (\Sigma X)_n \to XP(n)$ with $\varphi \mid \Sigma X = \iota_{n-1} \cdots \iota_1$. Here we give a proof of the following

THEOREM 2.2. *Let X be an associative H-space. Then X admits a C_n-form in the sense of* [25] *if and only if X admits a quasi C_n-form. Thus in particular the loop space of an H-space is a quasi C_∞-space.*

The above theorem is proved in §5.

The quasi C_2-space is closely related to the homotopy commutative H-space. In fact we have the following proposition which can be proved by [**19**, Th. 1.9] and [**6**, Prop. 3.4].

PROPOSITION 2.3. *A homotopy commutative H-space is a quasi C_2-space. Furthermore the converse holds if the multiplication is homotopy associative.*

Now as examples of the quasi C_n-space, we consider the p-localized spheres $S^t_{(p)}$, where p is a prime. Since no even dimensional spheres are H-spaces, we only consider the odd dimensional ones. Then we prove the following theorem which is the best possible since by the results on the existence of A_m-forms on the p-localized spheres ([**1**], [**20**, §5], [**22**, §4]).

THEOREM 2.4. (1) *$S^1_{(p)}$ admits a quasi C_∞-form for any p.*

(2) *$S^{2t-1}_{(p)}$ admits a quasi C_{p-1}-form for any p and $t \geq 1$.*

(3) *$S^3_{(2)}$ and $S^7_{(2)}$ admit no C_2-forms.*

(4) *Let t be a divisor of $p-1$ with $t > 1$. Put $n = (p-1)/t$. Then $S^{2t-1}_{(p)}$ with any A_∞-form admits a quasi C_n-form, and $S^{2t-1}_{(p)}$ with no A_p-form admits a quasi C_{n+1}-form.*

Proof. Since S^1 is the loop space of an H-space, (1) follows from Theorem 2.2. (This fact is noted and used by Toda [**24**].) Furthermore (3) follows by Theorem 1.1. Thus we prove (2) and (4) for $t > 1$.

(2) Put $X = S^{2t-1}_{(p)}$ and $\Omega = \Omega^2 S^{2t+1}_{(p)}$, and let $f\colon X \to \Omega$ be the natural map. Then X admits an A_{p-1}-form so that f preserves the A_{p-1}-forms (cf. [**26**, §1]). Now since Ω is a double loop space, it admits a quasi C_∞-form $\{\varphi_i\colon (\Sigma\Omega)_i \to \Omega P(i)\}_{i\leq\infty}$ by Theorem 2.2. Furthermore the homotopy fiber of the induced map $XP(i) \to \Omega P(i)$ is $(2tp-3)$-connected. Thus we have a quasi C_{p-1}-form on X which is a lift of $\{\varphi_i(\Sigma f)_i\}$.

(4) Suppose that t divides $p-1$. Then by considering the homotopy group of $X = S^{2t-1}_{(p)}$, we can easily show that if $it < p$, then both $XP(i)$ and $(\Sigma X)_i$ have the homotopy type of $S^{2t}_{(p)} \vee S^{4t}_{(p)} \vee \cdots \vee S^{2it}_{(p)}$. Thus a quasi C_n-form $\{\varphi_i\}_{i\leq n}$ is defined as the family of maps induced from the self maps of $S^{2t}_{(p)} \vee \cdots \vee S^{2it}_{(p)}$ which have degree $j!$ on $S^{2jt}_{(p)}$ $(j \leq i)$.

Next suppose to the contrary that X has an A_p-form admitting a C_{n+1}-form. It is well known that the cohomology $H^*(XP(i); \mathbf{Z}/p)$ is a truncated polynomial algebra of height $i+1$ generated by a single generator of dimension $2t$:

$$H^*(XP(i); \mathbf{Z}/p) = \mathbf{Z}/p[u]/(u^{i+1}).$$

Furthermore the homomorphism induced from the inclusion

$$\iota_{p-1} \cdots \iota_{n+1} \colon XP(n+1) \to XP(p)$$

preserves their generators. Now $\mathscr{P}^t u = u^p \neq 0$ in $H^*(XP(p); \mathbf{Z}/p)$. Thus

$$\mathscr{P}^1 u = c u^{n+1}$$

for some nonzero $c \in \mathbf{Z}/p$ in $H^*(XP(p); \mathbf{Z}/p)$, and also in $H^*(XP(n+1); \mathbf{Z}/p)$. Here by Lemma 4.8, which will be proved in §4, we have that

$$\mathscr{P}^1 u \in \mathscr{P}^1 DH^{2t}(XP(n+1); \mathbf{Z}/p) = 0,$$

where D denotes the decomposable module. This is a contradiction, and (4) is proved. □

3. Cohomology of A_n-space. In the rest of this paper p denotes a fixed prime, and $H^*(\cdot) = H^*(\cdot; \mathbf{Z}/p)$. Furthermore, if $p = 2$, we assume that $\mathscr{P}^n$ means Sq^{2n}.

Let X be a simply connected A_n-space with multiplication $\mu \colon X \times X \to X$. Suppose that the mod p cohomology of X is generated by finitely many odd dimensional generators:

(3.1) $$H^*(X) \cong \Lambda(x_1, \ldots, x_k), \qquad \dim x_i \colon \text{odd}.$$

Then we prove the following theorem which is a generalization of [**3**, Th. 6.6]:

THEOREM 3.2. *The generators* x_i, $1 \leq i \leq k$, *in* (3.1) *are chosen to be in the image of*

$$\sigma^{-1} \iota_1^* \cdots \iota_{n-2}^* \colon \widetilde{H}^*(XP(n-1)) \to \widetilde{H}^{*-1}(X),$$

where $\sigma \colon \widetilde{H}^{*-1}(\cdot) \to \widetilde{H}^*(\Sigma \cdot)$ *is the suspension isomorphism.*

Proof. The case of $n = 3$ is due to [**3**, Th. 6.6] since the theorem in this case is demanding that x_i, $1 \leq i \leq k$, are primitive. Thus we assume that $n > 3$ and x_i, $1 \leq i \leq k$, are primitive.

Let $\{E_r^{s,t}, d_r\}$ be the $\bmod p$ cohomology spectral sequence associated to the filtration $\Sigma X \subset XP(2) \subset \cdots \subset XP(n)$. Then

(3.3) $$E_2^{s,t} \cong \mathrm{Cotor}^{s,t}_{H^*(X)}(\mathbf{Z}/p, \mathbf{Z}/p) \quad \text{for } s \le n-1.$$

Furthermore, if we identify $\widetilde{H}^*(X)$ with $E_1^{1,*}$, then $x \in \widetilde{H}^*(X)$ is in the image of $\sigma^{-1}\iota_1^* \cdots \iota_j^*$ if and only if $d_r(x) = 0$ for $r \le j$ ([**20**, Th. 5.1]). Thus to prove the theorem we show that $d_r(x_i) = 0$ for $r \le n-2$.

First of all, $d_1(x_i) = 0$ since x_i is primitive. Furthermore if $2 \le r \le n-2$, then $E_r^{1+r,2s-r} = 0$ for any s by (3.3). Thus $d_r(x_i) = 0$ $(r \le n-2)$, and the theorem is proved. □

Now to state the next theorem we recall the spectral sequence used in the above proof. This spectral sequence is constructed by the following diagram:

(3.4)
$$\begin{tikzcd}[column sep=small]
0 & & \widetilde{H}^*(\Sigma X) \arrow[ll, "\iota_0"'] \arrow[dr, "\beta_1"] & & \cdots \arrow[ll, "\iota_1"'] & & \widetilde{H}^*(XP(n)) \arrow[ll, "\iota_{n-1}"'] \\
& \widetilde{H}^*(X) \arrow[ur, "\alpha_1"] & & \widetilde{H}^*(X)^{\otimes 2} \arrow[ur, "\alpha_2"] & \cdots \arrow[dr, "\beta_{n-1}"] & \widetilde{H}^*(X)^{\otimes n} \arrow[ur, "\alpha_n"] &
\end{tikzcd}$$

where $A^{\otimes t} = A \otimes \cdots \otimes A$ (t-folds) for any $\mathbf{Z}/p$-module A, $\deg \alpha_i = -\deg \beta_i = i$, $\beta_i\alpha_i = -\tilde{\mu}^* \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \tilde{\mu}^* \otimes \cdots \otimes 1 - \cdots$, and α_1 is the suspension isomorphism σ. We define a submodule $D(i)$ in $\widetilde{H}^*(X)^{\otimes i}$ by

$$D(i) = \sum_{0 \le j \le i-1} \widetilde{H}^*(X)^{\otimes j} \otimes DH^*(X) \otimes \widetilde{H}^*(X)^{\otimes i-j-1}.$$

Put $S(i) = \alpha_i(D(i)) \subset \widetilde{H}^*(XP(i))$. Then by Theorem 3.2 we have the following

THEOREM 3.5. *There exist classes $y(t)_i \in \widetilde{H}^*(XP(t))$ for $1 \le t \le n-1$ and $1 \le i \le k$ so that the following properties hold*:

(1) $\iota_{t-1}^*(S(t)) = 0$, *and* $S(t) \cdot \widetilde{H}^*(XP(t)) = 0$ *for* $1 \le t \le n$.

(2) $\iota_{t-1}^* y(t)_i = y(t-1)_i$ *and* $y(t)_{i(1)} \cdots y(t)_{i(t)} = \alpha_t(x_{i(1)} \otimes \cdots \otimes x_{i(t)})$.

(3) *For* $t \le n-1$, *we have the algebra splitting*:

$$H^*(XP(t)) \cong T^{t+1}[y(t)_1, \ldots, y(t)_k] \oplus S(t),$$

where $T^r[u_1, \ldots, u_s]$ denotes the truncated polynomial algebra of height r over $\mathbf{Z}/p$ with generators $\{u_i\}$.

(4) $$T^n[y(n-1)_1, \ldots, y(n-1)_k] \supset \mathrm{Im}\, \iota_{n-1}^* \supset DT^n[y(n-1)_1, \ldots, y(n-1)_k].$$

Proof. Since $\{x_i\}$ are in the image of $\sigma^{-1}\iota_1^* \cdots \iota_n^*$ by Theorem 3.2, (1)–(3) can be proved by the standard method (cf. [9]). Furthermore β_i is essentially induced from a map defined on a space homeomorphic to $\Sigma^i X^{\Lambda(i+1)}$. Thus we have $DH^*(XP(i)) \subset \operatorname{Ker}\beta_i$. The inclusion $\operatorname{Im}\iota_{n-1}^* \subset T^n[y(n-1)_1, \dots, y(n-1)_k]$ is clear, and (4) is proved. □

4. Proof of the main theorem. First we prove the following proposition which strengthens a result by Browder [4, Corollary 8.7].

PROPOSITION 4.1. *If $p = 2$, then for any simply connected quasi C_2-space X with finite* mod 2 *cohomology $H^*(X)$, $H^*(X)$ is an exterior algebra generated by finitely many odd dimensional generators.*

Proof. It is enough to prove that

$$PH^{2n}(X) = 0 \quad \text{for all } n, \tag{4.2}$$

where P denotes the primitive module. In fact the lowest dimensional nonzero square in $H^*(X)$ is even dimensional primitive. Thus (4.2) implies the proposition by [11].

Now suppose to the contrary that $PH^{2n}(X) \neq 0$ for some n. We choose n as the greatest such n. Take a nonzero $x \in PH^{2n}(X)$. Since x is primitive we have a class $y \in \widetilde{H}^{2n+1}(XP(2))$ with

$$\sigma^{-1}\iota_1^*(y) = x.$$

Here $\sigma^{-1}\iota_1^* \operatorname{Sq}^{2n}(y) = \operatorname{Sq}^{2n} x \in PH^{4n}(X) = 0$. Thus we have that

$$\operatorname{Sq}^{2n} y = \alpha_2 w \tag{4.3}$$

for some $w \in \widetilde{H}^*(X)^{\otimes 2}$. Let $\lambda\colon (\Sigma X)^2 \to XP(2)$ be the composition of φ_2 and the natural projection $(\Sigma X)^2 \to (\Sigma X)_2$. Write the element $\lambda^* y$ as

$$\lambda^* y = \sigma(x) \otimes 1 + 1 \otimes \sigma(x) + \sum \sigma(x_i) \otimes \sigma(x_i'),$$

where $\dim x_i + \dim x_i' = 2n - 1$. Then for dimensional reasons and by $\operatorname{Sq}^{2n} x = 0$ we have that $\lambda^* \alpha_2 w = \operatorname{Sq}^{2n} \lambda^* y = 0$, and so $w + \tau^* w = 0$ by Definition 2.1(3), where τ is the generator of $\mathscr{S}(2)$. Thus for any $u \in H_{2n}(X)$ we have that

$$\begin{aligned} \langle u \otimes u, \operatorname{Sq}^1 w\rangle &= \langle (1 + \tau_*)(u \operatorname{Sq}^1 \otimes u), w\rangle \\ &= \langle u \operatorname{Sq}^1 \otimes u, w + \tau^* w\rangle \\ &= 0. \end{aligned} \tag{4.4}$$

Now we notice that

$$\mathrm{Sq}^1\,\mathrm{Sq}^{2n}\,y = \mathrm{Sq}^{2n+1}\,y = y^2 = \alpha_2(x \otimes x) \qquad \text{(cf. [23, Th. 2.4])}.$$

Thus there is a class $z \in \widetilde{H}^*(X)$ with

$$\tilde{\mu}^*(z) = \mathrm{Sq}^1\,w - x \otimes x \tag{4.5}$$

by (4.3). Here by [11], we can write $x = x_0^{2^t}$ with $\dim x_0 = 2s+1$, $t \geq 1$. Thus $x = \mathrm{Sq}^1\,x_1$ with $x_1 = (\mathrm{Sq}^{2s}\,x_0)x_0^{2^t-2}$, and so

$$\tilde{\mu}^*\,\mathrm{Sq}^1\,z = 0$$

by (4.5). This means that

$$\mathrm{Sq}^1\,z \in PH^{4n+1}(X) \cap \mathrm{Im}\,\mathrm{Sq}^1 = 0.$$

Thus in the E_2-term $E_2^{*,*}$ of the Bockstein spectral sequence of $H^*(X)$, z represents a class which is primitive by (4.5). Let $v \in H_{2n}(X)$ be any class with $\langle v, x\rangle = 1$. Then

$$\langle v^2, z\rangle = \langle v \otimes v, \tilde{\mu}^* z\rangle = 1$$

by (4.4) and (4.5). These show that z represents an even dimensional nonzero class in $E_2^{*,*}$ since $v^2\,\mathrm{Sq}^1 = 0$. Thus we have a nonzero square in $E_2^{*,*}$ by Milnor-Moore [17]. On the other hand, according to [11] $H^*(X)$ has no even dimensional generators. Furthermore, the square of an odd dimensional class is in the image of Sq^1. Thus $E_2^{*,*}$ is an exterior algebra, and we have a contradiction. This proves (4.2), and the proposition is proved. □

REMARK 4.6. If we assume that the multiplication of X is homotopy associative, in addition, a similar result to the above proposition can also be proved for an odd prime p. But this case was already proved by [4, Cor. 8.9] using Proposition 2.3.

Let X be the A_n-space in §3. We use the notation $T(t)$ for $T^{t+1}[y(t)_1, \ldots, y(t)_k]$ for simplicity. Then Theorem 3.5 implies

$$H^*(XP(t)) \cong T(t) \oplus S(t).$$

Furthermore we assume that X has a quasi C_m-form

$$\{\varphi_i\colon (\Sigma X)_i \to XP(i)\}_{1\leq i\leq m} \quad (m \leq n).$$

Then we prove the following

LEMMA 4.7. *$\varphi_i^* \mid T(i)$ is monomorphic if $i \le \min\{n-1, m, p-1\}$.*

Proof. We prove by induction on i.

If $i = 1$, it is clear since $\varphi_1 = \mathrm{id}$.

Suppose that $2 \le i \le \min\{n-1, m, p-1\}$. Take $z \in T(i)$ with $\varphi_i^*(z) = 0$. Then by the inductive assumption we have that $\iota_{i-1}^*(z) = 0$, and so z is a linear combination of $\mathscr{Y} = \{y(i)_{k(1)} \cdots y(i)_{k(i)} \mid 1 \le k(1) \le \cdots \le k(i)\}$. Let $\lambda_i \colon (\Sigma X)^i \to XP(i)$ be the composition of φ_i and the projection $(\Sigma X)^i \to (\Sigma X)_i$. Then by Definition 2.1(3), we have that

$$\lambda_i^*(y(i)_{k(1)} \cdots y(i)_{k(i)}) = \sum_{\tau \in \mathscr{S}(i)} \tau^*(\sigma x_{k(1)} \otimes \cdots \otimes \sigma x_{k(i)}).$$

It is easy to prove that λ_i^* is a monomorphism on the submodule spanned by $\mathscr{Y}$ since $i \le p-1$, and so we have $z = 0$.

Now we prove the key lemma:

LEMMA 4.8. *Let $i \le \min\{n-1, m, p-1\}$. Then for any $z \in T(i)$ and $\theta \in \mathscr{A}(p)$ with $\iota_1^* \cdots \iota_{i-1}^* \theta z = 0$, there is a decomposable class $d \in DH^*(XP(i))$ with*

$$\theta z = \theta d,$$

where $\mathscr{A}(p)$ is the mod p *Steenrod algebra.*

Proof. We prove by induction on i.

If $i = 1$, the lemma is clear.

Suppose that $i \ge 2$. Here we notice that $DH^*(XP(i)) = DT(i)$ by Theorem 3.5. Then by the inductive assumption, we have that

$$\theta \iota_{i-1}^* z = \theta d'$$

for some $d' \in DT(i-1)$. Take $d'' \in DT(i)$ with $\iota_{i-1}^* d'' = d'$, and put

$$z' = z - d''.$$

Then since $\iota_{i-1}^* \theta z' = 0$, we have that

$$\theta z' = \alpha_i(v)$$

for some $v \in PH^*(X)^{\otimes i}$.

Now let $(\Sigma X)^{[i]}$ denote the fat wedge, i.e.,

$$(\Sigma X)^{[i]} = \left\{(x_1, \ldots, x_i) \in (\Sigma X)^i \mid x_j = * \text{ for at least one } j\right\}.$$

Let $\lambda_i\colon (\Sigma X)^i \to XP(i)$ be the map in the proof of Lemma 4.7. Since $\widetilde{H}^*((\Sigma X)^i)$ decomposes to the direct sum of submodules $\widetilde{H}^*((\Sigma X)^{[i]})$, $(\sigma PH^*(X))^{\otimes i}$ and $(\sigma \otimes \cdots \otimes \sigma)D(i)$, we can write

$$\lambda_i^* z' = w + (\sigma \otimes \cdots \otimes \sigma)(u_1' + u_2')$$

with $w \in \widetilde{H}^*((\Sigma X)^{[i]})$, $u_1' \in PH^*(X)^{\otimes i}$ and $u_2' \in D(i)$. Here $H^*((\Sigma X)^{[i]})$, $PH^*(X)^{\otimes i}$ and $D(i)$ are all closed under the action of $\mathscr{A}(p)$. Furthermore

$$\lambda_i^* \theta z' = (\sigma \otimes \cdots \otimes \sigma) \sum_{\tau \in \mathscr{S}(i)} (\operatorname{sgn} \tau)\tau^* v \in (\sigma PH^*(X))^{\otimes i}.$$

Thus $\theta w = \theta u_2' = 0$, and

$$\theta u_1' = (\sigma^{-1} \otimes \cdots \otimes \sigma^{-1})\lambda_i^* \theta z' = \sum_{\tau \in \mathscr{S}(i)} (\operatorname{sgn} \tau)\tau^* v \in PH^*(X)^{\otimes i}.$$

This implies that $\lambda_i^* \alpha_i \theta u_1' = (\sigma \otimes \cdots \otimes \sigma) i! \theta u_1' = \lambda_i^*(i! \theta z')$. Thus by using Lemma 4.7, we have that $\alpha_i \theta u_1' = i! \theta z'$, and hence

$$\theta z = \theta d,$$

where $d = d'' + \alpha_i (1/i!) u_1'$. This proves the lemma. □

LEMMA 4.9. *Suppose that $n \geq m \geq p$. Then for any t with $t \not\equiv 0 \bmod p$, we have that*

$$\iota_1^* \cdots \iota_{p-1}^* H^{2t}(XP(p)) = 0.$$

Proof. We prove by contradiction. Assume that the lemma is not true. Choose t to be the greatest integer such that

$$\iota_1^* \cdots \iota_{p-1}^* H^{2t}(XP(p)) \neq 0$$

with $t \not\equiv 0 \bmod p$. Take $x \in H^{2t}(XP(p))$ with $z = \sigma^{-1} \iota_1^* \cdots \iota_{p-1}^*(x) \neq 0$. Since $\dim \mathscr{P}^{t-1} x = 2(tp - p + 1)$, we have by the assumption that

$$\iota_1^* \cdots \iota_{p-1}^* \mathscr{P}^{t-1} x = 0.$$

Thus we have that

$$\mathscr{P}^{t-1} \iota_{p-1}^* x = \mathscr{P}^{t-1} d$$

for some $d \in DH^*(XP(p-1))$ by Lemma 4.8, and so

$$\mathscr{P}^{t-1} \iota_{p-1}^* x = 0$$

for dimensional reasons. This means that $(1/t)\mathscr{P}^{t-1}x = \alpha_p y$ for some $y \in \widetilde{H}^*(X)^{\otimes p}$, and so

$$x^p = \mathscr{P}^t x = \mathscr{P}^1(1/t)\mathscr{P}^{t-1}x = \alpha_p \mathscr{P}^1 y.$$

Here we notice that if $p = 2$, $x^2 = \mathrm{Sq}^{2t} x = \mathrm{Sq}^2 \mathrm{Sq}^{2t-2} x + \mathrm{Sq}^{2t-1}\mathrm{Sq}^1 x = \mathrm{Sq}^2 \mathrm{Sq}^{2t-2} x$ since $\mathrm{Sq}^1 \equiv 0$ on $H^*(X)$. Thus the above equation holds also for $p = 2$.

Now

$$x^p = \alpha_p(z \otimes \cdots \otimes z).$$

Thus

$$z \otimes \cdots \otimes z - \mathscr{P}^1 y \in \beta_{p-1} H^{2tp-1}(XP(p-1)).$$

But $H^{2tp-1}(XP(p-1)) \subset \operatorname{Im}\alpha_{p-1}$ since by Theorem 3.5(3). Thus

$$z \otimes \cdots \otimes z = \mathscr{P}^1 y + w$$

with $w \in \operatorname{Im}(\tilde{\mu}^* \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \tilde{\mu}^* \otimes 1 \otimes \cdots \otimes 1 + \cdots)$. Take $u \in PH_{2t-1}(X)$ with $\langle u, z\rangle \neq 0$. Then

$$\langle u \otimes \cdots \otimes u, z \otimes \cdots \otimes z\rangle \neq 0.$$

On the other hand, $\langle u \otimes \cdots \otimes u, w\rangle = 0$ since $u^2 = 0$ by [**10**, Lemma 2.5]. Furthermore

$$\begin{aligned}
&\langle u \otimes \cdots \otimes u, \mathscr{P}^1 y\rangle \\
&\quad = (1/(p-1)!)\left\langle \sum_{\tau \in \mathscr{S}(p)} (\operatorname{sgn}\tau)\tau_*(u\mathscr{P}^1 \otimes u \otimes \cdots \otimes u), y \right\rangle \\
&\quad = (1/(p-1)!)\langle u\mathscr{P}^1 \otimes u \otimes \cdots \otimes u, \lambda_p^* \alpha_p y\rangle \\
&\quad = (1/t(p-1)!)\langle u\mathscr{P}^1 \otimes u \otimes \cdots \otimes u, \mathscr{P}^{t-1}\lambda_p^* x\rangle \\
&\quad = 0
\end{aligned}$$

since $\lambda_p^* x \in H^{2t}((\Sigma X)^p)$ implies $\mathscr{P}^{t-1}\lambda_p^* x = 0$ for dimensional reasons. (We also use the fact that $\mathrm{Sq}^1 \equiv 0$ on $H^*(X)$ for $p = 2$.) This is a contradiction, and the lemma is proved. □

Now we prove our main theorem.

Proof of Theorem 1.1. First we notice that $H^*(X)$ is an exterior algebra generated by finitely many odd dimensional generators by Proposition 4.1 and Remark 4.6. Thus we assume that X satisfies (3.1).

Suppose to the contrary that $\widetilde{H}^*(X) \neq 0$. Let s be the smallest integer with $H^{2s-1}(X) \neq 0$. Then by (3.4) and Theorem 3.5(4), we have that

(4.10) $i_{p-1}^*: H^t(XP(p)) \to T(p-1)$ is isomorphic for $t < 2sp$, and epimorphic for $t < 2sp + 2s - 2$.

Now we prove that

(4.11) $\operatorname{Im}\theta \cap H^t(XP(p)) = 0$ for any $t \leq 2sp$ and for any $\theta \in \mathscr{A}(p)$.

In fact, (4.11) for the case that θ is the Bockstein operation follows, since

$$H^{2j-1}(XP(p)) = 0 \quad \text{for } 2j-1 \leq 2sp$$

by (4.10). Furthermore, by Lemma 4.9, we have that

$$\iota_1^* \cdots \iota_{p-1}^* \mathscr{P}^1 H^*(XP(p)) = 0.$$

Thus by Lemma 4.8 together with the inductive argument we have that

$$\iota_{p-1}^* \mathscr{P}^1 H^j(XP(p)) \subset \mathscr{P}^1 DH^j(XP(p-1)) = 0$$

for $j \leq 2sp - 2p + 2$. This shows that

$$\operatorname{Im}\mathscr{P}^1 \cap H^t(XP(p)) = 0 \quad \text{for } t < 2sp$$

by (4.10). Furthermore, since $\iota_1^* \cdots \iota_{p-1}^* H^{2(sp-p+1)}(XP(p)) = 0$ by Lemma 4.9, we have that $\iota_{p-1}^* H^{2(sp-p+1)}(XP(p)) \subset DT(p-1)$, and so $H^{2(sp-p+1)}(XP(p)) \subset DH^*(XP(p))$. Thus

$$\operatorname{Im}\mathscr{P}^1 \cap H^{2sp}(XP(p)) \subset \mathscr{P}^1 DH^*(XP(p)) \cap H^{2sp}(XP(p)) = 0.$$

This proves (4.11) for $\theta = \mathscr{P}^1$.

Now if p is an odd prime, (4.11) for the general case follows by Liulevicius [**13**] or Shimada-Yamanoshita [**18**]. For $p = 2$ we need to prove a little more. If $p = 2$, then by using the same method as in [**12**, Prop. 2.3], we can prove by Lemma 4.9 that

$$QH^{4k+1}(X) = 0, \quad \text{and} \quad \mathrm{Sq}^2 \equiv 0 \quad \text{on } H^*(X).$$

Then by induction on r we can prove that if $t = 2^r + 2^{r+1}k$, then

$$\iota_1^* H^t(XP(2)) = 0, \quad QH^{t-1}(X) = 0, \quad \text{and}$$
$$\mathrm{Sq}^{2^{r+1}} \equiv 0 \quad \text{on } H^*(X) \qquad \text{(cf. [**12**])}.$$

This proves (4.11) for the case that $p = 2$.

Now take $x \in H^{2s-1}(X)$ and $y \in H^{2s}(XP(p))$ with $\iota_1^* \cdots \iota_{p-1}^* y = \sigma x \neq 0$. Then by (4.11), we have that

$$\alpha_p(x \otimes \cdots \otimes x) = y^p = \mathscr{P}^s y = 0.$$

Since $\beta_{p-1}H^{\mathrm{odd}}(XP(p-1)) \subset \operatorname{Im}\beta_{p-1}\alpha_{p-1}$ with $\beta_{p-1}\alpha_{p-1} = \tilde{\mu}^* \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \tilde{\mu}^* \otimes 1 \otimes \cdots \otimes 1 + \cdots$, there is a class $w \in H^*(X)^{\otimes p}$ so that

$$x \otimes \cdots \otimes x = \beta_{p-1}\alpha_{p-1}w\,.$$

Then for any primitive class $u \in PH_{2s-1}(X)$ with $\langle u, x\rangle \neq 0$, we have that

$$\begin{aligned} 0 &\neq \langle u \otimes \cdots \otimes u, x \otimes \cdots \otimes x\rangle \\ &= \langle u \otimes \cdots \otimes u, \beta_{p-1}\alpha_{p-1}w\rangle \\ &= 0 \end{aligned}$$

since $u^2 = 0$ by [**10**, Lemma 2.5]. This is a contradiction, and the theorem is proved. □

As was shown in §2, $S^{2t-1}_{(p)}$ has an A_{p-1}-form which admits a quasi C_{p-1}-form. However, this A_{p-1}-form cannot be extended to an A_∞-form. Thus to show that our main theorem is the best possible, we have to find an example of a simply connected A_∞-space with non-trivial finite mod p cohomology which admits a C_{p-1}-form for each odd prime p. McGibbon [**14**] showed that $\mathrm{Sp}(2)_{(3)}$ is one of such examples for $p = 3$. For $p > 3$ the author does not know such examples. But it seems to be reasonable to conjecture that the space $B_1(p)_{(p)}$, which is a $S^3_{(p)}$-bundle over $S^{2p+1}_{(p)}$, is an A_∞-space admitting a C_{p-1}-form. In fact $\mathrm{Sp}(2)_{(3)}$ has the homotopy type of $B_1(3)_{(3)}$, and $B_1(p)_{(p)}$ is an A_∞-space for any odd prime p ([**5**, Th. 1]).

5. Proof of Theorem 2.2. In this section we prove Theorem 2.2. First we prepare some known facts.

Let $\mathbf{n}$ denote the set $\{1, 2, \ldots, n\}$ for any positive integer n. Then a partition of $\mathbf{n}$ is a sequence of nonempty disjoint subsets of $\mathbf{n}$, $\alpha = (A_1, \ldots, A_k)$, with $\bigcup_i A_i = \mathbf{n}$. We call the sequence $(\#A_1, \ldots, \#A_k)$ the type of α, where $\#$ denotes the cardinality. A partition $\alpha = (A_1, \ldots, A_k)$ of $\mathbf{n}$ of type $(n_1, \ldots, n_k)$ defines a shuffle τ of type $(n_1, \ldots, n_k)$ by $A_i = \{\tau(n_1 + \cdots + n_{i-1} + 1), \ldots, \tau(n_1 + \cdots + n_i)\}$. Here a shuffle of type $(m_1, \ldots, m_t)$ is a class ρ in $\mathscr{S}(m_1 + \cdots + m_t)$ so that $\rho(i) < \rho(i+1)$ if $m_1 + \cdots + m_j + 1 \leq i \leq m_1 + \cdots + m_{j+1}$ for some $j \leq t$. By this correspondence we consider any partition of $\mathbf{n}$ as an element in $\mathscr{S}(n)$. In particular, all partitions of $\mathbf{n}$ of type $(1, \ldots, 1)$ correspond to the elements in $\mathscr{S}(n)$ bijectively.

Let $C(n-1)$ be the convex hull of $\{\tau(s_n) \mid \tau \in \mathscr{S}(n)\}$, where $s_n = (1, 2, \ldots, n) \in \mathbf{R}^n$, and τ acts on $\mathbf{R}^n$ by $\tau(x_1, \ldots, x_n) = (x_{\tau^{-1}(1)}, \ldots, x_{\tau^{-1}(n)})$. Then $C(n-1)$ is an $n-1$ dimensional cell complex whose faces correspond to the partitions of $\mathbf{n}$ bijectively (see [25]). Thus we also identify a partition $\alpha = (A_1, \dots, A_k)$ of $\mathbf{n}$ with the inclusion of the corresponding face, $\alpha\colon C_\alpha \to C(n-1)$, where $C_\alpha = C(\#A_1 - 1) \times \cdots \times C(\#A_k - 1)$.

Let $\alpha = (A_1, \dots, A_k)$ be a partition of $\mathbf{n}$ of type $(n_1, \dots, n_k)$. Then for any t with $0 \le t \le k$ we define a partition $\alpha_t = (B_1, \dots, B_{k+1})$ of $\mathbf{n+1}$ by

$$B_j = \begin{cases} A_j & \text{if } j < k-t+1, \\ \{n+1\} & \text{if } j = k-t+1, \\ A_{j-1} & \text{if } j > k-t+1. \end{cases}$$

Here we define a map

$$g_\alpha\colon \Delta^k \times C_\alpha \to C(n)$$

by

$$g_\alpha\left(\sum_t a_t P_t, x_1, \dots, x_k\right) = \sum_t a_t \alpha_t(x_1, \dots, x_{k-t}, 1, x_{k-t+1}, \dots, x_k),$$

where α_t is considered as the inclusion $C_{\alpha_t} \to C(n)$, and Δ^k is the k-simplex with vertices $\{P_0, \dots, P_k\}$. Then the set $\{g_\alpha\}$ for all partitions α of $\mathbf{n}$ gives a decomposition of $C(n)$:

$$C(n) = \bigcup_\alpha \operatorname{Im} g_\alpha. \tag{5.1}$$

We also define a map

$$\tilde{h}(\alpha)\colon \Delta^k \times C_\alpha \to \Delta^n$$

by

$$\tilde{h}(\alpha)\left(\sum_t a_t P_t, x_1, \dots, x_k\right) = \sum_t a_t(y(t)_1, \dots, y(t)_n),$$

where

$$y(t)_i = \begin{cases} 0 & \text{if } \alpha^{-1}(i) > n_1 + \cdots + n_{k-1} + 1, \\ 1 & \text{if } \alpha^{-1}(i) \le n_1 + \cdots + n_{k-t}. \end{cases}$$

Then by using the decomposition (5.1), $\{\tilde{h}(\alpha)\}$ define a relative homeomorphism:

$$h_n\colon (C(n), \partial C(n)) \to (I^n, \partial I^n) \qquad (n \ge 0). \tag{5.2}$$

Now we recall the definition of Williams' C_n-form. Let X be an associative H-space. Then a C_n-form on X in the sense of [**25**] is defined as a family of maps $\{Q_i\colon C(i-1)\times X^i \to X\}_{1\le i\le n}$ satisfying the following conditions:

(1) $Q_1 = \mathrm{id}_X$ where $C(0)\times X$ is identified with X.

(2) Let α be a partition of $\mathbf{i}$ of type (r, s) $(r+s=i)$. Then

$$\begin{aligned} &Q_i(\alpha(\rho, \sigma), x_1, \dots, x_i) \\ &\qquad = Q_r(\rho, x_{\alpha(1)}, \dots, x_{\alpha(r)}) \cdot Q_s(\sigma, x_{\alpha(r+1)}, \dots, x_{\alpha(i)}), \end{aligned}$$

where $\rho \in C(r-1)$, $\sigma\in C(s-1)$, $x_1, \dots, x_i \in X$, and "$\cdot$" denotes the multiplication of X.

(3) If $x_j = *$, then

$$Q_i(\tau, x_1, \dots, x_i) = Q_{i-1}(D_j(\tau), x_1, \dots, \hat{x}_j, \dots, x_i),$$

where $D_j\colon C(i-1)\to C(i-2)$ is the degeneracy map (see [**16**, Lemma 4.5]).

Finally we recall the definition of the projective n-space $XP(n)$ of an associative H-space X. Stasheff [**20**] used his own complexes to define $XP(n)$. Here we use the n-simplex Δ^n since we get the equivalent one.

Let $\partial_i\colon \Delta^{n-1}\to\Delta^n$ $(0\le i\le n)$ and $s_i\colon \Delta^n\to\Delta^{n-1}$ $(1\le i\le n)$ be the boundary and the degeneracy operations, respectively:

$$\partial_i(P_j) = \begin{cases} P_j & \text{if } j<i, \\ P_{j+1} & \text{if } j\ge i, \end{cases} \qquad s_i(P_j) = \begin{cases} P_j & \text{if } j<i, \\ P_{j-1} & \text{if } j\ge i. \end{cases}$$

Then $XP(n)$ is defined inductively by the relative homeomorphism:

$$\xi_n\colon (\Delta^n\times X^n, \partial\Delta^n\times X^n \cup \Delta^n\times X^{[n]}) \to (XP(n), XP(n-1)),$$

where ξ_n is defined by

$$\begin{aligned} &\xi_n(\partial_i(\sigma), x_1, \dots, x_n) \\ &\qquad = \begin{cases} \xi_{n-1}(\sigma, x_2, \dots, x_n), & i=0, \\ \xi_{n-1}(\sigma, x_1, \dots, x_{n-1}), & i=n, \\ \xi_{n-1}(\sigma, x_1, \dots, x_i\cdot x_{i+1}, \dots, x_n), & 1\le i\le n-1, \end{cases} \\ &\xi_n(\sigma, x_1, \dots, x_n) = \xi_{n-1}(s_j(\sigma), x_1, \dots, \hat{x}_j, \dots, x_n) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_j = * \ (1\le j\le n). \end{aligned}$$

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. The second part is clear from the first part. So we only prove the first part by [**25**, Cor. 2.6].

Let X be an associative H-space with C_n-form $\{Q_i\}_{1\leq i\leq n}$. We construct a quasi C_n-form $\{\varphi\}_{i\leq n}$, inductively.

First we put $\varphi_1 = \mathrm{id}_{\Sigma X}$.

Next we suppose that $1 < m \leq n$ and $\{\varphi_i\}_{1\leq i\leq m-1}$ are constructed. Let $\alpha = (A_1, \dots, A_k)$ be a partition of $\mathbf{m}$ of type $(a_1, \dots, a_k)$. Then we consider the following composition:

$$\begin{aligned}\Delta^k \times C_\alpha \times X^m &\xrightarrow{\tau} \Delta^k \times C(a_1 - 1) \times X^{a_1} \times \cdots \times C(a_k - 1) \times X^{a_k} \\ &\xrightarrow{\eta} \Delta^k \times X^k \to XP(k) \subset XP(m),\end{aligned}$$

where τ is the appropriate switching map, and $\eta = 1 \times Q_{a_1} \times \cdots \times Q_{a_k}$. By considering the above maps for all partitions of $\mathbf{m}$, the decomposition $\{g_\alpha\}$ of $C(m+1)$ of (5.1) defines a map

$$C(m) \times X^m \to XP(m).$$

Then this map together with h_m of (5.2) defines a well defined map φ_m which satisfies the desired properties of quasi C_m-form since there is a natural relative homeomorphism

$$(I^m \times X^m, \partial I^m \times X^m \cup I^m \times X^{[m]}) \to ((\Sigma X)_m, (\Sigma X)_{m-1}).$$

Thus X is shown to have a quasi C_n-form.

Now suppose that X is an associative H-space with a quasi C_n-form $\{\varphi_i\}_{i\leq n}$. Let $\nu_i\colon (\Sigma X)_i \to BX$ be the composition of φ_i and the inclusion $XP(i) \to XP(\infty) = BX$. Then since $\nu_1\colon \Sigma X \to BX$ is the adjoint of the natural map $\varepsilon\colon X \to \Omega BX$, ν_i defines a map $Q'_i\colon C(i-1) \times X^i \to \Omega BX$ so that $\{Q'_i\}_{1\leq i\leq n}$ gives a C_n-commutativity of ε in the sense of [**25**, Def. 25]. Thus if $\psi\colon \Omega BX \to X$ denotes the natural A_∞-equivalence, then we have a C_n-form $\{\psi Q'_i\}_{i\leq n}$ on X. This completes the proof. □

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