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## **SUPERHARMONICITY OF CURVATURES FOR SURFACES OF CONSTANT MEAN CURVATURE**

WU HSIUNG HUANG

# SUPERHARMONICITY OF CURVATURES FOR SURFACES OF CONSTANT MEAN CURVATURE

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**For a hypersurface of constant mean curvature in an Euclidean space, we illustrate the superharmonicity of various forms of curvatures. Applications in parametric capillary surfaces, constant rank properties of surfaces and a Shiffman type theorem are discussed.**

Given a hypersurface  $M^n$  of constant mean curvature in  $R^{n+1}$ , we study its geometry by showing the superharmonicity of various forms of curvatures, such as the scalar curvature  $R$ , the Gauss-Kronecker curvature  $K_n$ , the  $\nu$ th mean curvature  $K_\nu$  and the level curvature  $L$ .

We first prove that, for  $M^n$  of positive sectional curvatures,  $R$  and  $\log K_n$  are superharmonic [Theorems 1 and 3]. When  $M^n$  has non-negative sectional curvatures,  $\log K_n$  is not everywhere defined, but  $K_n$  still satisfies the strong minimum principle that it never attains interior nonnegative minimum unless  $M^n$  is a portion of a sphere or of a cylinder. Such findings explain various results, known or unknown, on the geometry of  $M^n$  with a simple geometric version.

As an application, we consider a liquid contained but not full within a convex closed container left in outer space, i.e. a space of zero gravity. The Euler-Lagrange equation yields that the capillary surface of the liquid has constant mean curvature with constant contact angle  $\gamma$  against the wall of the container. Consider a  $C^2$ -deformation of the capillary surface by pouring in an extra amount of the liquid until a small vacuum bubble remained in the middle. The superharmonicity of  $\log K_n$  proves the convexity of capillary surfaces during the deformation, valid for  $\gamma = 0$  and arbitrary dimension  $n$  [Theorems 2 and 4]. This can be regarded as a generalization of the results by Chen-Huang [4] and Korevaar [11] where the nonparametric case was considered. Nonconvex examples of the problem were obtained by Finn [5] for nonzero  $\gamma$ . The strong minimum principle for  $K_n$  also reveals the intrinsic geometric effect of  $\gamma = 0$  on the convexity of the capillary surface, with the argument that  $K_n$  nonnegative on the boundary  $\partial M$  guaranteed by  $\gamma = 0$  is to ensure  $K_n$  positive in the interior.

As a second application of superharmonicity of curvatures, we consider a constant rank theorem established by Korevaar-Lewis [10] that the Hessian of the height function  $u = u(x)$ ,  $x \in \Omega \subset R^n$  has constant rank when the mean curvature  $H$  of a convex surface  $u = u(x)$  satisfies  $H^{-1}$  being strictly convex in  $u$ . Refining our previous work by considering the log-concavity of the  $\nu$ th mean curvature  $K_\nu$  defined by the  $\nu$ th symmetric function of the principal curvatures, we prove [Theorem 5] that given a convex hypersurface  $M^n$  in  $R^{n+1}$  with concave mean curvature  $H$  [the case that  $H \equiv \text{constant}$  is included—see §4.2], the second fundamental form  $(h_{ij})$  has constant rank.

We also point out that the superharmonicity of various forms of curvatures gives rise to classical known results, such as the rigidity of spheres, the classification of complete convex surfaces by Liebman-Klotz-Osserman [9] and the convexity theorem of level curves on minimal surfaces by Shiffman [14]. For the last one, we extend to surfaces of constant mean curvature in §5, deal with the “level curvature” (i.e. the curvature of level curves) and prove its superharmonicity in a sense [Theorem 8]. The computation is relatively tedious. We thereby conclude that deforming smoothly a given annular-like soap film with two ending curves lying in two respective horizontal planes to a vertical DeLaunay surface, all the level curves are convex during the deformation, if the outside pressure of the beginning soap film is no less than the inside pressure [Theorem 9]. This generalizes Shiffman’s theorem on minimal surfaces. Our method is free from using the complex analysis apparatus.

**1. The superharmonicity of scalar curvature.** Let  $M^n$  be an  $n$ -dimensional hypersurface of constant mean curvature immersed in  $R^{n+1}$ . We choose an orthonormal frame  $\{e_A\}$  such that  $\{e_1, \dots, e_n\}$  are tangent to  $M$  and  $e_{n+1}$  normal. Let the corresponding coframe be denoted by  $\{w_A\}$  and the connection forms by  $\{w_{AB}\}$ . The pull-backs of them through the immersion are still denoted by  $\{w_A\}$ ,  $\{w_{AB}\}$  in the abuse of notation. Therefore in particular,

$$(1.1) \quad w_{n+1} = 0.$$

We use the index convention as follows,

$$1 \leq i, j, k, \dots \leq n; \quad 1 \leq A, B, C, \dots \leq n + 1.$$

The second fundamental form  $B$  is defined by the matrix  $(h_{ij})$  with

$$(1.2) \quad w_{i, n+1} = h_{ij} w_j.$$

We have the curvature tensor  $R_{ijkl}$  expressed by

$$(1.3) \quad R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}.$$

The sectional curvature  $K_{ij}$  of the plane spanned by  $\{e_i, e_j\}$  is defined by

$$(1.4) \quad K_{ij} = R_{ijij} = h_{ii}h_{jj} - h_{ij}^2 = \lambda_i\lambda_j \quad (\text{if } i \neq j)$$

where  $\lambda_1, \dots, \lambda_n$  are the principal curvatures of  $M^n$  in  $R^{n+1}$ , and the scalar curvature  $R$  is defined by

$$(1.5) \quad R = \sum_{i,j} R_{ijij} = \sum_{i \neq j} \lambda_i\lambda_j.$$

We have

**THEOREM 1.** *Let  $M^n$  be a hypersurface of constant mean curvature immersed in  $R^{n+1}$ . Then the Laplacian of the scalar curvature  $R$  is*

$$(1.6) \quad \Delta_M R = - \sum_{i \neq j} (\lambda_i - \lambda_j)^2 K_{ij} - 2|\nabla B|^2,$$

where  $|\nabla B|^2 = \sum_{i,j,k} h_{ijk}^2$ . Furthermore if  $M$  is of non-negative sectional curvatures, then  $R$  is superharmonic on  $M$ .

*Proof.* We have a commutation formula,

$$(1.7) \quad h_{ijkl} = h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk},$$

while noting that  $h_{ijk}$  is symmetric in any two indices among the three. This gives

$$(1.8) \quad \begin{aligned} \Delta_M h_{ij} &= h_{ijkk} \quad (\text{using the summation convention}) \\ &= h_{ikjk} \\ &= h_{ikkj} + h_{mk}R_{imkj} + h_{im}R_{kmkj} \\ &= h_{kkij} + h_{mk}R_{imkj} + h_{im}R_{kmkj}. \end{aligned}$$

On the other hand, we have

$$(1.9) \quad \begin{aligned} \Delta_M R &= \sum_{i,j,k} (h_{ii}h_{jj} - h_{ij}^2)_{kk} \\ &= \sum_{i,j,k} (h_{iik}h_{jj} + h_{ii}h_{jjk} - 2h_{ij}h_{ijk})_k \\ &= \sum_{i,j,k} (h_{iikk}h_{jj} + h_{ii}h_{jjkk} - 2h_{ijkk}h_{ij}) \\ &\quad + 2(h_{iik}h_{jjk} - h_{ij}^2). \end{aligned}$$

Since the mean curvature  $H \equiv \sum_i h_{ii}$  is constant, it is clear that

$$(1.10) \quad \sum_i h_{iikk} h_{jj} = 0 = \sum_j h_{ii} h_{jjkk}$$

and

$$(1.11) \quad \sum_i h_{iik} = 0.$$

Hence we obtain

$$(1.12) \quad \Delta_M R = - \sum_{i,k} 2h_{iikk} h_{ii} - \sum_{i,j,k} 2h_{ij}^2$$

by suitably taking  $e_1, \dots, e_n$  to be the principal directions at a given point  $p$ . We continue the computation that at  $p$ ,

$$(1.13) \quad \begin{aligned} \Delta_M R &= -2 \sum_{i,k} (h_{kkii} + h_{mk} R_{imki} + h_{im} R_{kmki}) h_{ii} - 2|\nabla B|^2 \\ &= \left( 2 \sum_{i,k} h_{kk} h_{ii} R_{ikik} - 2 \sum_{i,k} h_{ii}^2 R_{kiki} \right) - 2|\nabla B|^2 \\ &= \left( 2 \sum_{i \neq k} (\lambda_k \lambda_i K_{ik}) - \sum_{i \neq k} (\lambda_i^2 + \lambda_k^2) K_{ik} \right) - 2|\nabla B|^2 \\ &= - \sum_{i \neq k} (\lambda_i - \lambda_k)^2 K_{ik} - 2|\nabla B|^2. \end{aligned}$$

This proves formula (1.6).

As a corollary for  $n = 2$ , we obtain a stronger form.

**COROLLARY 1.** *Let  $M$  be a surface of constant mean curvature in  $R^3$  and  $K$  be its Gaussian curvature. Then at non-umbilic points, we have*

$$(1.14) \quad \Delta_M K = -(\lambda_1 - \lambda_2)^2 K - \frac{4}{(\lambda_1 - \lambda_2)^2} |\nabla K|^2.$$

*And therefore  $K$  attains neither interior nonnegative minimum nor interior negative maximum, unless  $M$  is a sphere or a cylinder.*

*Proof.* We have

$$\begin{aligned}
 |\nabla K|^2 &= \sum_k ((h_{11}h_{22} - h_{12}^2)_k)^2 \\
 &= \sum_k (h_{11k}h_{22} + h_{11}h_{22k} - 2h_{12}h_{12k})^2 \\
 &= (h_{111}h_{22} + h_{11}h_{221})^2 + (h_{112}h_{22} + h_{11}h_{222})^2 \\
 &\quad \text{(by choosing frame so that } h_{12} = 0 \text{ at a given point)} \\
 &= (h_{11} - h_{22})^2(h_{221}^2 + h_{112}^2) \\
 &\quad \text{(since } h_{111} = -h_{221}, h_{222} = -h_{112})
 \end{aligned}$$

and

$$\begin{aligned}
 (1.15) \quad |\nabla B|^2 &= (h_{111}^2 + 2h_{121}^2 + h_{221}^2) + (h_{112}^2 + 2h_{112}^2 + h_{222}^2) \\
 &= 4(h_{221}^2 + h_{112}^2).
 \end{aligned}$$

Hence

$$(1.16) \quad |\nabla B|^2 = \frac{4}{(\lambda_1 - \lambda_2)^2} |\nabla K|^2.$$

By substituting  $R = 2K$  into (1.6), we obtain (1.14). The second assertion is then evident.

**REMARK 1.1.** For  $n = 2$ , the formula (1.14) can also be proved in a classical way. Consider the coefficient

$$\Phi = \frac{L - N}{2} - iM$$

of Hopf's differential, where  $L, M$  and  $N$  are the coefficients of the second fundamental form. By virtue of the Coddazzi equations,  $\Phi$  is holomorphic when the mean curvature is constant. Hence  $\log|\Phi^2|$  is harmonic on the surface. By computing  $\Delta(\log|\Phi^2|)$  with respect to the metric of the surface at non-umbilic points, we easily obtain formula (1.14). Since umbilic points attain maximal values for  $K$ , the superharmonicity around umbilic points is still valid.

**2. Parametric capillary surfaces.** Given a liquid  $V$  contained but not full in a bounded closed container  $C$ . Leave them in outer space, i.e. in the field of zero gravity. Let  $M$  denote the liquid surface inside  $C$  in equilibrium.

The Euler-Lagrange equation of energy function of the liquid implies that  $M$  has constant contact angle  $\gamma$  independent of the shape of  $C$ . This was originally observed by Pierre Simon Laplace [12],

Thomas-Young [18] and Carl Gauss [6], when the container is a vertical tube.

A natural question is raised. How does the shape of  $C$  affect the geometry of surface  $M$ ? For the case of tube with given cross section  $\Omega$  where  $M$  is expressed nonparametrically in  $u = u(x)$ ,  $x \in \Omega$ , related results have been obtained. In a joint work of the author with J. T. Chen [4], it was proved that for  $\gamma = 0$  the convexity of  $\Omega$  implies the strict convexity of the surface  $M$  in a strong sense that the Gaussian curvature is positive at the interior points. (In this paper we keep on using *strictly convex* to name the convexity in this strong sense.) The result was soon generalized by Korevaar [11] to higher dimensions of the domain  $\Omega$ , using his maximum principle of concavity function and a limiting argument that extends convexity results in the gravitational field to outer space. On the other hand, Finn [5] found counterexamples over trapezoids for each nonzero  $\gamma$ .

Why the assumption of  $\gamma = 0$  is that essential has been partially revealed in the proof of Korevaar and the examples of Finn. However, based on (1.14) and the continuity method we obtain the superharmonicity of nonnegative Gaussian curvature  $K$  for general parametric surfaces of constant mean curvature. This gives a more satisfactory and geometric explanation for the essentiality of vanishing  $\gamma$ . In fact, that  $\gamma = 0$  guarantees nonnegative Gaussian curvature on the boundary. The superharmonicity of  $K$  then shows its positiveness in the interior.

We make precise arguments of this for general parametric capillary hypersurfaces of dimension 2 and higher dimensions respectively in the subsequent theorems.

**THEOREM 2.** *Given a liquid contained inside a closed, strictly convex container  $C$  ( $C$  denoting the bounding wall) in outer space. We suppose that the material of the liquid and the container are made so that  $\gamma = 0$  wherever they contact. Pouring more liquid gradually into the container, it is assumed that the liquid surface  $M$  deforms smoothly until the small vacuum part left in the middle of the liquid. Then the interior of  $M$  is strictly convex in the beginning as well as during the deformation. Moreover, we have the minimum principle,*

$$(2.1) \quad K_M \geq \min K_C > 0$$

where  $K_M$  and  $K_C$  denote the Gaussian curvature of  $M$  and  $C$  respectively. And therefore the vacuum part at any stage is a strict convex set.

*Proof.* First we note that the small vacuum part left in the middle of the liquid is bounded by a closed surface of constant mean curvature embedded in  $R^3$ . By a theorem of Aleksandrov [1], it is known that the vacuum part is simply a sphere on which the Gaussian curvature is a positive constant. We reverse the deformation process that we begin with the sphere  $S^2(\varepsilon_0) \equiv M_0$  and deform  $M_0$  smoothly into  $M_1 \equiv M$  through  $\{M_t\}$ . Suppose that (2.1) is not satisfied for  $M$ , then there exists a smallest  $\tau$  such that an interior point  $P_0$  in  $M_\tau$  exists with  $K_{M_\tau}(p_0) = \min_{M_\tau} K_{M_\tau} > 0$ . But this is impossible by virtue of (1.14) with the strong minimum principle. Hence

$$(2.2) \quad K_M \geq \min K_C > 0$$

as required.

**REMARK.** For the non-parametric case, i.e. for the liquid contained in a tube with convex cross section  $\Omega$ , the above argument is modified to deform the tube into a circular tube in which the capillary surface is a lower hemisphere. We follow the reasoning in Theorem 2 except we have to exclude the possibility that  $K_{M_{t_0}}$  attains a zero minimum at a boundary point  $P_0$  for certain  $t_0 < 1$ . This can be argued by a refined strong maximum principle. The remark also implies that the assumption of strict convexity of  $C$  in Theorem 2 can weaken to convexity.

**3. The log-concavity of Gauss-Kronecker curvature.** For a hypersurface  $M^n$  immersed in  $R^{n+1}$ , the Gauss-Kronecker curvature  $K_n$  is defined by the product of principal curvature  $\{\lambda_i\}$ , i.e.  $K_n \equiv \lambda_1 \lambda_2 \cdots \lambda_n = \det h_{ij}$ . It describes the ratios of  $n$ -dimensional infinitesimal areas under the Gauss map

$$M^n \rightarrow \text{the unit sphere } S^n(1),$$

which sends a point  $p$  to the unit normal  $e_{n+1}$  at  $p$ .

**THEOREM 3.** *For a hypersurface  $M^n$  immersed in  $R^{n+1}$  with constant mean curvature and nonnegative sectional curvatures, we have the strong minimum principle for  $K_n$ , i.e.*

$$(3.1) \quad \min_M K_n = \min_{\partial M} K_n$$

with  $K_n$  never attaining minimum in the interior unless  $K_n$  is constant. Furthermore we have the following results: (i)  $\log K_n$  is superharmonic on the points where  $K_n > 0$ , (ii) On general points of  $M^n$ ,  $\log \tilde{K}_n$  is



superharmonic where  $\tilde{K}_n \equiv \det(h_{ij} + \varepsilon \delta_{ij})$ ,  $\varepsilon > 0$  and  $\tilde{h}_{ij} \equiv h_{ij} + \varepsilon \delta_{ij}$ ,  $\delta_{ij}$  being the Kronecker symbol.

REMARK. In the case that  $K_n$  is a positive constant,  $M^n$  is umbilic by (3.10) and (3.11) and therefore it is a portion of  $n$ -spheres. When  $K_n$  has a zero minimum,  $M^n$  is a cylinder by (4.21).

Proof. We consider locally on  $M_n$  an orthonormal moving frame  $e_1, e_2, \dots, e_n$ . For a function  $u$  on  $M^n$ , the Laplacian  $\Delta u$  of  $u$  is defined by

$$(3.2) \quad \Delta u = u_{kk}$$

where  $u_{ij}$  is the covariant derivative with respect to the frame and the summation convention on  $k$  is used here and likewise in the sequel. We recall that the Laplacian thus defined is the so-called Beltrami-Laplace operator defined by the metric of  $M_n$ . It follows straightforwardly that

$$(3.3) \quad \Delta \log \tilde{K}_n = \frac{\Delta \tilde{K}_n}{\tilde{K}_n} - \left| \frac{\nabla \tilde{K}_n}{\tilde{K}_n} \right|^2$$

and

$$(3.4) \quad \begin{aligned} \Delta \tilde{K}_n &= \sum_{\sigma} (-1)^{\sigma} (\tilde{h}_{1\sigma_1} \tilde{h}_{2\sigma_2} \cdots \tilde{h}_{n\sigma_n})_{kk} \\ &= \sum_{\sigma} (-1)^{\sigma} (\tilde{h}_{1\sigma_1 k} \tilde{h}_{2\sigma_2} \cdots \tilde{h}_{n\sigma_n} + \cdots + \tilde{h}_{1\sigma_1} \tilde{h}_{2\sigma_2} \cdots \tilde{h}_{n\sigma_n k})_k \\ &= \sum_{\sigma} (-1)^{\sigma} \{ [\tilde{h}_{1\sigma_1 k k} \tilde{h}_{2\sigma_2} \cdots \tilde{h}_{n\sigma_n} + \cdots + \tilde{h}_{1\sigma_1} \tilde{h}_{2\sigma_2} \cdots \tilde{h}_{n\sigma_n k k}] \\ &\quad + 2[\tilde{h}_{1\sigma_1 k} \tilde{h}_{2\sigma_2 k} \tilde{h}_{3\sigma_3} \cdots \tilde{h}_{n\sigma_n} + \cdots \\ &\quad + \tilde{h}_{1\sigma_1} \tilde{h}_{2\sigma_2} \tilde{h}_{3\sigma_3} \cdots h_{n-1\sigma_{n-1} k} h_{n\sigma_n k}] \} \end{aligned}$$

where  $\sigma = (\sigma_1, \dots, \sigma_n)$  is a permutation of  $(1, \dots, n)$ . By choosing the frame  $\{e_i\}$  so that  $h_{ij} = \lambda_i \delta_{ij}$  at a given point  $p$ , we have

$$(3.5) \quad \tilde{h}_{ij} = (\lambda_i + \varepsilon) \delta_{ij}$$

at  $p$ . Formula (3.4) is reduced to

$$(3.6) \quad \begin{aligned} \Delta \tilde{K}_n &= [\tilde{h}_{11kk} \tilde{h}_{22} \cdots \tilde{h}_{nn} + \cdots + \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{nnkk}] \\ &\quad + 2[(\tilde{h}_{11kk} \tilde{h}_{22k} - \tilde{h}_{12k}^2) \tilde{h}_{33} \cdots \tilde{h}_{nn} + \cdots \\ &\quad + \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{n-2, n-2} (\tilde{h}_{n-1, n-1, k} \tilde{h}_{nnk} - \tilde{h}_{n-1, n, k}^2)]. \end{aligned}$$

Hence we find

$$\begin{aligned}
 (3.7) \quad \Delta \log \tilde{K}_n &= \frac{1}{\tilde{K}_n} \{ [\tilde{h}_{11kk} \tilde{h}_{22} \cdots \tilde{h}_{nn} + \cdots + \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{nnkk}] \\
 &\quad + 2[(\tilde{h}_{11k} \tilde{h}_{22k} - \tilde{h}_{12k}^2) \tilde{h}_{33} \cdots \tilde{h}_{nn} + \cdots \\
 &\quad \quad \quad + \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{n-2, n-2} \\
 &\quad \quad \quad \times (\tilde{h}_{n-1, n-1, k} \tilde{h}_{nnk} - \tilde{h}_{n-1, n, k}^2)] \} \\
 &\quad - \frac{1}{\tilde{K}_n^2} [\tilde{h}_{11k} \tilde{h}_{22} \cdots \tilde{h}_{nn} + \cdots + \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{nnk}]^2.
 \end{aligned}$$

It is clear that

$$(3.8) \quad \tilde{h}_{ijk} = h_{ijk} \quad \text{and} \quad \tilde{h}_{ijkl} = h_{ijkl},$$

and therefore

$$(3.9) \quad \tilde{h}_{kkij} = h_{kkij} = 0.$$

By using the commutation formula (1.7) for  $\tilde{h}_{ijkl}$ , the first term of (3.7) at  $p$  becomes

$$\begin{aligned}
 (3.10) \quad &\frac{1}{\tilde{K}_n} \sum_{i, k} \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{iikk} \cdots \tilde{h}_{nn} \\
 &\quad (\tilde{h}_{iikk} \text{ appears at the } i\text{th position}) \\
 &= \frac{1}{\tilde{K}_n} \sum_{i, k} \tilde{h}_{11} \tilde{h}_{22} \cdots (\tilde{h}_{kkii} - \tilde{h}_{kk} R_{ikik} + \tilde{h}_{ii} R_{kiki}) \cdots \tilde{h}_{nn} \\
 &= \frac{1}{\tilde{K}_n} \sum_{i, k} \tilde{h}_{11} \tilde{h}_{22} \cdots ([-\tilde{h}_{kk} + \tilde{h}_{ii}] h_{ii} h_{kk}) \cdots \tilde{h}_{nn} \\
 &= \frac{\tilde{K}_n}{\tilde{K}_n} \sum_{i, k} \left[ -\frac{\tilde{h}_{kk}}{\tilde{h}_{ii}} + 1 \right] h_{ii} h_{kk} \\
 &= \sum_{i < k} \left[ -\frac{\tilde{h}_{kk}}{\tilde{h}_{ii}} + 2 - \frac{\tilde{h}_{ii}}{\tilde{h}_{kk}} \right] h_{ii} h_{kk} \\
 &= - \sum_{i < k} [\tilde{h}_{kk} - \tilde{h}_{ii}]^2 \left( \frac{h_{ii}}{\tilde{h}_{ii}} \right) \left( \frac{h_{kk}}{\tilde{h}_{kk}} \right) \leq 0
 \end{aligned}$$

and the equality holds if and only if  $p$  is an umbilic or a cylindric

point. The remaining terms of (3.7) equal

$$\begin{aligned}
 (3.11) \quad & \frac{2}{\tilde{K}_n} [(h_{11k}h_{22k} - h_{12k}^2)\tilde{h}_{33}\cdots\tilde{h}_{nn} + \cdots] \\
 & - \frac{1}{\tilde{K}_n^2} [h_{11k}\tilde{h}_{22}\tilde{h}_{33}\cdots\tilde{h}_{nn} + \tilde{h}_{11}h_{22k}\tilde{h}_{33}\cdots\tilde{h}_{nn} + \cdots]^2 \\
 & = - \left[ \frac{2h_{12k}^2}{\tilde{h}_{11}\tilde{h}_{22}} + \cdots \right] - \left[ \frac{h_{11k}^2}{\tilde{h}_{11}^2} + \cdots \right] \\
 & = - \sum_{i,j,k} \frac{h_{ijk}^2}{\tilde{h}_{ii}\tilde{h}_{jj}} \leq 0.
 \end{aligned}$$

Therefore at a given point  $p$  on which  $K_n \geq 0$ , we have shown (ii) of the theorem, i.e. we have proved the superharmonicity of  $\log \tilde{K}_n$ , or more precisely,

$$\begin{aligned}
 (3.12) \quad \Delta \log \tilde{K}_n &= - \sum_{i < k} (\lambda_i - \lambda_k)^2 \frac{\lambda_i \lambda_k}{(\lambda_i + \varepsilon)(\lambda_k + \varepsilon)} \\
 &\quad - \sum_{i,k} \frac{|\nabla h_{ik}|^2}{(\lambda_i + \varepsilon)(\lambda_k + \varepsilon)} \leq 0
 \end{aligned}$$

where  $\tilde{K}_n \equiv \det(h_{ij} + \varepsilon \delta_{ij})$ ,  $\varepsilon > 0$ , and the equality holds if and only if  $p$  is umbilic or cylindrical. The same computation follows evidently for  $\log K_n$  at  $p$  with  $K_n > 0$ . Thus we also obtain (i) of the theorem.

To show the strong minimum principle for  $K_n$  on general points of  $M$ , we note that  $\tilde{K}_n$  satisfies strong minimum principle for each  $\varepsilon > 0$ , by virtue of (3.12). However,

$$\begin{aligned}
 (3.13) \quad \tilde{K}_n &= \tilde{K}_n(\varepsilon) = \det(h_{ij} + \varepsilon \delta_{ij}) \\
 &= K_n + K_{n-1}\varepsilon + K_{n-2}\varepsilon^2 + \cdots + K_1\varepsilon^{n-1} + \varepsilon^n,
 \end{aligned}$$

where  $K_\nu$  is the  $\nu$ th mean curvature of  $M^n$  in  $R^{n+1}$ , defined by the  $\nu$ th symmetric function of principal curvatures  $\lambda_1, \lambda_2, \dots, \lambda_n$ .

Suppose that  $K_n$  attains a zero minimum at an interior point  $P_0$  of  $M^n$ . A contradiction is shown to follow unless  $K_n$  is constantly zero. The function  $K_n$  is real analytic on  $M^n$ , as  $M^n$  inherits a real analytic structure from  $R^{n+1}$  with all coordinate functions real analytic in  $M^n$ . The set  $Z$  of minimal points of  $K_n$  on  $M^n$  is a finite union of real analytic submanifolds around  $P_0$ . Notice that on  $Z$ ,

$$\log \tilde{K} \equiv \log \tilde{K}(\varepsilon) = \log \varepsilon + \log(K_{n-1} + \varepsilon K_{n-2} + \cdots + \varepsilon^{n-1})$$

while on  $M^n - Z$ ,

$$\log \tilde{K} \equiv \log \tilde{K}(\varepsilon) = \log[K_n + \varepsilon(K_{n-1} + \varepsilon K_{n-2} + \dots + \varepsilon^{n-2})].$$

Let  $\gamma$  be a geodesic segment of  $M^n$  with  $\gamma(0) = P_0$  and  $\gamma - \{P_0\}$  contained in  $M^n - Z$ . As  $\varepsilon$  tends to zero,  $\gamma(-t)$  and  $\gamma(t)$  have values of  $\log \tilde{K}_n$  bounded below, but at  $P_0$ ,  $\log \tilde{K}_n$  tends to  $-\infty$ . This gives that

$$\lim_{\varepsilon \rightarrow 0} \frac{d^2}{dt^2} (\log \tilde{K}_n \circ \gamma(t))(0) = +\infty.$$

On the other hand, the second derivatives of  $\log \tilde{K}_n$  along the directions tangent to  $Z$  as well as the gradient of  $\log \tilde{K}_n$  are convergent and therefore bounded when  $\varepsilon \rightarrow 0$ . It follows that as  $\varepsilon \rightarrow 0$ ,

$$\Delta \log \tilde{K}_n(\varepsilon) \rightarrow +\infty \quad \text{at } P_0.$$

This contradicts (3.12). The proof is completed.

**REMARK.** To avoid using the real analyticity of the zero set  $Z$ , a general rigorous argument of the above proof is given in [8].

Using Theorem 3, and following the argument in Theorem 2, we obtain

**THEOREM 4.** *The generalization of Theorem 2 to high dimensions is valid.*

**4. Log-concavity of  $\nu$ th mean curvature and a constant rank theorem.** Before considering  $\nu$ th mean curvatures, we mention that the superharmonicity of curvatures explained classical known results. For an example, Liebman-Klotz-Osserman [7] proved that a complete convex surface  $M$  of constant mean curvature in  $R^3$  is a cylinder, a plane or a sphere. Since the complete surface  $M$  of nonnegative curvature is parabolic, the superharmonicity of  $K$  when  $K \geq 0$  implies immediately that  $K$  is a constant. Hence  $M$  is a cylinder, a plane or a sphere. The version also explains why the standard sphere is rigid among the class of surfaces of a given constant mean curvature, while examples with  $K$  of alternating sign may possibly be deformed.

By considering superharmonicity of logarithm of the  $\nu$ th mean curvature  $K_\nu$  of  $M^n$  in  $R^{n+1}$ , we obtain a constant rank theorem as a further application. The  $\nu$ th mean curvature  $K_\nu$  is defined by the  $\nu$ th symmetric function of principal curvatures  $\lambda_1, \lambda_2, \dots, \lambda_n$ . For example,

$$(4.0) \quad \begin{aligned} K_1 &\equiv \lambda_1 + \lambda_2 + \dots + \lambda_n = \text{mean curvature } H, \\ K_2 &\equiv \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \dots + \lambda_{n-1} \lambda_n \end{aligned}$$

and

$$K_n \equiv \lambda_1 \lambda_2 \cdots \lambda_n = \text{Gauss-Kronecker curvature } K_n.$$

It is then clear that

$$(4.1) \quad \det(h_{ij} + \lambda \delta_{ij}) = K_n + K_{n-1} \lambda + \cdots + K_1 \lambda^{n-1} + \lambda^n.$$

**THEOREM 5.** *Let  $M^n$  be a connected convex hypersurface immersed in  $R^{n+1}$  with concave mean curvature  $H$ , i.e.*

$$(4.2) \quad \frac{d^2}{dt^2} H \leq 0$$

along any geodesic parametrized by  $t$ , or equivalently, the second covariant derivatives constitute a nonpositive matrix. Let the rank of the second fundamental form  $(h_{ij})$  attain a minimum  $r_0$  at an interior point  $p_0$  of  $M_n$ . Then (i) each  $K_\mu$  with  $r_0 < \mu \leq n$  vanishes on  $M^n$ , (ii) the rank of  $(h_{ij})$  is constantly equal to  $r_0$  on  $M^n$ , (iii) the logarithm of the  $r_0$ th mean curvature  $K_{r_0}$  is superharmonic at the points where  $K_{r_0} > 0$ , and (iv)  $K_{r_0}$  satisfies the strong minimum principle, i.e.

$$(4.3) \quad \min_M K_{r_0} = \min_{\partial M} K_{r_0}$$

where  $K_{r_0}$  has no interior minimum unless  $K_{r_0}$  is constant.

**REMARK 4.1.** The statement shows that the constant rank theorem is essentially a consequence of superharmonicity of  $\log \tilde{K}_\mu$ ,  $r_0 < \mu \leq n$  (for the meaning of  $\tilde{K}_\mu$ , see below). The details are contained in the following proof.

*Proof.* It suffices to prove the strong minimum principle for  $K_\nu$  with  $r_0 \leq \nu \leq n$ , since  $K_{r_0+1}, K_{r_0+2}, \dots, K_n$  then vanish on  $M$  as they attain a zero minimum at the interior point  $p_0$ . We use the induction on  $\nu$  backward until  $\nu$  equals  $r_0$ . For  $\nu = n$ , Theorem 3 has shown that the strong minimum principle holds for  $K_n$ . Now assuming  $K_{\nu+1}, K_{\nu+2}, \dots, K_n$  satisfies the strong minimum principle, we claim that  $K_\nu$  does also. However this induction hypothesis yields  $K_{\nu+1} = K_{\nu+2} = \cdots = K_n = 0$  at every point of  $M$ . Given an interior point  $p_1$  of  $M_n$ , we choose around  $p_1$  a local frame  $\{e_i\}$  on  $M^n$  by the following construction:

(1) letting

$$(4.4) \quad h_{ij} = \lambda_i \delta_{ij} \quad \text{at } p_1,$$

(2) defining  $e_{\nu+1}, e_{\nu+2}, \dots, e_n$  in a neighborhood of  $p_1$  by principal directions with zero principal curvature,

$$(4.5) \quad \lambda_{\nu+1} = \lambda_{\nu+2} = \dots = \lambda_n = 0.$$

It is direct to verify that the tangent subspace by  $\{e_{\nu+1}, \dots, e_n\}$  on a neighborhood of  $p_1$  is parallel in  $M^n$ . Hence we may construct a local frame  $\{e_1, \dots, e_n\}$  parallel along radial geodesics emanating from  $p_1$  by holding (1) and (2) valid. We now define  $\tilde{h}_{ij}$ , the *perturbed*  $h_{ij}$ , by

$$(4.6) \quad \tilde{h}_{ij} = h_{ij} + \varepsilon_i \delta_{ij}$$

where

$$\varepsilon_i = \begin{cases} \varepsilon; & i = 1, 2, \dots, \nu, \\ 0; & i = \nu + 1, \dots, n, \end{cases}$$

$\varepsilon$  being a small positive number. It is not difficult to see that

$$\tilde{h}_{ijk} = \tilde{h}_{ikj} + O(\varepsilon)$$

uniformly in a compact neighborhood of  $p_1$ . We also define  $\tilde{K}_\nu$ , the *perturbed*  $K_\nu$ , by

$$(4.8) \quad \det(\tilde{h}_{ij} + \lambda \delta_{ij}) = \tilde{K}_n + \tilde{K}_{n-1} \lambda + \tilde{K}_{n-2} \lambda^2 + \dots + \tilde{K}_1 \lambda^{n-1} + \lambda^n.$$

It is clear that for any given  $\mu$  with  $\nu + 1 \leq \mu \leq n$ ,

$$(4.9) \quad \tilde{h}_{\mu\mu}(p_1) = 0,$$

$$(4.10) \quad \tilde{h}_{\mu\mu k}(p_1) = 0$$

and

$$(4.11) \quad \begin{aligned} \tilde{h}_{\mu\mu k k}(p_1) &= (\tilde{h}_{kk\mu\mu} - \tilde{h}_{kk} R_{\mu k \mu k} + \tilde{h}_{\mu\mu} R_{k \mu k \mu})(p_1) + O(\varepsilon) \\ &= \tilde{h}_{kk\mu\mu}(p_1) + O(\varepsilon) = \frac{d^2}{dt^2} H + O(\varepsilon) \quad \text{at } p_1 \leq O(\varepsilon), \end{aligned}$$

$t$  being the parameter of the geodesic generated by  $e_\mu$  through  $p_1$ .

We now compute  $\Delta \log \tilde{K}_\nu$ . To avoid the complication of exhibiting long indices, we simplify notations of formulas when the meaning is obvious. We have

$$(4.12) \quad \tilde{K}_\nu = \sum_{I, \sigma_I} (-1)^{\sigma_I} \tilde{h}_{i_1 \sigma_{i_1}} \dots \tilde{h}_{i_\nu \sigma_{i_\nu}}$$

where the summation ranges over all  $I \equiv (i_1, i_2, \dots, i_\nu)$  with

$$(4.13) \quad 1 \leq i_1 < i_2 < \dots < i_\nu \leq n,$$

and over all the permutations  $\sigma_I$  of  $I$ . It is straightforward that

$$\begin{aligned}
 (4.14) \quad \Delta \log \tilde{K}_\nu &= \frac{\Delta \tilde{K}_\nu}{\tilde{K}_\nu} - \left| \frac{\nabla \tilde{K}_\nu}{\tilde{K}_\nu} \right|^2 \\
 &= \frac{1}{\tilde{K}_\nu} \left\{ \sum_{I, \sigma_I} (-1)^{\sigma_I} [(\tilde{h}_{i_1 \sigma_{i_1} k k} \tilde{h}_{i_2 \sigma_{i_2}} \cdots \tilde{h}_{i_\nu \sigma_{i_\nu}} + \cdots) \right. \\
 &\qquad \qquad \qquad \left. + 2(\tilde{h}_{i_1 \sigma_{i_1} k} \tilde{h}_{i_2 \sigma_{i_2} k} \tilde{h}_{i_3 \sigma_{i_3}} \cdots \tilde{h}_{i_\nu \sigma_{i_\nu}} + \cdots)] \right\} \\
 &\quad - \frac{1}{(\tilde{K}_\nu)^2} \left\{ \sum_{I, \sigma_I} (-1)^{\sigma_I} [\tilde{h}_{i_1 \sigma_{i_1} k} \tilde{h}_{i_2 \sigma_{i_2}} \cdots \tilde{h}_{i_\nu \sigma_{i_\nu}} + \cdots] \right\}^2.
 \end{aligned}$$

By virtue of (4.4), (4.5), (4.6), (4.9) and (4.10), most of the terms in (4.14) vanish at  $p_1$  and we have

$$\begin{aligned}
 (4.15) \quad \Delta \log \tilde{K}_\nu &\leq \frac{1}{\tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{\nu\nu}} \{ [\tilde{h}_{11kk} \tilde{h}_{22} \cdots \tilde{h}_{\nu\nu} + \cdots + \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{\nu\nu kk}] \\
 &\qquad \qquad \qquad + 2[(\tilde{h}_{11k} \tilde{h}_{22k} - \tilde{h}_{12k}^2) \tilde{h}_{33} \cdots \tilde{h}_{\nu\nu} + \cdots \\
 &\qquad \qquad \qquad + \tilde{h}_{11} \tilde{h}_{22} \cdots (\tilde{h}_{n-1, n-1, k} \tilde{h}_{nnk} - \tilde{h}_{n-1, n, k}^2)] \} \\
 &\quad - \frac{1}{(\tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{\nu\nu})^2} [\tilde{h}_{11k} \tilde{h}_{22} \cdots \tilde{h}_{\nu\nu} + \cdots + \tilde{h}_{11} \tilde{h}_{22} \cdots \tilde{h}_{\nu\nu k}]^2 \\
 &\quad + O(\varepsilon).
 \end{aligned}$$

For any given  $i$  with  $1 \leq i \leq n$ , we note that by (4.2),

$$\begin{aligned}
 (4.16) \quad \tilde{h}_{iikk} &= \tilde{h}_{kkii} - \tilde{h}_{kk} R_{ikik} + \tilde{h}_{ii} R_{kiki} + O(\varepsilon) \\
 &\leq 0 - \sum_{i, k=1}^\nu (\lambda_k + \varepsilon)(\lambda_i \lambda_k) + \sum_{i, k=1}^\nu (\lambda_i + \varepsilon)(\lambda_i \lambda_k) + O(\varepsilon).
 \end{aligned}$$

Letting the three parts of (4.15) corresponding to the three brackets in (4.15) be denoted respectively by I, II and III, we have

$$\begin{aligned}
 (4.17) \quad \text{I} &\leq - \sum_{i, k=1}^\nu \left( \frac{\lambda_k + \varepsilon}{\lambda_i + \varepsilon} - 1 \right) \lambda_i \lambda_k + O(\varepsilon) \\
 &= - \sum_{1 \leq i < k \leq \nu} (\lambda_k - \lambda_i)^2 \frac{\lambda_i \lambda_k}{(\lambda_i + \varepsilon)(\lambda_k + \varepsilon)} + O(\varepsilon) \leq O(\varepsilon).
 \end{aligned}$$

For the sum II + III, we cancel the terms of the sort,

$$(4.18) \quad 2(\tilde{h}_{11}\tilde{h}_{22}\cdots\tilde{h}_{\nu\nu})\tilde{h}_{11k}\tilde{h}_{22k}\tilde{h}_{33}\cdots\tilde{h}_{\nu\nu}$$

in II with the cross terms of the sort

$$(4.19) \quad -2(\tilde{h}_{11k}\tilde{h}_{22}\tilde{h}_{33}\cdots\tilde{h}_{\nu\nu})(\tilde{h}_{11}\tilde{h}_{22k}\tilde{h}_{33}\cdots\tilde{h}_{\nu\nu})$$

to obtain

$$(4.20) \quad \text{II} + \text{III} = -2 \left[ \frac{\tilde{h}_{12k}^2}{\tilde{h}_{11}\tilde{h}_{22}} + \cdots \right] - \left[ \frac{\tilde{h}_{11k}^2}{\tilde{h}_{11}^2} + \frac{\tilde{h}_{22k}^2}{\tilde{h}_{22}^2} + \cdots \right] \\ = - \sum_{i,j=1}^{\nu} \sum_{k=1}^{\nu} \frac{\tilde{h}_{ijk}^2}{(\lambda_i + \varepsilon)(\lambda_j + \varepsilon)}.$$

Combining (4.17) and (4.20), we obtain

$$(4.21) \quad \Delta \log \tilde{K}_{\nu} \leq - \sum_{1 \leq i < j \leq \nu} (\lambda_i - \lambda_j)^2 \frac{\lambda_i \lambda_j}{(\lambda_i + \varepsilon)(\lambda_j + \varepsilon)} \\ - \sum_{i,j=1}^{\nu} \sum_{k=1}^n \frac{\tilde{h}_{ijk}^2}{(\lambda_i + \varepsilon)(\lambda_j + \varepsilon)} + O(\varepsilon).$$

We therefore conclude that  $\Delta \log \tilde{K}_{\nu} \leq 0$  for small  $\varepsilon > 0$ . For otherwise  $\lambda_1 = \cdots = \lambda_{\nu}$  and  $\nabla \tilde{h}_{ij} = 0$  in a compact neighborhood  $U$  of  $p_1$ , since the term  $O(\varepsilon)$  tends to zero uniformly in compact neighborhood of  $p_1$ . But this means that in  $U$ ,  $\lambda_1 = \cdots = \lambda_{\nu} \equiv \lambda$  a positive constant and  $\lambda_{\nu+1} = \cdots = \lambda_n \equiv 0$ , which implies  $\tilde{K}_{\nu}$  is a positive constant in  $U$ , contradicting the assumption of  $\Delta \log \tilde{K}_{\nu} > 0$  at  $p_1$ . Hence  $\tilde{K}_{\nu}$  satisfies the strong minimum principle for  $r_0 \leq \nu \leq n$ . A similar argument at the end of the proof of Theorem 3 now applies and passes the strong minimum principle for  $\tilde{K}_{\nu} = \tilde{K}_{\nu}(\varepsilon)$  to that of the limit  $K_{\nu}$  as  $\varepsilon$  tends to zero (for more precisely details, see [8]). This completes the proof of Theorem 5.

**5. Superharmonicity for level curvatures.** Brascamp-Lieb proved in [2] that the first eigenfunction  $\mu$  of the Laplacian over a convex domain  $\Omega$  in  $R^2$  with zero boundary value on  $\partial\Omega$  has its level curves all convex. In the works of Lee-Wang [13] and Shih [15], the problem has been considered for convex domains in spheres or in hyperbolic planes, among which Shih used the expression

$$k = (-u_1^2 u_{22} + 2u_1 u_2 u_{12} - u_2^2 u_{11}) / |\nabla u|^3$$

for the curvature of level curves to illustrate a counterexample of the Brascamp-Lieb theorem in hyperbolic planes.



We consider in this section the level curves of surfaces of constant mean curvature. Shiffman [14] dealt with them for minimal surfaces using complex setting. In this note we tackle the problem by directly computing the Laplacian of the curvature of level curves and show its superharmonicity under  $C^2$ -deformation. This provides a more geometric version.

Let  $M^n$  be a hypersurface of constant mean curvature  $H$  immersed in  $R^{n+1}$  via

$$X: M^n \rightarrow R^{n+1},$$

and

$$u = u(X) = \langle X, \xi \rangle$$

be a height function relative to the direction  $\xi$  of  $R^{n+1}$ . For each real number  $a$ , the level set

$$N_a \equiv \{X \in M^n | u(X) = a\}$$

has Gauss-Kronecker curvature  $L$  in  $R^n_a \equiv \{X \in R^{n+1} | u(X) = a\}$  defined by the product of principal curvatures  $\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}$  of  $N_a$  in  $R^n_a$ . We call  $L$  the level curvature of  $M^n$  relative to  $\xi$  and a chosen normal field  $e_{n+1}$  of  $M^n$  in  $R^{n+1}$ . It is easy to see that

$$(5.1) \quad L = (u_i u_j H_{ij}) / |\nabla u|^3$$

where  $u_i, 1 \leq i \leq n$ , are the covariant derivatives of  $u$  relative to an orthonormal local frame  $\{e_i\}$  of  $M^n$  and  $H_{ij}$  the cofactor of the second fundamental form  $(h_{ij})$  with respect to  $e_{n+1}$  of  $M^n$  in  $R^{n+1}$ . We have

$$(5.2) \quad h_{ij} H_{jk} = \delta_{ik} \det(h_{ij}) = \delta_{ik} K_n$$

where  $K_n$  is the Gauss-Kronecker curvature of  $M^n$  in  $R^{n+1}$  considered in §§3, 4. On the other hand, the Hessian  $u_{ij}$  has the forms as follows:

$$(5.3) \quad \begin{aligned} u_{ij} &= \langle X, \xi \rangle_{ij} = \langle X_{ij}, \xi \rangle = \langle h_{ij} e_{n+1}, \xi \rangle \\ &= h_{ij} \langle e_{n+1}, \xi \rangle = \pm h_{ij} \sqrt{1 - |\nabla u|^2} = h_{ij} W \end{aligned}$$

where

$$(5.4) \quad W \equiv \pm \sqrt{1 - |\nabla u|^2} = \langle e_{n+1}, \xi \rangle, \quad \nabla u = \text{grad } u = \sum_{i=1}^n u_i e_i$$

and  $e_{n+1}$  is the unit normal of  $M^n$  in  $R^{n+1}$  with

$$(5.5) \quad h_{ij} = \langle X_{ij}, e_{n+1} \rangle.$$

We also remark that

$$(5.6) \quad \nabla u = \sum_{i=1}^n u_i e_i = \sum_{i=1}^n \langle X_i, \xi \rangle e_i = \sum_{i=1}^n \langle e_i, \xi \rangle e_i = \xi^T$$

which is the tangential component of  $\xi$  on tangent space of  $M^n$  and if we define

$$(5.7) \quad p \equiv |\nabla u|$$

then  $0 \leq p \leq 1$ , since

$$(5.8) \quad p^2 = |\nabla u|^2 = |\xi^T|^2 \leq |\xi|^2 = 1$$

and

$$(5.9) \quad \begin{aligned} |W| &= \sqrt{1 - p^2} = |\xi^N| \\ &= \text{the length of normal component of } \xi. \end{aligned}$$

To abbreviate notations, we define

$$(5.10) \quad L_\alpha = (u_i u_j H_{ij}) / |\nabla u|^\alpha,$$

by which we have

$$(5.11) \quad L_0 = u_i u_j H_{ij}, \quad \text{and} \quad L = L_3(u_i u_j H_{ij}) / |\nabla u|^3.$$

In the following, we first compute  $\Delta L_0$  and then  $\Delta L_3$ . Our idea is to express

$$\Delta L_a = A + B L_\alpha + C \cdot \nabla L_\alpha$$

so that  $A \leq 0, B \leq 0$ . Then deform  $M$  into a standard surface such as a catenoid to conclude  $L_\alpha \geq 0$  at every stage.

*Step 1.*

$$(5.12) \quad \begin{aligned} \Delta(u_i u_j H_{ij}) &= (u_i u_j H_{ij})_{kk} \quad (\text{summation convention used} \\ &\quad \text{unless otherwise declared}) \\ &= (2u_{ik} u_j H_{ij} + u_i u_j H_{ijk})_k \\ &= 2u_{ikk} u_j H_{ij} + 2u_{ik} u_{jk} H_{ij} \\ &\quad + 4u_{ik} u_j H_{ijk} + u_i u_j H_{ijkk} \end{aligned}$$

where the four terms are denoted respectively by

$$2\text{I}, 2\text{II}, 4\text{III}, \quad \text{and} \quad \text{IV}.$$

*Step 2.*

$$(5.13) \quad \begin{aligned} \text{I} &= u_{ikk} u_j H_{ij} = (u_{kki} + u_m R_{kmki}) u_j H_{ij} \\ &= [(h_{kk} W)_i + u_m R_{kmki}] u_j H_{ij} \quad [\text{by (5.3)}] \\ &= (h_{kki} W + h_{kk} W_i) u_j H_{ij} + u_m u_j H_{ij} R_{kmki}. \end{aligned}$$

However, we have

$$(5.14) \quad h_{kki}W = H_iW = 0 \quad \text{and} \\ W_i = \langle e_{n+1}, \xi \rangle_i = \langle -h_{il}e_l, \xi \rangle = -h_{il}u_l.$$

Since  $H$  is constant,

$$(5.15) \quad h_{kk}W_iu_jH_{ij} = -H(h_{il}u_l)u_jH_{ij} \quad [\text{by (5.3)}] \\ = -Hu_lu_j\delta_{lj}K_n \quad [\text{by (5.2)}] \\ = -|\nabla u|^2HK_n$$

and

$$(5.16) \quad u_mu_jH_{ij}R_{kmmki} = u_mu_jH_{ij}(h_{kk} - h_{mi} - h_{ki}h_{mk}) \\ = u_mu_j(\delta_{jm}K_n)H - u_mu_j(\delta_{jk}K_n)h_{mk} \\ = |\nabla u|^2HK_n - u_mu_jh_{mj}K_n \\ = [|\nabla u|^2H - B(\nabla u, \nabla u)]K_n$$

where  $B$  denotes the bilinear form defined by the second fundamental form  $(h_{ij})$  of  $M^n$  in  $R^{m+1}$ . It follows that

$$(5.17) \quad I = -B(\nabla u, \nabla u)K_n.$$

*Step 3.*

$$(5.18) \quad II = u_{ik}u_{jk}H_{ij} = W^2h_{ik}h_{jk}H_{ij} = W^2h_{ik}(\delta_{ki}K_n) \\ = W^2HK_n = (1 - |\nabla u|^2)HK_n.$$

*Step 4.* We will express III in terms of  $L$  and  $\nabla L$  when  $n = 2$ . Consider a frame  $\{e_1, e_2, \dots, e_n\}$  of  $M^n$  such that  $e_1$  is perpendicular to the level sets  $N_a$ ,  $a \in R$ . Then

$$(5.19) \quad \nabla u = u_1e_1.$$

Now at a given point  $p_0$  of  $M^n$ , we select  $e_2, e_3, \dots, e_n$  so that  $h_{ij} = \delta_{ij}$  at  $p_0$  for  $i, j \geq 2$ . We call such a frame a *level principal frame* of  $M^n$  at  $p_0$  with respect to  $\xi$ . It is evident that

$$(5.20) \quad III = u_{ik}u_jH_{ijk} = u_jWh_{ik}H_{ijk} = u_jW[(\delta_{kj}K_n)_k - h_{ikk}H_{ij}] \\ = u_jW(K_n)_j \quad [\text{since } h_{ikk} = h_{kki} = H_i = 0] \\ = W\nabla u \cdot \nabla K_n.$$

On the other hand, when  $n = 2$  we consider

$$(5.21) \quad \nabla L_0 = (L_0)_1e_1 + (L_0)_2e_2$$

where

$$\begin{aligned}
 (5.22) \quad (L_0)_1 &= (u_i u_j H_{ij})_1 = 2u_{i1} u_j H_{ij} + u_i u_j H_{ij1} \\
 &= 2W u_1 K_n + u_1^2 H_{111} \\
 &= 2|\nabla u| W K_n + |\nabla u|^2 h_{221}
 \end{aligned}$$

and

$$(5.23) \quad (L_0)_2 = (u_i u_j H_{ij})_2 = u_1^2 H_{112} = |\nabla u|^2 h_{222}.$$

Hence we have for  $n = 2$ ,

$$\begin{aligned}
 (5.24) \quad \text{III} &= u_{ik} u_j H_{ijk} = u_{ik} u_1 H_{i1k} \\
 &= u_1 W (h_{11} H_{111} + h_{12} (H_{112} + H_{211}) + h_{22} H_{212}) \\
 &= u_1 W (h_{11} h_{221} + h_{12} (h_{222} - h_{121}) - h_{22} h_{122}) \\
 &= |\nabla u| ((h_{11} - h_{22}) h_{221} + 2h_{12} h_{222})
 \end{aligned}$$

noting that  $h_{ijk}$  is symmetric with respect to each pair of  $i, j, k$ . By substituting (5.22) and (5.23) into (5.24), we obtain

$$(5.25) \quad \text{III} = \frac{W}{|\nabla u|} [(h_{11} - h_{22}) e_1 + 2h_{12} e_2] \cdot \nabla L_0 - 2W^2 \left( H - \frac{2L_0}{|\nabla u|^2} \right) K_n$$

where  $h_{22} = L_0/|\nabla u|^2$  has been considered. Therefore we have

$$\begin{aligned}
 (5.26) \quad \text{III} &= -2W^2 H K_n + \left( 4 \frac{W^2}{|\nabla u|^2} K_n \right) L_0 \\
 &\quad + \frac{W}{|\nabla u|} [(h_{11} - h_{22}) e_1 + 2h_{12} e_2] \cdot \nabla L_0
 \end{aligned}$$

for  $n = 2$ .

*Step 5.* Let  $\sigma_{ij}$  be a permutation

$$\begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_i & \cdots & \sigma_n \end{pmatrix}$$

with  $\sigma_i = j$ . Then

$$(5.27) \quad H_{ij} = \sum_{\sigma_{ij}} (-1)^{\sigma_{ij}} h_{1\sigma_1} \cdots \hat{h}_{i\sigma_i} \cdots h_{n\sigma_n}$$

where “ $\hat{\phantom{x}}$ ” denotes the deletion of the term, and we have

$$\begin{aligned}
 (5.28) \quad IV &= u_i u_j H_{ijkk} \\
 &= u_i u_j \left[ \sum_{\sigma_{ij}} (-1)^{\sigma_{ij}} h_{1\sigma_1} \cdots \hat{h}_{i\sigma_i} \cdots h_{n\sigma_n} \right]_{kk} \\
 &= u_i u_j \sum_{\sigma_{ij}} (-1)^{\sigma_{ij}} [h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} \\
 &\quad + \cdots + h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} k]_k \\
 &= u_i u_j \sum_{\sigma_{ij}} (-1)^{\sigma_{ij}} \{ [h_{1\sigma_1 k k} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} \\
 &\quad + \cdots + h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} k] \\
 &\quad + 2[h_{1\sigma_1 k} h_{2\sigma_2 k} \cdots \hat{h}_{ij} \cdots h_{n\sigma_n} + \cdots \\
 &\quad + h_{1\sigma_1} h_{2\sigma_2} \cdots \hat{h}_{ij} \cdots h_{n-1, \sigma_{n-1} k} h_{n\sigma_n k}] \}.
 \end{aligned}$$

By taking a level principal frame at  $p_0$ , we have

$$\begin{aligned}
 (5.29) \quad IV &= u_1^2 \{ [h_{22kk} h_{33} \cdots h_{nn} + \cdots + h_{22} h_{33} \cdots h_{nnkk}] \\
 &\quad + 2[(h_{22k} h_{33k} - h_{23k}^2) h_{44} \cdots h_{nn} \\
 &\quad + \cdots + h_{22} h_{33} \cdots (h_{n-1, n-1, k} h_{nnk} - h_{n-1, n, k}^2)] \}
 \end{aligned}$$

On the other hand, for each  $i$  with  $2 \leq i \leq n$ ,

$$\begin{aligned}
 (5.30) \quad h_{iikk} &= h_{kkii} + h_{mi} R_{kmki} + h_{km} R_{imki} \\
 &= 0 + h_{mi} (h_{kk} h_{mi} - h_{ki} h_{mk}) + h_{km} (h_{ik} h_{mi} - h_{ii} h_{mk}) \\
 &= H(h_{1i}^2 + h_{ii}^2) - h_{ii} \|B\|^2
 \end{aligned}$$

where the summation convention applies only on  $k$  while  $i$  is a fixed index. We obtain

$$\begin{aligned}
 (5.31) \quad IV &= |\nabla u|^2 H[h_{12}^2 h_{33} h_{44} \cdots h_{nn} + h_{22} h_{13}^2 h_{44} \cdots h_{nn} + \cdots] \\
 &\quad + |\nabla u|^2 H[h_{22} + h_{33} + h_{44} + \cdots + h_{nn}] h_{22} h_{33} h_{44} \cdots h_{nn} \\
 &\quad - \|B\|^2 L_0 + Q
 \end{aligned}$$

where  $Q$  denotes the second term of (5.29). However the first term of (5.31) equals to

$$(5.32) \quad |\nabla u|^2 H(h_{11} h_{22} \cdots h_{nn} - \det(h_{ij})).$$

Adding this to the second term of (5.31), it simplifies to

$$(5.33) \quad H^2 L_0 - |\nabla u|^2 H K_n.$$

We conclude with

$$(5.34) \quad \begin{aligned} \text{IV} &= H^2 L_0 - |\nabla u|^2 H K_n - \|B\|^2 L_0 + Q \\ &= 2K_2 L_0 - |\nabla u|^2 H K_n + Q \end{aligned}$$

where  $K_2$  is the 2nd mean curvature defined in (4.0) and the term  $Q$  has the properties:

- (i) For  $n = 2$ ,  $Q = 0$ ,
- (ii) At a zero minimal point  $p_0$  of  $L_\alpha$ ,  $\alpha \geq 0$  (for  $\alpha = 0, 1$ , also assume  $|\nabla u| \neq 0$  at  $p_0$ ), we have

$$(5.36) \quad Q \leq 0.$$

In fact, the frame  $\{e_i\}$  at  $p_0$  can be taken such that

$$(5.37) \quad h_{22}(p_0) = 0$$

and  $h_{33}(p_0), \dots, h_{nn}(p_0)$  all nonnegative, since

$$h_{22} h_{33} \cdots h_{nn} = |\nabla u|^{\alpha-2} L_\alpha \geq 0$$

and “ $= 0$ ” holds at  $p_0$ . By parallelly transporting  $\{e_i\}$  to neighboring points of  $p_0$  along radial geodesics from  $p_0$ , we have also

$$(5.38) \quad h_{22k}(p_0) = (dh_{22})(e_k) \quad \text{at } p_0 = 0.$$

Using (5.37) and (5.38), all the terms of  $Q$  vanish except the terms of the type

$$(5.39) \quad -h_{23k}^2 h_{44} \cdots h_{nn}$$

which are clearly not greater than zero. For  $n = 2$ , (5.34) reduces to

$$(5.40) \quad \text{IV} = -|\nabla u|^2 H K + 2K L_0.$$

**THEOREM 6.** *For a surface  $M$  of constant mean curvature  $H$  immersed in  $R^3$ , let  $L_0$  be defined by (5.11); then*

$$(5.41) \quad \begin{aligned} p^2 \Delta L_0 &= -3p^2(2 - p^2)HK + 4(4 - 3p^2)KL_0 \\ &\quad + 4p(1 - p^2)[(h_{11} - h_{22})e_1 + 2h_{12}e_2] \cdot \nabla L_0 \end{aligned}$$

where  $p = |\nabla u| \leq 1$ ,  $K$  is the Gaussian curvature of  $M$ , and  $(h_{ij})$  is the second fundamental form relative to the level principal frame  $\{e_1, e_2\}$ .

*Proof.* Combining (5.17), (5.18), (5.26), (5.40), we obtain

$$p^2 \Delta L_0 = 2p^2 \text{I} + 2p^2 \text{II} + 4p^2 \text{III} + p^2 \text{IV}$$

where

$$\begin{aligned}
 2p^2\text{I} &= -2p^2B(\nabla u, \nabla u)K = -2p^4h_{11}K = -2p^4(H - h_{22})K \\
 &= -2p^4HK + 2p^2KL_0, \\
 2p^2\text{II} &= 2p^2(1 - p^2)HK = 2p^2HK - 2p^4HK, \\
 4p^2\text{III} &= -8p^2(1 - p^2)HK + 16(1 - p^2)KL_0 \\
 &\quad + 4p(1 - p^2)[(h_{11} - h_{22})e_1 + 2h_{12}e_2] \cdot \nabla L_0, \\
 p^2\text{IV} &= -p^4HK + 2p^2KL_0.
 \end{aligned}$$

Hence (5.41) follows.

*Step 7.* In order to compute  $\Delta L_\alpha$ , we note that

$$(5.42) \quad L_0 = p^\alpha L_\alpha,$$

$$\begin{aligned}
 (5.43) \quad p^2\Delta L_0 &= p^2\Delta(p^\alpha L_\alpha) \\
 &= p^{\alpha+2}\Delta L_\alpha + 2p^2\nabla L_\alpha \cdot \nabla p^\alpha + p^2L_\alpha(\Delta p^\alpha)
 \end{aligned}$$

and

$$(5.44) \quad \Delta p^\alpha = (p^\alpha)_{kk} = (\alpha p^{\alpha-1}p_k)_k = (\alpha p^{\alpha-2}u_i h_{ik} \sqrt{1-p^2})_k.$$

A straightforward calculation implies

$$\begin{aligned}
 (5.45) \quad \Delta p^\alpha &= \alpha p^{\alpha-2}((\alpha - 2) - (\alpha - 1)p^2)(h_{11}^2 + h_{22}^2) \\
 &\quad + \alpha p^{\alpha-1}(1 - p^2)(h_{11}^2 + 2h_{12}^2 + h_{22}^2)
 \end{aligned}$$

and

$$(5.46) \quad (L_0)_k = (p^\alpha L_\alpha)_k = p^\alpha (L_\alpha)_k + \alpha p^{\alpha-1}(p_k)L_\alpha.$$

Substituting (5.41), (5.44), (5.45) and (5.46) into (5.43), we get for  $\alpha = 3$  the following main formula:

**THEOREM 7.** *For a surface  $M^2$  of constant mean curvature  $H$  immersed in  $R^3$ , let  $L$  be the level curvature of  $M^2$  relative to a given direction  $\xi$ . Then*

$$\begin{aligned}
 (5.47) \quad p^5\Delta L &= -3p^2(2 - p^2)HK \\
 &\quad + \{p^3K + 3p^3(2 - p^2)H^2 - 3p^4(3 - 2p^2)HL\}L \\
 &\quad + 2p^4\sqrt{1 - p^2}[-(h_{11} + 2h_{22})e_1 + 2h_{12}e_2] \cdot \nabla L
 \end{aligned}$$

where the notations are defined as in Theorem 6.

When the mean curvature  $H \equiv 0$ , we have the theorem:

**THEOREM 8.** *Let  $M^2$  be a compact surface with boundary  $\partial M$  and*

$$X: M^2 \rightarrow R^3$$

*be a minimal immersion of  $M^2$  into  $R^3$ . Given a unit vector  $\xi$  of  $R^3$ , consider the height function  $u$  defined by*

$$u(X) = \langle X, \xi \rangle.$$

*Let  $L$  denote the level curvature (i.e. the curvature of level curves) relative to the vertical direction  $\xi$ . Then*

$$(5.48) \quad |\nabla u|^2 \Delta L \equiv KL \pmod{\nabla L}$$

*and therefore  $L$  attains neither nonnegative interior minimum nor non-positive interior maximum.*

*Proof.* This is a direct consequence of (5.47).

As an application, we illustrate a plausible but simple geometric proof of Shiffman's theorem. Given an annular minimal surface  $M$  in  $R^3$  with its two boundary curves  $\Gamma_1, \Gamma_2$  convex in two horizontal planes respectively. Suppose there is a  $C^2$ -deformation of  $M$  into a catenoid as the two boundary curves are deformed into two circles on the given horizontal planes. Since  $K \leq 0$  on minimal surfaces, the formula (5.48) together with the continuity method shows by Shiffman's theorem that

$$(5.49) \quad L \geq 0.$$

In fact, we obtain a stronger estimate that

$$(5.50) \quad L \geq \min\{\text{curvature of } \Gamma_1, \Gamma_2\} \geq 0.$$

However with this argument, we have to assume additionally that  $|\nabla u| \neq 0$  during the deformation.

Using the main formula (5.47) we prove our main theorem of this section, which can be regarded as a partial generalization of Shiffman's theorem to surfaces of constant mean curvature.

**THEOREM 9.** *Given an annular surface  $M$  of constant mean curvature  $H$  in  $R^3$  such that  $\partial M = \Gamma_1 \cup \Gamma_2$ , each  $\Gamma_i$  being a convex curve in a plane  $\Pi_{i, i=1,2}$ , and  $\Pi_1$  is parallel to  $\Pi_2$ . Suppose that the mean curvature vector is outward from the tubular compartment  $V$  enclosed by  $\Pi_1, \Pi_2$  and  $M$ . This is equivalent to that  $H \leq 0$  in the above setting, or equivalent to that the exterior pressure of the soap film  $M$*



is no less than the inner pressure in  $V$ . Consider a  $C^2$ -deformation  $\{M_t; 0 \leq t \leq 1\}$ , each  $M_t$  having constant mean curvature  $H_t$ , of  $M = M_1$  with  $H_t$  increasing into a minimal surface  $M_0$  (for an example, we increase the inner pressure in  $V$  until both sides of the soap film reach equal pressures). Assume also that each  $M_t$  has no point tangent to a "horizontal" plane (i.e. a plane parallel to  $\Pi_i$ ) so that the curvature of level curves on  $M_t$  is well defined. Then the level curves of each  $M_t$ ,  $0 \leq t \leq 1$ , are all convex.

**REMARK 5.1.** We can also consider the deformation by taking  $M$  into a Delaunay surface corresponding to a rotating hyperbola, while deforming each  $\Gamma_i$  into a circle in  $\Pi_i$ .

Before proving the theorem, we consider a stronger form of constant rank theorem.

**LEMMA 5.1.** *Given a surface  $M$  of constant mean curvature  $H$  in  $R^3$ , let the level curvature  $L$  relative to a given vertical direction  $\xi$  attain zero minimum at a point  $x_0$ , interior in  $M$  or on its boundary  $\partial M$ . Additionally, if the tangent of the level curve at  $x_0$  is a principal direction of zero normal curvature, then  $M$  is either a circular cylinder or a plane.*

**REMARK 5.2.** This is a stronger form of constant rank theorem for  $n = 2$ , that instead of assuming  $K \geq 0$  around  $x_0$ , we assume the level curvature  $L \geq 0$  around  $x_0$ . I do not know whether the lemma is true for higher dimensions.

**REMARK 5.3.** Caffarelli-Friedman [3] have proved that for

$$\begin{cases} \Delta u = \gamma(u) \text{ in } \Omega; & \gamma(u) \nearrow \text{ in } u, \gamma(0) = 0, \\ u|_{\Gamma_1} = M, u|_{\Gamma_2} = m; & M > m > 0 \text{ constants,} \end{cases}$$

$\Gamma_1$  and  $\Gamma_2$  being inner and outer strictly convex  $C^{2,\alpha}$ -boundary curves of  $\Omega$ , the solution surface  $u = u(x)$  has all level curves strictly convex. Compared with the condition  $H \leq 0$  in our Theorem 9, the assumption  $\Delta u = \gamma(u) \geq \gamma(0) = 0$  are parallel in the linear case.

*Proof of Lemma 5.1.* We use the second-order comparison method developed as in [2]. Let  $Z$  be a cylinder of radius  $1/|H|$ , contacting  $M$  at  $x_0$  of at least second order. This is possible since

$$(5.51) \quad L(x_0) = 0 = K(x_0)$$

and therefore the horizontal direction tangent to  $M$  at  $x_0$  is a principal direction with zero principal curvature. On every horizontal plane  $\Pi$ , the straight line  $Z \cap \Pi$  intersects the convex level curve in at most two points unless  $M \equiv Z$ . This implies that the function  $u - v$  alternates its sign at most four times when turning around  $x_0$ , where  $u = u(x, y)$ ,  $v = v(x, y)$  are nonparametric functions of the surfaces  $M$  and  $Z$  on  $(x, y)$  of  $\Pi$ , respectively. However a theorem of Hopf shows that

$$(5.52) \quad u - v = R_e(\lambda(\xi + i\eta)^k) + o((\xi^2 + \eta^2)^{k/2})$$

where  $\lambda$  is a complex number,  $(\xi, \eta)$  is a coordinate change of  $(x, y)$  with  $(0, 0)$  corresponding to  $x_0$  and

$$(5.53) \quad k = \text{contact order} + 1.$$

By the above construction of  $Z$ ,  $k \geq 3$ , this leads to a contradiction. Hence  $M \equiv Z$ .

*Proof of Theorem 9.* Suppose there is  $M_t$  with some level curves nonconvex; then there would exist  $M_\tau$ ,  $\tau \in (0, t]$ , with zero minimal point  $x_0$  for the level curvature  $L$  on  $M_\tau$ . It is clear that

$$(5.54) \quad K(x_0) = h_{11}h_{22} - h_{12}^2 = h_{11} \cdot 0 - h_{12}^2 \leq 0$$

for  $h_{22} = L/p = 0$  at  $x_0$ ,  $p = |\nabla u|$ . By Lemma 5.1, unless  $M \equiv Z$  the possibility of  $K(x_0) = 0$  is also excluded. Hence  $K(x_0) < 0$  and the main formula (5.47) implies

$$(5.55) \quad p^5 \Delta L = -3p^2(2 - p^2)HK < 0 \quad \text{at } x_0$$

contradicting the minimality of  $L$  at  $x_0$ . This completes the proof.

**6. Concluding remarks.** Given a compact surface  $M$  of constant mean curvature in  $R^3$  with  $M$  convex and the boundary  $\partial M$ , is  $M$  then convex in the interior? i.e. does

$$(6.1) \quad K|_{\partial M} \geq 0 \text{ imply } K \geq 0 \text{ in } M,$$

where  $K$  is the Gaussian curvature of  $M$ ? The implication is seen not necessarily true in general by considering, for an example, one of Wente's immersed tori [13] and cutting off a small neighborhood of a point of positive  $K$ . Adding topological conditions such as embedding or restriction on topological type is insufficient. In fact, we consider a Delaunay surface defined by revolving around an axis the locus of one of the two foci of a given ellipse which rotates on the axis.

The Delaunay surface is a periodic surface of revolution. We take a segment of the Delaunay surface bounded by two largest meridians. It gives an example of embedded nonconvex surfaces of constant mean curvature in  $R^3$  which are convex around the boundary.

Certain stability seems required. A connection with capillary treatment leads to the capillary stability defined as follows. Let  $M^n$  be a hypersurface of constant mean curvature  $H_0$  embedded in  $R^{n+1}$ , and let  $\Omega^{n+1}$  be a bounded convex domain with  $C^2$ -boundary  $C \equiv \partial\Omega^{n+1}$ , such that  $M^n \subset \Omega^{n+1}$  and  $\partial M^n$  meets  $C$  in a constant angle  $\gamma_0$ . Let  $E$  be the energy function given by

$$E = \text{Area } M - (\cos \gamma_0)\text{Area } C_* + nH_0\text{Vol}(\Omega_*)$$

where  $\Omega_*$  is a component of  $\Omega - M$  away from which points the mean curvature vector of  $M$ . Also  $C_*$  denotes  $\bar{\Omega}_0 \cap C$ . The energy function  $E$  is indeed defined on the space  $\Sigma_{\gamma_0}$  of all the surfaces embedded in  $\Omega$  with their boundary meeting with  $C$  in the given constant angle  $\gamma_0$ . It is easily seen that  $E$  is critical on  $\Sigma_{\gamma_0}$  at the given surface  $M$  of constant mean curvature  $H_0$  with constant contact angle  $\gamma_0$  with  $C$  at the boundary points, or equivalently, the first variation of  $E$  at such  $M$  is zero for any perturbation of  $M$  in  $\Sigma_{\gamma_0}$ , i.e.

$$(6.2) \quad E' = 0 \quad \text{at } M.$$

We now call  $M$  *capillaryly stable* with respect to the container  $C$  and the given contact angle  $\gamma_0$  if the second variation of  $E$  at  $M$  is nonnegative for any perturbation of  $M$  in  $\Sigma_{\gamma_0}$ , i.e.

$$(6.3) \quad E'' \geq 0 \quad \text{at } M.$$

*Problem 1.* Let  $\gamma_0 = 0$  and suppose  $M$  minimizes  $E$  for a given convex container  $C$ . Does there exist a 1-parameter  $C^2$ -family  $\{M_t; 0 \leq t \leq 1\}$  of surfaces in  $\Sigma_{\gamma_0}$ , each  $M_t$  minimizing  $E$  with a corresponding  $H_0 = H_0(t)$  (but  $\gamma_0 \equiv 0$ , independent of  $t$ ) such that  $M_0 = M$  and  $M_1 =$  a small  $n$ -sphere? If it does, a capillary stable surface  $M$  with respect to a convex container  $C$  is convex in the interior by Theorems 2 and 4.

Whether the Dirichlet stability is related with capillary stability and is involved in order for (6.1) to be valid seems also interesting to know.

Another equally interesting *question* first proposed by Rosenberg [personal conversation, 1984 Stanford] is the following.

*Problem 2.* Does an embedded disk-type surface  $M$  of constant mean curvature in  $R^3$  having its boundary  $\partial M$  a convex closed curve

on a plane  $\Pi$  intersect with every plane parallel to  $\Pi$  (such a plane is called later “horizontal”) in a convex closed curve?

The problem is also a type of Shiffman’s problem. Suppose there is a  $C^2$ -deformation  $\{M_t\}$  of  $M$  into a spherical cap as  $\partial M$  deformed into a circle, each  $M_t$  having constant mean curvature. We reverse the process as in the proofs of Theorem 2 and Theorem 4 and notice that a point of zero Gaussian curvature is impossible to appear first in the interior. However it may show up on the boundary and whence spread into a region of concave set  $M_-$ , i.e. the set with  $K < 0$ . It is evident that for a while after a point of zero  $K$  sneaks in across the boundary, the set  $D$  of the directions of nonpositive normal curvature at a point of the concave set  $M_-$  does not contain a horizontal direction. Formula (5.47) yields that at an interior zero minimal point of the level curvature  $L$ ,  $\Delta L = -3p^{-3}(2 - p^2)HK > 0$ . This does not exclude the possibility that at a certain stage in the reversed deformation, the set  $D$  finally touches a horizontal direction. And then there may start to appear a region of negative  $L$ . With this observation, the existence of counter examples for Problem 2 perhaps should not be excluded.

Even for the annular type, the generalized Shiffman’s theorem established in Theorem 9 is not completed. In Theorem 9 we only consider the case  $H \leq 0$ , i.e. the inner pressure is not greater than the outer one. How is the case  $H > 0$ ? It nevertheless has no essential difference in this case and the case of Problem 2.

Finally, we would like to mention the following related questions of interest.

*Problem 3.* Is Shiffman’s theorem of  $H \leq 0$  valid for hypersurfaces of higher dimensions? Does there exist a dimension bound?

*Problem 4.* For a nonparametric surface  $M$  of constant mean curvature in  $R^3$  given by  $u = u(x)$ ,  $x \in \Omega \subset R^2$ , with  $u|_{\partial\Omega} = 0$ , what is the least pinching number of the curvature  $\kappa$  of the boundary curve  $\partial\Omega$  in order for the surface  $M$  to be convex? Here the pinching number of  $\kappa$  is defined by the ratio,  $\kappa_{\min}$  to  $\kappa_{\max}$ , the minimum to the maximum of  $\kappa$  of the curve  $\partial\Omega$  in  $R^2$ .

*Note.* A. N. Wang [16] has found, after the preprint of this paper was circulated, a counterexample showing that the level curves are not all convex when  $H > 0$  as mentioned in the comments of Problem 2. Therefore the assumption  $H \leq 0$  in Theorem 9 is essential.

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