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SOME REMARKS ON ACTIONS OF COMPACT MATRIX QUANTUM GROUPS ON C^*-ALGEBRAS

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In this paper we construct an action of a compact matrix quantum group on a Cuntz algebra or a UHF-algebra, and investigate the fixed point subalgebra of the algebra under the action. Especially we consider the action of $_\mu U(2)$ on the Cuntz algebra $\mathcal{O}_2$ and the action of $S_\mu U(2)$ on the UHF-algebra of type 2^∞. We show that these fixed point subalgebras are generated by a sequence of Jones' projections.

1. Compact matrix quantum groups and their actions. We use the terminology introduced by Woronowicz([**6**]).

DEFINITION. Let A be a unital C^*-algebra and $u=(u_{kl})_{kl}\in M_n(A)$, and $\mathscr{A}$ be the *-subalgebra of A generated by the entries of u. Then $G=(A, u)$ is called a compact matrix quantum group (a compact matrix pseudogroup) if it satisfies the following three conditions:

(1) $\mathscr{A}$ is dense in A.

(2) There exists a *-homomorphism Φ (comultiplication) from A to $A\otimes_\alpha A$ such that

$$\Phi(u_{kl})=\sum_{r=1}^{n} u_{kr}\otimes u_{rl} \qquad (1\le k, l\le n),$$

where the symbol $\otimes_\alpha$ means the spatial C^*-tensor product.

(3) There exists a linear, antimultiplicative mapping κ from $\mathscr{A}$ to $\mathscr{A}$ such that

$$\kappa(\kappa(a^*)^*)=a \qquad (a\in\mathscr{A})$$

and

$$\kappa(u_{kl})=(u^{-1})_{kl} \qquad (1\le k, l\le n).$$

We call $w\in B(C^N)\otimes A\cong M_N\otimes A$ a representation of a compact matrix quantum group $G=(A, u)$ on C^N if $w\ominus w=(\mathrm{id}\otimes\Phi)w$, where $\ominus$ is a bilinear map of $(M_N\otimes A)\times(M_N\otimes A)$ to $M_N\otimes A\otimes A$ as follows:

$$(l\otimes a)\ominus(m\otimes b)=lm\otimes a\otimes b$$

for any $l, m\in M_N$ and $a, b\in A$.

It is known that a compact matrix quantum group $G = (A, u)$ has the Haar measure h, that is, h is a state on A satisfying

$$(h \otimes \mathrm{id})\Phi(a) = (\mathrm{id} \otimes h)\Phi(a) = h(a)1 \quad \text{for any } a \in A.$$

So any finite dimensional representation is equivalent to a unitary representation. In this paper we only treat a unitary representation of a compact matrix quantum group.

DEFINITION. Let B be a C^*-algebra and π be a *-homomorphism from B to $B \otimes_\alpha A$. Then we call π an action of a compact matrix quantum group $G = (A, u)$ on B if $(\pi \otimes \mathrm{id}_A)\pi = (\mathrm{id}_B \otimes \Phi)\pi$.

Let w be a unitary representation of a compact matrix quantum group $G = (A, u)$ and belong to $M_N(A)$. We denote by $\mathcal{O}_N$ the Cuntz algebra which is generated by isometries $S_1, \dots, S_N$ satisfying $\sum_{i=1}^N S_i S_i^* = 1$ ([**1**]). Then we can construct an action of $G = (A, u)$ on $\mathcal{O}_N$ simultaneously to [**2**], [**3**].

THEOREM 1. *For a unitary representation $w \in M_N(A)$ of a compact matrix quantum group $G = (A, u)$, there exists an action φ of the compact matrix quantum group $G = (A, u)$ on the Cuntz algebra $\mathcal{O}_N$ such that*

$$\varphi(S_i) = \sum_{j=1}^N S_j \otimes w_{ji} \quad \textit{for any } 1 \le i \le N.$$

Proof. We set $T_i = \varphi(S_i) = \sum_{j=1}^N S_j \otimes w_{ji}$ for any $i = 1, 2, \dots, N$. By the relation $S_i^* S_j = \delta_{ij}$ and the unitarity of w, T_i's are isometries and $\sum_{i=1}^N T_i T_i^* = 1$. So φ can be extended to the *-homomorphism from $\mathcal{O}_N$ to $\mathcal{O}_N \otimes_\alpha A$. Then we have

$$(\varphi \otimes \mathrm{id})\varphi(S_i) = \sum_{j,k=1}^N S_k \otimes w_{kj} \otimes w_{ji} = (\mathrm{id} \otimes \Phi)\varphi(S_i)$$

for any $1 \le i \le N$. This implies that $(\varphi \otimes \mathrm{id})\varphi = (\mathrm{id} \otimes \Phi)\varphi$ on $\mathcal{O}_N$. □

REMARK 2. Let ε be a *-character from $\mathcal{A}$ to the algebra C of all the complex numbers such that

$$\varepsilon(u_{ij}) = \delta_{ij}$$

for any $1 \le i,\ j \le n$ ([6]). If the above unitary representation w belongs to $M_N(\mathscr{A})$, then the relation,

$$(id \otimes \varepsilon)\varphi = \mathrm{id}_{\mathcal{O}_N},$$

holds on the dense *-subalgebra of $\mathcal{O}_N$ generated by $S_1, S_2, \ldots, S_N$. □

We denote by M_N^K the K-times tensor product of the $N \times N$-matrix algebra M_N, and define a canonical embedding $\imath$ from M_N^K to $\mathcal{O}_N$ by

$$\imath(e_{i_1 j_1} \otimes \cdots \otimes e_{i_K j_K}) = S_{i_1} \cdots S_{i_K} S_{j_K}^* \cdots S_{j_1}^*,$$

where $\{e_{ij}\}_{i,j=1}^N$ is a system of matrix units of M_N. This embedding $\imath$ is compatible with the canonical inclusion of M_N^K into M_N^{K+1}. We denote by M_N^∞ the UHF-algebra of type N^∞, which is obtained as the inductive limit C^*-algebra of $\{M_N^K\}_{K=1}^\infty$. We may consider the UHF-algebra M_N^∞ as a C^*-subalgebra of $\mathcal{O}_N$ through the embedding.

COROLLARY 3. *Let φ be the action of a compact matrix quantum group $G = (A, u)$ on the Cuntz algebra $\mathcal{O}_N$ defined by the unitary representation $w \in M_N(A)$ as in Theorem 1. Then the restriction ψ of φ on the UHF-algebra M_N^∞ is also an action of $G = (A, u)$ on M_N^∞ satisfying*

$$\psi(e_{i_1 j_1} \otimes \cdots \otimes e_{i_K j_K}) = \sum_{\substack{a_1, \ldots, a_K \\ b_1, \ldots, b_K}} e_{a_1 b_1} \otimes \cdots \otimes e_{a_K b_K} \otimes w_{a_1 i_1} \cdots w_{a_K i_K} w_{b_K j_K}^* \cdots w_{b_1 j_1}^*$$

for any positve integer K.

REMARK 4. We define a bilinear map $\ominus$ of $(M_N \otimes A) \times (M_N \otimes A)$ to $M_N \otimes M_N \otimes A$ as follows:

$$(l \otimes a) \ominus (m \otimes b) = l \otimes m \otimes ab$$

for any $l, m \in M_N$ and $a, b \in A$. We denote $\overbrace{w \ominus \cdots \ominus w}^{K \text{ times}}$ by w^K. Then w^K is a unitary representation of a compact matrix quantum group $G = (A, u)$ if w is a unitary representation of $G = (A, u)$. The above action ψ is represented by the following form

$$\psi(x) = w^K(x \otimes 1_A)(w^K)^* \quad \text{for any } x \in M_N^K.$$

So we call the action ψ the product type action of $G = (A, u)$ on the UHF-algebra M_N^∞. □

DEFINITION. Let B be a C^*-algebra and π be an action of a compact matrix quantum group $G = (A, u)$ on B. We define the fixed point subalgebra B^π of B by π as follows :

$$B^\pi = \{x \in B | \pi(x) = x \otimes 1_A\}.$$

Let $\mathscr{P}_N$ be the dense $*$-subalgebra of $\mathscr{O}_N$ generated by $S_1, S_2, \ldots, S_N$ and $\mathscr{M}_N$ be the dense $*$-subalgebra $\bigcup_{K=1}^\infty M_N^K$ of M_N^∞.

LEMMA 5. *Let h be the Haar measure on a compact matrix quantum group $G = (A, u)$, and we define $E_\varphi = (\mathrm{id} \otimes h)\varphi$ and $E_\psi = (\mathrm{id} \otimes h)\psi$. Then E_φ (resp. E_ψ) is a projection of norm one from $\mathscr{O}_N$ onto $(\mathscr{O}_N)^\varphi$ (resp. from M_N^∞ onto $(M_N^\infty)^\psi$) such that*

$$E_\varphi(\mathscr{P}_N) \subset \mathscr{P}_N, \quad E_\psi(\mathscr{M}_N) \subset \mathscr{M}_N.$$

Proof. Clearly E_φ is a unital, completely positive map, $E_\varphi(x) = x$ for any $x \in (\mathscr{O}_N)^\varphi$, and $E_\varphi(\mathscr{P}_N) \subset \mathscr{P}_N$. By the property of the Haar measure, for any $x \in \mathscr{O}_N$, we have

$$\begin{aligned}
E_\varphi(E_\varphi(x)) &= (\mathrm{id} \otimes h \otimes \mathrm{id})(\varphi \otimes \mathrm{id})(\mathrm{id} \otimes h)\varphi(x) \\
&= (\mathrm{id} \otimes h \otimes h)(\varphi \otimes \mathrm{id})\varphi(x) \\
&= (\mathrm{id} \otimes h \otimes h)(\mathrm{id} \otimes \Phi)\varphi(x) = (\mathrm{id} \otimes (h \otimes h)\Phi)\varphi(x) \\
&= (\mathrm{id} \otimes h)\varphi(x) = E_\varphi(x).
\end{aligned}$$

So the assertion holds for E_φ.

Similarly the assertion also holds for E_ψ. □

We can easily get the following lemma.

LEMMA 6. *Let π be an action of a compact matrix quantum group $G = (A, u)$ on a C^*-algebra B and B_0 be a dense $*$-subalgebra of B. If E is a projection of norm one from B onto the fixed point subalgebra B^π of B by the action π such that $E(B_0) \subset B_0$, then $B_0 \cap B^\pi$ is dense in B^π.*

We define a $*$-endomorphism σ of $\mathscr{O}_N$ by $\sigma(X) = \sum_{i=1}^N S_i X S_i^*$ for any $X \in \mathscr{O}_N$. Then the restriction of σ to the UHF-algebra M_N^∞ of type N^∞ satisfies that $\sigma(X) = 1_{M_N} \otimes X$ for any $X \in M_N^\infty$.

LEMMA 7. (1) *If* $X \in (\mathcal{O}_N)^\varphi$, *then* $\sigma(X) \in (\mathcal{O}_N)^\varphi$.
(2) *If* $X \in (M_N^\infty)^\psi$, *then* $\sigma(X) \in (M_N^\infty)^\psi$.

Proof. (1) For $X \in (\mathcal{O}_N)^\varphi$, we have

$$\begin{aligned}\varphi(\sigma(X)) &= \sum_{i=1}^{N} \varphi(S_i X S_i^*) = \sum_{i=1}^{N} \varphi(S_i)(X \otimes 1_A)\varphi(S_i)^* \\ &= \sum_{i,j,k=1}^{N} S_j X S_k^* \otimes u_{ij} u_{ik}^* = \sum_{i=1}^{N} S_i X S_i^* \otimes 1_A = \sigma(X) \otimes 1_A.\end{aligned}$$

(2) The assertion follows that ψ is the restriction of φ. □

2. Jones' projections and compact matrix quantum groups $S_\mu U(2)$ **and** ${}_\mu U(2)$. We shall consider the actions of $S_\mu U(2)$ and ${}_\mu U(2)$ coming from their fundamental representations.

DEFINITION ([7]). A compact matrix quantum group $G = (A, u)$ is called $S_\mu U(2)$ if A is the universal C^*-algebra generated by α, γ satisfying

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + \mu^2\gamma\gamma^* = 1, \quad \gamma^*\gamma = \gamma\gamma^*,$$

$$\mu\gamma\alpha = \alpha\gamma, \quad \mu\gamma^*\alpha = \alpha\gamma^*, \quad \mu\alpha^*\gamma = \gamma\alpha^*, \quad \mu\alpha^*\gamma^* = \gamma^*\alpha^*,$$

where $-1 \le \mu \le 1$. Its fundamental representation u is as follows:

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(A).$$

The comultiplication Φ associated with $S_\mu U(2)$ is defined by

$$\Phi(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma, \quad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

We shall introduce the quantum $U(2)$ group ${}_\mu U(2)$, which is obtained by the unitarization of the quantum $\mathrm{GL}(2)$ group.

DEFINITION. A compact matrix quantum group $H = (B, v)$ is called ${}_\mu U(2)$ if B is the universal C^*-algebra generated by α, γ, D satisfying

$$D^*D = DD^* = 1, \quad \alpha D = D\alpha, \quad \gamma D = D\gamma, \quad \alpha^* D = D\alpha^*,$$

$$\gamma^* D = D\gamma^*, \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + \mu^2\gamma\gamma^* = 1, \quad \gamma^*\gamma = \gamma\gamma^*,$$

$$\mu\gamma\alpha = \alpha\gamma, \quad \mu\gamma^*\alpha = \alpha\gamma^*, \quad \mu\alpha^*\gamma = \gamma\alpha^*, \quad \mu\alpha^*\gamma^* = \gamma^*\alpha^*,$$

where $-1 \le \mu \le 1$. Its fundamental representation v is as follows:

$$v = \begin{pmatrix} \alpha & -\mu D\gamma^* \\ \gamma & D\alpha^* \end{pmatrix} \in M_2(B).$$

The comultiplication Ψ associated with ${}_\mu U(2)$ is defined by

$$\Psi(\alpha) = \alpha \otimes \alpha - \mu D\gamma^* \otimes \gamma, \quad \Psi(\gamma) = \gamma \otimes \alpha + D\alpha^* \otimes \gamma,$$

$$\Psi(D) = D \otimes D.$$

REMARK 8. The above C^*-algebra B associated with the compact matrix quantum group ${}_\mu U(2) = H = (B, v)$ is isomorphic to $A \otimes_\alpha C(T)$ as a C^*-algebra, where A is the C^*-algebra associated with the compact matrix quantum group $S_\mu U(2) = G = (A, u)$ and $C(T)$ is the algebra of all the continuous functions on the one dimensional torus T. The elements α and γ in H satisfy the same relation of α and γ in G. But the values of the comultiplication Ψ at α, γ differ from ones of the comultiplication Φ at α, γ. □

In the rest of the paper, we fix a number $\mu \in [-1, 1] \backslash \{0\}$.

We denote by φ_1 (resp. by φ_2) the action of the compact matrix quantum group ${}_\mu U(2) = (B, v)$ (resp. $S_\mu U(2) = (A, u)$) on the Cuntz algebra $\mathcal{O}_2$ coming from the fundamental representation v (resp. u) as in Theorem 1. We also denote ψ_1 (resp. ψ_2) the product type action of the compact matrix quantum group ${}_\mu U(2) = (B, v)$ (resp. $S_\mu U(2) = (A, u)$) on the UHF-algebra M_2^∞ of type 2^∞ coming from v (resp. u) as in Corollary 3.

From now on, we shall determine the fixed point subalgebras of the above actions.

In **[8]** Woronowicz defines the 4×4-matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & \mu & 1-\mu^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_2 \otimes M_2 \subset M_2^\infty$$

and shows that the algebra $\{x \in M_2^K | u^K(x \otimes 1_A) = (x \otimes 1_A)u^K\}$ is generated by $g_1, g_2, \ldots, g_{K-1}$, where $g_{i+1} = \sigma^i(g)$ $(i = 0, 1, \ldots, K-2)$.

We set

$$e_i = \frac{1}{1+\mu^2}(1 - g_i) \quad \text{for any } i = 1, 2, \ldots, K-1,$$

then the sequence $\{e_n\}_{n=1}^{\infty}$ of projections satisfies the Jones' relation

$$e_i e_{i\pm 1} e_i = \frac{\mu^2}{(1+\mu^2)^2} e_i, \quad e_i e_j = e_j e_i \quad (\text{if } |i-j|>1).$$

We denote by $C^*(\{e_n\}_{n=1}^{\infty})$ the unital C^*-algebra generated by the projections $\{e_n\}_{n=1}^{\infty}$.

PROPOSITION 9. *The fixed point subalgebra $(M_2^{\infty})^{S_\mu U(2)}$ of the UHF-algebra M_2^{∞} by the action ψ_2 of $S_\mu U(2)$ is generated by the above Jones' projections $\{e_n\}_{n=1}^{\infty}$.*

Proof. By Remark 4, $M_2^K \cap (M_2^{\infty})^{\psi_2} = \{x \in M_2^K | u^K(x \otimes 1_A) = (x \otimes 1_A)u^K\}$. So $M_2^K \cap (M_2^{\infty})^{\psi_2}$ is generated by $e_1, e_2, \ldots, e_{K-1}$. The assertion follows from Lemma 5 and Lemma 6. □

THEOREM 10. *The fixed point subalgebra $(\mathcal{O}_2)^{{}_\mu U(2)}$ of the Cuntz algebra $\mathcal{O}_2$ by the action φ_1 of ${}_\mu U(2) = (B, v)$ coincides with the fixed point subalgebra $(M_2^{\infty})^{S_\mu U(2)}$ of the UHF-algebra M_2^{∞} by the action ψ_2 of $S_\mu U(2) = (A, u)$.*

In particular,

$$(\mathcal{O}_2)^{{}_\mu U(2)} = (M_2^{\infty})^{{}_\mu U(2)} = (M_2^{\infty})^{S_\mu U(2)} = C^*(\{e_n\}_{n=1}^{\infty}).$$

Proof. It is clear that $(\mathcal{O}_2)^{{}_\mu U(2)} \supset (M_2^{\infty})^{{}_\mu U(2)}$. In order to show that $(\mathcal{O}_2)^{{}_\mu U(2)} \subset (M_2^{\infty})^{{}_\mu U(2)}$, it is sufficient to show that $\mathcal{P}_2 \cap (\mathcal{O}_2)^{\varphi_1} \subset (\mathcal{M}_2 \cap (M_2^{\infty})^{\psi_1})$ by Lemma 5 and Lemma 6. Let $x \in \mathcal{P}_2 \cap (\mathcal{O}_2)^{\varphi_1}$ and θ be a $*$-homomorphism of B onto $C^*(D)$ such that $\theta(\alpha) = D$, $\theta(\gamma) = 0$ and $\theta(D) = D^2$. The element x has the unique representation

$$x = \sum_{i>0} (S_1^*)^i A_{-i} + A_0 + \sum_{i>0} A_i (S_1)^i,$$

where each A_i $(i = 0, \pm 1, \pm 2, \ldots)$ belongs to $\mathcal{M}_2$ (**[1]**). Since $(\mathrm{id}_{\mathcal{O}_2} \otimes \theta)\varphi_1(S_i) = S_i \otimes D$ for any $i = 1, 2$,

$$\begin{aligned} x \otimes 1_B &= (\mathrm{id}_{\mathcal{O}_2} \otimes \theta)\varphi_1(x) \\ &= \sum_{i>0} (S_1^*)^i A_{-i} \otimes (D^*)^i + A_0 \otimes 1_B + \sum_{i>0} A_i (S_1)^i \otimes D^i. \end{aligned}$$

Hence $x = A_0 \in \mathcal{M}_2 \cap (M_2^{\infty})^{\psi_1}$. Therefore $(\mathcal{O}_2)^{{}_\mu U(2)} = (M_2^{\infty})^{{}_\mu U(2)}$.

We define a *-homomorphism η of B onto A such that $\eta(\alpha)=\alpha$, $\eta(\gamma)=\gamma$ and $\eta(D)=1$. Then the following diagram commutes

$$\begin{array}{ccc} M_2^\infty & \xrightarrow{\psi_1} & M_2^\infty \otimes_\alpha B \\ \| & & \downarrow{\scriptstyle \mathrm{id}\otimes\eta} \\ M_2^\infty & \xrightarrow{\psi_2} & M_2^\infty \otimes_\alpha A. \end{array}$$

So $(M_2^\infty)^{\mu U(2)} \subset (M_2^\infty)^{S_\mu U(2)}$.

We shall show that $(M_2^\infty)^{\mu U(2)} \supset (M_2^\infty)^{S_\mu U(2)}$. It is sufficient to show that $(M_2^\infty)^{\mu U(2)}$ contains $\{e_n\}_{n=1}^\infty$ by Proposition 9. We set

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D^2 \end{pmatrix} \in M_4(B) \cong M_2 \otimes M_2 \otimes B,$$

then

$$v \ominus v = \left(\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \ominus \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right) \tau$$

and

$$\tau(e_1 \otimes 1_B) = (e_1 \otimes 1_B)\tau.$$

Then we have

$$\begin{aligned} \psi_1(e_1) &= (v \ominus v)(e_1 \otimes 1_B)(v \ominus v)^* \\ &= \left(\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \ominus \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right) \tau(e_1 \otimes 1_B)\tau^* \left(\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \ominus \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right)^* \\ &= \left(\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \ominus \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right) (e_1 \otimes 1_B) \left(\begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \ominus \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right)^* \\ &= e_1 \otimes 1_B. \end{aligned}$$

By this fact and Lemma 7, $e_n \in (M_2^\infty)^{\mu U(2)}$ for any positive integer n.

So the theorem holds. □

REMARK 11. In the case $\mu = 1$,

$$e_i e_{i\pm1} e_i = \frac{\mu^2}{(1+\mu^2)^2} e_i = \frac{1}{4} e_i,$$

and the projection e_1 is represented as follows:

$$e_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the above theorem is a C^*-version of a deformation of Jones' result ([**2**], [**4**], [**5**]). □

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