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## **LEVEL SET MAXIMA AND QUASILINEAR ELLIPTIC PROBLEMS**

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**The present paper studies existence of solutions to the problem  $\rho A'(x) = B'(x)$  where  $A$  and  $B$  are Fréchet differentiable functionals on a Banach space. For every given value of  $A(x) = t$  we prove existence of a solution  $x$  and present an expression for the eigenvalue  $\rho = \rho(t)$ . The result is applied to quasilinear elliptic equations.**

**1. Introduction.** A typical problem of the second order studied below is

$$(1.1) \quad -\operatorname{div} \mathcal{A}'_{\xi}(x, \nabla u) = f(x, u), \quad u|_{\partial\Omega} = 0,$$

where  $\Omega \subset \mathbf{R}^n$  is an open bounded domain,  $\mathcal{A}(x, \xi): \Omega \times \mathbf{R}^n \rightarrow \mathbf{R}$  and “ellipticity” of  $\mathcal{A}$  is defined as:

$$(1.2) \quad \begin{aligned} &\mathcal{A}(x, \cdot) \text{ is convex for any } x \in \Omega, \\ &\mathcal{A}'_{\xi}(x, \xi) \cdot \xi \geq c|\xi|^p, \quad p > 1, \quad c > 0, \quad (x, \xi) \in \Omega \times \mathbf{R}^n. \end{aligned}$$

When  $\mathcal{A}$  is a quadratic form of  $\xi$  and  $p = 2$ , (1.1) becomes a semilinear equation. The study extends also to equations of higher order and to systems.

Our approach follows the approach of Browder [2] and Berger [1] with a subsequent refinement due to [3] and [4]. Let  $X$  be a Banach space,  $A, B \in C^1(X \rightarrow \mathbf{R})$ . We consider a critical point equation

$$(1.3) \quad \rho A'(x) = B'(x).$$

This equation might be associated with the maximum problem

$$(1.4) \quad \sigma(t) = \sup_{A(x)=t} B(x).$$

Under general compactness conditions, the maximum in (1.4) is attained and provides (1.3). Remarkably the eigenvalues  $\rho$  are derivatives of the function of critical values  $\sigma(t)$ . More precisely, since we prove that  $\sigma$  might have different right and left hand derivatives  $\sigma'_{\pm}(t)$ , both of them are eigenvalues. Under additional conditions we prove that (1.3) is solvable for any  $\rho$  between  $\inf_t \sigma'_{-}(t)$  and

$\sup_t \sigma'_+(t)$ . Thus, if the graph of  $\sigma(t)$  has a slope less than 1 and a slope greater than 1, then  $A'(u) = B'(u)$  will be solvable. Applications to quasilinear elliptic problems then follow.

**2. Maxima on the level sets.** Let  $X$  be a reflexive Banach space and let  $A, B \in C^1(X \rightarrow \mathbb{R})$ . Let us list several conditions to be used later. Let

$$(2.1) \quad S_t = \{u \in X : A(u) = t\},$$

$$(2.2) \quad \omega_t = \{u \in X : A(u) \leq t\}$$

and define for those  $t$  when  $S_t \neq \emptyset$ ,

$$(2.3) \quad \sigma(t) = \sup_{u \in S_t} B(u),$$

$$(2.4) \quad \beta(t) = \sup_{u \in \omega_t} B(u) = \sup_{\tau \leq t} \sigma(\tau).$$

**LEMMA 2.1.** *Assume the following conditions:*

(A1)  *$A$  is coercive, i.e.,  $\|u_j\| \rightarrow \infty \Rightarrow A(u_j) \rightarrow \infty$ .*

(A2)  *$A$  is weakly lower semicontinuous.*

(A3)  *$\langle A'(u), u \rangle > 0$  for  $u \in X \setminus \{0\}$ .*

(B1)  *$B$  has no local maxima on  $X$ .*

(B2)  *$B$  is weakly continuous.*

*Then*

$$(2.5) \quad \sigma(t) \text{ is increasing on } [0, \infty),$$

$$(2.6) \quad \sigma(t) \text{ is continuous on } [0, \infty),$$

$$(2.7) \quad \text{the maximum in (2.3) is attained for every } t \geq 0.$$

(An immediate consequence of (2.5) is that  $\sigma(t) = \beta(t)$ .)

Without loss of generality one can assume from now that

$$(2.8) \quad A(0) = 0.$$

Then, due to (A3)

$$\begin{aligned} (2.9) \quad A(u) &= \int_0^1 \frac{d}{dt} A(tu) dt = \int_0^1 \langle A'(tu), u \rangle dt \\ &= \int_0^1 t^{-1} \langle A'(tu), tu \rangle dt > 0, \quad \text{unless } u = 0. \end{aligned}$$

By (A1) the range of  $A$  is  $[0, \infty)$ .

*Proof of Lemma 2.1.* Consider a maximizing sequence  $u_j \in \omega_t$  in (2.4). By (A1) it is bounded in norm. Then there is a weakly convergent renamed subsequence  $u_j \xrightarrow{w} u_0$ . By (A2)  $u_0 \in \omega_t$ . By (B2)  $B(u_j) \rightarrow B(u_0) = \beta(t)$ . If  $u_0 \notin S_t$ , then it is a point of interior maximum for  $B$  in  $\omega_t$ . This contradicts (B1). Consequently,  $u_0 \in S_t$ . This implies in turn that  $\sigma(t) = \beta(t)$  and it is a monotone increasing function. For the given  $t$ , the point  $u_0$  is a point of maximum over  $S_t$ . Thus, (2.5) and (2.7) are proved.

Consider

$$(2.10) \quad \sigma(t_0 + 0) = \liminf_{t \rightarrow t_0, t > t_0} \sigma(t),$$

$$(2.11) \quad \sigma(t_0 - 0) = \limsup_{t \rightarrow t_0, t < t_0} \sigma(t).$$

Let  $t_j > t_0$ ,  $t_j \rightarrow t_0$  and let  $\sigma(t_j)$  be attained at  $u_j \in S_{t_j}$ . Then by (A1)  $u_j$  has a renamed weakly convergent subsequence:  $u_j \xrightarrow{w} u_0 \in \omega_{t_0}$ . Therefore,

$$(2.12) \quad \sigma(t_0 + 0) = B(u_0) \leq \sup_{u \in \omega_{t_0}} B(u) = \sigma(t_0).$$

The converse inequality is true by monotonicity of  $\sigma$ . Thus

$$(2.13) \quad \sigma(t_0 + 0) = \sigma(t_0).$$

Let now  $u_0$  be a point of maximum in (2.1) at  $t = t_0$ . Then by (A3)

$$(2.14) \quad t(s) := A(su_0)$$

is a monotone increasing function and

$$t(s) \rightarrow t_0, \quad t(s) < t_0 \quad \text{as } s \rightarrow 1, \quad s < 1.$$

Therefore,

$$(2.15) \quad \sigma(t_0 - 0) \geq \limsup_{\substack{s \rightarrow 1 \\ s < 1}} B(su_0) = \sigma(t_0).$$

The converse inequality is due to monotonicity of  $\sigma$ . Thus  $\sigma(t_0 - 0) = \sigma(t_0)$  and (2.6) is proved.  $\square$

**LEMMA 2.2.** *Assume the conditions of Lemma 2.1. Let  $u_0$  be a point of maximum in (2.3),  $t > 0$ . Then there is a  $\rho \geq 0$ , such that*

$$(2.16) \quad \rho A'(u_0) = B'(u_0).$$

*Proof.* Let  $v \in X$  be such that

$$(2.17) \quad \langle A'(u_0), v \rangle < 0.$$

By Lemma 2.1  $u_0$  is a point of maximum of  $B$  in  $\omega_t$  and

$$(2.18) \quad B(u_0 + \theta v) \leq B(u_0)$$

for  $\theta > 0$  sufficiently small. Thus,

$$(2.19) \quad \langle B'(u_0), v \rangle \leq 0$$

for any  $v$  satisfying (2.17). From (2.17), (2.19) a routine argument shows that  $A'(u_0)$  and  $B'(u_0)$  are parallel. By (A3),  $A'(u_0) \neq 0$  and (2.16) follows immediately.  $\square$

Let us define a set  $\Lambda_t \subset S_t \times [0, \infty)$ ,  $t \in (0, \infty)$ :

$$(2.20) \quad \Lambda_t = \{(u, \rho) : B(u) = \sigma(t), \rho A'(u) = B'(u)\}.$$

By Lemmas 2.1, 2.2  $\Lambda_t$  is nonempty. Let

$$(2.21) \quad \Lambda_t^{(\rho)} = \{\rho \geq 0 : \exists u \in S_t, (u, \rho) \in \Lambda_t\},$$

$$(2.22) \quad \Lambda_t^{(u)} = \{u \in S_t : \exists \rho \geq 0, (u, \rho) \in \Lambda_t\}.$$

By (2.20), one has also:

$$\Lambda_t^{(u)} = \{u \in S_t : B(u) = \sigma(t)\}$$

and

$$(2.23) \quad \Lambda_t^{(\rho)} = \{\rho = \langle B'(u), u \rangle / \langle A'(u), u \rangle, u \in \Lambda_t^{(u)}\}.$$

**LEMMA 2.3.** *Assume the conditions of Lemma 2.1 and in addition (A4) If  $u_j \xrightarrow{w} u_0$ , then*

$$A(u_j) \rightarrow A(u_0) \Leftrightarrow \langle A'(u_j), u_j \rangle \rightarrow \langle A'(u_0), u_0 \rangle.$$

(B3)  $\langle B'(u), u \rangle$  is weakly continuous.

Let  $T \subset (0, \infty)$  and

$$(2.24) \quad \Lambda_T = \bigcup_{t \in T} \Lambda_t.$$

If  $T$  is compact in  $(0, \infty)$ , then  $\Lambda_T$  is weakly compact in  $X \times (0, \infty)$ .

*Proof.* Let  $(u_j, \rho_j) \in \Lambda_{t_j}$ ,  $t_j \in T$ . Consider a renamed convergent subsequence  $t_j \rightarrow t_0 \in T$ . Then by (A1), (A2)  $u_j$  has a weakly convergent (renamed) subsequence  $u_j \xrightarrow{w} u_0 \in \omega_{t_0}$ . By (B2),  $B(u_j) \rightarrow B(u_0)$ , by Lemma 2.1  $B(u_0) = \sigma(t_0)$  and  $u_0 \in S_{t_0}$ , i.e.,

$u_0 \in \Lambda_{t_0}^{(u)}$ . By Lemma 2.2  $u_j$  satisfy (2.16) with eigenvalues, say,  $\rho_j$ ,  $u_0$  satisfies (2.16) with some  $\rho_0$ , and

$$(2.25) \quad \begin{aligned} \rho_j &= \langle B'(u_j), u_j \rangle / \langle A'(u_j), u_j \rangle, \\ \rho_0 &= \langle B'(u_0), u_0 \rangle / \langle A'(u_0), u_0 \rangle. \end{aligned}$$

Note now that  $\langle B'(u_j), u_j \rangle \rightarrow \langle B'(u_0), u_0 \rangle$  by (B3) and  $\langle A'(u_j), u_j \rangle \rightarrow \langle A'(u_0), u_0 \rangle$  by (A4). Thus  $\rho_j \rightarrow \rho_0$ .  $\square$

### 3. Critical values and eigenvalues.

**THEOREM 3.1.** *Assume*

(A1)  $A$  is coercive, i.e.,  $\|u_j\| \rightarrow \infty \Rightarrow A(u_j) \rightarrow \infty$ .

(A2)  $A$  is weakly lower semicontinuous.

(A3)  $\langle A'(u), u \rangle > 0$  for  $u \in X \setminus \{0\}$ .

(A4) If  $u_j \xrightarrow{w} u_0$ , then

$$A(u_j) \rightarrow A(u_0) \Leftrightarrow \langle A'(u_j), u_j \rangle \rightarrow \langle A'(u_0), u_0 \rangle.$$

(B1)  $B$  has no local maxima on  $X$ .

(B2)  $B$  is weakly continuous.

(B3)  $\langle B'(u), u \rangle$  is weakly continuous.

Then for every  $t > 0$  there exist left and right derivatives  $\sigma'_\pm(t)$ ,  $\sigma'_\pm(t) \leq \sigma'_t(t)$ . Moreover,

$$(3.1) \quad \sigma'_+(t) = \sup\{\rho \in \Lambda_t^{(\rho)}\},$$

$$(3.2) \quad \sigma'_-(t) = \inf\{\rho \in \Lambda_t^{(\rho)}\}.$$

Before we prove the theorem, we wish to note that supremum in (3.1) and infimum in (3.2) are attained on some  $u_\pm \in \Lambda_t^{(u)}$  due to Lemma 2.3. As a result one has

**THEOREM 3.2.** *Under conditions of Theorem 3.2 for every  $t > 0$  there exist  $u_\pm \in S_t$ , such that*

$$(3.3) \quad \sigma'_+(t)A'(u_+) = B'(u_+),$$

$$(3.4) \quad \sigma'_-(t)A'(u_-) = B'(u_-).$$

*Proof of Theorem 3.1.* 1. Let  $u_0 \in \Lambda_{t_0}^{(u)}$ . Let  $\theta_j \rightarrow 1$ ,  $t_j = A(\theta_j u_0)$ . Then by continuity of  $A$ ,  $t_j \rightarrow A(u_0) = t_0$ . Moreover,

$$(3.5) \quad t_j - t_0 = A(\theta_j u_0) - A(u_0) = \langle A'(u_0), u_0 \rangle (\theta_j - 1) + o(\theta_j - 1).$$

Consequently,

$$\begin{aligned}
 (3.6) \quad \sigma(t_j) - \sigma(t_0) &\geq B(\theta_j u_0) - B(u_0) \\
 &= \langle B'(u_0), u_0 \rangle (\theta_j - 1) + o(\theta_j - 1) \\
 &= (\langle B'(u_0), u_0 \rangle / \langle A'(u_0), u_0 \rangle) (t_j - t_0) + o(t_j - t_0).
 \end{aligned}$$

Fro (3.6) and (2.23) one has immediately,

$$(3.7) \quad D_+ \sigma(t_0) \geq \sup\{\rho \in \Lambda_{t_0}^{(\rho)}\},$$

$$(3.8) \quad D^- \sigma(t_0) \leq \inf\{\rho \in \Lambda_{t_0}^{(\rho)}\}.$$

2. Let now  $u_j \in \Lambda_{t_j}^{(u)}$ ,  $t_j \rightarrow t_0$ . By Lemma 2.3 a renamed sequence  $u_j$  converges weakly to  $u_0 \in \Lambda_{t_0}^{(u)}$ . Let us define  $\theta_j > 0$  by

$$(3.9) \quad A(\theta_j u_j) = t_0.$$

By (A3), the function  $\theta \rightarrow A(\theta u_j)$  is monotone for any  $u \neq 0$  and by (A1) the range of it is  $[0, \infty)$ . Thus, for given  $t_0 > 0$  and  $u_j \neq 0$ , (3.9) has a unique solution  $\theta_j > 0$ . Since  $A \in C^1$ , there exist  $\eta_j \in [\theta_j, 1]$ , such that

$$(3.10) \quad t_j - t_0 = A(u_j) - A(\theta_j u_j) = \langle A'(\eta_j u_j), u_j \rangle (1 - \theta_j).$$

From (3.9) it follows that  $\theta_j$  is a bounded sequence. Let us consider a renamed convergent subsequence:  $\theta_j \rightarrow \theta_0$ . Then  $\theta_j u_j \xrightarrow{w} \theta_0 u_0$  and, necessarily,  $\theta_0 = 1$ . Therefore  $\eta_j \rightarrow 1$  and

$$(3.11) \quad t_j - t_0 = \langle A'(u_j), u_j \rangle (1 - \theta_j) + o(1 - \theta_j).$$

Similarly,

$$(3.12) \quad B(u_j) - B(\theta_j u_j) = \langle B'(u_j), u_j \rangle (1 - \theta_j) + o(1 - \theta_j).$$

Therefore

$$\begin{aligned}
 (3.13) \quad \sigma(t_j) - \sigma(t_0) &\leq \langle B'(u_j), u_j \rangle (1 - \theta_j) + o(1 - \theta_j) \\
 &= (\langle B'(u_j), u_j \rangle / \langle A'(u_j), u_j \rangle) (t_j - t_0) + o(t_j - t_0).
 \end{aligned}$$

We have to note only that  $\langle A'(u_j), u_j \rangle \rightarrow \langle A'(u_0), u_0 \rangle$  by (A4) and  $\langle B'(u_j), u_j \rangle \rightarrow \langle B'(u_0), u_0 \rangle$  by (B2). Then from (3.13) follows:

$$(3.14) \quad D^+ \sigma(t_0) \leq \sup\{\rho \in \Lambda_{t_0}^{(\rho)}\},$$

$$(3.15) \quad D_- \sigma(t_0) \geq \inf\{\rho \in \Lambda_{t_0}^{(\rho)}\}.$$

3. Let us combine (3.7) and (3.14). Then

$$\sup\{\rho \in \Lambda_{t_0}^{(\rho)}\} \leq D_+ \sigma(t_0) \leq D^+ \sigma(t_0) \leq \sup\{\rho \in \Lambda_{t_0}^{(\rho)}\},$$

i.e.,  $\sigma'_+(t_0)$  exists and satisfies (3.1). Similarly, (3.2) follows from (3.8) and (3.15).  $\square$

**4. Continua of solutions. Range of solvability.** In this section we assume the conditions of Theorem 3.1.

**PROPOSITION 4.1.** *Assume that for every  $t \in (s_1, s_2) \subset (0, \infty)$  the set  $\Lambda_t^{(u)}$  consists of a single element  $u_t$ . Then the map  $t \rightarrow u_t$  is weakly continuous on  $(s_1, s_2)$ .*

*Proof.* Let  $t_0 \rightarrow t_0 \in (s_1, s_2)$ . Then  $\overline{\{t_j\}}$  is compact in  $(0, \infty)$  and by Lemma 2.3  $u_{t_j}$  has a (renamed) weakly convergent subsequence  $u_{t_j} \xrightarrow{w} u_0 \in \Lambda_{t_0}^{(u)}$ . Since  $\Lambda_{t_0}^{(u)} = \{u_{t_0}\}$ , the original sequence  $u_{t_j}$  must be weakly convergent to the same element  $u_{t_0}$ .  $\square$

**COROLLARY 4.2.** *Under conditions of Proposition 4.1 the problem*

$$(4.1) \quad \rho A'(u) = B'(u)$$

*possesses a weakly continuous family of eigenfunctions  $t \rightarrow u_t$  corresponding to eigenvalues  $\rho_t = \langle B'(u_t), u_t \rangle / \langle A'(u_t), u_t \rangle$  continuous in  $t \in (s_1, s_2)$ . The function (2.3) has a continuous derivative on  $(s_1, s_2)$  and  $\rho_t = \sigma'(t)$ .*

The proof follows from Theorems 3.1 and 3.2.

**PROPOSITION 4.3.** *Let  $h: X^2 \rightarrow X$  be a map, such that*

(Ah)  $A(h(u, v)) > \max\{A(u), A(v)\}$  for  $u \neq v$ .

(Bh)  $B(h(u, v)) \geq \min\{B(u), B(v)\}$  for  $u, v \in X$ .

*Then the assertions of Proposition 4.1 and Corollary 4.2 hold for any  $t > 0$ .*

*Proof.* Let  $u, v \in \Lambda_t^{(u)}$ ,  $u \neq v$ . Then by (Ah),  $h(u, v) \in \omega_{t'}$ ,  $t' < t$ , but by (Bh)  $B(h(u, v)) \geq B(u) = \sigma(t)$ . This contradicts Lemma 2.1. Thus the conditions of Lemma 4.1 are satisfied at any  $t > 0$ .  $\square$

A simple example when (Ah), (Bh) are satisfied, can be provided by

**COROLLARY 4.4.** *Let  $A$  be strictly convex and  $B$  be concave on  $X$ . Then the assertions of Proposition 4.1 and Corollary 4.2 hold for  $t \in (0, \infty)$ .*

*Proof.* Take  $h(u, v) = \lambda u + (1 - \lambda)v$ ,  $\lambda \in (0, 1)$ .  $\square$



If  $\sigma \in C_{\text{loc}}^1(0, \infty)$ , then (4.1) is solvable (with  $u \neq 0$ ) for any  $\rho \in I(A, B)$ , where

$$(4.2) \quad I(A, B) = \left( \inf_t \sigma'_-(t), \sup_t \sigma'_+(t) \right).$$

However,  $\sigma$  does not generally have a continuous derivative (cf. [4]). Thus we wish to answer the question, for what subsets of  $I(a, b)$  does (4.1) still have a non-zero solution.

**PROPOSITION 4.5.** *Let the function  $\rho t - \sigma(t)$  have a local minimum on  $(0, \infty)$ . Then (4.1) has a non-zero solution.*

*Proof.* Let  $\rho t - \sigma(t)$  have a local minimum at  $t_0 > 0$ . Then  $\sigma'_-(t_0) \geq \rho \geq \sigma'_+(t_0)$ . By Theorem 3.1,  $\sigma'_-(t_0) \leq \sigma'_+(t_0)$ . Thus,  $\sigma$  is differentiable at  $t_0$  and  $\sigma'(t_0) = \rho$ . Then by Theorem 3.2, (4.1) has a solution with  $A(u) = t_0$ .  $\square$

In order to get a more extensive result we use a mountain pass theorem from [5].

**THEOREM 4.6.** *Let  $G \in C^1(X \rightarrow \mathbf{R})$  and let  $u_0 \in X \setminus \{0\}$ . Let  $N \subset X$  be an open bounded set, such that  $0 \in N$  but  $u_0 \notin \overline{N}$ . Assume that*

$$(4.3) \quad G(u) \geq 0, \quad u \in \partial N,$$

$$(4.4) \quad G(0) \leq 0, \quad G(u_0) \leq 0.$$

*Then there is a sequence  $u_k \in X$  and  $\gamma \geq 0$  such that*

$$(4.5) \quad G(u_k) \rightarrow \gamma,$$

$$(4.6) \quad \|G'(u_k)\|_{X^*} \|u_k\|_X \rightarrow 0.$$

Let now

$$(4.7) \quad A_\theta = \theta \langle A'(u), u \rangle - A(u),$$

$$(4.8) \quad B_\theta = \theta \langle B'(u), u \rangle - B(u),$$

$$(4.9) \quad \Phi = \{\theta \in \mathbf{R} : |A_\theta| \text{ is coercive}\},$$

and let

$$(4.10) \quad Q(\theta) \text{ be set of limit points for } B_\theta(u)/A_\theta(u) \\ \text{when } \|u\| \rightarrow \infty, \theta \in \Phi.$$

Set now

$$(4.11) \quad Q_* = \bigcap_{\theta \in \Phi} Q(\theta).$$

**THEOREM 4.7.** *Assume, in addition to conditions of Theorem 3.1, that:*

(ABw)  $A', B'$  are continuous from  $X$  to  $X^*$  with regard to respective weak topologies.

Then for every  $\rho \in I(A, B) \setminus Q_*$  the equation (4.1) has a nonzero solution.

*Proof.* 1. Let  $\rho \in I(A, B) \setminus Q_*$ , i.e.,  $\rho$  is a slope of a secant to the graph of  $\sigma(t)$ , the functional  $|A_\theta|$  is coercive for some  $\theta$  and  $\rho$  is not a limit point of  $B_\theta/A_\theta$  at infinity. If  $\rho t - \sigma(t)$  has a local minimum at  $t \neq 0$ , then  $\rho$  is an eigenvalue (Proposition 4.5). Thus we would consider the case when  $\rho t - \sigma(t)$  has no local minimum. If  $\rho = \sigma'_+(t_0)$  or  $\rho = \sigma'_-(t_0)$  for some  $t_0 > \theta$ , then it is an eigenvalue by Theorem 3.2. The remaining case is: for some  $t_0 > 0$ ,  $\sigma'(t_0) < \rho \sigma'_+(t_0)$ . This implies that  $t_0$  is a point of local maximum of  $\rho t - \sigma(t)$ . Since we assume that  $\rho t - \sigma(t)$  has no local minimum,  $t_0$  is a point of strict global maximum. In particular,  $\delta := \rho t_0 - \sigma(t_0) > 0 - \sigma(0) = 0$ , and there exists  $t_1 > t_0$ , such that  $\delta_1 := \rho t_1 - \sigma(t_1) < \delta$ .

2. Let

$$(4.12) \quad G(u) = \rho A(u) - B(u) - \delta_1.$$

Then all the conditions of Theorem 4.6 are fulfilled with  $N = \{u \in X, A(u) \leq t_0\}$  and  $u_0 \in S_{t_1}$ . Let  $u_k \in X$ ,  $\gamma \geq 0$  satisfy

$$(4.13) \quad \rho A(u_k) - B(u_k) \rightarrow \delta_1 + \gamma \geq \delta_1,$$

$$(4.14) \quad \|\rho A'(u_k) - B'(u_k)\|_{X^*} \|u_k\|_X \rightarrow 0.$$

Then

$$(4.15) \quad \rho \langle A'(u_k), u_k \rangle - \langle B'(u_k), u_k \rangle \rightarrow 0$$

and, consequently,

$$(4.16) \quad \rho A_\theta(u_k) - B_\theta(u_k) \rightarrow -\gamma - \delta.$$

If  $\|u_k\| \rightarrow \infty$ , then  $|A_\theta(u_k)| \rightarrow \infty$ ,  $\rho = \lim B_\theta(u_k)/A_\theta(u_k)$ , which contradicts the assumptions. Thus the sequence  $u_k$  is bounded in norm. Let now  $u_k$  be a renamed weakly convergent sequence, and  $u_0 = w\text{-}\lim u_k$ . Then by (ABw) from (4.14) follows:

$$(4.17) \quad \rho A'(u_0) = B'(u_0).$$

Moreover, by (B3), (4.14), (4.17)

$$(4.18) \quad \begin{aligned} \lim \rho \langle A'(u_k), u_k \rangle &= \lim \langle B'(u_k), u_k \rangle = \langle B'(u_0), u_0 \rangle \\ &= \rho \langle A'(u_0), u_0 \rangle. \end{aligned}$$

Then by (A4)

$$(4.19) \quad \lim A(u_k) = A(u_0).$$

Thus by (4.13), (B2),

$$0 < \delta_1 + \gamma = \lim \rho A(u_k) - B(u_k) = \rho A(u_0) - B(u_0)$$

which proves that  $u_0 \neq 0$ .  $\square$

**COROLLARY 4.8.** *Let  $\rho_* = \inf_{\theta \in \Phi} \limsup_{\|u\| \rightarrow \infty} B_\theta(u)/A_\theta(u)$ . Then (4.1) has a non-zero solution for*

$$\rho \in I(A, B) \cap (\rho_*, \infty).$$

**5. Applications to quasilinear elliptic problems.** Let  $\Omega \subset \mathbf{R}^n$  be an open bounded set  $X = W_0^{l,p}(\Omega)^k$ ,  $p > 1$ ,  $l \in \mathbf{N}$ ,  $k \in \mathbf{N}$ . Let  $\nu(l)$  be the number of multi-indices of length not exceeding  $l$ . Assume that  $\mathcal{A}(x, \{y_\alpha\}_{|\alpha| \leq l})$  and  $\mathcal{B}(x, \{y_\alpha\}_{|\alpha| \leq l-1})$  are  $C^1$  real valued functions of  $\{y_\alpha\}$  whose derivatives are Carathéodory functions of  $(x, \{y_\alpha\})$ . Without loss of generality we assume that

$$(5.1) \quad \mathcal{A}(x, 0) = 0, \quad \mathcal{B}(x, 0) = 0.$$

We require for the function  $\mathcal{A}$  the following ellipticity condition:

$$(5.2) \quad \mathcal{A}(x, \cdot) \text{ is convex for almost every } x \in \Omega$$

and the following coercivity condition:

$$(5.3) \quad \frac{d}{dt} \mathcal{A}(x, \{ty_\alpha\})|_{t=1} \geq c \sum_{|\alpha| \leq l} |y_\alpha|^p, \quad c > 0, x \in \Omega.$$

We also require the following upper bounds for  $\mathcal{A}'$  and  $\mathcal{B}'$ :

$$(5.4) \quad |\mathcal{A}'_{y_\alpha}(x, \{y_\alpha\})| \leq c \left( 1 + \sum_{|\alpha| \leq l} |y_\alpha|^{p-1} \right), \quad c > 0,$$

$$(5.5) \quad |\mathcal{B}'_{y_\alpha}(x, \{y_\alpha\})| \leq C \sum_{\substack{|\beta| \leq l-1 \\ |\beta| \geq l-n/p}} |y_\beta|^{q_{\alpha\beta}} + V_\alpha(\{y_\beta\}_{|\beta| < l-n/p}) + W_\alpha(x),$$

where

$$(5.6) \quad \begin{aligned} C &> 0 \\ W_\alpha &\in L^{r_\alpha}, r_\alpha = 1 \quad \text{if } l - |\alpha| > n/p, \\ r_\alpha &> pn/(pn - n - p(l - |\alpha|)) \quad \text{if } l - |\alpha| \leq n/p, \\ V_\alpha &\text{ is a continuous function,} \end{aligned}$$

$$q_{\alpha\beta} > \frac{pn}{n - p(l - |\beta|)} \quad \text{if } l - |\alpha| > np,$$

$$q_{\alpha\beta} < \frac{pn - n + p(l - |\alpha|)}{n - p(l - |\beta|)}, \quad \text{if } l - |\alpha| \leq n/p.$$

**PROPOSITION 5.1.** *Assume (5.2)–(5.6). Then the functionals*

$$(5.7) \quad A(u) = \int_{\Omega} \mathcal{A}(x, \{\partial^\alpha u\}) dx,$$

$$(5.8) \quad B(u) = \int_{\Omega} \mathcal{B}(x, \{\partial^\alpha u\}) dx$$

satisfy (A1–A4), (B2), (B3), (ABw) on  $X = W_0^{l,p}(\Omega)^k$ .

*Proof.* The verification of continuity and differentiability properties is standard and based on compactness in the Sobolev embedding theorem. We wish to make remarks on only a few details.

1. Relation (5.3) implies (A3) and also (A1), since it immediately gives

$$(5.9) \quad \frac{d}{dt} \mathcal{A}(x, \{ty_\alpha\}) \geq ct^{p-1} \sum_{|\alpha|=l} |y_\alpha|^p, \quad t > 0.$$

2. Relation (5.2) implies that the set (2.2) is convex. Thus it is weakly closed and  $A$  is lower semicontinuous.

3. Due to (5.9), (5.4) weak convergence of a sequence  $u_j$  together with convergence of either  $A(u_j)$  or of  $\langle A'(u_j), u_j \rangle$  is equivalent to convergence in norm.  $\square$

To verify the condition (B1) in most of the applications it suffices to prove that  $B'(u) \neq 0$  unless  $u = 0$  and that  $u = 0$  is not a point of maximum. Two particular cases are given below.

**LEMMA 5.2.** *Let*

$$(5.10) \quad \sum_{\alpha} \mathcal{B}'_{y_\alpha}(x, \{y_\alpha\}) y_\alpha > 0 \quad \text{for } \{y_\alpha\} \neq 0.$$

*Then (B1) holds.*

*Proof.* From (5.10) it follows that  $\langle B'(u), u \rangle > 0$  unless  $u = 0$ . The point  $u = 0$  is not a point of maximum, but rather of minimum, since for every  $u \in X \setminus \{0\}$ ,  $B$  increases along the line  $t \rightarrow tu$ ,  $t > 0$ .  $\square$

LEMMA 5.3. Let  $k = 1$ ,  $b(x, \{y_\alpha\}) = F(y_0)$  and assume that with some  $\varepsilon > 0$ ,

$$(5.11) \quad \begin{aligned} F'(y_0) &> 0 \quad \text{for } y_0 \in (0, \varepsilon) \quad \text{and} \\ F'(y_0) &\neq 0 \quad \text{for } y_0 \in (-\varepsilon, 0). \end{aligned}$$

Then (B1) holds.

*Proof.* Let  $u$  be the point of maximum of  $b$  and let  $\bar{u}$  be a decreasing spherical rearrangement for  $u$ . Then  $\bar{u}$  is a  $W_0^{l,p}$ -function dependent on the radial variable only. Therefore  $\bar{u}$  is continuous away from the origin, the range of  $\bar{u}$  is a closed interval  $I$  containing zero. Moreover, the range of  $u$  is dense in  $I$ , for if  $(s, t)$  is not in the range of  $u$ ,  $(s, t)$  is not in the range of  $\bar{u}$ . If  $u$  is a point of maximum for  $B$ , then  $F' = 0$  on the range of  $u$ , and since  $F'$  is continuous  $F' = 0$  on  $I$ . By (5.11), therefore,  $I \cap (-\varepsilon, \varepsilon) \subset \{0\}$ . However,  $u = 0$  is not a maximum: one can perturb 0 by a function  $v \geq 0$  of an arbitrarily small norm so that  $B(v) > B(0) = 0$ .  $\square$

The following statement is now an immediate corollary of Theorems 3.1, 3.2.

THEOREM 5.4. Assume (5.1–5.6) and (B1). Then for every  $t > 0$  there exists a semistrong solution  $u_t^\pm \in W_0^{l,p}(\Omega)^k$  satisfying the respective equations:

$$(5.12) \quad \begin{aligned} \sigma'_\pm(t) \sum_{|\alpha| \leq l} (-1)^{|\alpha|} \partial^\alpha \mathcal{A}'_{y'_\alpha}(x, \{\partial^\gamma u(x)\}) \\ = \sum_{|\beta| \leq l-1} (-1)^{|\beta|} \partial^\beta \mathcal{B}'_{y'_\beta}(x, \{\partial^\gamma u(x)\}). \end{aligned}$$

Moreover,

$$(5.13) \quad A(u_t^\pm) = t$$

and the function  $\sigma$  is given by

$$(5.14) \quad \sigma(t) = \sup_{A(u)=t} B(u).$$

We now will look for realization of conditions (Ah), (Bh) to get continuous curves of eigenfunctions.

**THEOREM 5.5.** *Assume (5.1–5.6) and (B1). Let  $\mathcal{B}(x, \cdot)$  be concave for a.e.  $x \in \Omega$ . Then there is a continuous family  $t \rightarrow (\rho_t, u_t) \in (0, \infty) \times W_0^{1,p}(\Omega)^k$ , such that*

$$(5.15) \quad \begin{aligned} \rho_t \sum_{|\alpha| \leq l} (-1)^{|\alpha|} \partial^\alpha \mathcal{A}'_{y_\alpha}(x, \{\partial^\gamma u_t(x)\}) \\ = \sum_{|\alpha| \leq l-1} (-1)^{|\alpha|} \partial^\alpha \mathcal{B}'_{y_\alpha}(x, \{\partial^\gamma u_t(x)\}). \end{aligned}$$

*Proof.* Apply Corollary 4.4. Then  $u_t$  satisfies (5.15) and  $t \rightarrow u_t$  is weakly continuous. However, by (5.4), (5.9) convergence of  $t_j = A(u_{t_j})$  together with weak convergence of  $u_{t_j}$  is equivalent to convergence in norm and the family  $(\rho_t, u_t)$  is continuous.  $\square$

**THEOREM 5.6.** *Let  $\mathcal{A}(x, \{\partial^\alpha u\}) = |\nabla u|^p$ ,  $\mathcal{B}(x, \{y_\alpha\}) = F(y_0)$ ,  $k = 1$ . Assume also that  $F \in C^1$ ,*

$$(5.16) \quad F'(s) > 0 \text{ for } s > 0 \text{ and } F'(s) = 0 \text{ for } s \leq 0,$$

*and that the map*

$$(5.17) \quad s \rightarrow F(s^{1/p}) \text{ is concave for } s > 0.$$

*Then there is a continuous family  $t \rightarrow (\rho_t, u_t)$  satisfying*

$$(5.18) \quad -\rho_t \operatorname{div}(|\nabla u_t|^{p-2} \nabla u_t) = F'(u_t), \quad u_t \geq 0.$$

*Proof.* Let

$$(5.19) \quad h_\theta(u_1, u_2) = (\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p}, \quad 0 \in (0, 1).$$

Then, applying Hölder inequality with  $\frac{1}{p} + \frac{1}{p'} = 1$ ,

$$(5.20) \quad \begin{aligned} & |\nabla(\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p}|^p \\ &= \left| \frac{1}{p} \frac{\theta p |u_1|^{p-1} \nabla|u_1|^p + (1-\theta)p |u_2|^{p-1} \nabla|u_2|^p}{(\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p'}} \right|^p \\ &= \left| \frac{\theta^{1/p+1/p'} |u_1|^{p/p'} \nabla|u_1|^p + (1-\theta)^{1/p+1/p'} |u_2|^{p/p'} \nabla|u_2|^p}{(\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p'}} \right|^p \\ &\leq \theta |\nabla|u_1||^p + (1-\theta) |\nabla|u_2||^p \\ &= \theta |\nabla u_1|^p + (1-\theta) |\nabla u_2|^p. \end{aligned}$$

The relation (5.20) makes sense a.e. when  $u_1, u_2 \in C_0^1(\Omega)$  and the equality holds only if  $u_1 = u_2$  or one of them vanishes. Then the following will be true on  $W_0^{1,p}(\Omega)$ :

$$(5.21) \quad A(h_\theta(u_1, u_2)) \leq \theta A(u_1) + (1-\theta) A(u_2)$$

with the same remark on equality. By (5.16), (5.17),

$$(5.22) \quad B(h_\theta(u_1, u_2)) \leq \theta B(u_1) + (1 - \theta)B(u_2).$$

Thus (Ah), (Bh) are satisfied and the assertion follows from Theorem 5.11. One has only to note that  $t_j \rightarrow t_0$  and  $u_{t_j} \xrightarrow{w} u_{t_0}$  implies  $u_{t_j} \rightarrow t_0$ .  $\square$

Now we wish to find some realizations of Theorem 4.7.

LEMMA 5.7. Assume (5.1–5.6) and (B1). Let

$$(5.23) \quad \mathcal{B}'_{y_\alpha}(x, \{y_\alpha\}) = o\left(\sum_{|\alpha| \leq l-1} |y_\alpha|^{p-1}\right) \quad \text{uniformly in } \Omega.$$

Then

$$(5.24) \quad Q_* \subset \{0\}.$$

*Proof.* By (5.3), (5.4),  $A_\theta \geq \|u\|^p$  with  $\theta$  sufficiently large. By (5.23),  $B_\theta(u) = o(\|u\|^p)$ . Thus  $Q_* \subset \{0\}$ .  $\square$

THEOREM 5.8. Assume (5.1–5.6), (B1) and (5.23). Then for every

$$(5.25) \quad \rho \in \left(0, \sup_t \sigma'_+(t)\right)$$

there is a non-zero solution of

$$(5.26) \quad \begin{aligned} \rho \sum_{|\alpha| \leq l} (-1)^{|\alpha|} \partial^\alpha \mathcal{A}(x, \{\partial^\gamma u(x)\}) \\ = \sum_{|\beta| \leq l-1} (-1)^{|\beta|} \partial^\beta \mathcal{B}(x, \{\partial^\gamma u(x)\}). \end{aligned}$$

Moreover,

$$(5.27) \quad \sup_t \sigma'_+(t) \geq \sup_{u \neq 0} B(u)/A(u).$$

*Proof.* By Theorem 4.7, (5.26) is solvable for  $\rho \in I(a, b)$ . The lower bound in  $I(a, b)$  is less than  $\sigma(t)/t$  which goes to zero when  $t$  tends to  $\infty$ . The upper bound of  $I(a, b)$  is greater or equal to any given slope of a secant line to the graph of  $\sigma$ , e.g.  $\sigma(t)/t$ , which implies (5.27).  $\square$

THEOREM 5.9. Assume that  $k = 1$ ,

$$\mathcal{B}(x, \{y_\alpha\}) = F(y_0), \quad \mathcal{A}(x, \{\partial_x^\alpha u\}) = |\nabla u|^p.$$

Let  $F \in C^1$  and if  $n \geq p$ , let  $F'(s) = o(|s|^{q-1})$ ,  $q < pn/(n-p)$ . Assume that for some  $\varepsilon > 0$

(5.28)  $F'(s)/s^{p-1+\varepsilon}$  is an increasing function  
in a neighbourhood of  $+\infty$ ,

$$F'(s) = o(|s|^{p-1}) \quad \text{as } s \rightarrow -\infty.$$

Then for every

$$(5.29) \quad \rho \in \left( \inf_t \sigma'_-(t), \infty \right)$$

there is a solution  $u \neq 0$  for

$$(5.30) \quad -\rho p \operatorname{div}(|\nabla u|^{p-2} \nabla u) = F'(u).$$

*Proof.* Note that  $\mathcal{A}$ ,  $\mathcal{B}$  satisfy (5.1–5.6) and B1.

From (5.28) it follows that  $\mathcal{B}_\theta(u) \leq o(\|u\|^p)$  for  $\theta > 1/(p+\varepsilon)$ . Thus  $Q_* \cap (0, \infty) = \emptyset$ , and one can apply Theorem 4.7. By (5.8)  $\sup \sigma'_+(t) \geq \sup_t \sigma(t)/t = \infty$ .  $\square$

As a general realization of 4.7 we state:

**THEOREM 5.10.** Assume (5.1–5.6) and (B1). Then (5.26) has a non-zero solution  $u$  for

$$(5.31) \quad \rho \in (\inf \sigma'_-(t), \sup \sigma'_+(t)) \setminus Q_*.$$

**6. Examples.** The following examples illustrate the solvability results of this paper.

**EXAMPLE 6.1.** Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set. Consider

$$(6.1) \quad \begin{cases} -\rho \operatorname{div}(|\nabla|^{p-2} \nabla u) = u^\alpha + u^\beta, & u \geq 0, p > 1, \\ u|_{\partial\Omega} = 0. \end{cases}$$

*Case 1.*  $0 < \alpha < \beta < p-1$ . Then by Theorem 5.6 there is a continuous family  $(\rho_t, u_t)$  of eigenfunctions. By Theorem 5.4, the range of eigenvalues  $\rho$  is  $I(A, B)$  which is here  $(0, \infty)$ .

*Case 2.*  $0 < \alpha < p-1 < \beta$ . If  $n > p$  assume also  $\beta < np/(n-p) - 1$ . Then (6.1) has a solution for every  $\rho \in (\rho_0, \infty)$ ,  $\rho_0 > 0$  and

$$(6.2) \quad \rho_0 \leq \inf_t \sup_{\|u\|_{1,p}=1} t^{-1} \int_{\Omega} \left( \frac{p}{\alpha+1} u^{\alpha+1} t^{(\alpha+1)/p} + \frac{p}{\beta+1} u^{\beta+1} t^{(\beta+1)/p} \right) dx.$$



Solvability for  $\rho \in I(A, B)$  is provided by Theorem 5.9.

*Case 3.*  $p - 1 < \alpha < \beta$  and if  $n > p$ ,  $\beta < np/(n - p) - 1$ . Similarly, (6.1) is solvable with  $\rho \in (0, \infty)$  by Theorem 5.9.

*Case 4.*  $0 < \alpha < \beta = p - 1$ . The argument is like in Case 1, only  $I(A, B) = (0, \rho_0)$ ,

$$(6.3) \quad \rho_0 = \sup_{\|u\|_{L_p}=1} \frac{1}{p} \int |u|^p.$$

*Case 5.*  $0 < \alpha = p - 1 < \beta$ , if  $n > p$ ,  $\beta < np/(n - p) - 1$ . The argument follows one of Case 3, but  $I(a, b) = (\rho_0, \infty)$ , where  $\rho_0$  is like in (6.3).

**EXAMPLE 6.2.** Let  $\Omega \subset \mathbf{R}^n$ ,  $n < 6$ , be an open bounded domain. Consider a system:

$$\begin{cases} -3\rho \operatorname{div}(|\nabla u| + \frac{1}{2}|\nabla v|^{3/2}|\nabla u|^{-1/2})\nabla u = 5u^4, \\ -3\rho \operatorname{div}(|\nabla v| + \frac{1}{2}|\nabla u|^{3/2}|\nabla v|^{-1/2})\nabla v = 4v^3, \end{cases} \quad u, v|_{\partial\Omega} = 0.$$

This system corresponds to

$$\begin{aligned} A(u, v) &= \int_{\Omega} (|\nabla u|^3 + |\nabla v|^3 + |\nabla u|^{3/2}|\nabla v|^{3/2}) dx, \\ B(u, v) &= \int_{\Omega} (u^5 + v^4) dx. \end{aligned}$$

By Theorem 5.9, it is solvable for  $\rho \in I(A, B)$  and  $I(A, B) = (0, \infty)$ .

## REFERENCES

- [1] M. S. Berger, *Nonlinearity and Functional Analysis*, Academic Press 1977.
- [2] F. E. Browder, *Infinite dimensional manifolds and non-linear elliptic eigenvalue problems*, Ann. of Math., **82** (1965), 459–477.
- [3] M. Schechter and K. Tintarev, *Spherical maxima in Hilbert space and semilinear elliptic problems*, Differential Equations, **3(5)** (1990), 889–899.
- [4] —, *Eigenvalues for semilinear boundary value problem*, Arch. Rational Mech. Anal., **113** (1991), 197–208.
- [5] M. Schechter, *A variation of the mountain pass lemma and applications*, J. London Math. Soc., (to appear).

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