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Let X be a Hausdorff continuous image of an ordered continuum. Mardešić proved that X has a basis of open sets with metrizable boundaries. We use T-set approximations to obtain bases of open sets for X whose boundaries satisfy a variety of conditions. In particular, we prove that

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X = \max\{1, \sup\{\dim Y : Y \subset X \text{ is metrizable and closed}\}\}.$$

1. Introduction. In this paper we study the rim-properties of images of ordered continua and, more generally, of compact ordered spaces. Mardešić proved in [M1] that a Hausdorff space which is a continuous image of a compact ordered space is rim-metrizable. In [N3], the first author proved that every hereditarily locally connected continuum is a continuous image of an ordered continuum. Then he used the approximation by T-sets of cyclic elements in images of ordered continua to prove that every hereditarily locally connected continuum is rim-countable. We use the techniques of [N3] to improve the result of Mardešić and to answer a question of Mardešić and Papić [MP] about dimension-theoretic properties of continuous images of ordered continua and ordered compacta. We improve a result of Simone [Si1] by proving that if X is a continuous image of an ordered continuum and X contains no nondegenerate metric continuum, then it is rim-finite. We also prove that if a rim-scattered space is a continuous image of an ordered compactum, then it is rim-countable.

All spaces in this paper are Hausdorff. A *continuum* is a compact connected (Hausdorff) space. An *ordered compactum* is a compact space which admits a linear ordering such that the order topology is the given topology. Ordered continua are locally connected; they are often called *arcs*.

A point p of a connected set X is a *separating point* of X if $X-\{p\}$ is not connected. We let $E(X)$ denote the set of all separating points of X.

Let X be a locally connected continuum. A connected subset Q of X is a *cyclic element* of X if Q is maximal with respect to containing no separating points of itself. Each cyclic element of X is a locally connected continuum. The theory of cyclic elements is presented in [**Wh1**, Ch. 4] for the case of metric locally connected continua. We shall use some extensions of this theory to the non-metric setting as set out in [**Wh2**] and [**C**], see also [**N4**].

A collection $\mathbf{A}$ of subsets of a compact space X is said to be a *null-family* in X if, for every open covering $\mathbf{U}$ of X, the subcollection $\{B \in \mathbf{A} : B$ is not contained in any $V \in \mathbf{U}\}$ is finite.

Let A be a subset of a locally connected continuum X. We let $K(X-A)$ denote the set of all components of $X-A$. We will say that A is a T-*set* in X if A is closed and each component of $X-A$ has a two-point boundary.

Let Y be a cyclic element of a locally connected continuum X. We say that a sequence $\{A_1, A_2, \dots, A_n, \dots\}$ of T-subsets of Y T-*approximates* Y if

(1) A_1 is metrizable,
(2) $A_n \subset A_{n+1}$,
(3) if $Z \in K(Y-A_n)$, then $E(\mathrm{Cl}(Z)) \subset A_{n+1}$,
(4) if $Z \in K(Y-A_n)$ and C is a nondegenerate cyclic element of $\mathrm{Cl}(Z)$, then $C \cap A_{n+1}$ is a metrizable set which contains at least three points.

Note that the conditions of the above definition imply that $\mathrm{Cl}(\bigcup_{n=1}^{\infty} A_n) = Y$ (see [**N1**, Lemma 3.4]).

In [**N1**], there are given several characterizations of continuous Hausdorff images of ordered continua. One of them is the following:

THEOREM 1 [**N1**, 1.1]. *Let X be a locally connected continuum. Then the following are equivalent*:

(1) *X is a continuous image of an ordered continuum,*
(2) *if Y is a nondegenerate cyclic element of X, then there is a sequence $\{A_1, A_2, \dots\}$ of T-sets in Y which T-approximates Y.*

Further properties of continuous images of arcs and ordered compacta can be found in survey articles [**M3**], [**TrW**] and [**N4**]; see also [**N1**].

Let $\mathbf{P}$ be a property of sets. A space X is said to be rim-$\mathbf{P}$ if it has a basis of open sets whose boundaries have property $\mathbf{P}$. A set is said to be *scattered* if each of its non-empty closed subsets has an isolated point. Recall that compact, metrizable, scattered spaces are countable. For definitions of dimensions dim , Ind and ind , the reader is referred to [**E**].

For a compact space X, we define

$$\alpha(X) = \sup\{\dim Z : Z \text{ is a closed metrizable subset of } X\}.$$

We let $\alpha - 1 = \infty$ if $\alpha = \infty$.

We shall need the following lemmas.

LEMMA 1 [**Tr2**]. *Let X be a locally connected continuum and A a T-set in X. There exists an upper semi-continuous decomposition G_A of X into closed sets such that if X_A denotes the quotient space and $f : X \to X_A$ is the quotient map, then:*

(1) *$f|_A$ is one-to-one and $f(A)$ is a T-set in X_A,*

(2) *each $Z \in K(X_A - f(A))$ is homeomorphic to $]0, 1[$,*

(3) *for each $Z \in K(X_A - f(A))$ there exists a unique $P_Z \in K(X - A)$ such that $f(P_Z) \subset \mathrm{Cl}(Z)$, and each component of $X - A$ is a P_Z for some $Z \in K(X_A - f(A))$.*

In the above lemma, $f(A)$ is a T-set in X_A, and we call f a T-*map* with respect to A. The space X_A is uniquely determined by X and A. If the set A is metrizable it follows, by local connectedness of X, that $K(X - A)$ is countable, [**N1**, 4.1].

LEMMA 2. *Let X be a locally connected continuum and, for every cyclic element Y of X, let $\mathbf{B}_Y$ be a basis for Y. Then X has a basis $\mathbf{B}$ such that, for each $U \in \mathbf{B}$, there exist a family $\mathbf{A}$ of cyclic elements of X, non-negative integers m and n, nondegenerate cyclic elements $Y_1, \dots, Y_m$ of X, sets $U_1 \in \mathbf{B}_{Y_1}, \dots, U_m \in \mathbf{B}_{Y_m}$, and separating points $x_1, \dots, x_n$ of X such that*

$$U = \left(\bigcup \mathbf{A}\right) \cup U_1 \cup \cdots \cup U_m \quad \textit{and}$$
$$\mathrm{Bd}(U) = \mathrm{Bd}_{Y_1}(U_1) \cup \cdots \cup \mathrm{Bd}_{Y_m}(U_m) \cup \{x_1, \dots, x_n\}.$$

Proof. The lemma follows from the generalization, by Cornette [**C**, p. 225-6], of Whyburn's cyclic chain approximation theorem [**Wh1**, IV.7.1, p. 73] to the case of locally connected Hausdorff continua. □

LEMMA 3. *Let γ be an infinite cardinal number and let $\mathbf{P}$ be a hereditary property of compact sets that is preserved under unions of fewer than γ compact sets. Let X be a locally connected continuum,*

$\{A_i\}_{i=1}^{\infty}$ an increasing sequence of closed subsets of X, and $\{\mathbf{V}_i\}_{i=1}^{\infty}$ a sequence of collections of sets such that:

(1) *$\mathbf{V}_i$ is a basis of open sets for A_i,*
(2) *$\mathrm{Bd}(K)$ has property $\mathbf{P}$ for each $K \in K(X - A_i)$,*
(3) *$V \in \mathbf{V}_i$ implies $\mathrm{Bd}_{A_i}(V)$ has property $\mathbf{P}$,*
(4) *$V \in \mathbf{V}_i$ implies $\{K \in K(X - A_i) : \mathrm{Bd}(K) \cap V \neq \emptyset$ and $\mathrm{Bd}(K) \not\subset \mathrm{Cl}(V)\}$ has cardinality less than γ,*
(5) *for each open cover $\mathbf{W}$ of X there is an integer i such that $K(X - A_i)$ refines $\mathbf{W}$.*

Then X admits a basis of open sets whose boundaries have property $\mathbf{P}$.

Proof. Let $x \in X$ and let U be an open neighbourhood of x. Let W be an open neighbourhood of x such that $\mathrm{Cl}(W) \subset U$.

Suppose that $x \notin \bigcup_{n=1}^{\infty} A_n$. For every n let $K_n \in K(X - A_n)$ be such that $x \in K_n$. Then $K_{n+1} \subset K_n$. By (5), there is an integer i such that K_i is contained either in U or in $X - \mathrm{Cl}(W)$. Since $x \in \mathrm{Cl}(W) \cap K_i$, it follows that $K_i \subset U$. Since X is locally connected, K_i is an open set. By (2), $\mathrm{Bd}(K_i)$ has property $\mathbf{P}$.

Now suppose that $x \in A_n$ for some integer n. By (5), we may take n to be such that no component of $X - A_n$ meets both $\mathrm{Cl}(W)$ and $X - U$. Let $V \in \mathbf{V}_n$ be such that $x \in V \subset \mathrm{Cl}(V) \subset W$. Let $V' = V \cup \bigcup\{K \in K(X - A_n) : \mathrm{Bd}(K) \cap V \neq \emptyset\}$. Then $V' \subset U$. Since X is locally connected, V' is open and

$$\mathrm{Bd}(V') \subset \mathrm{Bd}_{A_n}(V) \cup \bigcup\{\mathrm{Bd}(K) : K \in K(X - A_n),\ \mathrm{Bd}(K) \cap V \neq \emptyset \text{ and } \mathrm{Bd}(K) \not\subset V\}.$$

By (3), (2) and (4), it follows that $\mathrm{Bd}(V')$ has property $\mathbf{P}$. □

2. Main results. The proof of the following lemma uses some ideas from the proof of [**N3**, Theorem 4.1].

LEMMA 4. *Let Y be a continuum with no separating point which is a continuous image of an ordered continuum. Let $\alpha = \max\{1, \alpha(Y)\}$. Then Y has a basis $\mathbf{V}$ of open sets whose boundaries are metrizable sets of $\dim \leq \alpha - 1$. Moreover, if Y admits a basis of open sets with scattered boundaries, then the boundaries of members of $\mathbf{V}$ are countable.*

Proof. Let $\{A_1, A_2, \dots\}$ be a sequence of T-sets in Y which T-approximates Y. For each n, let $f_n : Y \to Y_{A_n} = Y_n$ be a T-map with respect to A_n (see Lemma 1). We let $B_n^m = f_n(A_m) \subset Y_n$ provided $m \leq n$. Notice that Y_n has no separating point, each B_n^m is a T-set in Y_n provided $m \leq n$, $f_n|_{A_m} : A_m \to B_n^m$ is a homeomorphism, and every component of $Y_n - B_n^n$ is homeomorphic to $]0, 1[$. Since Y_n has no separating point, it follows that if P is a component of $Y_n - B_n^m$, $\mathrm{Bd}(P) = \{a, b\}$, then $\mathrm{Cl}(P)$ is a cyclic chain from a to b (in the case when $m = n - 1$, all cyclic elements of $\mathrm{Cl}(P)$ are metrizable—see below).

First, we use an induction to show that, for $n = 1, 2, \dots$, Y_n has a basis $\mathbf{B}_n$ such that $\mathrm{Bd}_{Y_n}(V)$ is metrizable and $\dim(\mathrm{Bd}_{Y_n}(V)) \leq \alpha - 1$ for each $V \in \mathbf{B}_n$.

Note that $Y_1 = B_1^1 \cup (Y_1 - B_1^1)$ is a metrizable space which is a union of the compact metrizable set B_1^1 (which is homeomorphic to A_1) and a countable family of copies of $]0, 1[$. By [**E**, 1.5.3, p. 42], $\dim Y_1 \leq \max\{1, \dim B_1^1\} \leq \alpha$. Hence, Y_1 has a basis $\mathbf{B}_1$ as required.

Suppose that the required basis $\mathbf{B}_n$ for Y_n has been already defined. Let $y \in Y_{n+1}$ and let V be an open neighbourhood of y in Y_{n+1}. If $y \notin B_{n+1}^n$, then $y \in Q$ for some $Q \in K(Y_{n+1} - B_{n+1}^n)$. Let $\mathrm{Bd}(Q) = \{a, b\}$. Then $\mathrm{Cl}(Q)$ is a cyclic chain from a to b and $E(\mathrm{Cl}(Q)) \subset B_{n+1}^{n+1}$. If Z is a nondegenerate cyclic element of $\mathrm{Cl}(Q)$, then $B_Z = B_{n+1}^{n+1} \cap Z$ is a metrizable T-set in Z, $Z \cap (E(\mathrm{Cl}(Q)) \cup \{a, b\})$ consists of exactly two points, and each component of $Z - B_Z$ is homeomorphic to $]0, 1[$. Hence, $K(Z - B_Z)$ is countable and Z is metrizable. Now, it is easy to find an open neighbourhood W of y in Y_{n+1} such that $W \subset V \cap Q$, $\mathrm{Bd}_{Y_n}(W)$ is contained in two cyclic elements Z_1 and Z_2 of $\mathrm{Cl}(Q)$ and for $i = 1, 2$

$$\begin{aligned}\dim(\mathrm{Bd}_{Y_n}(W) \cap Z_i) &\leq \dim Z_i - 1 \leq \max\{1, \dim B_{Z_i}\} - 1 \\ &\leq \max\{1, \dim A_{n+1}\} - 1 \leq \alpha - 1\end{aligned}$$

provided Z_i is nondegenerate (the case when Z_i is degenerate is trivial). Thus we have $\dim(\mathrm{Bd}_{Y_n}(W)) \leq \alpha - 1$.

Now, suppose that $y \in B_{n+1}^n$. Let x denote the unique point of A_n such that $f_{n+1}(x) = y$. For every $P \in K(Y_{n+1} - B_{n+1}^n)$ let $Q_P \in K(Y - A_n)$ be a component such that $f_{n+1}(Q_P) \subset \mathrm{Cl}(P)$ and let $R_P \in K(Y_n - B_n^n)$ be such that $f_n(Q_P) \subset \mathrm{Cl}(R_P)$. Set $\mathrm{Bd}_{Y_{n+1}}(P) = \{a_P, b_P\}$ and $\mathrm{Bd}_{Y_n}(R_P) = \{a'_P, b'_P\}$, where $f_{n+1}^{-1}(a_n) \cap A_n = f_n^{-1}(a'_n) \cap A_n$, and let $\leq$ denote the natural ordering on $\mathrm{Cl}(R_P)$ from a'_P to b'_P. Choose $r_P \in R_P$ and let $I_P = \{r \in R_P : r < r_P\}$ and $J_P = \{r \in R_P : r_P < r\}$.

Let

$$\begin{aligned} V' = {} & f_n(f_{n+1}^{-1}(V) \cap A_n) \\ & \cup \bigcup \{R_P : P \in K(Y_{n+1} - B_{n+1}^n) \text{ and } \mathrm{Cl}(P) \subset V\} \\ & \cup \bigcup \{I_P : P \in K(Y_{n+1} - B_{n+1}^n) \text{ and } a_P \in V\} \\ & \cup \bigcup \{J_P : P \in K(Y_{n+1} - B_{n+1}^n) \text{ and } b_P \in V\}. \end{aligned}$$

Since $\{\mathrm{Cl}(R_P) : P \in K(Y_{n+1} - B_{n+1}^n)\}$ is a null-family, V' is an open subset of Y_n. Moreover, $f_n(x) \in V'$. By the inductive hypothesis, there is a connected open set W' in Y_n such that $f_n(x) \in W' \subset V'$, $\mathrm{Bd}_{Y_n}(W')$ is metrizable and $\dim(\mathrm{Bd}_{Y_n}(W')) \leq \alpha - 1$. Let

$$\mathbf{H}_1 = \{P \in K(Y_{n+1} - B_{n+1}^n) : a'_P \in W' \text{ and } R_P \not\subset W'\},$$
$$\mathbf{H}_2 = \{P \in K(Y_{n+1} - B_{n+1}^n) : b'_P \in W' \text{ and } R_P \not\subset W'\}$$

and

$$\mathbf{H}_3 = \{P \in K(Y_{n+1} - B_{n+1}^n) : R_P \subset W'\}.$$

Note that if $P \in \mathbf{H}_1 \cup \mathbf{H}_2$, then $R_P \cap \mathrm{Bd}_{Y_n}(W')$ is a non-empty open subset of $\mathrm{Bd}_{Y_n}(W')$. Since $\mathrm{Bd}_{Y_n}(W')$ is compact and metrizable, $\mathbf{H}_1 \cup \mathbf{H}_2$ is countable. For every $P \in \mathbf{H}_1$ (resp. $P \in \mathbf{H}_2$), let W_P^1 (resp. W_P^2) be an open subset of $\mathrm{Cl}(P)$ such that $a_P \in W_P^1 \subset V$ (resp. $b_P \in W_P^2 \subset V$), $\mathrm{Bd}_{\mathrm{Cl}(P)}(W_P^1)$ is metrizable and $\dim(\mathrm{Bd}_{\mathrm{Cl}(P)}(W_P^1)) \leq \alpha - 1$ (resp. $\mathrm{Bd}_{\mathrm{Cl}(P)}(W_P^2)$ is metrizable and $\dim(\mathrm{Bd}_{\mathrm{Cl}(P)}(W_P^2)) \leq \alpha - 1$). Note that $\mathrm{Bd}_{\mathrm{Cl}(P)}(W_P^i)$ may be assumed to be contained in one cyclic element Z of $\mathrm{Cl}(P)$. By the fact that $K(Z - B_Z)$ is countable, it follows that Z is metrizable and $\dim Z \leq \alpha$. Let

$$W = f_{n+1}(f_n^{-1}(W') \cap A_n) \cup \bigcup_{P \in \mathbf{H}_1} W_P^1 \cup \bigcup_{P \in \mathbf{H}_2} W_P^2 \cup \bigcup \mathbf{H}_3.$$

Since $K(Y_{n+1} - B_{n+1}^n)$ is a null-family, W is open in Z. A straightforward argument shows that $y \in W \subset V$ (because if $P \in K(Y_{n+1} - B_{n+1}^n)$ is not contained in V, then $r_P \notin V'$ and so $R_P \not\subset W'$) and

$$\begin{aligned} \mathrm{Bd}_{Y_{n+1}}(W) = {} & f_{n+1}(f_n^{-1}(\mathrm{Bd}_{Y_n}(W') \cap A_n)) \\ & \cup \bigcup_{P \in \mathbf{H}_1} \mathrm{Bd}_{\mathrm{Cl}(P)}(W_P^1) \cup \bigcup_{P \in \mathbf{H}_2} \mathrm{Bd}_{\mathrm{Cl}(P)}(W_P^2). \end{aligned}$$

Thus $\mathrm{Bd}_{Y_{n+1}}(W)$ is a union of countably many compact metrizable sets of $\dim \leq \alpha - 1$. It is well-known that each compact space which can be covered by countably many closed and metrizable subsets is metrizable. Hence, $\mathrm{Bd}_{Y_{n+1}}(W)$ is metrizable. By [E, 1.5.3, p. 42], $\dim(\mathrm{Bd}_{Y_{n+1}}(W)) \leq \alpha - 1$. The inductive argument is complete.

Let **P** be the following property of compact spaces: a space is metrizable of dimension $\leq \alpha - 1$. Let $\gamma = \aleph_1$ be the first uncountable cardinal number. Note that Y satisfies all the assumptions of Lemma 3. Indeed, the condition (2) of Lemma 3 follows immediately from the definition of a T-set. Let $\mathbf{V}_n = \{A_n \cap f_n^{-1}(U) : U \in \mathbf{B}_n\}$ for $n = 1, 2, \ldots$. Then $\mathbf{V}_n$ is a basis for A_n which satisfies the conditions (1) and (3). The condition (4) follows from [**N1**, 4.1], and the condition (5) is a consequence of [**N1**, 3.4]. By Lemma 3, Y has a basis **V** of open sets with metrizable boundaries of dimension $\leq \alpha - 1$.

Suppose that Y is rim-scattered. Then Y_1 is metrizable and rim-scattered. Hence, Y_1 has a basis of open sets with countable boundaries. It is now easy to modify the above argument to show that each Y_n has a basis of open sets with countable boundaries. By Lemma 3, Y has a basis of open sets with countable boundaries. □

Simone, [**Si1**] and [**Si2**], proved that if X is a continuum with degree of cellularity $\aleph_0$, which is a continuous image of an ordered continuum and which contains no nondegenerate metric subcontinuum, then X has a basis of open sets with finite boundaries. Simone's theorem can be improved as follows:

THEOREM 2. *Let X be a continuum which is a continuous image of an arc and which contains no nondegenerate metric subcontinuum. Then X has a basis of open sets with finite boundaries.*

Proof. Let Y be a nondegenerate cyclic element of X. Since having a basis of open sets with finite boundaries is a cyclically extensible property (see Lemma 2), it suffices to prove that Y is rim-finite.

Let $\{A_1, A_2, \ldots\}$ be a sequence of T-sets in Y which T-approximates Y and, for $n = 1, 2, \ldots$, let $f_n : Y \to Y_n$ be a T-map with respect to A_n (see Lemma 1). Since A_1 is metrizable, and, hence, zero-dimensional, Y_1 has a basis of open sets with finite boundaries (see [**N1**, 4.3]). If U is an open set in Y_1 which has a finite boundary, then all but at most finitely many components of $Y_1 - A_1$ whose closures meet $U \cap A_1$ are contained in $\mathrm{Cl}(U)$. An inductive argument similar to the one given in the proof of Lemma 4 shows that each Y_n is rim-finite. Taking **P** to be the property of being a finite set and

$\gamma = \aleph_0$ in Lemma 3, it follows that Y has a basis of open sets with finite boundaries. □

THEOREM 3. *If X is a nondegenerate continuous image of an ordered continuum, then*

$$\max\{1, \alpha(X)\} = \dim X = \operatorname{Ind} X = \operatorname{ind} X.$$

Proof. Let $\alpha = \max\{1, \alpha(X)\}$. Since X is a nondegenerate continuum, $\operatorname{ind} X \geq 1$. By general facts (see [E, 3.1.4 on p. 209, 2.2.1 on p. 170, and 1.1.2 on p. 4]), it follows that $\dim X \geq \dim Z$, $\operatorname{Ind} X \geq \operatorname{Ind} Z$ and $\operatorname{ind} X \geq \operatorname{ind} Z$ for each closed subspace Z of X. Hence $\dim X$, $\operatorname{Ind} X$, $\operatorname{ind} X \geq \alpha$. For each normal space X, we have $\operatorname{ind} X \leq \operatorname{Ind} X$ [E, 1.6.3, p. 52] and $\dim X \leq \operatorname{Ind} X$ [E, 3.1.28, p. 220]. Thus it suffices to show that $\operatorname{Ind} X \leq \alpha$.

Let $x \in X$ and V be an open neighbourhood of x. By Lemmas 4 and 2, there exists an open set W such that $x \in W \subset V$, $\operatorname{Bd}(W)$ is contained in the union of a finite collection $\{Z_1, \dots, Z_n\}$ of cyclic elements of X, $\operatorname{Bd}(W) \cap Z_i$ is metrizable and $\dim(\operatorname{Bd}(W) \cap Z_i) \leq \alpha - 1$ for $i = 1, \dots, n$. Hence, $\operatorname{Bd}(W)$ is metrizable and $\operatorname{Ind}\operatorname{Bd}(W) = \dim \operatorname{Bd}(W) \leq \alpha - 1$. By the sum theorem for separable metric spaces, [E, 1.5.3, p. 42], we have $\operatorname{Ind} X \leq \alpha$. □

REMARK. In Theorem 3, if $\alpha(X) = 0$, then X is rim-finite by Theorem 2.

THEOREM 4. *Let X be a continuum which is a continuous image of an arc. If X has a basis of open sets with scattered boundaries, then it has a basis of open sets with countable boundaries.*

Proof. By Lemma 4, each cyclic element of X is rim-countable. The theorem follows by Lemma 2. □

The following theorem answers a question of Mardešić and Papić ([MP], see also [N4, Problem 4]):

THEOREM 5. *Let Z be a continuous image of a compact ordered space. Then*

(1) $\dim Z = \operatorname{Ind} Z = \operatorname{ind} Z$. *If, moreover,* $\dim Z > 0$ *then* $\dim Z = \max\{1, \alpha(Z)\}$.
(2) *If Z is rim-scattered, then it is rim-countable.*

Proof. For every compact space T, $\operatorname{Ind} T = 0$ iff $\dim T = 0$ iff $\operatorname{ind} T = 0$, [E, 3.1.30, p. 221]. Thus we may assume that Z is not zero-dimensional. Let $\alpha = \max\{1, \alpha(Z)\}$.

By [**N2**, Theorem 2], see also [**M1**, Lemma 8], there exists a space X such that X is a continuous image of an arc, $Z \subset X$, Z is a T-set in X, and each component of $X - Z$ is homeomorphic to $]0, 1[$. If Y is a closed metrizable subset of X, then Y is a union of $Z \cap Y$ and at most countably many closed sets which are homeomorphic to subsets of $]0, 1[$. Hence, $\dim Y \leq \max\{1, \dim(Y \cap Z)\}$. By Theorem 3, $\alpha = \dim X = \operatorname{Ind} X = \operatorname{ind} X$. Since Z is not zero-dimensional, $\alpha \leq \dim Z, \operatorname{Ind} Z, \operatorname{ind} Z$. However, $\dim Z \leq \dim X$, $\operatorname{Ind} Z \leq \operatorname{Ind} X$ and $\operatorname{ind} Z \leq \operatorname{ind} X$. This completes the proof of (1). A similar argument together with Theorem 4 show that (2) holds. □

REMARKS. 1. In the case when $\alpha(Z) = 0$, the result (1) of Theorem 5 was obtained by Mardešić [**M2**, Corollary, p. 425].

2. The proofs of Lemma 4 and Theorems 3 and 5 show that if a space X is a continuous image of an ordered compactum, then it has a basis $\mathbf{B}$ such that $\operatorname{Bd}(U)$ is metrizable and $\dim \operatorname{Bd}(U) \leq \dim X - 1$ for each $U \in \mathbf{B}$. This improves results of [**M1**].

3. Problems. Filippov gave in [**F**] an example of a locally connected continuum which admits a basis of open sets with metrizable zero-dimensional and perfect boundaries and which is not a continuous image of any ordered compactum.

In general, rim-scattered continua are not continuous images of ordered compacta. For example: the space $X = L \times S/_{\{0\}\times S}$, where L denotes the long interval and $S = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$, is a rim-countable continuum which is a continuous image of no ordered compactum. In fact, X contains a non-metric product of infinite compact spaces—see [**Tr1**]. However, the space X is not locally connected. In [**Tu**], it was proved that rim-scattered locally connected continua do not contain a non-metric product of nondegenerate continua. Hence we may ask the following question:

Question 1. Is every locally connected rim-scattered continuum a continuous image of an ordered continuum?

Filippov's example shows that rim-scattered locally connected continua are the largest possible class of spaces defined with the use of rim-properties that could be contained in the class of continuous images of ordered continua. Recall the following weaker question which is still open (see [**N3**] and [**N4**]).

Question 2. Is every locally connected, rim-countable continuum a continuous image of an ordered continuum?

Let us also pose the following problem:

Question 3. Is every locally connected and rim-scattered continuum a rim-countable space?

Recall that, by Theorem 4, Question 3 has a positive answer provided the space under consideration is a continuous image of an arc.

Added in proof. Recently the authors answered questions 1 and 2 in the negative in the paper: J. Nikiel, H. M. Tuncali, and E. D. Tymchatyn, *A locally connected rim-countable continuum which is the continuous image of no arc*, Topology Appl. (to appear). L. B. Treybig proved a result which implies Theorem 2 in Proc. Amer. Math. Soc. **74** (1979), 326–328.

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