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THE PERIOD MATRIX OF BRING'S CURVE

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In genus four there is only one Riemann surface admitting the symmetric group of order five as group of automorphisms: we compute its Riemann matrix. On the other hand, we show that there is a one complex parameter family of Jacobians admitting the same group of automorphisms and using the Schottky relation we give a non-trivial equation vanishing exactly on the matrix of the surface.

1. Introduction. When studying the general equation of degree five, Bring constructed a curve of genus four admitting the symmetric group $\mathscr{S}_5$ as a group of automorphisms. Namely he considered the equations

$$\mathrm{B}\colon \sum_{i=1}^{5} x_i = 0, \quad \sum_{i=1}^{5} x_i^2 = 0, \quad \sum_{i=1}^{5} x_i^3 = 0$$

in homogeneous coordinates in $\mathbb{P}_4$ where the group acts by permutations of the coordinates.

As this is an example of a curve with a maximal group of automorphisms in genus 4 just as Klein's curve is in genus 3, cf. [7]–[8], or as Fermat's curves are in other genus, cf. [5], one can ask whether this group can help in computing the Riemann matrix of the curve. The answer however involves considerably more difficulty than in the previous known examples and comes from the fact, as we will show, that there is a one complex parameter family of matrices in Siegel's space fixed by a group in $\mathrm{Sp}(8, \mathbb{Z})$ isomorphic to $\mathscr{S}_5$. To decide which one of these matrices effectively corresponds to Bring's curve involves a transcendental equation equivalent to the vanishing of Schottky's relation and this provides a direct way of proving that this relation is not trivial (cf. [1]).

The methods we use are novel in the sense that they stem from the universal cover of B and not from this curve viewed as a covering of $\mathbb{P}_1$, and are readily generalized to higher genus and to the study, to be pursued elsewhere, of families of surfaces. We would like to thank Professors A. Beauville and H. Clemens for their kind interest in this work.

2. The universal cover of Bring's curve. If $f\colon B \to B$ is an automorphism of a compact Riemann surface into itself, it induces an integral

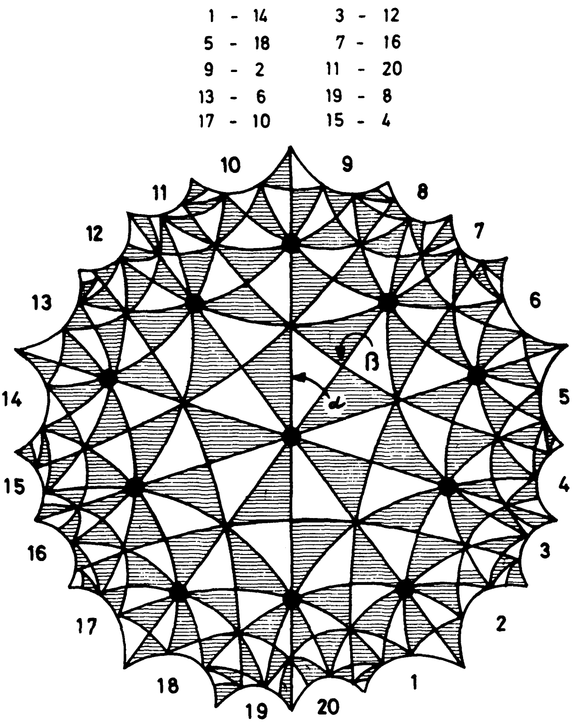

FIGURE 1

matrix in homology $[f]_*: H_1(B, \mathbb{Z}) \to H_1(B, \mathbb{Z})$ in terms of a canonical homology basis. To construct such a basis is not altogether a straightforward matter and to represent a given automorphism in this basis can be troublesome. This is the main reason why we start from the universal cover of the surface.

Consider the non-euclidean polygon of 20 sides in the unit disc Δ as in Figure 1, the pattern of triangles with interior angles $\pi/5$, $\pi/4$, $\pi/2$. We let α denote the counterclockwise conformal rotation of order 5 at the center and β the adjacent rotation of order 2.

We then have

$$\alpha^5 = 1, \quad \beta^2 = 1, \quad (\beta\alpha)^4 = 1$$

as defining relations for a group $\Gamma^* = \langle \alpha, \beta \rangle$; that is, Γ^* is a $(2, 4, 5)$ triangle group, for which any pair of adjacent triangles in Figure 1 form a fundamental domain for Γ^*, whence Γ^* is discrete.

Let $\varphi: \Gamma^* \to \mathscr{S}_5$ be the group homomorphism given by $\varphi(\alpha) = (12345)$, $\varphi(\beta) = (12)$. This homomorphism is onto and gives rise to the exact sequence

$$1 \to \Gamma \to \Gamma^* \to \mathscr{S}_5 \to 1$$

where $\Gamma = \operatorname{Ker}\varphi$ so that $\Gamma \lhd \Gamma^*$, $\Gamma^*/\Gamma = \mathscr{S}_5$. Notice that Γ has no torsion, since all torsion elements of Γ^* are conjugate to powers of α, β or $\beta\alpha$.

Furthermore, if we consider the Cayley diagram (graph of the group) associated to the presentation for $\mathscr{S}_5$ given by

$$\langle A, B\colon A^5 = B^2 = (BA)^4 = 1 = (BA^{-2}BA^2)^2\rangle$$

and take its dual, we obtain a fundamental polygon for Γ, tessellated by fundamental domains for Γ^*, as in Figure 1.

The side identifications indicated there allow us to compute that there are three cycles of vertices on the boundary, so that the genus of $B = \Delta/\Gamma$ is 4 and $\mathscr{S}_5$ acts on B as a group of automorphisms. Furthermore it can be shown by other means, cf. González and Rodríguez **[9]**, that this is the only surface of genus 4 admitting $\mathscr{S}_5$ as a group of automorphisms and that it is the full group.

We now build a homology basis in $H_1(B, \mathbb{Z})$ so that α and β are represented by 8×8 integer values matrices that preserve the intersection matrix. It is natural to set

$$\alpha_1 = 1 + 2, \qquad \alpha_2 = 3 + 4$$

and act on them via the rotation α so that

$$\begin{aligned} \alpha_3 &= 5 + 6, & \alpha_4 &= 7 + 8, & \alpha_5 &= 9 + 10, \\ \alpha_6 &= 11 + 12, & \alpha_7 &= 13 + 14, & \alpha_8 &= 15 + 16. \end{aligned}$$

The intersection matrix of this basis is the matrix C whose entries are the intersection numbers $c_{ij} = \alpha_i \circ \alpha_j$, and which can be found by careful considerations at the vertices of the polygon:

$$C = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

The automorphisms α, β then act in $H_1(B, \mathbb{Z})$ via the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We have $C' = -C$, $\det C = +1$, $A'CA = C$, $B'CB = C$ and $A^5 = I$, $B^2 = I$, $(AB)^4 = I$, so that they give an integer valued representation of the group $\mathscr{S}_5$. (We do not know why these matrices have only $0, 1, -1$ as entries as it happens in all known examples.)

What we need at this point is a change of basis by a unimodular transformation so that the new basis is canonical and so that A, B transform into manageable matrices. A check of the character of this representation against a list of all characters of $\mathscr{S}_5$ in [3] gives this representation as twice the following irreducible representation, x_6 in [3]:

$$(1) \qquad a = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

This means that under an integral change of basis Ω, for instance,

$$\Omega = \begin{pmatrix} 2 & 1 & 0 & 1 & 2 & 0 & -1 & 0 \\ 1 & -1 & -2 & -1 & 1 & -1 & -1 & -1 \\ 2 & -1 & -1 & 1 & 1 & -2 & -1 & 1 \\ 0 & -2 & -1 & 1 & -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 2 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ -1 & -2 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix},$$

the matrices A, B, C transform into

$$\Omega^{-1}A\Omega=\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix},\quad \Omega^{-1}B\Omega=\begin{pmatrix} b & 0\\ 0 & b\end{pmatrix},\quad \Omega' C\Omega=\begin{pmatrix} 0 & c\\ -c & 0\end{pmatrix}.$$

Here

$$c=\begin{pmatrix} 4 & 1 & -1 & 1\\ 1 & 4 & 1 & -1\\ -1 & 1 & 4 & 1\\ 1 & -1 & 1 & 4\end{pmatrix},\quad \text{with } c^{-1}=\frac{1}{5}\begin{pmatrix} 2 & -1 & 1 & -1\\ -1 & 2 & -1 & 1\\ 1 & -1 & 2 & -1\\ -1 & 1 & -1 & 2\end{pmatrix}$$

is actually the only matrix invariant under the required actions of a, b. The new change of coordinates

$$\Lambda=\begin{pmatrix} c^{-1} & 0\\ c^{-1} & -I\end{pmatrix}$$

transforms these matrices into

$$(*)\qquad \begin{pmatrix} cac^{-1} & 0\\ 0 & a\end{pmatrix},\quad \begin{pmatrix} cbc^{-1} & 0\\ 0 & b\end{pmatrix},\quad \begin{pmatrix} 0 & I\\ -I & 0\end{pmatrix}.$$

Notice that $\Omega\Lambda$ is unimodular (with integer entries), and that $cac^{-1} = (a')^{-1}$ and $cbc^{-1}=(b')^{-1}$.

In order to study the action of $\mathscr{S}_5$ on Siegel's space, we embed $\mathscr{S}_5$ in $\mathrm{Sp}(8, \mathbb{Z})$ by sending α and β respectively to

$$(2)\qquad A_\alpha=\begin{pmatrix} (a')^{-1} & 0\\ 0 & a\end{pmatrix}\quad\text{and}\quad A_\beta=\begin{pmatrix} (b')^{-1} & 0\\ 0 & b\end{pmatrix}$$

and denote the image group by $\mathrm{Sp}\mathscr{S}_5$; we also embed $\Gamma_0(5)$ in $\mathrm{Sp}(8, \mathbb{Z})$ by sending

$$(3)\qquad \begin{pmatrix} m & n\\ p & q\end{pmatrix}\quad\text{to}\quad \begin{pmatrix} mI & nc\\ pc^{-1} & qI\end{pmatrix}$$

and denote the image group by $\mathrm{Sp}\,\Gamma_0(5)$. With this notation, we prove

LEMMA 1. *If M is symplectic and commutes with $\mathrm{Sp}\mathscr{S}_5$, then*

$$M=\begin{pmatrix} mI & nc\\ pc^{-1} & qI\end{pmatrix},\quad \textit{where}\ \begin{pmatrix} m & n\\ p & q\end{pmatrix}\in \mathrm{SL}(2, \mathbb{R}).$$

If in addition M is integral, then m, n, p and q are integers and 5 divides p; i.e. $M\in \mathrm{Sp}\,\Gamma_0(5)$.

Proof. Let $M=\left(\begin{smallmatrix} E & F\\ G & H\end{smallmatrix}\right)$ be a symplectic matrix commuting with every $A_g=\left(\begin{smallmatrix} (g')^{-1} & 0\\ 0 & g\end{smallmatrix}\right)$ in $\mathrm{Sp}\mathscr{S}_5$.

Equivalently, we obtain the equations

$$E(a')^{-1} = (a')^{-1}E, \qquad Fa = (a')^{-1}F,$$
$$G(a')^{-1} = aG, \qquad Ha = aH,$$

and similar ones for b, with a and b the generators for the representation as in equation (1). Since this representation and the contragradient representation are irreducible and c is the "only" matrix satisfying $a'ca = c$ and $b'cb = c$, the first result follows.

If in addition we assume M is integral, then m, n, p, q are integers and since $G = pc^{-1}$ has integer coefficients, 5 divides p.

Now we can prove

THEOREM 1. *There is a one complex parameter family—Bring's half-plane—of Riemann matrices Z in Siegel's upper half space fixed under the action of* $\mathrm{Sp}\,\mathcal{S}_5$. *These are the matrices of the form $Z = \tau\cdot c$, $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$.*

The stabilizer of Bring's half plane in $\mathrm{Sp}(8, \mathbb{Z})$ *is the product of the groups* $\mathrm{Sp}\,\mathcal{S}_5$, *which fixes the set pointwise, and* $\mathrm{Sp}\,\Gamma_0(5)$, *which acts by $\tau c \to \tau^* c$, where*

$$\tau^* = \frac{q\tau + n}{p\tau + m}.$$

If $\tau_0 c$ is a Riemann matrix for Bring's curve and M is a matrix in $\mathrm{Sp}(8, \mathbb{Z})$ *that maps $\tau_0 c$ to a point $\tau' c$ in the half-plane, then M stabilizes Bring's half-plane.*

Proof. We use the fact that a symplectic change of basis $\left(\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right)$ acts on a period matrix Z by sending it to $(E + ZG)^{-1}(F + ZH)$, so that the action of $\mathrm{Sp}\,\Gamma_0(5)$ on Bring's half-plane is as stated and hence $\mathrm{Sp}\,\Gamma_0(5)$ is contained in the stabilizer as claimed.

Furthermore, it also follows that the (right) action of $\mathrm{Sp}\,\mathcal{S}_5$ on Siegel's space is generated by $Z \to a'Za$, $Z \to b'Zb$; now, if a matrix Z is fixed by this latter action, it can be checked by a direct computation that $Z = \tau c$, with τ in $\mathbb{C}$; since the imaginary part of Z is positive definite, we conclude that $\operatorname{Im}\tau$ is positive.

Now let M be an element of $\mathrm{Sp}(8, \mathbb{Z})$ that sends a Riemann matrix $\tau_0 c$ for Bring's curve B to a point $\tau' c$ in Bring's half-plane; in particular, M may be any element of the stabilizer. Then for each A_g in $\mathrm{Sp}\,\mathcal{S}_5$ we have that MA_gM^{-1} fixes $\tau_0 c$; considering that it follows from Torelli's theorem (as in Weil's proof, **[6]**–**[10]**) that any automorphism of the Jacobian comes from an automorphism of the curve, and because B does not admit more automorphisms, we obtain

$$M \cdot A_g = A_{\lambda(g)} \cdot M ,$$

where $\lambda\colon \mathscr{S}_5 \to \mathscr{S}_5$ is an automorphism of $\mathscr{S}_5$. Since every automorphism of $\mathscr{S}_5$ is interior, there is some h in $\mathscr{S}_5$ such that $A_{h^{-1}}M$ commutes with all matrices A_g; then it follows from Lemma 1 that $M = A_h\gamma$ with $\gamma \in \operatorname{Sp}\Gamma_0(5)$.

3. The Riemann matrix. We now compute the values of the complex number τ so that the matrix $\tau \cdot c$ is the period matrix of Bring's curve; it is the only value modulo $\Gamma_0(5)$ that comes from a Riemann surface.

Let us identify the automorphisms α, β with their classes (12345) and (12) in $\mathscr{S}_5$. Then $\beta\alpha$, that is (2345), is represented in homology by twice the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This special matrix tells us not only that $B/\langle(2345)\rangle = E_1$ is an elliptic curve but shows how the homology basis projects to the natural basis of the torus.

In fact, if we draw a sphere with four tori attached to it, $\beta\alpha$ acts as a rotation of order four and a canonical pair of curves on each handle projects to a canonical basis for E_1.

Thus if dw is the differential in $H^{1,0}(B, \mathbb{C})$ invariant by $\beta\alpha$ that projects to a differential on the elliptic curve with modulus μ, its periods are

$$1, 1, 1, 1, \mu, \mu, \mu, \mu.$$

We may now complete dw to a basis of differentials on B using $\alpha^* dw$, $\alpha^{2*} dw$, $\alpha^{3*} dw$, and since we know the action of α on homology via cac^{-1} and a, we can compute the period matrix $Z = (\mu/5) \cdot c$.

But there is among all possible quotients of B another interesting elliptic curve, namely $B/\langle(24)(35), (25)(34)\rangle = E_2$. Starting from the same marked sphere with four tori, E_2 is obtained by identifying pairs of opposite handles, that is, if a canonical pair of curves on a handle projects to a canonical basis for E_2 then the corresponding pair on the adjacent handle projects to the same curves but with opposite orientation.

Therefore if dw is the differential on B invariant by this subgroup we compute its periods to be

$$1, -1, 1, -1, \tau, -\tau, \tau, -\tau,$$

where τ denotes the elliptic modules of E_2.

Completing dw to a basis of $H^{1,0}(B, \mathbb{C})$ as before gives $Z = \tau \cdot c$; hence $\mu = 5\tau$.

With these two tori we can now establish

THEOREM 2. *The period matrix of Bring's curve is* $Z = \tau_0 \cdot c$, *where the class of* τ_0 *modulo* $\Gamma_0(5)$ *is characterized by the two values*

$$j(\tau_0) = -\frac{25}{2}, \qquad j(5\tau_0) = -\frac{29^3 \times 5}{2^5}.$$

Proof. We first compute the quotient of B by $(\beta\alpha)^2 = (24)(35)$. It is a curve of genus two and this curve will project to the two elliptic curves.

In the equations defining B as in the introduction, we set $u_1 = x_1/2$, $u_2 = (x_2 + x_4)/2$, $u_4 = (x_2 - x_4)/2$, $u_3 = (x_3 + x_5)/2$, $u_5 = (x_3 - x_5)/2$, so that the action of (24)(35) is given in projective space by $u_4 \to -u_4$, $u_5 \to -u_5$, where the curve is now

$$3u_2^2 + 3u_3^2 + 4u_2u_3 + u_4^2 + u_5^2 = 0,$$
$$u_2^3 + u_3^3 + 4u_2^2u_3 + 4u_2u_3^2 - u_2u_4^2 - u_3u_5^2 = 0$$

in homogeneous coordinates in $\mathbb{P}_3$. Let $\pi(u) = (u_2, u_3)$ be the projection $\pi: B \to \mathbb{P}_1$: it establishes $B/\langle(24)(35)\rangle$ as a two-to-one covering of $\mathbb{P}_1$ branched at the six branch points where $u_4 = 0$ or $u_5 = 0$. Thus this curve of genus two is branched at the three roots a_1, a_2, a_3 of $x^3 + 7x^2 + 8x + 4 = 0$ and the three roots $1/a_1, 1/a_2, 1/a_3$ of $4x^3 + 8x^2 + 7x + 1 = 0$.

The quotient $B/\langle(24)(35)\rangle$ is then realized as

$$y^2 = \prod_{i=1}^{3}(x - a_i)(x - a_i^{-1}),$$

where the action of $(\beta\alpha)$ is $(x, y) \to (x^{-1}, -y)$. With $z = (x-1)/x+1)$, this is equivalent to

$$w^2 = \prod_{i=1}^{3}(z^2 - c_i^2),$$

where $c_1^2 + c_2^2 + c_3^2 = -10$, $c_1^2c_2^2 + c_1^2c_3^2 + c_2^2c_3^2 = 25$, $c_1^2c_2^2c_3^2 = 100$.

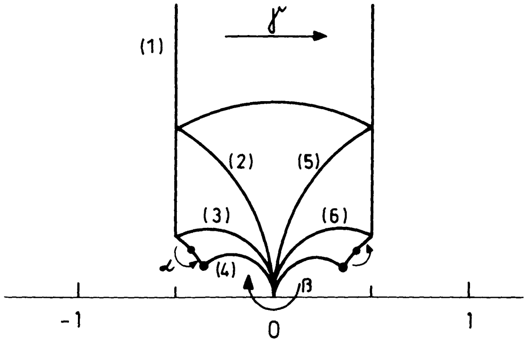

FIGURE 2

The torus E_1 is then obtained from this last equation by the identification $(z, w) \leftrightarrow (-z, -w)$, while the torus E_2 is obtained by the other identification $(z, w) \leftrightarrow (-z, w)$.

The first torus is therefore given by

$$E_1 \colon v^2 = 4u^3 - \frac{145}{3}u - \left(5 \times \frac{421}{27}\right), \qquad j(5\tau_0) = -\frac{29^3 \times 5}{2^5}$$

and the second torus by

$$E_2 \colon v^2 = 4u^3 - \frac{100}{3}u - \left(4 \times \frac{2940}{27}\right), \qquad j(\tau_0) = -\frac{25}{2}.$$

Finally we have to establish the fact that these values of $j(5\tau_0)$ and $j(\tau_0)$ determine a unique class of τ_0 modulo $\Gamma_0(5)$. To do so we shall locate τ_0 in a fundamental region for this group.

We consider then, as in Figure 2, the polygon in the upper half-plane built from six copies of a fundamental region for $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$: the side identifications marked by the arrows generate the discrete group $\Gamma_0(5)$.

Since $j(\tau_0)$ is a real negative number, we know that τ_0 lies in one of the arcs labeled (1) to (6). If we show that it actually lies in only one of them, we will have fixed the class of τ_0 mod $\Gamma_0(5)$, for $j(\tau_0)$ fixes the class of τ mod $\Gamma(1)$.

The following two values of j are well known from the theory of complex multiplication:

$$j\left(\frac{-1 + i\sqrt{7}}{2}\right) = -(3.5)^3, \qquad j\left(\frac{-1 + i\sqrt{11}}{2}\right) = -2^{15},$$

whence we have the inequalities

$$j\left(\frac{-1+i\sqrt{11}}{2}\right) < j(5\tau_0) < j\left(\frac{-1+i\sqrt{7}}{2}\right) < j(\tau_0) < 0.$$

Suppose τ_0 is in the arc labeled (1); then $\tau_0 = (-1+it)/2$, with $\sqrt{3} < t$, and $5\tau_0 \equiv (-1+i5t)/2 \bmod \Gamma(1)$. From $j((-1+i\sqrt{7})/2) < j((-1+it)/2) < 0$ we obtain $\sqrt{3} < t < \sqrt{7}$ and from $j((-1+i\sqrt{7})/2) < j((-1+it)/2) < 0$ we obtain $\sqrt{7} < 5t < \sqrt{11}$. Both inequalities cannot hold and we have a contradiction in this case.

Suppose τ_0 is in the arc labeled (2); then $\tau_0 = -1+\cos\theta + i\sin\theta$, $0 < \theta < \pi/3$, and $5\tau_0 \equiv 5\cos\theta + i5\sin\theta \bmod \Gamma(1)$. Since $j(5\tau_0)$ is a negative real number, there are only two possibilities for $5\tau_0$: $5\tau_0 \equiv (9+it)/2$ or $5\tau_0 \equiv (7+it)/2$, with $\sqrt{3} < t$. In the first instance, $\tau_0 = (-1+\sqrt{19})/10$ and $5\tau_0 = (-1+i\sqrt{19})/2$, obtaining the contradiction $j(5\tau_0) = j((-1+i\sqrt{19})/2) < j((-1+i\sqrt{11})/2)$. The other case is similar, with $\tau_0 = (-3+\sqrt{51})/10$ and $5\tau_0 = (-3+i\sqrt{51})/2$, giving a contradiction as before.

Suppose τ_0 is in the arc labeled (3); then

$$\tau_0 = (-1+\cos\theta + i\sin\theta)/2, \qquad 0 < \theta < 2\pi/3.$$

Since $j(5\tau_0)$ is a real negative number, there are two possibilities for $5\tau_0$: $\Re 5\tau_0 = -\frac{1}{2}$ or $-\frac{3}{2}$. In one case $\cos\theta = \frac{7}{10}$, $\sin\theta = \sqrt{51}/10$, $5\tau_0 \equiv (-1+i\frac{5}{3}\sqrt{51})/2$ which is impossible and in the other case $\cos\theta = \frac{1}{10}$, $\sin\theta = \frac{3}{10}$, $5\tau_0 \equiv (-1+i15)/2$ which is again impossible.

The cases for arcs labeled (5) and (6) are similar to cases (2) and (3), and we are left with just one possibility: τ_0 lies in the arc labeled (4), that is,

$$|\tau_0 + \tfrac{1}{5}| = \tfrac{1}{5}, \qquad -\tfrac{5}{14} < \Re\tau_0 < -\tfrac{1}{3}.$$

Observation. It was known that the abelian variety of Bring's curve is isogeneous to the product of four times the elliptic curve

$$y^2 = x^3 - 675x - 79650, \qquad j = -25/2$$

by Serre (cf. [4]). We shall show in the next section that via a natural isogeny every matrix $Z = \tau \cdot c$ is equivalent to the diagonal $(5\tau, 5\tau, 5\tau, \tau)$.

4. The Schottky relations. The theta function with characteristics ε, η is defined by the series

$$\Theta\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}(z, Z) = \sum_{n\in\mathbb{Z}^4} \exp \pi i(\tau(n+\varepsilon/2)'c(n+\varepsilon/2) + 2(n+\varepsilon/2)'(z+\eta/2)),$$

where $Z = \tau \cdot c$, and transposition is denoted by primes. To obtain a formula for Θ as a sum of theta functions of one variable we diagonalize the matrix c: with

$$m = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}$$

we have

$$m'cm = \begin{pmatrix} 5 & 0 & 0 & 0\\ 0 & 5 & 0 & 0\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \hat{c}.$$

Note that then

$$2m(I, \tau c) = (I, \tau\hat{c})\begin{pmatrix} 2m & 0\\ 0 & 2m\end{pmatrix}.$$

The above matrix m has been well known since Riemann and it is curious to see it appear in a purely group representation context; for us it is convenient, since we can almost copy the usual computations for theta functions (cf. [**1**, Chapter 6]).

We let L_1, L_2, L be the three lattices in $\mathbb{R}^4$:

$L_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4$,

$L_2 =$ [submodule of $\frac{1}{2}L_1$ generated by the vectors $e_j + e_k$, $1 \le j, k \le 4$ and $\frac{1}{2}(e_1+e_2+e_3+e_4)$],

$L = L_1 + L_2$

so that $[L\colon L_1] = 2$, $[L\colon L_2] = 2$ and m gives an isomorphism

$L_1 \cong L_2$ as scalar product spaces. Then

$$\begin{aligned}
\Theta\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}(z, Z) &= \sum_{n\in L_1} \exp \pi i(\tau(n+\varepsilon/2)' c(n+\varepsilon/2) \\
&\qquad\qquad + 2(n+\varepsilon/2)'(z+\eta/2)) \\
&= \tfrac{1}{2}\sum_{n\in L} \exp \pi i(\tau(n+\varepsilon/2)' c(n+\varepsilon/2) \\
&\qquad\qquad + 2(n+\varepsilon/2')(z+\eta/2)) \\
&\quad + \exp(-\pi i\varepsilon'\cdot e_1)\exp \pi i(\tau(n+\varepsilon/2)' c(n+\varepsilon/2) \\
&\qquad\qquad + 2(n+\varepsilon/2)'(z+e_1+\eta/2)) \\
&= \tfrac{1}{2}\sum_{n\in L_2} \exp \pi i(\tau(n+\varepsilon/2)' c(n+\varepsilon/2) \\
&\qquad\qquad + 2(n+\varepsilon/2)'(z+\eta/2)) \\
&\quad + \exp \pi i(\tau(n+e_1+\varepsilon/2)' c(n+e_1+\varepsilon/2) \\
&\qquad\qquad + 2(n+e_1+\varepsilon/2)'(z+\eta/2)) \\
&\quad + \exp \pi i(\tau(n+\varepsilon/2)' c(n+\varepsilon/2) \\
&\qquad\qquad + 2(n+\varepsilon/2)'(z+e_1+\eta/2))\exp(-\pi i(\varepsilon'\cdot e_1)) \\
&\quad + \exp \pi i(\tau(n+\varepsilon/2)' c(n+e_1+\varepsilon/2) \\
&\qquad\qquad + 2(n+e_1+\varepsilon/2)'(z+e_1+\eta/2))\exp(-\pi i(\varepsilon'\cdot e_1)).
\end{aligned}$$

We now change the summation index via m so that we obtain

$$\begin{aligned}
\Theta\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}(z, Z) = \tfrac{1}{2}\sum_{n\in L_1} & \exp \pi i(\tau(n+m\varepsilon/2)' \hat{c}(n+m\varepsilon/2) \\
&\qquad + 2(n+m\varepsilon/2)'(mz+m\eta/2)) \\
& + \exp \pi i(\tau(n+me_1+m\varepsilon/2)' \hat{c}(n+me_1+m\varepsilon/2) \\
&\qquad + 2(n+me_1+m\varepsilon/2)'(mz+m\eta/2)) \\
& + \exp \pi i(\tau(n+m\varepsilon/2)' \hat{c}(n+m\varepsilon/2) \\
&\qquad + 2(n+m\varepsilon/2)'(mz+me_1+m\eta/2)) \\
& \cdot \exp(-\pi i(\varepsilon'\cdot e_1)) \\
& + \exp \pi i(\tau(n+me_1+m\varepsilon/2)' \hat{c}(n+me_1+m\varepsilon/2) \\
&\qquad + 2(n+me_1+m\varepsilon/2)'(mz+me_1+m\eta/2)) \\
& \cdot \exp(-\pi i(\varepsilon' e_1)).
\end{aligned}$$

Since $\hat{c}$ is a diagonal matrix, each summand is a product of four theta functions of one variable, three with modulus 5τ, one with modulus τ. For instance

$$\begin{aligned}
2\Theta&\begin{bmatrix}0&0&0&0\\0&0&0&0\end{bmatrix}(z\,,Z)\\
&=\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(\hat{z}\,;5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\hat{z}\,;\tau)+\vartheta\begin{bmatrix}1\\0\end{bmatrix}^3(\hat{z}\,;5\tau)\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\hat{z}\,;\tau)\\
&\quad+\vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(\hat{z}\,;5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(\hat{z}\,;\tau)\\
&\quad+\vartheta\begin{bmatrix}1\\1\end{bmatrix}^3(\hat{z}\,;5\tau)\vartheta\begin{bmatrix}1\\1\end{bmatrix}(\hat{z}\,;\tau),
\end{aligned}$$

where if $z=(z_1\,,z_2\,,z_3\,,z_4)$, we have set $\hat{z}=(z_1+z_2+z_3+z_4)/2$.

The general Schottky relation

$$\begin{aligned}
\prod_{j,k,l=0,1}\Theta\begin{bmatrix}0&0&0&0\\j&k&l&0\end{bmatrix}^{1/2}&=\prod_{j,k,l=0,1}\Theta\begin{bmatrix}0&0&0&1\\j&k&l&0\end{bmatrix}^{1/2}\\
&\quad+\prod_{j,k,l=0,1}\Theta\begin{bmatrix}0&0&0&0\\j&k&l&1\end{bmatrix}^{1/2}
\end{aligned}$$

is an identity that has to be satisfied by the Riemann matrices that come from surfaces and it was shown by Igusa [2] that this equation gives exactly the variety of these matrices. What we do now is show in this example how this relation is first, non trivial, and second, that it has only one zero for the value of τ_0 as in Theorem 2.

Let us first recall some of the usual notations from the theory of elliptic functions:

$$\kappa=\sqrt{\frac{e_2-e_3}{e_1-e_3}}=\left(\frac{\vartheta\begin{bmatrix}1\\0\end{bmatrix}(0\,,\tau)}{\vartheta\begin{bmatrix}0\\0\end{bmatrix}(0\,,\tau)}\right)^2,$$

$$J=\frac{4}{27}\frac{(1-\kappa^2+\kappa^4)^3}{\kappa^4(1-\kappa^2)^2}=\frac{1}{1728}j.$$

When the value of τ changes via the group $\Gamma(1)=\mathrm{PSL}(2\,,\mathbb{Z})$, the value of κ^2 change via the group $\mathscr{S}_3$; namely from the transformation formulae

$$\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau+1)=e^{i\pi/4}\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau),$$

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau+1)=\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau+1)=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau),$$

$$\vartheta\begin{bmatrix}1\\0\end{bmatrix}\left(-\frac{1}{\tau}\right)=\sqrt{\frac{\tau}{i}}\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(-\frac{1}{\tau}\right)=\sqrt{\frac{\tau}{i}}\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau),$$

$$\vartheta\begin{bmatrix}0\\1\end{bmatrix}\left(-\frac{1}{\tau}\right)=\sqrt{\frac{\tau}{i}}\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau)$$

it follows that

$$\kappa^2(\tau+1)=\frac{\kappa^2}{\kappa^2-1},\qquad \kappa^2\left(-\frac{1}{\tau}\right)=1-\kappa^2.$$

Thus, for a given value of J (or j) there are six possible values of κ^2,

$$\kappa^2,\quad \frac{\kappa^2}{\kappa^2-1},\quad 1-\kappa^2,\quad \frac{1}{\kappa^2},\quad \frac{\kappa^2-1}{\kappa^2},\quad \frac{1}{1-\kappa^2},$$

forming a group isomorphic to $\mathscr{S}_3$. In all, we have an exact sequence of discrete groups

$$1\to\Gamma^*\to\Gamma(1)\to\mathscr{S}_3\to 1$$

so that even though κ^2 is not an automorphic function for $\Gamma(1)$, it is one for Γ^*.

THEOREM 3. *The Schottky relation restricted to Bring's half-plane defines a non trivial algebraic equation in the theta-nullwerte of one complex variable with moduli* 5τ, τ.

There is an exact sequence of discrete groups

$$1 \to N \to \Gamma_0(5) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1$$

and Schottky's relation defines a differential form of degree 4 *for* N.

This differential form has one zero in the upper half-plane Δ *modulo* $\Gamma_0(5)$: *the point* τ_0 *corresponding to Bring's curve.*

Proof. We first shorten Schottky's relation using the action of $\mathscr{S}_5$: if $g \in \langle a, b\rangle$ then

$$\Theta\begin{bmatrix} g\varepsilon \\ (g')^{-1}\eta \end{bmatrix} = \Theta\begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}, \qquad \varepsilon, \eta \in (\mathbb{Z}/2\mathbb{Z})^4.$$

We obtain

$$\prod_{i,j,k=0,1} \Theta\begin{bmatrix} 0 & 0 & 0 & 0 \\ i & j & k & 0 \end{bmatrix} = \Theta\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Theta\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}^6 \cdot \Theta\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\prod_{i,j,k=0,1} \Theta\begin{bmatrix} 0 & 0 & 0 & 0 \\ i & j & k & 1 \end{bmatrix} = \Theta\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}^4 \Theta\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}^4,$$

$$\prod_{i,j,k=0,1} \Theta\begin{bmatrix} 0 & 0 & 0 & 1 \\ i & j & k & 0 \end{bmatrix} = -\Theta\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Theta\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}^3 \cdot \Theta\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^3 \Theta\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Here we used for instance

$$\Theta\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \Theta\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = -\Theta\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

with $g = a$.

According to the general formula for $\Theta\left[\begin{smallmatrix} \varepsilon \\ \eta \end{smallmatrix}\right](z, Z)$ derived earlier we can compute each thetanullwerte:

$$2\Theta\begin{bmatrix}0&0&0&0\\0&0&0&0\end{bmatrix}=\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)+\vartheta\begin{bmatrix}1\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau)$$
$$+\vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$2\Theta\begin{bmatrix}0&0&0&0\\1&1&0&0\end{bmatrix}=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}^2(5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)$$
$$+\vartheta\begin{bmatrix}0\\0\end{bmatrix}^2(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$2\Theta\begin{bmatrix}0&0&0&0\\1&1&1&1\end{bmatrix}=\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)-\vartheta\begin{bmatrix}1\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau)$$
$$+\vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$2\Theta\begin{bmatrix}1&1&1&1\\0&0&0&0\end{bmatrix}=\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)+\vartheta\begin{bmatrix}1\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau)$$
$$-\vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$-2\Theta\begin{bmatrix}1&1&1&1\\0&1&0&1\end{bmatrix}=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}^2(5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)$$
$$-\vartheta\begin{bmatrix}0\\0\end{bmatrix}^2(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$2\Theta\begin{bmatrix}1&1&1&1\\0&1&1&0\end{bmatrix}=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}^2(5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)$$
$$-\vartheta\begin{bmatrix}0\\0\end{bmatrix}^2(5\tau)+\vartheta\begin{bmatrix}0\\1\end{bmatrix}(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau),$$

$$2\Theta\begin{bmatrix}1&1&1&1\\1&1&1&1\end{bmatrix}=\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)-\vartheta\begin{bmatrix}1\\0\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau)$$
$$-\vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(5\tau)\vartheta\begin{bmatrix}0\\1\end{bmatrix}(\tau).$$

We then set

$$\vartheta\begin{bmatrix}i\\j\end{bmatrix}=\vartheta\begin{bmatrix}i\\j\end{bmatrix}(0,5\tau),\quad \eta\begin{bmatrix}i\\j\end{bmatrix}=\vartheta\begin{bmatrix}i\\j\end{bmatrix}(0,\tau)\qquad(i,j=0,1)$$

so that we can write

$$\begin{aligned} f(\tau) = {} & \left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3\eta\begin{bmatrix}0\\0\end{bmatrix} + \vartheta\begin{bmatrix}1\\0\end{bmatrix}^3\eta\begin{bmatrix}1\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^{1/2} \\ & \cdot\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}\vartheta\begin{bmatrix}0\\1\end{bmatrix}^2\eta\begin{bmatrix}0\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\0\end{bmatrix}^2\vartheta\begin{bmatrix}0\\1\end{bmatrix}\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^3 \\ & \cdot\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3\eta\begin{bmatrix}0\\0\end{bmatrix} - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^3\eta\begin{bmatrix}1\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^{1/2} \\ & -\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}\vartheta\begin{bmatrix}0\\1\end{bmatrix}^2\eta\begin{bmatrix}0\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\0\end{bmatrix}^2\vartheta\begin{bmatrix}0\\1\end{bmatrix}\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^2 \\ & \cdot\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3\eta\begin{bmatrix}0\\0\end{bmatrix} - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^3\eta\begin{bmatrix}1\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^2 \\ & -\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3\eta\begin{bmatrix}0\\0\end{bmatrix} + \vartheta\begin{bmatrix}1\\0\end{bmatrix}^3\eta\begin{bmatrix}1\\0\end{bmatrix} - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^{1/2} \\ & \cdot\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}\vartheta\begin{bmatrix}0\\1\end{bmatrix}^2\eta\begin{bmatrix}0\\0\end{bmatrix} - \vartheta\begin{bmatrix}0\\0\end{bmatrix}^2\vartheta\begin{bmatrix}0\\1\end{bmatrix}\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^3 \\ & \cdot\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3\eta\begin{bmatrix}0\\0\end{bmatrix} - \vartheta\begin{bmatrix}1\\0\end{bmatrix}^3\eta\begin{bmatrix}1\\0\end{bmatrix} - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^{1/2}. \end{aligned}$$

We examine now the behaviour of the differential form of 4th degree $f(\tau)(d\tau)^4$ under the generators of $\Gamma_0(5)$. These generators are

$$\alpha(\tau) = \frac{-2\tau - 1}{5\tau + 2}, \quad \beta(\tau) = \frac{\tau}{-5\tau + 1}, \quad \gamma(\tau) = \tau + 1$$

so that we may present this group by

$$\Gamma_0(5) = \langle \alpha, \beta, \gamma; \alpha^2 = (\gamma\alpha\beta^{-1})^2 = 1 \rangle.$$

Since $f(\tau)$ has an expansion with only even powers of q, the differential form is invariant under γ. We define then a group epimorphism from $\Gamma_0(5)$ to

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle u, v; u^2 = v^2 = (uv)^2 = 1 \rangle$$

via $\lambda(\alpha) = u$, $\lambda(\beta) = v$, $\lambda(\gamma) = 1$. This epimorphism defines an exact sequence

$$1 \to N \to \Gamma_0(5) \xrightarrow{\lambda} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1,$$

where Δ/N and $\Delta/\Gamma_0(5)$ are surfaces of genus 0 and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ acts as a covering group on $\overline{\Delta/\Gamma_0(5)} \simeq \mathbb{P}_1$ by the transformations

$$z, u(z) = z^{-1}, v(z) = -z, (uv)(z) = -z^{-1}.$$

To prove that $f(\tau)\,d\tau^4$ is a differential form invariant by the action of the group N we have to look at its behaviour under β in a neighborhood of 0. Using the transformation $\tau \to -1/\tau$ we are led to the expansion

$$\begin{aligned} g(\tau) = &\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(\tau/5)\eta\begin{bmatrix}0\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(\tau/5)\eta\begin{bmatrix}0\\1\end{bmatrix}\right. \\ &\qquad\left. + \vartheta\begin{bmatrix}1\\0\end{bmatrix}^3(\tau/5)\eta\begin{bmatrix}1\\0\end{bmatrix}\right)^{1/2} \\ &\cdot\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau/5)\vartheta\begin{bmatrix}1\\0\end{bmatrix}^2(\tau/5)\eta\begin{bmatrix}0\\0\end{bmatrix}\right. \\ &\qquad\left. + \vartheta\begin{bmatrix}0\\0\end{bmatrix}^2(\tau/5)\vartheta\begin{bmatrix}1\\0\end{bmatrix}(\tau/5)\eta\begin{bmatrix}1\\0\end{bmatrix}\right)^3 \\ &\cdot\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^3(\tau/5)\eta\begin{bmatrix}0\\0\end{bmatrix} - \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3(\tau/5)\eta\begin{bmatrix}0\\1\end{bmatrix}\right. \\ &\qquad\left. + \vartheta\begin{bmatrix}1\\0\end{bmatrix}^3(\tau/5)\eta\begin{bmatrix}1\\0\end{bmatrix}\right)^{1/2} \\ &- \cdots \\ &- \cdots \\ = {}& 2\cdot 4^3\left(\sqrt{6}\frac{22}{3} - 18\right)q^{3/5}(1+\cdots), \qquad q = e^{\pi i\tau}. \end{aligned}$$

Since by conjugation by $-1/\tau$, $\beta(\tau)$ transforms to $\tau + 5$ we obtain that $g(\tau)(d\tau)^4$ is invariant under $\tau + 10$, or that $f(\tau)(d\tau)^4$ is invariant under β^2.

The zeros of $f(\tau)$ are exactly the points where Schottky's relation holds, establishing the first statement of the theorem.

To show that this relation is not trivial we will expand f in a power series of $q = e^{\pi i \tau}$ in a neighborhood of $i \cdot \infty$. From the expansions

$$\eta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0; \tau) = 2q^{1/4}(1 + q^2 + q^6 + q^{12} + \cdots),$$

$$\eta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0; \tau) = 1 + 2(q + q^4 + q^9 + \cdots),$$

$$\eta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0; \tau) = 1 - 2(q - q^4 + q^9 + \cdots) \qquad (q = e^{\pi i \tau})$$

it follows that

$$\begin{aligned}
\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^3 \eta \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= 1 + 2q + 2q^4 + 6q^5 + 12q^6 + \cdots , \\
\vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix}^3 \eta \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= 16q^4(1 + q^2 + q^6 + \cdots), \\
\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}^3 \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= 1 - 2q + 2q^4 - 6q^5 + 12q^6 - \cdots , \\
\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \eta \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= 1 + 2q + 2q^4 - 2q^5 - 4q^6 + \cdots , \\
\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^2 \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= 1 - 2q + 2q^4 + 2q^5 - 4q^6 + \cdots ,
\end{aligned}$$

and then

$$f(\tau) = 4^5 \cdot q^{10} + \cdots ,$$

proving that f is not constantly zero.

To find the number of zeros of this differential form on Δ/N we have to study its behaviour at the six punctures:

$$\infty, \; 0, \; \beta(\infty) = -\tfrac{1}{5}, \; \alpha(0) = -\tfrac{1}{2}, \; \alpha(\infty) = -\tfrac{2}{5}, \; (\alpha\beta)(\infty) = -\tfrac{3}{5}.$$

If Z is the number of zeros and P the number of poles on Δ/N we have the formula

$$Z - P = 8(g - 1) = -8.$$

At ∞ a local parameter is $p = q^2 = e^{2\pi i \tau}$ so that

$$\begin{aligned}
f(\tau)\, d\tau^4 &= (4^5 p^5 + \cdots) \left(\frac{dp}{2\pi i p} \right)^4 \\
&= \left(\frac{4^3}{\pi^4} p + \cdots \right) dp^4 ,
\end{aligned}$$

proving that the differential form has a simple zero there.

At 0, using $-1/\tau$ as before, a local parameter is $p = q^{1/5} = e^{\pi i\tau/5}$ so that

$$\begin{aligned} f(\tau)\,d\tau^4 &= \left(2\cdot 4^3\left(\sqrt{6}\frac{22}{3} - 18\right)q^{3/5} + \cdots\right)\left(\frac{dq}{2\pi i q}\right)^4 \\ &= \left(\frac{40}{\pi^4}\left(\sqrt{6}\frac{22}{3} - 18\right)p^3 + \cdots\right)\frac{1}{p^4}\,dp^4, \end{aligned}$$

proving that the differential form has a simple pole there.

At $-\frac{1}{5}$ we use the transformation $\beta(\tau) = \tau/(-5\tau+1)$ so that

$$\begin{aligned} \vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(\frac{5\tau}{-5\tau+1}\right) &= \vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(\frac{-1}{1-1/5\tau}\right) \\ &= \sqrt{\frac{1-1/5\tau}{i}}\vartheta\begin{bmatrix}0\\0\end{bmatrix}\left(1-\frac{1}{5\tau}\right) \\ &= \sqrt{\frac{1-1/5\tau}{i}}\vartheta\begin{bmatrix}0\\1\end{bmatrix}\left(-\frac{1}{5\tau}\right) \\ &= \sqrt{1-5\tau}\vartheta\begin{bmatrix}1\\0\end{bmatrix}(5\tau), \end{aligned}$$

and similar formulas for $\vartheta\left[\begin{smallmatrix}i\\j\end{smallmatrix}\right]$, $\eta\left[\begin{smallmatrix}i\\j\end{smallmatrix}\right]$. The expansion

$$\begin{aligned} h(\tau) = {} & \left(\vartheta\begin{bmatrix}1\\0\end{bmatrix}^3\eta\begin{bmatrix}1\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\0\end{bmatrix}^3\eta\begin{bmatrix}0\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^{1/2} \\ & \cdot\left(i\vartheta\begin{bmatrix}1\\0\end{bmatrix}\vartheta\begin{bmatrix}0\\1\end{bmatrix}^2\eta\begin{bmatrix}1\\0\end{bmatrix} - i\vartheta\begin{bmatrix}1\\0\end{bmatrix}^2\vartheta\begin{bmatrix}0\\1\end{bmatrix}\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^3 \\ & \cdot\left(\vartheta\begin{bmatrix}1\\0\end{bmatrix}^3\eta\begin{bmatrix}1\\0\end{bmatrix} - \vartheta\begin{bmatrix}0\\0\end{bmatrix}^3\eta\begin{bmatrix}0\\0\end{bmatrix} + \vartheta\begin{bmatrix}0\\1\end{bmatrix}^3\eta\begin{bmatrix}0\\1\end{bmatrix}\right)^{1/2} \\ & -\cdots \\ & -\cdots \end{aligned}$$

gives, using the local parameter $p = q^2 = e^{2\pi i\tau}$,

$$f(\tau)\,d\tau^4 = \left(\frac{4^2}{\pi^4} + \cdots\right)\frac{dp^4}{p^2},$$

proving that it has a pole of order 2 there.

At $-\frac{1}{2}, -\frac{2}{5}, -\frac{3}{5}$ similar expansions show that there are poles of orders 4, 2, 4 so that in all we have

$$P = 1 + 2 + 4 + 2 + 4 = 13 \text{ poles}.$$

To determine its zeros we recall the proof of Theorem 2 where the elliptic curve E_1 has equation

$$y^2 = x^3 - 10x^2 + 25x - 100, \qquad j(5\tau) = -\frac{29^3 \times 5}{2^5}$$

and roots c_1^2, c_2^2, c_3^2 with

$$\kappa^4(5\tau) = \frac{c_2^2 - c_3^2}{c_1^2 - c_3^2} = \frac{e_2 - e_3}{e_1 - e_3}.$$

Solving the cubic equation gives

$$c_1^2 = A + B, \quad c_2^2 = \rho A + \rho^2 B, \quad c_3^2 = \rho^2 A + \rho B,$$

where as usual $\rho = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and

$$A = \sqrt[3]{50}\sqrt[3]{\frac{49}{27} + \sqrt{\frac{22}{27}}}, \qquad B = \sqrt[3]{50}\sqrt[3]{\frac{49}{27} - \sqrt{\frac{22}{27}}}.$$

With a similar value for $\kappa(\tau)$ there are in all $6 \times 6 = 36$ pairs of values that give the same invariants $j(5\tau)$, $j(\tau)$. One such pair gives the values of

$$\vartheta\begin{bmatrix}1\\0\end{bmatrix}/\vartheta\begin{bmatrix}0\\0\end{bmatrix}, \qquad \eta\begin{bmatrix}1\\0\end{bmatrix}/\eta\begin{bmatrix}0\\0\end{bmatrix}$$

that make Schottky's relation hold since for these values the Riemann matrix corresponds to Bring's curve.

Since acting by α, β corresponds to a change in a homology basis for Bring's curve, Schotty's relation holds also at the three points $\alpha(\tau_0)$, $\beta(\tau_0)$, $\alpha\beta(\tau_0)$. We thus have $Z = 1 + 4 = 5$ zeros at least. But then

$$Z - P = 5 - 13 = -8$$

says that there cannot be more zeros establishing the theorem. Furthermore, Schottky's relation vanishes at τ_0 to the first order.

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