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## **A REMARK ON THE SYMMETRY OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS**

JI MIN

# A REMARK ON THE SYMMETRY OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS

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**This note gives a necessary and sufficient condition for solutions of second order elliptic equations to be radially symmetric.**

## 1. Introduction.

1.1. In an elegant paper [GNN], Gidas-Ni-Nirenberg proved that the positive solutions of

$$(1) \quad \begin{cases} \Delta u = f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u \in C^2(\overline{B}), \end{cases}$$

must be radially symmetric. Here  $f$  is  $C^1$  and  $B$  is the  $n$ -dimensional ball:  $\{x \in R^n; |x| < 1\}$ . Obviously a symmetric solution of (1) is not necessary to be positive. In this note, we give a necessary and sufficient condition for symmetric solutions of (1). The main result is the following

**THEOREM 1.** *Suppose  $n \geq 2$ . A solution  $u$  of (1) is radially symmetric if and only if its nodal set  $\{x \in \overline{B}; u(x) = 0\}$  is radially symmetric.*

**REMARK.** It is interesting to note that Theorem 1 need not hold in case  $n = 1$ . For,  $u = \sin x$  solves

$$u'' = -u \quad \text{in } [-\pi, \pi]$$

with the symmetric nodal set  $\{0\} \cup \{-\pi, \pi\}$ , but  $u$  is not radially symmetric since  $\sin(-x) = -\sin x$ .

It is clear that the result of [GNN] is a special case of Theorem 1 since the nodal set of a positive solution to (1) is the sphere  $\partial B$ .

In order to prove Theorem 1, we need the following two preliminary results.

**THEOREM 2.** *Let  $u \in C^2(\overline{B})$  satisfy*

$$(2) \quad \Delta u = f(u) \quad \text{in } B.$$

If the nodal set of  $u$  consists of spheres with the center 0, then these spheres must be isolated unless  $u \equiv 0$ .

**THEOREM 3.** Let  $n \geq 2$  and  $u \in C^2(\overline{B})$  satisfy

$$(3) \quad \begin{cases} \Delta u = f(u) & \text{in } B, \\ u > 0 & \text{in } B \setminus \{0\}, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Then  $u > 0$  in  $B$ .

**REMARK.** In case  $n = 1$ , Theorem 3 need not hold. For example, let  $u(x) = \sin(x - \frac{\pi}{2}) + 1$  for  $x \in [-2\pi, 2\pi]$ , we have

$$\begin{cases} u'' = 1 - u & \text{in } (-2\pi, 2\pi), \\ u > 0 & \text{in } (-2\pi, 2\pi) \setminus \{0\}, \\ u = 0 & \text{at } x = 0, -2\pi, 2\pi. \end{cases}$$

1.2. The proof of Theorem 3 is based on Lemma 12.1 in [GNN], we rewrite it in the form.

**LEMMA A.** Let  $p = (p^1, p^2, \dots, p^n) \in \partial B$  with  $p^1 > 0$ . Assume for some  $\varepsilon > 0$  that  $u$  is a  $C^2$  function satisfying equation (2) in  $\overline{\Omega}_\varepsilon$  where  $\Omega_\varepsilon = B \cap \{x; |x - p| < \varepsilon\}$ ,  $u > 0$  in  $\overline{\Omega}_\varepsilon \setminus \partial B \cap \{x; |x - p| < \varepsilon\}$  and  $u = 0$  on  $\partial B \cap \{x; |x - p| < \varepsilon\}$ . Then there exists  $\delta > 0$  such that in  $B \cap \{x; |x - p| < \delta\}$ ,  $\frac{\partial u}{\partial x_1} < 0$ .

## 2. Proofs.

2.1. *Proof of Theorem 2.* We may assume that the nodal set of  $u$  is  $\bigcup_{\lambda \in \Lambda} S(\lambda)$  where  $\Lambda \subset [0, 1]$  and  $S(\lambda) = \{x \in R^n; |x| = \lambda\}$ . It needs to be proved that the set  $\Lambda$  contains only isolated points unless  $u \equiv 0$ . Suppose that there is a sequence  $\{\lambda_i\} \subset \Lambda$  with  $\lambda_i \rightarrow \bar{\lambda}$ . Using the polar coordinates  $x = r\xi$  where  $\xi \in S^{n-1}$  and  $r^2 = x_1^2 + x_2^2 + \dots + x_n^2$ , we obtain that  $u = \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} = 0$  for  $r = \bar{\lambda}$ , which implies that

$$u(0) = \frac{\partial u}{\partial x_i}(0) = \frac{\partial^2 u}{\partial x_i^2}(0) = 0 \quad (i = 1, 2, \dots, n)$$

when  $\bar{\lambda} = 0$ , and that  $u = D_\xi^2 u = D_\xi^2 u = 0$  on  $S(\bar{\lambda})$  when  $\bar{\lambda} > 0$ . Thus, in both cases,  $u = \Delta u = 0$  on  $S(\bar{\lambda})$ , and, from (2) we conclude that  $f(0) = 0$ . Set

$$c(x) = \int_0^1 f'(tu(x)) dt.$$

In case  $\bar{\lambda} > 0$ , we have

$$\begin{cases} \Delta u - c(x)u = 0 & \text{in } \{x; |x| < \bar{\lambda}\}, \\ u = \frac{\partial u}{\partial r} = 0 & \text{on } S(\bar{\lambda}), \end{cases}$$

and obtain  $u = 0$  in  $B$  by uniqueness of solutions to Cauchy's problem of linear elliptic equations. Now it remains to consider the case  $\bar{\lambda} = 0$ . Set

$$w(x) = \cos Nx_1 \cdot \cos Nx_2 \cdot \dots \cdot \cos Nx_n,$$

where  $N$  is taken to be large enough so that

$$(4) \quad c(x) + N^2 \geq 0.$$

Put  $u = w \cdot v$  for  $|x| < \frac{\pi}{2N}$ . It is easy to see that

$$\begin{cases} \Delta w = -N^2 w \\ w > 0 \end{cases} \quad \text{in } \left\{x; |x| < \frac{\pi}{2N}\right\}$$

and  $S(\lambda_i) \subset \{x; |x| < \frac{\pi}{2N}\}$  for  $i$  large enough since  $\lambda_i \rightarrow 0$  as  $i \rightarrow \infty$ . On account of (2), it follows

$$\begin{cases} \Delta v + \frac{\nabla W}{W} \nabla v - (c(x) + N^2)v = 0 & \text{in } \{x; |x| < \lambda_i\}, \\ v = 0 & \text{on } S(\lambda_i). \end{cases}$$

Because of (4), a well-known maximum principle for second order linear elliptic equations can be applied, and that  $v = 0$  is obtained, so  $u = 0$  for  $|x| < \lambda_i$ , and in turn  $u = 0$  in  $B$ . The proof is completed.

**2.2. Proof of Theorem 3.** Suppose for contradiction that  $u(0) = 0$ . Automatically  $\nabla u(0) = 0$ . For  $0 \leq \lambda < 1$ , denote  $\Sigma_\lambda = \{x \in B; x_1 > \lambda\}$ ;  $T_\lambda = \{x \in B; x_1 = \lambda\}$ , and for  $x \in \Sigma_\lambda$ , denote by  $x^\lambda$  the reflexion of  $x$  with respect to  $T_\lambda$ , denote by  $\Sigma'_\lambda$  the reflexion of  $\Sigma_\lambda$  with respect to  $T_\lambda$ . Set

$$\Lambda = \left\{ \lambda \in (0, 1); u(x^\lambda) > u(x) \text{ in } \Sigma_\lambda, \frac{\partial u}{\partial x_1} < 0 \text{ on } T_\lambda \right\},$$

which is not empty by Lemma A and a similar argument to [GNN]. First of all we prove  $\inf \Lambda \in \Lambda$ . Indeed, there holds

$$\begin{cases} u(x^\alpha) \geq u(x) & \text{in } \Sigma_\alpha, \\ \frac{\partial u}{\partial x_1} \geq 0 & \text{on } T_\alpha \end{cases}$$

where  $\alpha = \inf \Lambda$ . Letting  $w(x) = u(x^\alpha)$  for  $x \in \Sigma_\alpha$  and

$$c(x) = \int_0^1 f'(u + t(w - u)) dt,$$

we have

$$\begin{cases} \Delta(w - u) - c(x)(w - u) = 0, \\ (w - u) \geq 0 \quad \text{in } \Sigma_\alpha, \\ (w - u) = 0 \quad \text{on } T_\alpha. \end{cases}$$

Then for  $K > 0$ ,

$$\Delta(w - u) - (K + c(x)) \cdot (w - u) = -K(w - u) \leq 0 \quad \text{in } \Sigma_\alpha.$$

Taking  $K$  large enough, we may apply the Hopf maximum principle to  $(w - u)$  and obtain that either

$$(5) \quad (w - u) = 0 \quad \text{in } \Sigma_\alpha$$

or

$$(6) \quad \begin{cases} w(x) > u(x) \quad \text{in } \Sigma_\alpha, \\ \frac{\partial}{\partial \bar{n}}(w - u)(p) < 0, \end{cases}$$

where  $p \in \partial \Sigma_\alpha$  such that  $(w - u)(p) = 0$  and  $\bar{n} = \bar{n}(p)$  is the outward normal vector of  $\partial \Sigma_\alpha$  at  $p$ . Then (5) cannot hold since  $n \geq 2$  and  $u = 0$  on  $\partial B$ ;  $u > 0$  in  $B \setminus \{0\}$ . Now (6) holds, then  $u(x^\alpha) > u(x)$  in  $\Sigma_\alpha$ , and on  $T_\alpha$ ,

$$2 \frac{\partial u}{\partial x_1} = \frac{\partial}{\partial(-x_1)}(w - u) < 0$$

since  $(w - u) = 0$ , which means  $\alpha \in \Lambda$ . Next it is easy to see that  $\alpha \geq \frac{1}{2}$ . If  $\alpha = \frac{1}{2}$ , let  $p_0 = (1, 0, \dots, 0) \in \partial B$ , then  $p_0^\alpha = 0$ , and

$$(w - u)(p_0) = u(p_0^\alpha) - u(p_0) = 0.$$

By (6) we have

$$\frac{\partial}{\partial x_1}(w - u)(p_0) < 0, \quad \text{i.e.} \quad -\frac{\partial u}{\partial x_1}(0) - \frac{\partial u}{\partial x_1}(p_0) < 0.$$

Then we get

$$\frac{\partial u}{\partial x_1}(0) > -\frac{\partial u}{\partial x_1}(p_0) \geq 0,$$

a contradiction since  $\nabla u(0) = 0$ . Thus  $\alpha > \frac{1}{2}$ . In this case we claim that there exists  $\alpha_0 < \alpha$  such that  $\alpha_0 \in \Lambda$ , which will contradict the assumption  $\alpha = \inf \Lambda$  and our proof would then be completed. To

this end, we assume again for contradiction that there exists a sequence  $\{\alpha_i\}$  with  $\alpha_i \rightarrow \alpha$  but  $\alpha_i \notin \Lambda$  which means that either

$$(7) \quad u(a_i^{\alpha_i}) \leq u(a_i) \quad \text{for some } a_i \in \Sigma_{\alpha_i}$$

or

$$(8) \quad \frac{\partial u}{\partial x_1}(b_i) \geq 0 \quad \text{for some } b_i \in T_{\alpha_i}.$$

The latter cannot always remain true for any subsequence of  $\{i\}$  since, otherwise, it implies that  $\frac{\partial u}{\partial x_1} \geq 0$  at some point on  $T_\alpha$  when  $\{b_i\}$  do not approach  $\partial B$ , contradicting  $\alpha \in \Lambda$ , and that there exists a point in any neighborhood of  $b$  such that  $\frac{\partial u}{\partial x_1} \geq 0$  when  $b_i \rightarrow b \in \partial B$ , contradicting Lemma A since  $b = (b^1, \dots, b^n)$  with  $b^1 = \alpha > 0$ . Now let  $a_i \rightarrow \bar{a} \in \bar{\Sigma}_\alpha$ . From (7)  $u(\bar{a}^\alpha) \leq u(\bar{a})$ , and  $\bar{a} \in \partial \Sigma_\alpha$  by  $\alpha \in \Lambda$ . But because  $\alpha > \frac{1}{2}$ , for  $x \in \partial \Sigma_\alpha \setminus \bar{T}_\alpha \subset \partial B$ , where  $\bar{T}_\alpha$  is the closure of  $T_\alpha$ , obviously  $u(x^\alpha) > 0 = u(x)$ . Thus we further have  $\bar{a} \in \bar{T}_\alpha$ . Let  $L_i$  be the segment joining  $a_i^{\alpha_i}$  and  $a_i$ , having  $(1, 0, \dots, 0)$  as the tangent vector. From (7) it is seen that there exists  $y_i \in L_i$  such that  $\frac{\partial u}{\partial x_1}(y_i) \geq 0$ . Since  $\bar{a} \in \bar{T}_\alpha$ ,  $y_i$  must also tend to  $\bar{a}$ . And automatically  $\frac{\partial u}{\partial x_1}(\bar{a}) \geq 0$ , which leads to a contradiction when  $\bar{a} \in T_\alpha$ . Then  $\bar{a} \in \partial \bar{T}_\alpha \subset \partial B$ . But we have seen that  $\frac{\partial u}{\partial x_1}(y_i) \geq 0$  and  $y_i \rightarrow \bar{a}$ , which contradicts Lemma A. Thus we complete the proof.

**2.3. Proof of Theorem 1.** Denote  $B(\lambda) = \{x \in R^n; |x| < \lambda\}$ . The necessity is obvious. For sufficiency, by Theorem 2, the nodal set of  $u$  must be  $\bigcup_{i=1}^k S(\lambda_i)$  where  $0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k = 1$ . We further prove  $\lambda_1 > 0$ .

Indeed suppose  $\lambda_1 = 0$ , i.e.  $u(0) = 0$ . We see that there are no nodal points of  $u$  in  $B(\lambda_2) \setminus \{0\}$ , which, together with the fact that  $B(\lambda_2) \setminus \{0\}$  is path-connected (since  $n \geq 2$ ), implies that  $u$  is positive (or negative) in  $B(\lambda_2) \setminus \{0\}$ . Then from Theorem 3 we have  $u(0) > 0$  (or  $u(0) < 0$ ) also. It contradicts  $u(0) = 0$ , which shows  $\lambda_1 > 0$ .

Now in  $B(\lambda_1)$ ,  $u$  is positive (or negative). It allows us to apply the result of [GNN] to conclude that  $u$  is radially symmetric in  $B(\lambda_1)$ . It is clear that

$$(9) \quad \frac{\partial u}{\partial r} = \text{const.} \quad \text{on } S(\lambda_1).$$

Let  $T: R^n \rightarrow R^n$  be any rotation transform. Since equation (2) is invariant under the transform  $T$ ,  $v = u(Tx)$  also solves (2). On

$S(\lambda_1)$ , obviously  $v = u$ , and  $\frac{\partial v}{\partial r} = \frac{\partial u}{\partial r}$  by (9). Then  $(v - u)$  is a solution to the Cauchy problem

$$\Delta w = \left( \int_0^1 f'(tv + (1-t)u) dt \right) \cdot w \quad \text{in } B,$$

$$w = \frac{\partial w}{\partial r} = 0 \quad \text{on } S(\lambda_1)$$

and constantly equals 0 by the uniqueness of the Cauchy problem, i.e.  $u(x) = u(Tx)$  in  $B$  for any rotation transforms  $T$ , which means  $u$  is radially symmetric in  $B$ . We finish the proof of our main theorem.

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