DOMAINS IN RIEMANNIAN MANIFOLDS AND INVERSE SPECTRAL GEOMETRY

Harold Gerard Donnelly and Jeffrey Marc Lee

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The asymptotic formula of Weyl, $(\lambda_k)^{n/2} \sim c(n)k/\mathrm{vol}(D)$, shows that the volume of a bounded domain D in an n dimensional Riemannian manifold is determined by the Dirichlet spectrum, $\{\lambda_k\}$, of the domain. Also, the asymptotic expansion for the trace of the Dirichlet heat kernel of a smooth bounded domain shows that the volume of the boundary is determined by the spectrum of the domain. However, these asymptotic expressions do not tell us, in themselves, how many eigenvalues one needs in order to approximate the volume of the domain or its boundary to within a prescribed error. We give several results which answer this question, for certain types of domains, in terms of the geometry of the ambient manifold. Some knowledge of the domain is needed. In particular, the distance from the boundary to the boundary's cut locus in the ambient manifold is relevant. Thus, we also prove a purely differential geometric structure theorem relating the distance from the boundary of the domain to the interior part of its cut locus, to the principal curvatures of the boundary.

1. **Introduction.** We consider the Laplace operator Δ defined on a smooth relatively compact domain D in a Riemannian manifold M^n. By domain we will always mean connected open set. Let $\{-\lambda_i\}$ be the set of eigenvalues or spectrum of Δ with Dirichlet boundary conditions indexed in increasing order and including multiplicity:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \uparrow \infty.$$

The spectrum of D determines certain aspects of the geometry of D. For instance Weyl's formula:

$$\lim_{k \to \infty} \frac{\lambda_k}{k^{2/n}} = \frac{c_n}{\mathrm{vol}(D)^{2/n}}$$

shows that the spectrum determines the volume of D. More information is given by the asymptotic formula for $\mathrm{trace}(e^{t\Delta})$, for sufficiently small t,

$$\sum_{i=1}^{\infty} e^{-\lambda_i t} = \frac{1}{(4\pi t)^{n/2}} \left\{ \mathrm{vol}(D) - \frac{\sqrt{\pi}}{2}\, \mathrm{vol}(\partial D) t^{1/2} + O(t) \right\}.$$

In particular, $\mathrm{vol}(\partial D)$ is determined by the spectrum of Δ.

An important point is that these asymptotic expressions, in themselves, do not tell us how many eigenvalues one would need to estimate the given geometric quantity (e.g. volume) to within a prescribed error. This provides motivation for much of what we do here. One would like to be able to estimate geometric invariants without assuming too much a priori knowledge of the geometry of D. For example, it would be unfortunate if in order to tell how large to take k in Weyl's formula to know the volume to within some $\varepsilon > 0$, we needed to know the volume itself. Along these lines, there is the following result [T] of Li and Yau for convex domains in $\mathbb{R}^n$.

THEOREM 1.1 (*Li and Yau*). *Let* D *be a convex domain in* $\mathbb{R}^n$. *Given* $\varepsilon > 0$ *there exists an* N_0 *depending only on* ε, n, λ_1 *and* $\mathscr{M}(\beta\lambda_1)$ *such that for* $k > N_0$

$$\left| \frac{\lambda_k}{k^{2/n}} - \frac{c_n}{(\mathrm{vol}(D))^{2/n}} \right| < \varepsilon$$

where $\beta > 8\pi^{-2}n(n+4)$ *and* $\mathscr{M}(\beta\lambda_1) = \max\{j : \lambda_j \leq \beta\lambda_1\}$.

Notice that $\mathscr{M}(\lambda_1\beta)$ is determined by a finite part of the spectrum. It is worth pointing out that $\mathscr{M}(\lambda_1\beta)$ really occurs in the proof as part of an upper bound for the out-radius:

$$(1.2) \qquad\qquad R_{\mathrm{out}} \leq \frac{\pi\mathscr{M}(\lambda_1\beta)}{\sqrt{\lambda_1}}.$$

The derivation of this estimate uses properties of Euclidean space that fail in spaces of negative curvature. One can, however, give a result analogous to the above theorem for geodesically convex domains in hyperbolic space if one is willing to replace $\mathscr{M}(\lambda_1\beta)$ by some assumed crude upper bound on the out-radius [L].

We employ heat kernel remainder estimates to extend Theorem 1.1 for domains in manifolds of variable curvature. The authors of the present paper have obtained some results of this type [D-L]. In [D-L], as in [L] and Theorems 2.3 and 2.13 below, one must in general assume an a priori estimate on the out-radius as part of the data. In case the ambient manifold is compact this is not so much to ask since it is more acceptable to assume something about the ambient manifold and we can replace the out-radius bound by a bound on the diameter of the ambient manifold. The assumed out-radius bound provides an upper bound for the surface area and volume which is needed to obtain the results in [D-L]. In [D-L] the authors give a way of estimating $\mathrm{vol}\, D$

up to arbitrary accuracy from a finite part of the spectrum of Δ on D. As in Theorem 1.1 above, D is required to satisfy an appropriate convexity condition. Whereas the results of [D-L] deal only with the volume of the domain, in the present paper we obtain analogous results for the volume of the boundary. In particular, Theorem 2.13 below shows that one can estimate $\mathrm{vol}\,\partial D$ up to arbitrary accuracy from a finite part of the Dirichlet spectrum of Δ on D. However in Theorem 2.13 we do not need convexity. This is true for most results of §2. In [D-L] we must assume some knowledge of both D and the geometry of the ambient manifold M. The situation for Theorem 2.13 below is similar, and in particular, we need a lower bound for the distance from ∂D to its cut-locus. In other words, we need to add to our data a lower bound for $c_{\partial D}$, where $c_{\partial D} > 0$ is the supremum of all numbers such that the normal exponential map $\exp_{\partial D}^{\perp} : \nu(\partial D) \to M$ is a diffeomorphism on $\Gamma = \{\nu \in \nu(\partial D) : |\nu| < c_{\partial D}\}$. Once a positive lower bound for $c_{\partial D}$ is allowed as data, the out-radius estimate can be eliminated. This is shown in Theorems 2.16 and 2.17 below. In fact, we will show in Theorem 2.15 that, in this case, an upper bound for the surface area and volume of D can be determined from a finite part of the spectrum of D. In Theorem 2.14 an upper bound for the out-radius itself is given.

Finally, it is desirable to have a way of obtaining the bounds on $c_{\partial D}$ in terms of boundary curvature and the geometry of M. This is also motivated by the fact that it is boundary curvature that plays a role in the asymptotic expansion of the heat kernel for D. The estimation of the distance to the part of the cut locus of ∂D interior to D is of interest in itself and is the subject of the last half of the present paper. The following fact also provides motivation for the results of §3. Let D be a convex domain in $\mathbb{R}^n$ with ∂D having principal curvatures bounded above by $\frac{1}{R}$. If $x \in \partial D$ then there is a ball of radius R tangent to x and contained in D. This is a classical result of Blaschke [Bl]. We can view Blaschke's theorem in terms of the minimum distance from ∂D to its interior cut locus. In these terms the result says that for convex domains in $\mathbb{R}^n$ a positive lower bound for this distance can be given once we have appropriate bounds on the curvature of the boundary ∂D. That things are not so simple for boundary convex domains in a general manifold is clear once we consider the domain on an infinite cylinder $S^1 \times R^1$ bounded by two geodesic circles a distance δ apart. Although the curvature of the boundary of such a domain is zero, we cannot guarantee that it will contain a ball of a given size. There even exist counterexamples

in spaces of negative curvature where the domain is contractable. The final section of this paper gives results that show how far the result of Blaschke quoted above can be generalized to domains in manifolds. That there must be exceptional cases is made clear by the example of a ring on a cylinder given above. For domains in manifolds with nonnegative sectional curvature the exceptional cases are shown to have a simple structure not unlike the ring example given here.

Some of the results of this paper and those of [D-L] require that some kind of convexity condition on D holds. We give the relevant definitions. Given a submanifold N of codimension 1 in M, the shape operator with respect to a (local) unit normal field ν on N is defined by

$$S_\nu^N(u) \equiv \nabla_u \nu$$

for $u \in TN$. We need only have ν smoothly defined along a curve with tangent u. We will be concerned mainly with the case for which N is ∂D or an inner parallel set for a portion of ∂D. Let ν_{in} denote the inward normal field to ∂D.

DEFINITION. We will call a domain D in a Riemannian manifold *boundary-convex* if $-\langle S_{\nu_{\text{in}}}^{\partial D}(u), u \rangle \geq 0$ for all $u \in T\partial D$. We write $0 \leq -S_{\nu_{\text{in}}}^{\partial D}$ to express this.

We use this terminology to distinguish this type of convexity from the idea of geodesically convex.

DEFINITION. A domain D in a Riemannian manifold is called *geodesically convex* if every two of its points are connected by a minimizing geodesic contained completely within D. If the minimizing geodesic is unique we call the set *strongly convex*.

DEFINITION. D is *weakly convex* (in the sense of [D-L]) if whenever two points in ∂D are connected by a unique minimizing geodesic segment then that segment is contained in the closure of D.

For domains with smooth boundary it is true that all strongly convex domains are geodesically convex, all geodesically convex domains are weakly convex, and all weakly convex domains are boundary convex.

2. Eigenvalues, the heat kernel, and geometry. If M is noncompact, then assume $D \subset B(p, R)$, a geodesic ball of fixed radius R. For convenience of notation, set $R = \operatorname{diam} M + 1$, when M is compact. Thus R is an upper bound for the out-radius of D. Finally, we

assume that we are given a number ϱ such that $\exp^{\perp}_{\partial D} : \nu(\partial D) \to M$ is a diffeomorphism in a tube $\Gamma = \{v \in \nu(\partial D) \,|\, \|v\| < \varrho\}$. This means that ϱ is a lower bound for $c_{\partial D}$.

Let ϱ_1 be such that $(r, p) \mapsto \exp^{\partial D}_{\perp} r\nu_{\text{in}}(p)$ is a diffeomorphism on $(0, \varrho_1) \times \partial D$. Thus ϱ_1 gives the size of an inner tube. Similarly let ϱ_2 give the width of an outer tube. For weakly convex domains such a ϱ_2 can be determined solely from the geometry of M. This is shown in [D-L]. We may as well assume that $\varrho = \min\{\varrho_1, \varrho_2\}$. In all that follows we will assume that M has positive injectivity radius i_M. It should also be noted that when i_M appears below it may be replaced by simply a positive lower bound for i_M.

REMARK. Most of the results in this section involve ϱ_1 as given in the hypotheses. However, for suitably convex domains in manifolds of nonnegative curvature, ϱ_1 can be estimated in terms of the geometry of the ambient manifold and an upper bound on the principal curvatures of the boundary of the domain. This is done in §3. For this reason each of the main results of this section gives rise to a corresponding result for convex domains where the reference to ϱ_1 is dropped from the hypothesis while assumptions on the boundary curvature are included. Actually, for a certain class of exceptional domains, which we identify in §3, we also need to know λ_1 to get the needed estimate on ϱ_1. This is clearly not a drawback when applied to the results of this section since we use a finite number of eigenvalues anyway. The way in which λ_1 comes in to play is explained near the end of §3.

LEMMA 2.1. (i) *Let $D \subset M$ have a proper inner tube of width ϱ_1 as described above. Then we have $h \leq S^{\partial D}_{\nu_{\text{in}}}$ where h depends only on n, ϱ_1, i_M, and a lower bound $-a$ for the sectional curvature of M. We also have $\overline{h} \leq \text{trace} \, S^{\partial D}_{\nu_{\text{in}}}$ where $\overline{h}$ depends only on n, ϱ_1, i_M, and a lower bound $-(n-1)a$ for the Ricci curvature of M.*

(ii) *Let $D \subset M$ have a proper outer tube of width ϱ_2 as described above. Then we have $H \geq S^{\partial D}_{\nu_{\text{in}}}$ where H depends only on n, ϱ_2, i_M, and a lower bound $-a$ for the sectional curvature of M. We also have $\overline{H} \geq \text{trace} \, S^{\partial D}_{\nu_{\text{in}}}$ where $\overline{H}$ depends only on n, ϱ_2, i_M, and a lower bound $-(n-1)a$ for the Ricci curvature of M.*

Proof. (i) First assume $\varrho_1 \leq i_M$. There is a ball B of radius $\varrho_1/2$ contained inside D with center a distance $\varrho_1/2$ from a given point

$p \in \partial D$. Now, ∂B is smooth at p and tangent to ∂D at p and thus

$$S_{\overline{\nu}_{\text{in}}}^{\partial B}(p) \le S_{\nu_{\text{in}}}^{\partial D}(p),$$

where $\overline{\nu}_{\text{in}}$ is the inward normal for ∂B. If $-a$ is a lower bound for the sectional curvature then by comparing with a ball of radius $\varrho_1/2$ in a hyperbolic space of constant curvature $-a$ (or a Euclidean space if $a = 0$) we have

$$h \le S_{\overline{\nu}_{\text{in}}}^{\partial B},$$

where h depends on ϱ_1, n, and a. If we are only given a lower bound on the Ricci curvature we can only conclude that

$$\overline{h} \le \text{trace } S_{\overline{\nu}_{\text{in}}}^{\partial B}.$$

The first result follows. For the general case we just replace ϱ_1 by $\min\{\varrho_1, i_M\}$.

(ii) The proof of (ii) is analogous.

LEMMA 2.2. *Let D be a domain with smooth boundary in M.*

(i) *Let $-a(n-1)$ be a lower bound on the Ricci curvature of M where $a \ge 0$. Then $\text{vol } D \le c_1$ where c_1 depends only on n, a, and R.*

(ii) *Let $-a$ be a lower bound on the sectional curvature of M and b an upper bound for the sectional curvature of M, where $a, b \ge 0$. Then $\text{vol } \partial D \le c_2$ where c_2 depends only on n, R, a, b, i_M, and either ϱ_1 or ϱ_2.*

Proof. The proof of (i) is immediate since $\text{vol } D \le \text{vol } B(p, R+1) \le \text{vol } B_{-a}(R+1)$, where $B_{-a}(R+1)$ is a ball of radius $R+1$ in the simply connected hyperbolic space of constant curvature $-a$ if $-a < 0$, and in $\mathbb{R}^n$ if $a = 0$. This last inequality is Bishop's volume comparison for balls.

For the proof of (ii) we first consider the case where we are given ϱ_1. Let Γ_1 denote the inner proper tube of width ϱ_1 on which $\exp_{\perp}^{\partial D}$ is a diffeomorphism. By the comparison theory of [H-K] or [W] we can find a c depending only on n, ϱ_1, b, and a lower bound for $S_{\nu_{\text{in}}}^{\partial D}$ such that

$$\text{vol}(\partial D) \le c \text{ vol } \Gamma_1.$$

But by Lemma 2.1(i) we can take c to depend only on n, ϱ_1, i_M, b, and a. Now $\text{vol } \Gamma_1 \le \text{vol } B(p, R+1)$ for some $p \in D$ and $\text{vol } B(p, R+1) \le \text{vol } B_{-a}(R+1)$ where $B_{-a}(R+1)$ is a ball of radius

$R + 1$ in the simply connected hyperbolic space of constant curvature $-a$ if $-a < 0$ and in $\mathbb{R}^n$ if $a = 0$. The result in the case where we are given ϱ_1 follows. The case where we are given ϱ_2 is proved similarly using an outer tube and Lemma 2.1(ii).

THEOREM 2.3. *Let M be a (possibly noncompact) complete Riemannian manifold with sectional curvature K and positive injectivity radius i_M. Let D be a bounded domain with smooth compact boundary. Suppose also that $-a \leq K \leq b$, where $a, b \geq 0$. Then given $0 < \delta < 1$ there is a number $N(\delta)$ such that*

$$\left| (4\pi\delta)^{n/2} \sum_{i=1}^{N(\delta)} e^{-\lambda_i \delta} - \mathrm{vol}(D) \right| \leq c\delta^{1/2},$$

where $N(\delta)$ and c depend only on n, ϱ_2, a, b, i_M, and R.

Proof. Apply Theorem 3.5 of [D-L] . This reads

$$(2.4) \qquad \left| (4\pi t)^{n/2} \, \mathrm{Tr} \, E_D(t) - \mathrm{vol} \, D \right|$$

$$\leq \alpha_1 \left(\int_0^\infty e^{\alpha_2 s} e^{-\alpha_3 s^2 / at} ds \right) \mathrm{vol}(\partial D) C$$

$$+ \alpha_4 (a + b) t \, \mathrm{vol}(D) + dt \, \mathrm{vol}(D) t^{n/2},$$

where E_D is the heat kernel for D with Dirichlet boundary conditions. Also, α_1, α_2, α_3 and α_4 are universal constants and d depends only on b and the injectivity radius i_M of M and C depends on a and some upper bound for the mean curvature trace $S_{\nu_{\mathrm{in}}}^{\partial D}$. But, by Lemma 2.1(ii), this upper bound can be given in terms of n, a, ϱ_2, and i_M. By Lemma 2.2 we see that the right-hand side of (2.4) is less than $C_3 t^{1/2}$ for $0 < t < 1$, where C_3 depends only on n, a, b, i_M, ϱ_2, and R. Now, by a well known result [D-Li], there exist constants m and C_5 so that $\lambda_i(D) \geq C_5 i^{2/n}$ for $i \geq m$ and where m and C_5 depend only on n, a, b, i_M, and R.

Now choose $N(\delta) \geq m$ so that

$$\sum_{i=N(\delta)+1}^{\infty} e^{-\lambda_i \delta} \leq \sum_{i=N(\delta)+1}^{\infty} e^{-C_5 i^{2/n} \delta} < C_6 \delta^{1/2}.$$

Then we have

$$\left| (4\pi\delta)^{n/2} \sum_{i=1}^{N(\delta)} e^{-\lambda_i\delta} - \text{vol}(D) \right|$$

$$\leq C_6\delta^{1/2} + \left| (4\pi\delta)^{n/2}\, \text{Tr}\ E_D(\delta) - \text{vol}(D) \right|$$

$$\leq (C_6 + C_3)\,\delta^{1/2}.$$

Theorem 2.3 above should be compared to Theorem 4.3 of [D-L]. The difference is that in Theorem 2.3 we do not need to have D weakly convex and we need only the injectivity radius as opposed to the convexity radius of M. However, we do need to know the bound ϱ_2 which was not needed in [D-L].

Next we study the role of the boundary. In all statements below, c_i denote constants depending upon just R, ϱ, and the geometry of M unless otherwise stated. By the geometry of M we mean dim $M = n$, an upper bound b and a lower bound $-a$ on the sectional curvature of M, and a lower bound on its injectivity radius. Here we take $a, b \geq 0$. We may suppose that $\varrho \leq 1$ and that ϱ is less than the injectivity radius of $B(p, R+2)$. Let A be a domain with smooth boundary satisfying $B(p, R+2) \subseteq A \subseteq B(p, R+3)$. Consider a point $x \in D$ of distance $d(x) \leq \varrho/2$ from ∂D. The point, in ∂D, which is closest to x will be denoted as y. There are balls B and B_1, each of radius $\varrho/2$, which are respectively internally and externally tangent, at y, to ∂D. Let $C = A - B_1$. We have that $B \subset D \subset C$. Consequently, Duhamel's principle gives the monotonicity of heat kernels:

$$(2.5) \qquad E_B(x, x, t) \leq E_D(x, x, t) \leq E_C(x, x, t).$$

It follows from the standard parametrix construction [M-S, R-S] of the heat kernel, carried out in harmonic coordinates, that

$$(2.6) \qquad \max\left\{ \left| E_B(x, x, t) - (4\pi t)^{-n/2}(1 - e^{-d^2(x)/t}) \right|, \right.$$

$$\left. \left| E_C(x, x, t) - (4\pi t)^{-n/2}(1 - e^{-d^2(x)/t}) \right| \right\}$$

$$\leq c_4 e^{-d^2(x)/4t} t^{-n/2+1/2} + c_5 t^{-n/2+1} + c_6 t.$$

Combining (2.5) and (2.6), we get, for $d(x) \leq \varrho/2$,

$$(2.7) \qquad \left| E_D(x, x, t) - (4\pi t)^{-n/2}(1 - e^{-d^2(x)/t}) \right|$$

$$\leq c_4 e^{-d^2(x)/4t} t^{-n/2+1/2} + c_5 t^{-n/2+1} + c_6 t.$$

If $d(x) \geq \varrho/2$, then from [D-L], one has

$$(2.8) \qquad |E_D(x, x, t) - (4\pi t)^{-n/2}| \leq c_7 t^{-n/2+1} + c_8 t.$$

It is now easy to derive a remainder estimate for E_D. If $M = R^n$, this was proved earlier by van den Berg [B]. One has

THEOREM 2.9.
$$\left| \text{Tr } E_D(t) - \frac{\text{vol } D}{(4\pi t)^{n/2}} + \frac{\text{vol } \partial D}{4(4\pi t)^{(n-1)/2}} \right| \leq c_9 t^{-n/2+1} + c_{10} t.$$

Proof. One computes the trace of the heat kernel by integrating its restriction to the diagonal:

$$\text{Tr } E_D(t) = \int_D E_D(x, x, t) \, dx.$$

Let W denote the set where $d(x) \leq \varrho/2$. Then

$$(2.10) \qquad \text{Tr } E_D(t) = \int_W E_D(x, x, t) \, dx + \int_{D-W} E_D(x, x, t) \, dx.$$

Using (2.8) and Lemma 2.2, we may write

$$(2.11) \qquad \left| \int_{D-W} E_D(x, x, t) - (4\pi t)^{-n/2} \text{vol}(D - W) \right|$$
$$\leq c_{11} t^{-n/2+1} + c_{12} t.$$

Similarly from (2.7) and Lemma 2.2,

$$\left| \int_W E_D(x, x, t) - (4\pi t)^{-n/2} \text{vol } W + (4\pi t)^{-n/2} \int_W e^{-d^2(x)/t} \right|$$
$$\leq c_{13} t^{-n/2+1} + c_{14} t.$$

Comparison theory, for Jacobi fields along ∂D, [H-K], and the upper bound of $\text{vol}(\partial D)$, were employed in the most recent estimate.

Invoking comparison principles again, along with the elementary formula, [S], $\int_0^\infty e^{-s^2} ds = \sqrt{\pi}/2$, gives

$$(2.12) \qquad \left| \int_W E_D(x, x, t) - (4\pi t)^{-n/2} \text{vol } W + \frac{\text{vol } \partial D}{4(4\pi t)^{n-1/2}} \right|$$
$$\leq c_{17} t^{-n/2+1} + c_{18} t.$$

Theorem 2.9 follows by combining (2.10), (2.11), and (2.12).

One may apply Theorem 2.9 to inverse spectral theory. In Theorem 2.3 we determined $\text{vol } D$ up to arbitrary accuracy from a finite part of the spectrum of Δ. It is now possible to derive a similar result for

vol ∂D:

THEOREM 2.13. *Let D be a bounded domain with smooth boundary in M. Let $-a$ be a lower bound on the sectional curvature of M and b an upper bound for the sectional curvature of M, where $a, b \geq 0$. For any given $\varepsilon > 0$, there exists a constant c_{19}, depending upon ε, so that:* $\mathrm{vol}(\partial D)$ *is determined, up to error of order ε, by those λ_i with $i \leq c_{19}$. The number c_{19} depends only on $\varepsilon, a, b, i_M, R$, and ϱ.*

Proof. From Theorem 2.9, we may choose δ suitably small, so that

$$\left| 4(4\pi\delta)^{\frac{n-1}{2}} \mathrm{Tr}\ E_D(\delta) - \frac{2\ \mathrm{vol}\, D}{(\pi\delta)^{1/2}} + \mathrm{vol}\, \partial D \right| \leq c_{20}\delta^{1/2} < \varepsilon/3.$$

Fix such a choice of δ.

By Theorem 2.3, knowledge of the λ_i, for $i \leq c_{21}$, determines $\mathrm{vol}\, D$ up to error $\frac{1}{2}(\pi\delta)^{\frac{1}{2}}\varepsilon/3$. Let $\mathrm{vol}_\varepsilon D$ be such an approximate value for $\mathrm{vol}\, D$. Then

$$\left| 4(4\pi\delta)^{\frac{n-1}{2}} \mathrm{Tr}\ E_D(\delta) - \frac{2\ \mathrm{vol}_\varepsilon D}{(\pi\delta)^{1/2}} + \mathrm{vol}\, \partial D \right| \leq 2\varepsilon/3.$$

Recall that $\mathrm{Tr}\ E_D(\delta) = \sum_{i=1}^{\infty} e^{-\delta\lambda_i}$. As shown in the proof of Theorem 2.3, we can choose c_{22} so that

$$4(4\pi\delta)^{\frac{n-1}{2}} \sum_{i=c_{22}}^{\infty} e^{-\delta\lambda_i} \leq \varepsilon/3.$$

Combining these ingredients gives for $c_{19} \geq \max(c_{21}, c_{22})$:

$$\left| 4(4\pi\delta)^{\frac{n-1}{2}} \sum_{i=1}^{c_{19}} e^{-\delta\lambda_i} - \frac{2\ \mathrm{vol}_\varepsilon D}{(\pi\delta)^{1/2}} + \mathrm{vol}\, \partial D \right| \leq \varepsilon.$$

Since $\mathrm{vol}_\varepsilon D$ is determined by the first c_{21} eigenvalues, the proof of Theorem 2.13 is complete.

We may estimate the out-radius for geodesically convex domains. Let $\mathcal{M}(\cdot)$ be the eigenvalue counting function as in Theorem 1.1.

THEOREM 2.14. *Let $D \subset M$ be a geodesically convex domain with smooth boundary. Assume that the Ricci curvature of M satisfies $-a(n-1)|X|^2 \leq \mathrm{Ric}(X, X)$, $a \geq 0$, for all $X \in TM$. Let ϱ_1 be given as above. Then*

$$R_{\mathrm{out}} \leq 4\varrho_1\left(\mathcal{M}(\beta(n, a, \varrho_1) + 1) \right)$$

where $\beta(n, a, \varrho_1)$ depends only on a, ϱ_1, and $\dim M = n$.

Proof. Let B_0 be a ball of radius ϱ_1 with center p_0, for $p_0 \in D$. We take p_0 to be a distance ϱ_1 from ∂D and then it is not hard to see from the definition of ϱ_1 that $B_0 \subset D$. Let q be the furthest point from p_0 in $\overline{D}$. We have that $R_{\text{out}} \leq \text{dist}(p_0, q)$, where dist denotes the geodesic distance. Let $\gamma: [0, 1] \rightarrow \overline{D}$ be a minimizing geodesic from $p_0 = \gamma(0)$ to $q = \gamma(1)$. Let $k \geq 0$ be the largest integer such that there exist $t_0, t_1, \dots, t_k$ with $0 = t_0 \leq t_1 < t_2 < \cdots < t_k \leq 1$ and with $\text{dist}(\gamma(t_i), \gamma(t_{i+1})) = 4\varrho_1$.

We have

$$k \geq \left\lceil \frac{\text{dist}(p_0, q)}{4\varrho_1} \right\rceil \geq \frac{R_{\text{out}}}{4\varrho_1} - 1.$$

Now if $\text{dist}(\gamma(t_i), \partial D) \geq \varrho_1$ then let $B_i \equiv B(\gamma(t_i), \varrho_1)$. If $\text{dist}(\gamma(t_i), \partial D) < \varrho_1$, let x_i be the closest point of ∂D to $\gamma(t_i)$ and let B_i be the ball of radius ϱ_1 centered at $\exp_{\perp}^{\partial D} \varrho_1 \nu_{\text{in}}(x_i)$. We have that $\gamma(t_i) = \exp_{\perp}^{\partial D} r\nu_{\text{in}}(x_i)$ for some $0 \leq r < \varrho_1$ and hence the distance from $\gamma(t_i)$ to the center of B_i is less than ϱ_1. The triangle inequality gives that $B_i \cap B_j = \varnothing$ for $i \neq j$. Also $B_i \subset D$ for $0 \leq i \leq k$.

Let $\mathbb{H}^n_{-a}$ be the simply connected space of constant curvature $-a \leq 0$ and dimension n, let $B_{-a}(r)$ be a geodesic ball of radius r in that space, and $\lambda_1(B_{-a}(r))$ its first Dirichlet eigenvalue. Choose

$$\beta(n, a, \varrho_1) \geq \lambda_1(B_{-a}(\varrho_1)).$$

Note that $\beta(n, a, \varrho_1)$ may be chosen explicitly by using the upper bounds for $\lambda_1(B_{-a}(r))$ given for example in [Ga]. In particular if $a = 0$ we may let $\beta = n(n + 4)/2\varrho_1^2$ as in [T]. We now show that $k \leq \mathscr{M}(\beta(n, a, \varrho_1))$. The Poincaré minimum principle asserts that

$$\lambda_k(D) = \inf \frac{\int_D |df|^2}{\int_D f^2},$$

where the infimum is taken over all $f \in H_0^1$ such that f is L^2-orthogonal to the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_{k-1}$. Let f^i be the first eigenfunction of B_i. We can choose constants a_i so that $\sum a_i f^i$ satisfies the orthogonality conditions needed for the minimum principle. Then we have

$$\lambda_k(D) \leq \frac{\int_D |df|^2}{\int_D f^2} \leq \frac{\sum a_i^2 \int_{B_i} |df^i|^2}{\sum a_i^2 \int_{B_i} (f^i)^2}$$

$$\leq \sup_i \frac{\int_{B_i} |df^i|^2}{\int_{B_i} (f^i)^2} \leq \sup_i \lambda_1(B_i) \leq \lambda_1(B_{-a}(\varrho_1)),$$

where we have used Cheng's eigenvalue comparison [C]. Thus, by the definition of $\mathcal{M}$, we have $k \leq \mathcal{M}(\beta(n, a, \varrho_1))$. Hence, $\frac{R_{\text{out}}}{4\varrho_1} - 1 \leq \mathcal{M}(\beta(n, a, \varrho_1))$ which implies the result.

The following theorem should be compared to Lemma 2.2. Both results give upper bounds for $\text{vol}\,\partial D$ and $\text{vol}\,D$. We point out some of the differences. First, the constants of Lemma 2.2 depend on R, and ϱ_1 or ϱ_2 while in Theorem 2.15 we need only ϱ_1. Secondly, we point out that the upper bounds below involve the counting function $\mathcal{M}$ and thus a finite part of the spectrum becomes involved.

THEOREM 2.15. *Let D be a domain with compact smooth boundary in a complete Riemannian manifold M. Assume that the sectional curvature satisfies $-a \leq K \leq b$, where $a, b \geq 0$. In particular, D might not be convex. If ϱ_1 is a lower bound on the distance from D to the interior part of the cut locus of ∂D as before then*

$$\text{vol}(\partial D) \leq C(n, \varrho_1, i_M, a, b)\mathcal{M}(\beta_1(n, a, \varrho_1)).$$

If we weaken the lower bound on sectional curvature to a lower bound on Ricci curvature, $-a(n-1) \leq \text{Ric}$, then

$$\text{vol}(D) \leq C_1(n, \varrho_1, a)\mathcal{M}(\beta_1(n, a, \varrho_1)).$$

The constants depend only on the indicated quantities. Also, C above may be taken to depend on a lower bound for $S_{\nu_{\text{in}}}^{\partial D}$ instead of i_M and in this case we can take the lower bound on curvature to be the Ricci curvature bound $-a(n-1) \leq \text{Ric}$.

Proof. Let Γ be the region inside $\overline{D}$ given by

$$\Gamma = \left\{ \exp_{\perp}^{\partial D}(r\nu_{\text{in}}(p)) : p \in \partial D, \ 0 \leq r \leq \frac{\varrho_1}{2} \right\}.$$

Now, fit as many balls $B_i(\varrho_1/4)$ inside Γ as possible without intersecting interiors and with centers a distance $\varrho_1/4$ from ∂D. Let this number be N. Define ρ on $\nu(\partial D)$ by $\rho(\xi) = \langle \xi, \xi \rangle^{1/2}$ for $\xi \in \nu(\partial D)$. Now using the fact that for points inside Γ the distance to the boundary is given by $\rho \circ \exp_{\perp}^{-1}$, and using the triangle inequality, we see easily that a ball of radius $\varrho_1/4$ centered at a point in D at a distance $\varrho_1/4$ from the boundary ∂D is contained completely inside Γ. It follows from the maximality of N that every point of Γ is at most $3\varrho_1/4$ from the center of some B_i. Thus, by Bishop's

volume comparison for balls,

$$\mathrm{vol}(\Gamma) \le N \,\mathrm{vol}\, B_{-a}\left(\frac{3\varrho_1}{4}\right).$$

Note that so far we only need the lower bound on Ricci curvature. Now by the comparison theory of [H-K] again we can find a c depending only on n, ϱ_1, b, and a lower bound for $S^{\partial D}_{\nu_{\mathrm{in}}}$ such that

$$\mathrm{vol}(\partial D) \le c \,\mathrm{vol}\,\Gamma.$$

But now if $-a$ is taken as a lower bound on sectional curvature then by Lemma 2.1 we can take c to depend only on n, ϱ_1, i_M, and a. Hence,

$$\mathrm{vol}(\partial D) \le Nc \,\mathrm{vol}\, B_{-a}\left(\frac{3\varrho_1}{4}\right).$$

Let $\beta_1(n, a, \varrho_1) \ge \lambda_1(B_{-a}(\varrho_1/4))$. As in the proof of Theorem 2.14 we can use the Poincaré minimum principle to show that

$$\lambda_N(D) \le \lambda_1\left(B_{-a}\left(\frac{\varrho_1}{4}\right)\right).$$

Thus, $N \le \mathscr{M}(\beta_1(n, a, \varrho_1))$. Finally, the above inequalities combine to give

$$\mathrm{vol}(\partial D) \le C(n, \varrho_1, i_M, a, b)\mathscr{M}(\beta_1(n, a, \varrho_1)).$$

Now we derive the volume inequality. We only need the lower bound on Ricci curvature. In addition to the N balls of radius $\varrho_1/4$ inside Γ, fit as many disjoint balls B'_j of radius $\varrho_1/4$ inside $D-\Gamma$ as possible. Call this number N'. Every point of $D - \Gamma$ must be either at a distance of not greater than $\varrho_1/2$ from the center of one of the B'_j or at a distance of not greater than $\varrho_1/4$ from Γ since otherwise we could fit another ball of radius $\varrho_1/4$ into $D - \Gamma$. Since a point that has distance less than or equal to $\varrho_1/4$ of Γ is within a distance ϱ_1 of the center one of the $B_i \subset \Gamma$, we see that the $N + N'$ balls say $\widetilde{B}_i$ and $\widetilde{B}'_i$, with centers coinciding with those of the B_i and B'_j but with radii ϱ_1, cover D. Hence, we have

$$\mathrm{vol}(D) \le (N + N')\sup_{i,j}\{\mathrm{vol}(\widetilde{B}_i), \mathrm{vol}(\widetilde{B}'_j)\} \le (N + N')\,\mathrm{vol}(B_{-a}(\varrho_1)).$$

Now, arguing with the minimum principle as before, we easily obtain

$$\lambda_{N+N'}(D) \le \lambda_1\left(B_{-a}\left(\frac{\varrho_1}{4}\right)\right).$$

Thus, as before, we can conclude that

$$N + N' \le \mathscr{M}(\beta_1(n, a, \varrho_1))$$

and the above combine to give the second inequality of the theorem. To get the final statement we simply forego the use of Lemma 2.1.

Theorems 2.16 and 2.17 below should be compared to Theorems 2.3 and 2.13. Notice that the out-radius bound R makes no appearance in the results which follow.

THEOREM 2.16. *Let M be a (possibly noncompact) complete Riemannian manifold with sectional curvature K and positive injectivity radius i_M. Let D be a bounded connected domain with smooth boundary. Suppose also that $-a \leq K \leq b$, where $a, b \geq 0$. Then given $0 < \delta < 1$ there is a number $N(\delta)$ such that*

$$\left| (4\pi\delta)^{n/2} \sum_{i=1}^{N(\delta)} e^{-\lambda_i \delta} - \text{vol}(D) \right| \leq c\delta^{1/2}$$

where $N(\delta)$ and c depend only on n, ϱ, a, b, i_M and $\mathscr{M}(\beta_1(n, a, \varrho_1))$.

Also, if we are given an upper bound h_2 for the mean curvature trace $S_{\nu_{in}}^{\partial D}$ then we may replace ϱ by ϱ_1 and h_2.

Proof. The proof is like that of Theorem 2.3 except that we use Theorem 2.15 instead of Lemma 2.2. To get the final statement we simply forego the use of Lemma 2.1(ii).

THEOREM 2.17. *Let D be a bounded domain with smooth boundary in M. Let $-a$ be a lower bound on the sectional curvature of M and b an upper bound for the sectional curvature of M where $a, b \geq 0$. For any given $\varepsilon > 0$, there exists a constant C_{19}, depending upon ε, so that: $\text{vol}(\partial D)$ is determined, up to error of order ε, by those λ_i with $i \leq C_{19}$. The constant C_{19} depends only on $\varepsilon, n, \varrho, a, b, i_M$, and $\mathscr{M}(\beta_1(n, a, \varrho_1))$.*

Proof. The proof follows the proof of Theorem 2.13 except that we use Theorem 2.15 instead of Lemma 2.2. Also Theorem 2.16 is cited to replace Theorem 2.3.

3. ϱ_1 and the geometry of convex domains. We now show how to estimate ϱ_1 in terms of the geometry of M and the boundary curvature of D. We have already used the submanifold comparison theory found for example in [H-K], [G] and [W]. In what follows we will need a part of these results which we now explain. Let D

be a domain in a Riemannian manifold M of dimension n. We assume throughout that ∂D is smooth and compact. Let K denote the sectional curvature of M. If N is any regular submanifold of M, let $\pi\colon \nu(N) \to N$ denote the associated normal bundle. Given a curve $\xi\colon (-\varepsilon, \varepsilon) \to \nu(N)$ with $\|\xi(t)\| = 1$ for all $t \in (-\varepsilon, \varepsilon)$ and letting $\gamma = \pi \circ \xi$, $\gamma(0) = p$, $\gamma'(0) = u \in T_p N$ we can define an N-Jacobi field by:

$$Y(s) = \frac{d}{dt}\, \exp_\perp^N s\xi(t)\big|_{t=0}$$

or

$$Y(s) = d\, \exp_\perp^N U(s)$$

where

$$U(s) = \frac{d}{dt} s\xi(t)\big|_{t=0}.$$

A short calculation gives

$$\frac{d}{ds}\, \log \|Y(s)\|\big|_{s=0} = \frac{\langle S_\xi^N(u),\, u\rangle}{\|u\|^2}$$

where $\xi = \xi(0)$. Let ν be a unit normal field for N. In case $N = \partial D$ we agree that $\nu = \nu_{\mathrm{in}}$ and for a point in question $p \in \partial D$ and $u \in T_p \partial D$ that ξ be defined by $\xi = \nu_{\mathrm{in}} \circ \gamma$ where γ is some curve in ∂D with $\gamma(0) = p$ and $\gamma'(0) = u$. Of course, the N-Jacobi fields $Y(s)$ depend on the choice of p and u but this will remain tacit for notational brevity.

Let $s_1, s_2, \ldots, s_{n-1}$ denote the eigenvalues of S_ν^N. Now, according to the comparison theory of [W] or [H-K] if we have another situation where we are given a manifold $\overline{M}$ with sectional curvature $\overline{K}$, a submanifold $\overline{N}$, and associated $\overline{U}(s)$, $\overline{Y}(s)$, $\overline{s}_i$ etc. then we have the following:

I. If $\inf_M K \geq \overline{K}$ and $\max_i\{s_i\} \leq \min_i\{\overline{s}_i\}$ then

(3.1)
$$\frac{\|Y(s)\|}{\|U(s)\|} \leq \frac{\|\overline{Y}(s)\|}{\|\overline{U}(s)\|}$$

and

(3.2)
$$\frac{d}{ds}\, \log \|Y(s)\| \leq \frac{d}{ds}\, \log \|\overline{Y}(s)\|$$

for all nonnegative s not greater than the first focal distance of N along the curve $s \mapsto \exp_\perp^N s\xi(0)$.

II. If $\overline{K} \geq \sup_M K$ and $\max_i\{\overline{s}_i\} \leq \min_i\{s_i\}$ then

(3.3)
$$\frac{\|Y(s)\|}{\|U(s)\|} \geq \frac{\|\overline{Y}(s)\|}{\|\overline{U}(s)\|}$$

and

(3.4) $$\frac{d}{ds} \log \|Y(s)\| \geq \frac{d}{ds} \log \|\overline{Y}(s)\|$$

for all nonnegative s not greater than the first focal distance of $\overline{N}$ along the curve $s \mapsto \exp_{\perp}^{\overline{N}} s\overline{\xi}(0)$.

Now, if $\exp_{\perp}^{N}$ is a diffeomorphism from a neighborhood $\widetilde{\mathscr{U}}$ in $\nu(N)$ onto $\mathscr{U} \in \mathscr{M}$ then for $\alpha \geq 0$ define

$$N_\alpha \equiv \exp_{\perp}^{N}(\{\alpha\nu(x) : x \in N\} \cap \widetilde{\mathscr{U}}).$$

This is a regular submanifold of codimension 1 whenever it is nonempty. On $\mathscr{U}$ define the function $r \equiv \rho \circ \exp_{\perp}^{N}|_{\widetilde{\mathscr{U}}}^{-1}$. Here N_α is just the $r = \alpha$ level submanifold. The associated vector field on $\mathscr{U}$ is $\partial_r = \text{grad } r$. For nonempty N_α we have that $\partial_r|_{N_\alpha}$ is a unit normal field for N_α. In this case, we also have

$$\frac{d}{ds} \log \|Y(s)\| \Big|_{s=\alpha} = \frac{\langle S_{\partial_r}^{N_\alpha} v, v \rangle}{\|v\|^2},$$

where

$$v = Y(\alpha).$$

LEMMA 3.5. *Let $\widetilde{\mathscr{U}}$ be a connected open set in $\nu(N)$ on which $\exp_{\perp}^{N}$ is a diffeomorphism onto $\mathscr{U}$. Let $N_\alpha \subset \mathscr{U}$ and ∂_r be as above. Suppose that N_α is nonempty and let L be a submanifold of codimension 1 in $\mathscr{U}$ tangent to N_α at q and with normal field ν such that $-\nu = \partial_r$ at q. Suppose also that L lies in the $r \geq \alpha$ part of $\exp_{\perp}^{N} \mathscr{U}$. Then we have the following.*
   A. If

$$S_{\partial_r}^{N_\alpha} < 0$$

then

$$-S_\nu^L < 0$$

at q.
   B. Assume that L is connected. Suppose that

$$S_{\partial_r}^{N_\alpha} \leq 0$$

and $r(p) > \alpha$ for some $p \in L$. Then there is a point $q_1 \in L$ such that

$$\langle -S_\nu^L w, w \rangle < 0$$

for some $w \in T_{q_1} L$.

Proof. For A we just take a normal coordinate system $\{x^i\}$ around q with $\frac{\partial}{\partial x^n} = \partial_r$ at q and $\frac{\partial}{\partial x^i}$ tangent to N_α at q for $1 \le i \le n-1$. Let γ_1 be a curve in N_α with $\gamma_1'(0) = u$ and γ_2 a curve in L with $\gamma_2'(0) = \gamma_1'(0) = u$. Now, since the Christoffel symbols vanish at p, we have

$$-\langle S_{\partial_r}^{N_\alpha} \gamma_1', \gamma_1' \rangle_p = \left\langle \frac{\partial}{\partial x^n}, \nabla_{\gamma_1'} \gamma_1' \right\rangle_p = \frac{d^2 x^n(\gamma_1(t))}{dt^2}\bigg|_{t=0}$$

and similarly for S_ν^L. Thus, the proof reduces to showing that

$$\frac{d^2 x^n(\gamma_1(t))}{dt^2}\bigg|_{t=0} > 0, \quad \frac{dx^n(\gamma_1(t))}{dt}\bigg|_{t=0} = \frac{dx^n(\gamma_2(t))}{dt}\bigg|_{t=0} = 0,$$

and $x^n(\gamma_2(t)) \ge x^n(\gamma_1(t))$ for small t imply

$$\frac{d^2 x^n(\gamma_2(t))}{dt^2}\bigg|_{t=0} > 0$$

which is elementary.

Now we prove B. We have $p \in L$ with $r(p) > \alpha$. Join p to q with a smooth curve in $\mathscr{U} \cap L$ starting at $q = \gamma(0)$. Let $q^* = \gamma(t_0)$ where $t_0 = \sup\{t : r(\gamma(t)) = \alpha\}$. Replacing q by q^* we then have that $r(\gamma(t)) > \alpha$ for all t such that $0 < t \le \delta$ and some small $\delta \le \frac{1}{2}$. Now replacing p by $\gamma(t)$ for some t close to t_0 if necessary we can join p to q^* by a unit speed geodesic $\overline{\gamma}$ in L (with respect to the induced metric on L) with $\overline{\gamma}(0) = q^*$. If we let $\overline{t}_0 = \sup\{t : r(\overline{\gamma}(t)) = \alpha\}$ and $q^{**} = \gamma(\overline{t}_0)$ then $r(\overline{\gamma}(t)) > \alpha$ for $t > \overline{t}_0$ and $r(q^{**}) = \alpha$. Thus, we see that, after making the above replacements if necessary and a reparameterization of γ, we may assume that p is connected to q by a unit speed geodesic in L with respect to the induced metric on L and with $\gamma(0) = q$, $\gamma(\delta) = p$ and $r(\gamma(t)) > \alpha$ for $0 < t \le \delta \le \frac{1}{2}$.

Now the hypothesis $S_{\partial_r}^{N_\alpha} \le 0$ means that if we choose coordinates $x^1, \ldots, x^{n-1}$ on N_α and set $x^n = r - \alpha$ then $x^1, \ldots, x^n$ give a coordinate system on $\mathscr{U}$ such that Hess $x^n \le 0$, when $x^n = 0$. Let $\varphi : \mathscr{U} \to V \subset \mathbb{R}^n$ denote this coordinate map. Now on a possibly smaller neighborhood $\mathscr{U}'$ of q we may express L as the graph of a function $f(x^1, \ldots, x^{n-1})$, i.e. as the zero set of $g = f - x^n$ in M. Let $f(t) = f(x^1 \circ \gamma(t), \ldots, x^{n-1} \circ \gamma(t))$. Then, since image $(\gamma) \subset L$,

we have $f(t) = x^n(\gamma(t))$. We have

$$f''(t) = \text{Hess } f\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\right)f$$

$$= \text{Hess } g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}f + \text{Hess } x^n\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right).$$

However, since γ is a geodesic in L,

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}f = \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}, \frac{\nabla g}{|\nabla g|}\right\rangle df(\nabla g/|\nabla g|)$$

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}f = \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}, \frac{\nabla g}{|\nabla g|}\right\rangle \frac{\langle dg, df\rangle}{|\nabla g|}$$

$$= -\left\langle \frac{\partial}{\partial t}, \nabla_{\frac{\partial}{\partial t}}\left(\frac{\nabla g}{|\nabla g|}\right)\right\rangle \frac{\langle dg, df\rangle}{|\nabla g|}$$

$$= -\left\langle \frac{\partial}{\partial t}, \nabla_{\frac{\partial}{\partial t}}\nabla g\right\rangle \frac{\langle dg, df\rangle}{|\nabla g|^2}.$$

$$= -\text{Hess } g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)\frac{\langle dg, df\rangle}{|\nabla g|^2}.$$

Thus, since $x^n = r - \alpha$, we have

$$f''(t) - \text{Hess } r\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \left(1 - \frac{\langle dg, df\rangle}{|\nabla g|^2}\right)\text{Hess } g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right).$$

Let $\mathcal{W} \subset\subset \mathcal{U}'$ be an open set with compact closure $\overline{\mathcal{W}} \subset \mathcal{U}'$ such that $\gamma([0, \delta]) \subset \mathcal{W}$. Let $T^1\mathcal{W}$ be the unit tangent bundle over $\mathcal{W}$ and set $\widetilde{\mathcal{W}} = \varphi(\mathcal{W})$. Let X be a smooth section of $T^1\mathcal{W}$ which extends $\frac{\partial}{\partial t}$ along γ. Define the function $H_X = \text{Hess } r(X, X) \circ \varphi^{-1}$ on $\widetilde{\mathcal{W}}$. We have

$$H_X(u^1, \ldots, u^n) = H_X(u^1, \ldots, u^{n-1}, 0)$$
$$+ \int_0^{u^n}\frac{\partial}{\partial s}H_X(u^1, \ldots, u^{n-1}, s)\,ds$$

for $(u^1, \ldots, u^n) \in \widetilde{\mathcal{W}}$.

Thus,

$$H_X(u^1, \ldots, u^n) - H_X(u^1, \ldots, u^{n-1}, 0) \leq Cu^n,$$

where C is independent of $(u^1, \ldots, u^n) \in \widetilde{\mathcal{W}}$. In fact,

$$C = \sup_{\widetilde{\mathcal{W}}}\left\{\frac{\partial}{\partial u^n}H_X(u^1, \ldots, u^n)\right\}.$$

We have
$$H_X(u^1, \ldots, u^{n-1}, 0) = \text{Hess } r(X, X) \leq 0.$$
Consequently, if $u^1 = x^1(\gamma(t)), \ldots, u^n = x^n(\gamma(t))$ and $X = \frac{\partial}{\partial t}$ then

$$H_X(x^1(\gamma(t)), \ldots, x^n(\gamma(t))) = \text{Hess } r\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$$
$$\leq Cx^n(\gamma(t)) = Cf(t).$$

Thus we have

$$f''(t) - Cf(t) \leq \left(1 - \frac{\langle dg, df \rangle}{|\nabla g|^2}\right) \text{Hess } g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right).$$

Now by choosing δ smaller if necessary we can assume that

$$\langle dg, df \rangle / |\nabla g|^2 \leq \frac{1}{2}$$

since $df|_{\gamma(0)} = 0$, because L and N_α are tangent at p. Thus it remains to show that $f''(t) - Cf(t) > 0$ for some t with

$$0 < t < \delta \leq \frac{1}{2}.$$

We may assume $C = 1$ since we can replace $f(t)$ by $f(\sqrt{C^{-1}}t)$ if necessary. Now, with the aim of reaching a contradiction assume that

$$f''(t) \leq f(t)$$

for all t, $0 < t < \delta \leq \frac{1}{2}$. Note that $f(0) = f'(0) = 0$ since L and N_α are tangent at p.

Let $t_0 \in (0, \delta)$. By the mean value theorem there is a $t^* \in (0, t_0)$ so that
$$f'(t^*) = \frac{f(t_0) - f(0)}{t_0} = \frac{f(t_0)}{t_0}.$$
Note that $f(t_0) > 0$ by our choice of γ and δ. By another application of the mean value theorem there is a t_1 with $0 < t_1 < t^* < t_0$ so that from above we have

$$f''(t_1) = \frac{f'(t^*)}{t^*} = \frac{f(t_0)}{t^* t_0}.$$

Our assumption, and the fact that t^*, $t_0 < \delta \leq \frac{1}{2}$, gives

$$f(t_1) \geq f''(t_1) \geq 2f(t_0).$$

Repeat this procedure with t_1 replacing t_0 to get a $t_2 < t_1$ with

$$f(t_2) \geq 4f(t_0).$$

Continuing this procedure gives

$$f(t_k) \geq 2^k f(t_0).$$

This is a contradiction since the continuity of f on $[0, \delta]$ implies that f is uniformly bounded. Thus we conclude that $f''(t') - Cf(t') > 0$ for some $t' \in (0, \delta)$. This together with our previous computations implies that

$$0 < \text{Hess } g \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right)\Big|_{t'}.$$

But

$$\text{Hess } g \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right)\Big|_{t'}$$

is just $|\nabla g|\langle S_\nu^L \omega, \omega\rangle$ where $\omega = \gamma_* \frac{\partial}{\partial t}|_{\gamma(t')}$ and $\nu = \frac{\nabla g}{|\nabla g|}$, and Lemma 3.5 holds.

In the following we establish some notation and record some facts to be used later.

3.6. Using the notation of the beginning of the section where the comparison theory was described, consider the case in which $\overline{M} = S^n(\frac{1}{\sqrt{b}})$ and $\overline{N} = \partial B_r$, where B_r is a proper geodesic ball of radius r. We use the inward normal $\overline{\nu}$. Then, for any $\xi(t)$ as above, and for $0 \leq s < r < \pi/\sqrt{b}$, we have

$$\frac{\|\overline{Y}(s)\|}{\|\overline{U}(s)\|} = \frac{\sin \sqrt{b}(r - s)}{\sin \sqrt{b}r}$$

and

$$\frac{d}{ds} \log \|\overline{Y}(s)\| = -\sqrt{b} \, \cot(\sqrt{b}(r - s)).$$

3.7. Take $\overline{M} = \mathbb{H}^n_{-a}$ the hyperbolic space of constant curvature $-a$. Let $\mathbb{H}^{n-1}_{-a}$ denote a totally geodesic submanifold of $\mathbb{H}^n_{-a}$ and let $\overline{N} = P_r$ where

$$P_r = \{\exp_\perp^{\mathbb{H}^{n-1}_{-a}} r\nu(p) : p \in \mathbb{H}^{n-1}_{-a}\}.$$

Take $\overline{\nu}$ to point toward $\mathbb{H}^{n-1}_{-a}$ i.e. let

$$\overline{\nu}_q = -\frac{d}{dt}\Big|_{t=r} \exp_\perp^{\mathbb{H}^{n-1}_{-a}} t\nu(p)$$

where $q = \exp_\perp^{\mathbb{H}^{n-1}_{-a}} r\nu(p)$. By symmetry, it suffices to take $n = 2$ and use coordinates x, y with metric

$$ds^2 = dx^2 + \cosh(\sqrt{a}x)dy^2.$$

We can take $\mathbb{H}^{n-1}_{-a} = \{x = 0\}$ and $P_r = \{x = r\}$. An easy computation in these coordinates gives for $0 \leq s < r$,

$$\frac{\|\overline{Y}(s)\|}{\|\overline{U}(s)\|} = \frac{\cosh \sqrt{a}(r - s)}{\cosh \sqrt{a}r},$$

$$\frac{d}{ds} \log \|\overline{Y}(s)\| = -\sqrt{a} \tanh(\sqrt{a}(r - s)).$$

3.8. Now let r, $\widetilde{\mathscr{U}}$, $\mathscr{U}$, ν, N, $N_\alpha \neq \varnothing$ etc. be as in the paragraph before Lemma 3.5. Suppose that the ambient manifold M has non-negative sectional curvature. Then $S^N_\nu \leq 0$ implies $S^{N_\alpha}_{\partial_r} \leq 0$ (recall that $\alpha > 0$). To prove this, we can use the comparison theory quoted above with the choices $\xi(t) = \nu \circ \gamma(t)$, $\overline{M} = \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ and $\overline{N} = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. We use (3.2) and the fact that

$$\frac{d}{ds} \log \|Y(s)\|\big|_{s=\alpha} = \langle S^{N_\alpha}_{\partial_r}(v), v \rangle / \|v\|^2,$$

where v can be made equal to any element of TN_α by an appropriate choice of $\gamma(t)$. We also use that in this case the right-hand side of (3.2) will be zero.

3.9. We make the following definitions

$$F(\kappa_1, a) \equiv \begin{cases} \dfrac{\operatorname{arc tanh}\left(\frac{\kappa_1}{\sqrt{a}}\right)}{\sqrt{a}} & \text{if } 0 < \kappa_1 < \sqrt{a}, \\[2mm] \infty & \text{if } \kappa_1 \geq \sqrt{a} \geq 0, \end{cases}$$

$$G(\kappa_2, b) \equiv \begin{cases} \dfrac{\operatorname{arc cot}\left(\frac{\kappa_2}{\sqrt{b}}\right)}{\sqrt{b}} & \text{if } b > 0, \kappa_2 > 0, \\[2mm] \dfrac{1}{\kappa_2} & \text{if } b = 0, \kappa_2 > 0, \\[2mm] \dfrac{\pi}{2\sqrt{b}} & \text{if } b > 0, \kappa_2 = 0, \\[2mm] \infty & \text{if } b = \kappa_2 = 0. \end{cases}$$

Here our convention is that $0 < \operatorname{arc cot}(\kappa_2/\sqrt{b}) < \pi/2$. Also, $G(.,.): \mathbb{R}^+ \times \mathbb{R}^+ \to \overline{\mathbb{R}}$ is continuous, where $\mathbb{R}^+$ is the nonnegative real numbers and $\overline{\mathbb{R}}$ is the extended real numbers.

3.10. If N is a compact manifold and $\tau : N \to N$ is an isometric involution without fixed points then τ can be extended to $[0, s_0] \times N$ by $\tau(s, x) = (s_0 - s, \tau(x))$. This action also has no fixed points and so the group of 2 elements generated by τ acts by isometries on $[0, s_0] \times N$. This action gives a quotient manifold with boundary which we denote by N_{τ, s_0}.

Theorems 3.11 and 3.22 below allow us to replace ϱ_1 in the spectral results of §2 by κ_1 and κ_2 in some cases. If the hypothesis of Theorem 3.22(ii) below holds ϱ_1 can be replaced by boundary curvature bounds κ_2 and λ_1. Note that although most results in §2 use both ϱ_1 and ϱ_2, the number ϱ_2 was bounded for weakly convex domains in [**D-L**].

THEOREM 3.11. *Let D be a boundary-convex domain in a Riemannian manifold M. Suppose that ∂D is smooth and compact. Let K be the sectional curvature of M.*
  (i) *If for $\kappa_1, \kappa_2, a, b \geq 0$ we have*

$$0 < \kappa_1 \leq -S_{\nu_{\mathrm{in}}}^{\partial D} \leq \kappa_2$$

and

$$-a \leq K \leq b$$

then $(r, p) \mapsto \exp_{\perp}^{\partial D} r\nu_{\mathrm{in}}(p)$ is a diffeomorphism on $(0, r_*) \times \partial D$ where

$$r_* = \min\{F(\kappa_1, a), G(\kappa_2, b)\}.$$

  (ii) *If $0 \leq -S_{\nu_{\mathrm{in}}}^{\partial D} \leq \kappa_2$ and $0 \leq K \leq b$ then one of the following holds:*
  (a) *$(r, p) \mapsto \exp_{\perp}^{\partial D} r\nu_{\mathrm{in}}(p)$ is a diffeomorphism on $(0, G(\kappa_2, b)) \times \partial D$ or*
  (b) *There is an isometric involution τ of ∂D without fixed points such that $\overline{D}$ is isometric to $\partial D_{\tau, s_0}$ where $s_0 < 2G(\kappa_2, b)$.*

Proof of Theorem 3.11. We will need several lemmas during the proof the first of which is

LEMMA 3.12. *The map $(r, p) \mapsto \exp_{\perp}^{\partial D} r\nu_{\mathrm{in}}(p)$ is a local diffeomorphism on $(0, R_0) \times \partial D$ where $R_0 = G(\kappa_2, b)$ and $0 \leq \kappa_2$.*

First we assume $b > 0$ and $\kappa_2 > 0$. Let $R_0 = \frac{\mathrm{arc\,cot}(\kappa_2/\sqrt{b})}{\sqrt{b}}$. Invoke the comparison theory mentioned above with the choices $\overline{M} = S^n(\frac{1}{\sqrt{b}})$, $\overline{N} = \partial B_{R_0}$ and $\overline{\nu}$ inward. Here B_{R_0} is a geodesic ball of radius R_0 in $S^n(\frac{1}{\sqrt{b}})$. We have

$$S_{\overline{\nu}}^{\partial B_{R_0}} = -\sqrt{b} \cot \sqrt{b} R_0 = -\kappa_2 \leq S_{\nu_{\mathrm{in}}}^{\partial D}$$

and $\overline{K} = b$ so that (3.3) holds and gives

$$\frac{\|Y(s)\|}{\|U(s)\|} \geq \frac{\|\overline{Y}(s)\|}{\|\overline{U}(s)\|} = \frac{\sin \sqrt{b}(R_0 - s)}{\sin \sqrt{b} R_0} > 0$$

for $0 < s < R_0$. Now since $\|Y(s)\| = \|d \exp_\perp^{\partial D} U(s)\|$ and $d \exp_\perp^{\partial D}$ is already nonsingular in the radial direction we conclude that $d \exp_\perp^{\partial D}$ is nonsingular at any $\xi \in v(\partial D)$ with $\xi = r\nu_{in}(x)$ as long as $0 \leq r < R_0$. By the inverse function theorem, $\exp_\perp^{\partial D}$ is a diffeomorphism on some neighborhood of any such point. The cases where $b = 0$ or $\kappa_2 = 0$ follow by continuity of G.

We now show that (i) of the theorem holds. Using Lemma 3.12 above we will show next, as an intermediate result, that each segment S_x given by the image of

$$r \mapsto \exp_\perp^{\partial D} r\nu_{in}(x), \qquad 0 < r < r_*,$$

does not intersect ∂D. Note the strict inequalities for r above. We start by noticing that the set $F = (\exp_\perp^{\partial D})^{-1}(\partial D)$ is closed and contains the zero section of $\nu(\partial D)$ as a union of connected components. Let F_0 be this union and let F_1 be the part of $F - F_0$ contained in the ν_{in} component(s) of $\nu(\partial D) - F_0$. Now on $\nu(\partial D)$ we have a smooth function ρ given by $\rho(\xi) = \|\xi\|$. Let $\rho|_{F_1}$ take a minimum value at some $\xi_1 \in F_1 \subset v(\partial D)$. Now if $\rho(\xi_1) \equiv \alpha_1 \geq r_*$ we have our intermediate result. We will show that the assumption $\alpha_1 < r_*$ leads to a contradiction.

From Lemma 3.12, there is a small ball $\widetilde{B}_{\xi_1}(\delta) = \widetilde{B}$ containing ξ_1 as its center so that $\exp_\perp^{\partial D}|_{\widetilde{B}}$ is a diffeomorphism onto its image say $\mathscr{U} \subset M$. Under this diffeomorphism $\widetilde{B} \cap F_1$ maps to $\mathscr{U} \cap \partial D$. By taking $\widetilde{B}$ smaller if necessary we can guarantee that $\widetilde{B} \cap F_0 = \varnothing$. Now, since at ξ_1 the function ρ restricted to F_1 achieves the minimum $\rho(\xi_1) = \alpha_1 < r_*$, we must have that at $q = \exp_\perp^{\partial D} \xi_1$, $\rho \circ (\exp_\perp^{\partial D}|_{\widetilde{\mathscr{U}}})^{-1}|_{\mathscr{U} \cap \partial D} \equiv r|_{\mathscr{U} \cap \partial D}$ achieves the minimum α_1 and so $\mathscr{U} \cap \partial D$ is tangent with N_{α_1} (defined before where here $N = \partial D$ and $\widetilde{\mathscr{U}} = \widetilde{B}$) at q and every point $p \in \mathscr{U} \cap \partial D$ has $r(p) \geq \alpha_1$. Let ∂_r be the vector field associated with r on $\mathscr{U}$. Then at q we must have either

$$\partial_r|_q = \nu_{in}(q) \quad \text{or} \quad \partial_r|_q = -\nu_{in}(q).$$

In the first case we see that for t a little smaller than α_1 the point $\exp_\perp^{\partial D} t\nu_{in}(p_*)$ is contained in $M - \overline{D}$ where $p_* = \pi(\xi_1) \in \partial D$. This means that $\exp_\perp^{\partial D} t\nu_{in}(p_*)$ must have crossed ∂D for some $0 < t < \alpha_1$ contradicting the minimality of ρ occurring at ξ_1. Thus, $\partial_r = -\nu_{in}(q)$. We will use the comparison theory again to show that

$$(3.13) \qquad \langle S_{\partial_r}^{N_{\alpha_1}} X_q, X_q \rangle < 0$$

for $X_q \in T_q N_{\alpha_1} = T_q(\mathscr{U} \cap \partial D)$. This together with Lemma 3.5A implies $\langle S_{\nu_{\text{in}}}^{\partial D} X_q, X_q \rangle > 0$ contradicting our hypothesis that $S_{\nu_{\text{in}}}^{\partial D} \leq 0$. This then implies the desired intermediate result.

Claim. (3.13) holds i.e.

$$\frac{\langle S_{\partial_r}^{N_{\alpha_1}} v, v \rangle}{\|v\|^2} < 0$$

for any $v \in T_q N_{\alpha_1}$.

Proof of Claim. We can arrange that $\xi(t)$ (see the first paragraph of §3) is chosen so that

$$\frac{d}{ds} \log \|Y(s)\|\Big|_{s=\alpha_1} = \frac{\langle S_{\partial_r}^{N_{\alpha_1}} v, v \rangle}{\|v\|^2}.$$

We will need to show that

$$\frac{d}{ds} \log \|Y(s)\|\Big|_{s=\alpha_1} < 0.$$

First assume $a > 0$. Use the comparison theory with $\overline{M} = \mathbb{H}^n_{-a}$, P_{r_0} and $\overline{\nu}$ as above (see 3.7), where we now let

$$r_0 = \frac{1}{\sqrt{a}} \operatorname{arc tanh}\left(\frac{\kappa_1^*}{\sqrt{a}}\right)$$

and $\kappa_1^* < \min\{\sqrt{a}, \kappa_1\}$.

We have that

$$S_{\overline{\nu}}^{P_{r_0}} = -\sqrt{a} \tanh \sqrt{a} r_0 = -\kappa_1^* > -\kappa_1 \geq S_{\nu_{\text{in}}}^{\partial D}$$

and, since $\overline{K} = -a \leq K$, we have that (3.2) holds and gives

$$\frac{d}{ds} \log \|Y(s)\| \leq -\sqrt{a} \tanh \sqrt{a}(r_0 - s) < 0$$

for $0 \leq s < \min\{G(\kappa_2, b), r_0\}$. We have that $\min\{G(\kappa_2, b), r_0\} \to r_*$ as $\kappa_1^* \to \min\{\sqrt{a}, \kappa_1\}$ and so we may choose κ_1^* so that $\alpha_1 < \min\{G(\kappa_2, b), r_0\}$. The claim for $a > 0$ now follows by letting $\kappa_1^* \to \min\{\sqrt{a}, \kappa_1\}$. The case $a = 0$ is proven similarly by using the comparison theory with $\mathbb{R}^n$ instead of $\mathbb{H}^n_{-a}$ and a sphere of radius $1/\kappa_1$ instead of P_{r_0}.

We now have the intermediate result that the segments S_x described above are contained in D. This implies that L_r given by the image of ∂D under

$$x \mapsto \exp_\perp^{\partial D} r\nu_{\text{in}}(x)$$

for $0 < r < r_*$ is contained in D (and does not intersect ∂D).

Now we are in a position to show that $(r, p) \to \exp_{\perp}^{\partial D} r\nu_{\text{in}}(P)$ is one to one on $(0, r_*) \times \partial D$. This fails if there exists a point p, and $r_2 > r_1$ with

$$\exp_{\perp}^{\partial D} r_1 \nu_{\text{in}}(p) = \exp_{\perp}^{\partial D} r_2 \nu_{\text{in}}(p)$$

or points $p_1 \neq p_2$ with

$$\exp_{\perp}^{\partial D} r_1 \nu_{\text{in}}(p_1) = \exp_{\perp}^{\partial D} r_2 \nu_{\text{in}}(p_2).$$

We can reduce to the case where $r_1 < r_2$. To see this consider the case where $p_1 \neq p_2$ and $r_1 = r_2$. In case the two geodesic segments connecting p_1 and p_2 to $p_3 \equiv \exp_{\perp}^{\partial D} r_1 \nu_{\text{in}}(p_1) = \exp_{\perp}^{\partial D} r_2 \nu_{\text{in}}(p_2)$ do not meet tangentially or antitangentially then we may extend the segment $\overline{p_2 p_3}$ to a slightly longer geodesic segment say $\overline{p_2 p_4}$ of length $r_2 + \varepsilon$. Because $\overline{p_1 p_3}$ and $\overline{p_3 p_4}$ meet with an angle other than 0 or π the usual corner cutting argument shows that $r_5 \equiv \text{dist}(p_4, \partial D) < r_1 + \varepsilon$. Thus, if we choose a point p_5 in ∂D a distance r_5 from p_4 then we can consider the segments $\overline{p_2 p_4}$ and $\overline{p_5 p_4}$ as our replacements. In other words, since $r_5 < r_2 + \varepsilon$ we just replace p_1 by p_5, r_2 by $r_2 + \varepsilon$, and r_1 by r_5.

In case $\overline{p_1 p_3}$ and $\overline{p_2 p_3}$ meet antitangentially the reduction is trivial, we just extend $\overline{p_1 p_3}$ slightly. If $\overline{p_1 p_3}$ and $\overline{p_2 p_3}$ meet tangentially then $r_1 = r_2$ forces $p_1 = p_2$ by uniqueness of geodesics with equal tangents.

Thus, all we need to do is show that assuming

$$(3.14) \qquad \exp_{\perp} r_1 \nu_{\text{in}}(p_1) = \exp_{\perp}^{\partial D} r_2 \nu_{\text{in}}(p_2), \qquad r_1 < r_2 < r_*,$$

leads to a contradiction . We proceed with this assumption.

Consider the set $\widetilde{L}_{r_2}$ in $\nu(\partial D)$ given by

$$\widetilde{L}_{r_2} = \{\xi \in \nu(\partial D) : \xi = r_2 \nu_{\text{in}}(x), \, x \in \partial D\}.$$

Define L_{r_2} and $\widetilde{\widetilde{L}}_{r_2}$ by $L_{r_2} = \exp_{\perp}^{\partial D}(\widetilde{L}_{r_2})$ and

$$\widetilde{\widetilde{L}}_{r_2} = (\exp_{\perp}^{\partial D})^{-1} L_{r_2} \cap \{r\nu_{\text{in}}(p) : \, r \geq 0, \, p \in \partial D\}.$$

Since ∂D is compact and $\widetilde{\widetilde{L}}_{r_2}$ is closed without intersecting the zero section F_0, we see that $\rho|_{\widetilde{\widetilde{L}}_{r_2}}$ must achieve a minimum α_2 at some $\xi \in \widetilde{\widetilde{L}}_{r_2}$ which must have $\alpha_2 = \rho(\xi) \leq r_1$ since (3.14) holds. Let $\widetilde{\mathscr{U}}$ be a small neighborhood of ξ such that $\exp_{\perp}^{\partial D}|_{\widetilde{\mathscr{U}}}$ is a diffeomorphism onto $\mathscr{U} \subset D$. Then $r = \rho \circ \exp_{\perp}^{\partial D}|_{\widetilde{\mathscr{U}}}^{-1}$ constrained to $S_{r_2} = \exp_{\perp}^{\partial D}(\widetilde{\mathscr{U}} \cap \widetilde{\widetilde{L}}_{r_2})$

achieves a minimum at the point $q = \exp_\perp \xi$. Now, by our definition of $\widetilde{L}_{r_2}$ and S_{r_2} there must be a $\xi^* \in \widetilde{L}_{r_2}$ with $q = \exp_\perp \xi^*$ and for some small neighborhood $\widetilde{V}$ of ξ^* the restricted map $\exp_\perp^{\partial D}|_{\widetilde{V}}$ is a diffeomorphism onto an open set V. Define a function r' on V by $r' = \rho \circ \exp_\perp^{\partial D}|_{\widetilde{V}}^{-1}$. Notice that $q \in V$ and $r'(q) = r_2$. Thus, we have an r_2 level set for the function r' in V which we denote by N'_{r_2}. We have that $q \in \exp_\perp^{\partial D}(\widetilde{V} \cap \widetilde{L}_{r_2}) = N'_{r_2} \subset S_{r_2}$. Hence, $r|_{N'_{r_2}}$ achieves a minimum α_2 at $q \in N'_{r_2}$ and so N'_{r_2} must lie in the part of $\mathscr{U}$ corresponding to values of r greater than or equal to $\alpha_2 \leq r_1$. Thus, we once again have two submanifolds tangent at a point q. Namely, the r level set N_{α_2} and the r' level set N'_{r_2}. The submanifold N'_{r_2} lies on the side of N_{α_2} corresponding to $r \geq \alpha_2$. N_{α_2} has an associated normal ∂_r and N'_{r_2} has normal field $\partial_{r'} = \text{grad } r'$. Now, if $\partial_{r'} = \partial_r$ at q then the two geodesics

$$ t \mapsto \exp_\perp t\nu_{\text{in}}(\pi(\xi^*)), \qquad 0 < t \leq r_2, $$

and

$$ t \mapsto \exp_\perp t\nu_{\text{in}}(\pi(\xi)), \qquad 0 < t \leq \alpha_2, $$

are tangent at $q = \exp_\perp \alpha_2 \nu_{\text{in}}(\pi(\xi)) = \exp_\perp r_2 \nu_{\text{in}}(\xi^*)$. But, by uniqueness of geodesics with equal tangents and the fact that $r_2 > r_1 \geq \alpha_2$, the geodesic $t \mapsto \exp_\perp t\nu_{\text{in}}(\pi(\xi^*))$, $0 \leq t \leq r_2$, must hit ∂D at $\pi(\xi)$ for t with $0 < t \leq r_2$, a contradiction of the intermediate result obtained in the first part of the proof since $r_2 < r_*$. Now, the case $\partial_{r'} = -\partial_r$ is impossible since by (3.13) and Lemma 3.5A (choose $\nu = \partial_{r'}$ and $L = N'_{r_2}$) we would have $S^{N'_{r_2}}_{\partial_{r'}} > 0$. But (3.13) applies to N'_{r_2} also so that $S^{N'_{r_2}}_{\partial_{r'}} < 0$, a contradiction. Thus (3.14) cannot hold and this completes the proof of (i).

The following elementary lemma plays a crucial role in the proof of Theorem 3.11(ii).

LEMMA 3.15. *Let* $(\partial D)_0$ *be a connected component of* ∂D *and let* $\widetilde{D}_\alpha$ *be the band in* $\nu(\partial D)$ *given by*

$$ \widetilde{D}_\alpha = \{r\nu_{\text{in}}(p) : p \in (\partial D)_0, \; 0 \leq r \leq \alpha\}. $$

Suppose that $\exp_\perp^{\partial D}$ *is a local diffeomorphism on* $\widetilde{D}_\alpha$. *If* $\exp_\perp^{\partial D}$ *maps* $\partial \widetilde{D}_\alpha$ *into* ∂D *and* interior($\widetilde{D}_\alpha$) *into* interior(D) *then in fact*

$\exp_{\perp}^{\partial D}(\widetilde{D}_{\alpha}) = \overline{D}$. *In particular ∂D has at most 2 connected components.*

Proof. Give $\overline{D}$ the relative topology inherited from M. We show that $\exp_{\perp}^{\partial D}(\widetilde{D}_{\alpha})$ is open and closed in this topology. $\exp_{\perp}^{\partial D}(\widetilde{D}_{\alpha})$ is closed since $\widetilde{D}_{\alpha}$ is compact and $\exp_{\perp}^{\partial D}$ is continuous. Let $x \in \exp_{\perp}^{\partial D}(\widetilde{D}_{\alpha})$. We show that $\exp_{\perp}^{\partial D} \widetilde{D}_{\alpha}$ contains an open neighborhood of x in the $\overline{D}$ relative topology. If x is interior to D then for $p \in (\exp_{\perp}^{\partial D})^{-1}(x)$ it must be that p is interior to $\widetilde{D}_{\alpha}$ since otherwise our hypotheses guarantee that $x \in \partial D$. Thus we can find an open neighborhood $O \subset \text{interior}(\widetilde{D}_{\alpha})$ containing p so that $\exp_{\perp}^{\partial D}|_{O}$ is a diffeomorphism. Thus, since $\exp_{\perp}^{\partial D}(O) \subset \text{interior}(D)$, we see that $\exp_{\perp}^{\partial D}(O)$ is a $\overline{D}$-open set containing x and contained in $\exp_{\perp}^{\partial D}(\widetilde{D}_{\alpha})$. Now, if $x \in \partial D$ then $p \in (\exp_{\perp}^{\partial D}|_{\widetilde{D}_{\alpha}})^{-1}(x)$ is in $\partial \widetilde{D}_{\alpha}$ and we can find an open set O' containing p on which $\exp_{\perp}^{\partial D}$ is a diffeomorphism. Our hypotheses and continuity guarantee that for small O', $x \in \exp_{\perp}^{\partial D}(O' \cap \widetilde{D}_{\alpha}) = \exp_{\perp}^{\partial D}(O') \cap \overline{D}$.

Proof of (ii). Once again we have that $\rho|_{F_1}$ achieves a minimum. Call this minimum value α_1 as before. The proof of (ii) is similar to but more difficult than that of (i) above. The program will be first to strive to prove the intermediate result that the geodesic segments S_x given by
$$r \mapsto \exp_{\perp}^{\partial D} r\nu_{\text{in}}(x), \qquad 0 < r < R_0,$$
do not intersect ∂D for any $x \in \partial D$. Note that now the segments S_x have length $R_0 \equiv G(\kappa_2, b)$. Secondly we strive to show that
$$(r, p) \mapsto \exp_{\perp}^{\partial D} r\nu_{\text{in}}(p)$$
is one to one on $(0, R_0) \times \partial D$. That this map is a local diffeomorphism was proved in Lemma 3.12. We shall see that the only case in which we shall not be able to carry this program out is if the domain D has the structure described in (iib) of the statement of the theorem.

Let A be the set of points $\xi \in F_1$ such that $\rho(\xi) = \alpha_1$. We have two cases, namely $\pi(A) \neq \partial D$ and $\pi(A) = \partial D$.

Case I. $\pi(A) \neq \partial D$.

We shall prove that in this case (iia) holds. We first prove the intermediate result that the segments S_x described above do not intersect ∂D.

LEMMA 3.16. $\pi(A) \neq \partial D$ and $\alpha_1 < R_0$ implies $\pi(A)$ has boundary points in ∂D, where ∂D has the relative topology inherited from M.

The set A is compact since it is closed and ∂D is compact. By continuity $\pi(A)$ is compact and therefore closed in the Hausdorff space ∂D. We wish to show that in the case at hand $\pi(A)$ cannot also be open (in the induced topology on ∂D). To this end let us assume that $\pi(A)$ is open and seek a contradiction. We are assuming $\alpha_1 < R_0$.

Since $\pi(A)$ is open and closed, we must have that $\pi(A)$ contains a connected component of ∂D say $(\partial D)_0$. If ∂D were connected so that $(\partial D)_0 = \partial D$ we would be in Case II; $\partial D = \pi(A)$. However, even if ∂D is not connected, we can use the connectedness of D to show that $(\partial D)_0 \subset \pi(A)$ implies $\pi(A) = \partial D$ (Case II). This is a contradiction since we are in Case I. We have $(\partial D)_0 \subset \pi(A)$. The definition of A implies that the set $\widetilde{A}_{\alpha_1}$ defined by

$$\widetilde{A}_{\alpha_1} = \{\alpha_1 \nu_{\text{in}}(p) : p \in (\partial D)_0 \subset \pi(A)\}$$

is contained in F_1. Identify ∂D with its image in $\nu(\partial D)$ under the zero section. With this notation, we have that $\widetilde{A}_{\alpha_1}$ is diffeomorphic to $(\partial D)_0$ and $\widetilde{A}_{\alpha_1} \cup (\partial D)_0$ forms the boundary of $\widetilde{D}_{\alpha_1}$, where

$$\widetilde{D}_{\alpha_1} = \{r\nu_{\text{in}}(p) : p \in (\partial D)_0 ,\ 0 \leq r \leq \alpha_1\}.$$

Since $\alpha_1 < R_0$, Lemmas 3.12 and 3.15 imply that

$$\exp_\perp^{\partial D}(\widetilde{D}_{\alpha_1}) = \overline{D}.$$

Let $x \in \partial D$. The above considerations show that we can find $\xi \in (\partial D)_0 \cup \widetilde{A}_{\alpha_1}$ with $\exp_\perp^{\partial D} \xi = x$. If $\xi \in (\partial D)_0$ then since $(\partial D)_0 \subset \pi(A)$ we have $x \in \pi(A)$. If $\xi \in \widetilde{A}_{\alpha_1}$ then by the definition of $\widetilde{A}_{\alpha_1}$ and the established fact $\widetilde{A}_{\alpha_1} \subset F_1$ we see that there is a geodesic of length α_1 connecting $\pi(\xi) \in \partial D$ to $\exp_\perp^{\partial D} \xi \in \partial D$, given by

$$t \mapsto \exp_\perp^{\partial D} t\nu_{\text{in}}(\pi(\xi)).$$

By the Gauss lemma and the fact that $\widetilde{A}_{\alpha_1} \subset F_1$, we see that this geodesic meets ∂D perpendicularly. Also, since α_1 is a minimum of ρ on F_1 and $\exp_\perp^{\partial D}$ preserves radial distance the above geodesic can only meet ∂D at its endpoints. But this means that $x = \exp_\perp^{\partial D} \xi$ must also be in $\pi(A)$. In fact, the geodesic $t \mapsto \exp_\perp^{\partial D} t\nu_{\text{in}}(x)$, $0 \leq t \leq \alpha_1$, is just the reverse of the former geodesic and we see that $\alpha_1 \nu_{\text{in}}(x) \in A$

and so $x \in \pi(A)$. Summarizing what we have done so far, we have shown that $\pi(A)$ open in ∂D implies $\partial D = \pi(A)$ unless $\alpha_1 \geq R_0$. Since we are in Case I, we conclude that in this case $\alpha_1 < R_0$ is impossible unless $\pi(A)$ has boundary points in the relative topology on ∂D. This proves Lemma 3.16.

Next we shall see even $\pi(A)$ having boundary points in ∂D is, in Case I, incompatible with $\alpha_1 < R_0$ so that in Case I, $\alpha_1 \geq R_0$ and we will have our intermediate result. We continue to assume $\alpha_1 < R_0$ and will once again reach a contradiction. Let $x_0 \in \partial D$ be a boundary point of $\pi(A) \subset \partial D$. Let $\xi = \alpha_1 \nu_{\text{in}}(x_0)$. Let $\widetilde{\mathscr{U}} \subset \nu(\partial D)$ be a connected open neighborhood of ξ such that $\exp^{\partial D}_\perp |_{\widetilde{\mathscr{U}}}$ is a diffeomorphism onto a (connected) neighborhood $\mathscr{U}$ of $q = \exp^{\partial D}_\perp \xi$. We define $r = \rho \circ (\exp^{\partial D}_\perp |_{\widetilde{\mathscr{U}}})^{-1}$ on $\mathscr{U}$. Now under this diffeomorphism $F_1 \cap \widetilde{\mathscr{U}}$ maps onto $\partial D \cap \mathscr{U}$ and $(\widetilde{L}_{\alpha_1} \cap \widetilde{\mathscr{U}}, \xi)$ maps onto the pair (N_{α_1}, q) where $N_{\alpha_1} \subset L_{\alpha_1}$ is the level $r = \alpha_1$ submanifold as before. Since the minimum of $r|_{\partial D \cap \mathscr{U}}$ occurs at q (because $\xi \in A$) we must have

$$T_q \partial D = T_q N_{\alpha_1}.$$

Also, just as in the proof of part (i) of the theorem $\partial_r = -\nu_{\text{in}}$ at q. Now π is a submersion with $d\pi$ an isomorphism when restricted to $T_\xi(F_1 \cap \widetilde{\mathscr{U}}) = T_\xi(\widetilde{L}_{\alpha_1} \cap \widetilde{\mathscr{U}})$. It follows, after taking $\widetilde{\mathscr{U}}$ smaller if necessary, that $\pi|_{F_1 \cap \widetilde{\mathscr{U}}}$ a diffeomorphism onto its image $\mathscr{W} \subset \partial D$ where $x_0 \in \mathscr{W}$. Now since x_0 is a boundary point of $\pi(A)$, there must be a point $x_0' \in \mathscr{W}$ with $x_0' \notin \pi(A)$. This implies that $\xi' = \pi^{-1}|_{F_1 \cap \widetilde{\mathscr{U}}}(x_0')$ must satisfy $\rho(\xi') > \alpha_1$. Thus the point $p = \exp^{\partial D}_\perp \xi' \in \partial D \cap \mathscr{U}$ must satisfy $r(p) > \alpha_1$. Hence we are in a position to apply (3.8) and Lemma 3.5B to conclude that there exists $q_1 \in \partial D \cap \mathscr{U}$ and $w \in T_{q_1} \partial D$ with $\langle -S^{\partial D}_{\nu_{\text{in}}} w, w \rangle < 0$. This contradicts our hypothesis on the curvature of the boundary and so we are forced to conclude that, in Case I, we must have $\alpha_1 \geq R_0 = G(\kappa_2, b)$. This implies the intermediate result in this case.

Next we show that, in Case I, the map $(r, p) \mapsto \exp^{\partial D}_\perp r \nu_{\text{in}}(p)$ is one to one on $(0, R_0) \times \partial D$. Arguing as in the proof of (i), we see that we just need to show that assuming

$$(3.17) \qquad \exp^{\partial D}_\perp r_1 \nu_{\text{in}}(p_1) = \exp^{\partial D}_\perp r_2 \nu_{\text{in}}(p_2), \qquad r_1 < r_2 < R_0,$$

leads to a contradiction. As before, we consider the sets $\widetilde{L}_{r_2}$, L_{r_2}, and $\widetilde{\widetilde{L}}_{r_2}$. With $\widetilde{L}_{r_2} = (\exp^{\partial D}_\perp)^{-1} L_{r_2} \subset \nu(\partial D)$ as before we still have

$\widetilde{\widetilde{L}}_{r_2} \cap F_0 = \varnothing$ where F_0 is the zero section of $\nu(\partial D)$ (identified with ∂D). Now $\rho|_{\widetilde{\widetilde{L}}_{r_2}}$ achieves a minimum which we again call α_2. Let A' be the set of points $\xi \in \widetilde{\widetilde{L}}_{r_2}$ for which $\rho(\xi) = \alpha_2$. Now $\pi(A')$ is closed. If $\pi(A')$ is also open then $(\partial D)_0 \subset \pi(A')$, where $(\partial D)_0$ is a connected component of ∂D. Assume then that this is the case. Then we see that the set

$$\widetilde{A}' = \{\alpha_2 \nu_{\text{in}}(x) : x \in (\partial D)_0\}$$

is contained in $\widetilde{\widetilde{L}}_{r_2}$ and hence $\exp^{\partial D}_\perp \widetilde{A}' \subset L_{r_2}$. Also, we have seen in the proof of (i) that we have, for all $x \in (\partial D)_0$,

$$-\frac{d}{dt} \exp^{\partial D}_\perp t\nu_{\text{in}}(x)\big|_{t=r_1} = \frac{d}{dt} \exp^{\partial D}_\perp t\nu_{\text{in}}(\pi(\xi^*))\big|_{t=r_2}$$

where $\xi^* \in L_{r_2}$ and $\exp^{\partial D}_\perp \xi^* = q$ as before. Thus we have the following fact:

Fact 3.18. For any $x \in (\partial D)_0$ the geodesic

$$\gamma : t \mapsto \exp^{\partial D}_\perp t\nu_{\text{in}}(x), \qquad 0 < t \leq \alpha_2 + r_2,$$

connects $x \in \pi(A')$ to ∂D and is perpendicular to ∂D at its endpoints.

The above geodesic never intersects ∂D for $0 < t < \alpha_2 + r_2$ by the intermediate result since $\alpha_2, r_2 < R_0$. Thus it must be that $\alpha_2 + r_2 \geq \alpha_1$. In particular $\alpha_1 < 2R_0$. Let us agree to denote $\alpha_2 + r_2$ by α_3. Notice that since $(\partial D)_0 \subset \pi(A')$ the number $\alpha_3 = \alpha_2 + r_2$ is independent of our choice of $x \in (\partial D)_0$, the starting point of the above geodesic. Thus $(r, p) \mapsto \exp^{\partial D}_\perp r\nu_{\text{in}}(p)$ maps the boundary (resp. interior) of $[0, \alpha_3] \times (\partial D)_0$ into the boundary (resp. interior) of $\overline{D}$. Now apply the following lemma, with $\alpha = \alpha_3$ and $U = (\partial D)_0$.
Let

$$\mathscr{L}(x) = \inf\{t : \exp^{\partial D}_\perp t\nu_{\text{in}}(x) \in \partial D\}.$$

LEMMA 3.19. *If U is an open subset of ∂D contained in $\{x \in \partial D : \mathscr{L}(x) = \alpha\}$, $\alpha < 2R_0$ and*

$$\frac{d}{dt} \exp^{\partial D}_\perp t\nu_{\text{in}}(x)\big|_{t=\alpha}$$

is perpendicular to ∂D for all $x \in U$ then $(r, p) \mapsto \exp_\perp^{\partial D} r\nu_{\text{in}}(p)$ is a local diffeomorphism on $[0, \alpha] \times U$.

Proof. It suffices to show that if $\xi = r\nu_{\text{in}}(p)$ for some $p \in U$ and $0 \le r \le \alpha$ then $d \exp_\perp^{\partial D}|_\xi$ is a linear isomorphism. Since $d \exp_\perp^{\partial D}$ is always nonsingular in the radial direction we just need to show for any curve $\xi(t)$ in $\nu(\partial D)$, $\|\xi(t)\| = \text{constant} \le \alpha$, $v = \frac{d}{dt}\xi(t)|_{t=0} \ne 0$ and $\xi(0) = \xi$ inward that

$$d \exp_\perp^{\partial D}|_\xi v \ne 0.$$

Consider the curve $\theta(t)$ in ∂D given by $\pi(\xi(t))$. Let $\tilde{\theta}(t) = \exp_\perp^{\partial D} \alpha\nu_{\text{in}}(\theta(t))$. Notice that $\tilde{\theta}(t)$ is a curve in ∂D. Since the geodesic

$$t \mapsto \exp_\perp^{\partial D} t\nu_{\text{in}}(x)$$

where $0 \le t \le \alpha$, is normal to ∂D at both end points, we can consider the reverse geodesic and we have

(3.20) $$\exp_\perp^{\partial D} s\nu_{\text{in}}(\theta(t)) = \exp_\perp^{\partial D}(\alpha - s)\nu_{\text{in}}(\tilde{\theta}(t)).$$

From this we see that $\exp_\perp^{\partial D} \alpha\nu_{\text{in}}(\tilde{\theta}(t)) = \theta(t)$. Hence, $\theta'(0) \ne 0 \Rightarrow \tilde{\theta}'(0) \ne 0$. Let $Y(s)$ be the ∂D-Jacobi field given by

$$Y(s) = \frac{d}{dt} \exp_\perp^{\partial D} s\nu_{\text{in}}(\theta(t))|_{t=0}$$

and let $\tilde{Y}(s)$ be given by

$$\tilde{Y}(s) = \frac{d}{dt} \exp_\perp^{\partial D} s\nu_{\text{in}}(\tilde{\theta}(t))|_{t=0}.$$

Then from (3.20), we have $\tilde{Y}(\alpha - s) = Y(s)$. Now $d \exp_\perp^{\partial D}|_\xi v = Y(r_3)$ where $r_3 = |\xi|$. If $r_3 \le \alpha/2$ $(< R_0)$ then by (3.12), $Y(r_3) \ne 0$. If $r_3 > \alpha/2$ then $\alpha - r_3 < R_0$ and so

$$Y(r_3) = \tilde{Y}(\alpha - r_3) \ne 0$$

since

$$\tilde{Y}(\alpha - r_3) = \frac{d}{dt} \exp_\perp^{\partial D}(\alpha - r_3)\nu_{\text{in}}(\tilde{\theta}(t))|_{t=0}$$

and $\tilde{\theta}'(0) \ne 0$. This proves the lemma.

In particular, we have, in the present case, that

$$(r, p) \mapsto \exp_\perp^{\partial D} r\nu_{\text{in}}(p)$$

is a local diffeomorphism from $[0, \alpha_3] \times (\partial D)_0$ into $\overline{D}$. From this and the comment just before Lemma 3.19, we see that Lemma 3.15 can be applied to conclude that

$$\{\exp_{\perp}^{\partial D} r\nu_{\text{in}}(p) : (r, p) \in [0, \alpha_3] \times (\partial D)_0\} = \overline{D}.$$

But this implies that in fact $\alpha_3 = \alpha_1$. It follows as before that

$$\pi(A) = \partial D$$

which is Case II. Since we are in Case I, we are led to conclude that $\pi(A')$ is not open in ∂D and thus contains a boundary point. Thus, arguing as in the paragraph before (3.17), once again we are in a position to use (3.8) and Lemma 3.5B to reach a contradiction. Thus $(r, p) \mapsto \exp_{\perp}^{\partial D} r\nu_{\text{in}}(p)$ is a diffeomorphism on $(0, R_0) \times \partial D$.

Case II. $\pi(A) = \partial D$.

LEMMA 3.21. *In case* $\pi(A) = \partial D$ *if* $\alpha_1 \geq 2R_0$ *then* $(r, p) \mapsto \exp_{\perp}^{\partial D} r\nu_{\text{in}}(p)$ *is a diffeomorphism on* $[0, R_0) \times \partial D$.

Proof. We already have the local diffeomorphism property by 3.12 so it suffices to show that the map is one to one. We once again show that (3.17) leads to a contradiction if $r_2 < R_0$. This time it is much easier. Let L_{r_2}, $\tilde{L}_{r_2}$, and $\tilde{\tilde{L}}_{r_2}$ be defined as before. If $\rho|_{\tilde{\tilde{L}}_{r_2}}$ achieves the minimum α_2 at say ξ then the geodesic $t \mapsto \exp_{\perp}^{\partial D} t\nu_{\text{in}}(x)$, $0 < t \leq \alpha_2 + r_2$ with $x = \pi(\xi)$ meets ∂D for the first time at $t = \alpha_2 + r_2$. However $\alpha_2 + r_2 < 2R_0 \leq \alpha_1$ a contradiction to the meaning of α_1.

We must finally consider the case $\pi(A) = \partial D$ and $\alpha_1 < 2R_0$. This is the only case where the conclusions of (iia) of the theorem have not been shown to hold. It is this situation in fact where (iib) holds.

Let $r_3 = \frac{\alpha_1}{2} - \varepsilon$ and $r_2 = \frac{\alpha_1}{2} + \varepsilon$ for some ε, where $0 < \varepsilon < (2R_0 - \alpha_1)/2$. Then $r_3 + r_2 = \alpha_1$ and $r_3 < r_2 < R_0$. We may choose ε small enough so that $r_3, r_2 > 0$. Let $\tilde{L}_{r_2}$ be as before and consider the minimum α_2 of ρ on $\tilde{L}_{r_2}$. Clearly, $\alpha_2 \leq r_2 < R_0$. The distance from $p = \exp_{\perp}^{\partial D} r_2\nu_{\text{in}}(x)$ to ∂D, for any $x \in \partial D$, is (by assumption) less than or equal to r_3. Thus we see, by considering a geodesic from p to ∂D whose length realizes this distance, that in fact $\alpha_2 \leq r_3 < r_2$. Let this minimum occur at some point $\xi \in \tilde{L}_{r_2}$. Then arguing in a manner similar to the paragraph of 3.14, we see that the geodesic

$$t \to \exp_{\perp}^{\partial D} t\nu_{\text{in}}(\pi(\xi)),$$

$0 < t \leq \alpha_2 + r_2$, meets ∂D for the first time when $t = \alpha_2 + r_2$ and is normal to ∂D there. Hence, $\alpha_2 = r_3$ since we must have $\alpha_2 + r_2 = \alpha_1$. Let $x \in \partial D$. By assumption the geodesic,

$$t \to \exp_{\perp}^{\partial D} t\nu_{\text{in}}(x),$$

$0 \leq t \leq \alpha_1$, hits ∂D only at its end points. Consider the point $p \in D$ given by $p = \exp_{\perp}^{\partial D} r_2\nu_{\text{in}}(x)$. Let $q \in \partial D$ be a closest point of ∂D to p and let $r = \text{dist}(p, \partial D)$. Since $\alpha_1 = r_3 + r_2$ we have that $r \leq r_3$ so that in particular $q \neq x$. On the other hand since $\alpha_2 = r_3$ we must have $r_3 \leq r$ and thus $r = r_3$. But r is the distance from p to ∂D and this implies that the geodesic segment $t \mapsto \exp_{\perp}^{\partial D} t\nu_{\text{in}}(x)$, $r_2 \leq t \leq r_3 + r_2$, hits ∂D normally. This means that the geodesic segment $t \mapsto \exp_{\perp}^{\partial D} t\nu_{\text{in}}(x)$, $0 \leq t \leq \alpha_1 = r_3 + r_2$, hits ∂D normally at both end points. Since x was an arbitrary point in ∂D, we can apply Lemma 3.19 to get that $(r, p) \to \exp_{\perp}^{\partial D} r\nu_{\text{in}}(p)$ is a local diffeomorphism from $[0, \alpha_1] \times \partial D$ onto $\overline{D}$. Next we show that this map is a local isometry.

By the Gauss lemma, it suffices to show that for all $v \in T_p\partial D$, we have

$$\|v\| = \|Y(r)\|$$

where $Y(r) = \frac{d}{dt} \exp_{\perp}^{\partial D} r\nu_{\text{in}}(\theta(t))\big|_{t=0}$ and $\theta'(0) = v$. As in the proof of Lemma 3.18 we define $\tilde{\theta}$ and $\widetilde{Y}$ by

$$\tilde{\theta}(t) = \exp_{\perp}^{\partial D} \alpha_1\nu_{\text{in}}(\theta(t))$$

and

$$\widetilde{Y}(s) = \frac{d}{dt} \exp_{\perp}^{\partial D} s\nu_{\text{in}}(\tilde{\theta}(t))\big|_{t=0}.$$

Notice that since we are in Case II, $\tilde{\theta}(t) \in \partial D$. We can use the comparison theory with $\overline{M} = \mathbb{R}^n$, $\overline{N} = \mathbb{R}^{n-1} \subset \mathbb{R}^n$ to conclude, using (3.2), that

$$\frac{2\langle Y'(s), Y(s)\rangle}{\|Y(s)\|^2} = \frac{d}{ds} \log \|Y(s)\|^2 \leq 0,$$

i.e.

$$\frac{d}{ds} \langle Y(s), Y(s)\rangle = 2\langle Y'(s), Y(s)\rangle \leq 0.$$

However a similar statement is true for $\widetilde{Y}$. We have $\widetilde{Y}(\alpha_1 - s) = Y(s)$ and

$$0 \geq \frac{d}{ds} \langle Y(s), Y(s)\rangle = \frac{d}{ds} \langle \widetilde{Y}(\alpha_1 - s), \widetilde{Y}(\alpha_1 - s)\rangle$$

$$= -\frac{d}{dr}\langle \widetilde{Y}(r), \widetilde{Y}(r)\rangle\big|_{r=\alpha_1-s} \geq 0.$$

Thus $\frac{d}{ds}\|Y(s)\|^2 = 0$, for $0 \le s \le \alpha_1$. And $\|Y(s)\| = \|Y(0)\| = \|v\|$
for $0 \le s \le \alpha_1$. Thus $(r, p) \mapsto \exp_\perp^{\partial D} r\nu_{\text{in}}(p)$ is a local isometry on
$[0, \alpha_1] \times \partial D$. Define a map $\tau : \partial D \to \partial D$ by $p \mapsto \exp_\perp^{\partial D} \alpha_1 \nu_{\text{in}}(p)$.
This is clearly a local isometry, even more, it is one to one and hence
an isometry. In fact, it is clear that τ is an involution and hence
one to one. We can extend τ to an isometry of $[0, \alpha_1] \times \partial D$ by
$\tau(r, p) = (\alpha_1 - r, \tau(p))$. This action is free. $Q = [0, \alpha_1] \times \partial D/\langle\tau\rangle$
is the quotient manifold (with boundary). It is clear that the map
$Q \mapsto \overline{D}$ induced by $(r, p) \mapsto \exp_\perp^{\partial D} r\nu_{\text{in}}(p)$ is a local isometry. We
just need to show that $Q \to \overline{D}$ is 1-1. Notice that Q is compact and
so $Q \to \overline{D}$ is a covering space with a finite number of sheets. Let this
number be k. Thus $\text{vol}\,Q = k\,\text{vol}\,D$ as well as $\text{vol}\,\partial Q = k\,\text{vol}\,\partial D$.
However, it is clear from our construction that ∂Q is isometric to ∂D
and so $\text{vol}\,\partial Q = \text{vol}\,\partial D$. It follows that $k = 1$. This is equivalent to
$Q \to \overline{D}$ being one to one. This completes the proof of Theorem 3.11.

THEOREM 3.22. *Let D be a boundary-convex domain in a Riemann-
ian manifold M. Assume that ∂D is compact and smooth. Assume
$0 \le -S_{\nu_{\text{in}}}^{\partial D} \le \kappa_2$ and $0 \le K \le b$. We then have the following*

(i) *If one of (a), (b) or (c) below holds then $(r, p) \mapsto \exp_\perp^{\partial D} r\nu_{\text{in}}(p)$
is a diffeomorphism on $(0, G(\kappa_2, b)) \times \partial D$.*

    (a)   *$K > 0$ on all 2-planes in $T_q D$, for some point $q \in D$.*

    (b)   *$0 \ne S_{\nu_{\text{in}}}^{\partial D}$ at some point in ∂D.*

    (c)   *M has no compact totally geodesic submanifolds.*

(ii) *If there is a point $p \in \partial D$ such that the ball of radius R_1
centered at $\exp_\perp^{\partial D} R_1 \nu_{\text{in}}(p)$ is contained in $\overline{D}$ then*

$$(r, p) \mapsto \exp_\perp^{\partial D} r\nu_{\text{in}}(p)$$

is a diffeomorphism on $(0, \min\{G(\kappa_2, b), R_1\}) \times \partial D$.

Notice that we may take R_1 to be any number less than the in-
radius of D. In case D is boundary convex we have, by a standard
result [L-Y], that the in-radius is greater than $\frac{\pi}{2\sqrt{\lambda_1(D)}}$ so in this case
we may take R_1 to be this number. This fact is relevant if one applies
these results to the inverse spectral project of §2.

Proof. The hypotheses of Theorem 3.11(ii) hold. Now (a), (b) or
(c) exclude the possibility that $\overline{D} = \partial D_{\tau, s_0}$ and so the conclusions of
Theorem 3.11(iia) hold. Thus (i) holds. For (ii) notice that even in
case $\overline{D} = \partial D_{\tau, s_0}$ holds we must have $R_1 \le \frac{s_0}{2}$ and the result follows
easily.

COROLLARY 3.23. *Let* $p \in \partial D$ *and let* $p_0 = \exp_{\perp}^{\partial D} s_0 \nu_{\mathrm{in}}(p)$. *Let* $B(p_0, s_0)$ *denote the geodesic ball of radius* s_0 *and center* p_0. *The number* s_0 *need not be less than the injectivity radius of* M. *If the hypotheses of Theorem* 3.11(i) *hold then* $B(p_0, s_0) \subset \overline{D}$ *if* $s_0 \leq r_*$. *If the hypotheses of Theorem* 3.11(ii) *hold, and* $D \neq \partial D_{\tau, s}$ *for any* $s < 2G(\kappa_2, b)$ *and any* τ, *then* $B(p_0, s_0) \subset \overline{D}$ *with* $s_0 = G(\kappa_2, b)$.

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