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THE ADJOINT REPRESENTATION L -FUNCTION FOR $GL(n)$

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Ideas underlying the proof of the “simple” trace formula are used to show the following. Let F be a global field, and \mathbb{A} its ring of adèles. Let π be a cuspidal representation of $GL(n, \mathbb{A})$ which has a supercuspidal component, and ω a unitary character of $\mathbb{A}^\times/F^\times$. Let s_0 be a complex number such that for every separable extension E of F of degree n , the L -function $L(s, \omega \circ \text{Norm}_{E/F})$ over E vanishes at $s = s_0$ to the order $m \geq 0$. Then the product L -function $L(s, \pi \otimes \omega \times \tilde{\pi})$ vanishes at $s = s_0$ to the order m . This result is a reflection of the fact that the tensor product of a finite dimensional representation with its contragredient contains a copy of the trivial representation.

Let F be a global field, \mathbb{A} its ring of adèles and \mathbb{A}^\times its group of ideles. Denote by \underline{G} the group scheme $GL(n)$ over F , and put $G = \underline{G}(F)$, $\mathbb{G} = \underline{G}(\mathbb{A})$, and $Z \simeq F^\times$, $\mathbb{Z} \simeq \mathbb{A}^\times$ for the corresponding centers. Fix a unitary character ε of \mathbb{Z}/Z , and signify by π a cuspidal representation of \mathbb{G} whose central character is ε . For almost all F -places v the component π_v of π at v is unramified and is determined by a semi-simple conjugacy class $t(\pi_v)$ in $\widehat{G} = \underline{G}(\mathbb{C})$ with eigenvalues $(z_i(\pi_v); 1 \leq i \leq n)$. Given a finite dimensional representation r of \widehat{G} , and a finite set V of F -places containing the archimedean places and those where π_v is ramified, one has the L -function

$$L^V(s, \pi, r) = \prod_{v \notin V} \det(I - q_v^{-s} r(t(\pi_v)))^{-1}$$

which converges absolutely in some right half plane $\text{Re}(s) \gg 1$. Here q_v is the cardinality of the residue field of the ring R_v of integers in the completion F_v of F at v .

In this paper we consider the representation r of \widehat{G} on the $(n^2 - 1)$ -dimensional space M of $n \times n$ complex matrices with trace zero, by the adjoint action $r(g)m = \text{Ad}(g)m = gmg^{-1}$ ($m \in M, g \in \widehat{G}$). More generally we can introduce the representation Adj of $G \times \mathbb{C}^\times$ by $\text{Adj}((g, z)) = zr(g)$, and hence for any character ω of \mathbb{Z}/Z the

L -function

$$L^V(s, \pi, \omega, \text{Adj}) = \prod_{v \notin V} \det(I - q_v^{-s} t(\omega_v) r(t(\pi_v)))^{-1}.$$

Here V contains all places v where π_v or the component ω_v of ω is ramified, and $t(\omega_v) = \omega_v(\underline{\pi}_v)$; $\underline{\pi}_v$ is a generator of the maximal ideal in R_v .

In fact the full L -function is defined as a product over all v of local L -functions. These are introduced in the p -adic case as (a quotient of) the “greatest common denominator” of a family of integrals whose definition is recalled from [JPS] after Proposition 3 below. The local L -functions in the archimedean case are introduced below as a quotient of the L -factors studied in [JS1]. We denote by $L(s, \pi, \dots)$ the full L -function.

More precisely, we have

$$L^V(s, \pi, \omega, \text{Adj}) = L^V(s, \pi \otimes \omega \times \tilde{\pi}) / L^V(s, \omega),$$

where $L^V(s, \pi_1 \times \pi_2)$ denotes the partial L -function attached to the cuspidal $\text{GL}(n_i, \mathbb{A})$ -modules π_i ($i = 1, 2$) and the tensor product of the standard representation of $\widehat{G}_1 = \text{GL}(n_1, \mathbb{C})$ and $\widehat{G}_2 = \text{GL}(n_2, \mathbb{C})$. This provides a natural definition for the complete function $L(s, \pi, \omega, \text{Adj})$ globally, and also locally. This definition permits using the results of [JPS] and [JS1] mentioned above. In particular, for any cuspidal \mathbb{G} -module π , the L -function $L(s, \pi, \omega, \text{Adj})$ has analytic continuation to the entire complex s -plane.

To simplify the notations we shall assume, when $\omega \neq 1$, that ω does not factorize through $z \mapsto \nu(z) = |z|$; this last case can easily be reduced to the case of $\omega = 1$. Indeed, $L(s, \pi, \omega \otimes \nu^{s'}, \text{Adj}) = L(s + s', \pi, \omega, \text{Adj})$. Our main result is the following.

1. THEOREM. *Suppose that the cuspidal \mathbb{G} -module π has a supercuspidal component, and ω is a character of \mathbb{Z}/Z of finite order for which the assumption (Ass; E, ω) below is satisfied for all separable field extensions E of F of degree n . Then the L -function $L(s, \pi, \omega, \text{Adj})$ is entire, unless $\omega \neq 1$ and $\pi \otimes \omega \simeq \pi$. In this last case the L -function is holomorphic outside $s = 0$ and $s = 1$. There it has simple poles.*

To state (Ass; E, ω) note that given any separable field extension E of degree n of F there is a finite galois extension K of F , containing E , such that ω corresponds by class field theory to a character, denoted again by ω , of the galois group $J = \text{Gal}(K/F)$.

Denote by $H = \text{Gal}(K/E)$ the subgroup of J corresponding to E , and by $\omega|E$ the restriction of ω to H . It corresponds to a character, denoted again by $\omega|E$, of the idele class group $\mathbb{A}_E^\times/E^\times$ of E . When E/F is galois, and $N_{E/F}$ is the norm map from E to F , then $\omega|E = \omega \circ N_{E/F}$. Our assumption is the following.

(Ass; E, ω) *The quotient $L(s, \omega|E)/L(s, \omega)$ of the Artin (or Hecke, by class field theory) L -functions attached to the characters $\omega|E$ of $\text{Gal}(K/E) = H$ and ω of $\text{Gal}(K/F) = J$, is entire, except at $s = 0$ and $s = 1$ when $\omega \neq 1$ and $\omega|E = 1$.*

If E/F is an abelian extension, (Ass; E, ω) follows by the product decomposition $L(s, \omega|E) = \prod_{\zeta} L(s, \omega\zeta)$, where ζ runs through the set of characters of $\text{Gal}(E/F)$. More generally, (Ass; E, ω) is known when E/F is galois, and when the galois group of the galois closure of E over F is solvable, for $\omega = 1$ (see, e.g., [CF], p. 225, and the survey article [W]). For a general E we have

$$L(s, \omega|E) = L(s, \text{Ind}_H^J(\omega|E)) = L(s, \omega)L(s, \rho),$$

where the representation $\text{Ind}_H^J(\omega|E)$ of $J = \text{Gal}(K/F)$ induced from the character $\omega|E$ of H , contains the character ω with multiplicity one (by Frobenius reciprocity); ρ is the quotient by ω of $\text{Ind}_H^J(\omega|E)$. Artin's conjecture for J now implies that $L(s, \rho)$ is entire, unless $\omega|E = 1$ and $\omega \neq 1$, in which case $L(s, \rho)$ is holomorphic except at $s = 0, 1$, where it has a simple pole. When $[E : F] = n$, $\omega = 1$ and K is a galois closure of E/F , then $J = \text{Gal}(K/F)$ is a quotient of the symmetric group S_n . Artin's conjecture is known to hold for S_3 and S_4 , hence (Ass; $E, 1$) holds for all E of degree 3 or 4 over F , and Theorem 1 holds unconditionally (when $\omega = 1$) for $GL(3)$ and $GL(4)$, as well as for $GL(2)$.

The conclusion of Theorem 1 can be rephrased as asserting that $L(s, \omega)$ divides $L(s, \pi \otimes \omega \times \tilde{\pi})$ when $\pi \otimes \omega \neq \pi$ or $\omega = 1$, namely the quotient is entire, and that the quotient is holomorphic outside $s = 0, 1$, if $\pi \otimes \omega \simeq \pi$ and $\omega \neq 1$; of course we assume (Ass; E, ω) for all separable extensions E of F of degree n . Note that the product L -function $L(s, \pi_1 \times \pi_2)$ has been shown in [JS], [JS1], [JPS] and (differently) in [MW] to be entire unless $\pi_2 \simeq \tilde{\pi}_1$. In this last case the L -function is holomorphic outside $s = 0, 1$, and has a simple pole at $s = 0$ and $s = 1$. This pole is matched by the simple pole of $L(s, \omega)$ when $\omega = 1$. Hence $L(s, \pi, 1, \text{Adj})$ is also entire.

Another way to state the conclusion of Theorem 1 is that if $L(s, \omega)$ vanishes at $s = s_0$ to the order $m \geq 0$, then so does $L(s, \pi \otimes \omega \times \tilde{\pi})$,

provided that $(\text{Ass}; E, \omega)$ is satisfied for all separable extensions E of F of degree n . Note that $L(s, \omega)$ does not vanish on $|\text{Re } s - \frac{1}{2}| \geq \frac{1}{2}$.

Yet another restatement of the Theorem: Let π be a cuspidal \mathbb{G} -module with a supercuspidal component, and ω a unitary character of \mathbb{Z}/\mathbb{Z} . Let s_0 be a complex number such that for every separable extension E of F of degree n , the L -function $L(s, \omega|E)$ vanishes at $s = s_0$ to the order $m \geq 0$. Then $L(s, \pi \otimes \omega \times \tilde{\pi})$ vanishes at $s = s_0$ to the order m . This is the statement which is proven below. Note that the assumption that ω is of finite order was put above only for convenience. Embedding \mathbb{A}_E^\times as a torus in \mathbb{G} , the character $\omega|E$ can be defined also by $(\omega|E)(x) = \omega(\det x)$ on $x \in \mathbb{A}_E^\times \subset \mathbb{G}$. In general ω would be a character of a Weil group, and not a finite galois group.

When $n = 2$ the three dimensional representation Adj of $\text{GL}(2, \mathbb{C})$ is the symmetric square Sym^2 representation, and the holomorphy of the L -function $L(s, \omega \otimes \text{Sym}^2 \pi)$ ($s \neq 0, 1$ if $\pi \otimes \omega \simeq \pi$, $\omega \neq 1$) is proven in [GJ] using the Rankin-Selberg technique of Shimura [Sh], and in [F1] using a trace formula. Another proof was suggested by Zagier [Z] in the context of $\text{SL}(2, \mathbb{R})$ and generalized by Jacquet-Zagier [JZ] to the context of π on $\text{GL}(2, \mathbb{A})$. This last technique is the one extended to the context of cuspidal π with a supercuspidal component and arbitrary $n \geq 2$, in the present paper.

The path followed in [Z] and [JZ] is to compute the integral

$$\int K_\varphi(x, x)E(x, \Phi, \omega, s) dx$$

on x in $\mathbb{Z}\mathbb{G} \backslash \mathbb{G}$, where $E(x, \Phi, \omega, s)$ is an Eisenstein series, and $K_\varphi(x, y)$ the kernel representing the cuspidal spectrum in the trace formula. The computation shows that the integral is a sum of multiples of $L(s, \omega|E)$ (with $[E : F] = 2$ in the case of [Z] and [JZ]), and on the other hand of (a sum of multiples of) $L(s, \pi \otimes \omega \times \tilde{\pi})$, from which the conclusion is readily deduced. However, [Z] and [JZ] computed all terms in the integral, and reported about the complexity of the formulae. To generalize their computations to $\text{GL}(n)$, $n \geq 3$, considerable effort would be required.

To bypass these difficulties in this paper we use the ideas employed in [FK] and [F2] to establish various lifting theorems by means of a simple trace formula. In particular we use a special class of test functions φ , with one component supported on the elliptic regular set, and another component is chosen to be supercuspidal. The first choice reduces the conjugacy classes contributing to $K_\varphi(x, y)$ to elliptic ones only, while the second guarantees the vanishing of the non-cuspidal

terms in the spectral kernel. The first choice does not restrict the applicability of our formulae. Thus our Theorem 1 is offered as another example of the power and usefulness of the ideas underlying the simple trace formula.

For a "twisted tensor" analogue of this paper see [F4].

We shall work with the space $L(G)$ of smooth complex valued functions ϕ on $G \backslash \mathbb{G}$ which satisfy (1) $\phi(zg) = \varepsilon(z)\phi(g)$ ($z \in \mathbb{Z}$, $g \in \mathbb{G}$), (2) ϕ is absolutely square integrable on $\mathbb{Z}G \backslash \mathbb{G}$. The group \mathbb{G} acts on $L(G)$ by right translation: $(r(g)\phi)(h) = \phi(hg)$. The action is unitary since ε is. The function $\phi \in L(G)$ is called *cuspidal* if for each proper parabolic subgroup \underline{P} of \underline{G} over F with unipotent radical \underline{N} we have $\int \phi(n g) dn = 0$ ($n \in \underline{N} \backslash \underline{N}$) for all $g \in \mathbb{G}$. Let r_0 be the restriction of r to the space $L_0(G)$ of cusp forms in $L(G)$. The space $L_0(G)$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary \mathbb{G} -modules called *cuspidal* \mathbb{G} -modules.

Let φ be a complex valued function on \mathbb{G} with $\varphi(g) = \varepsilon(z)\varphi(zg)$ ($z \in \mathbb{Z}$), compactly supported modulo \mathbb{Z} , smooth as a function on the archimedean part $G(F_\infty)$ of \mathbb{G} , and bi-invariant by an open compact subgroup of $G(\mathbb{A}_f)$; here \mathbb{A}_f is the ring of adèles without archimedean components, and F_∞ is the product of F_v over the archimedean places. Fix Haar measures dg_v on G_v/Z_v ($G_v = \underline{G}(F_v)$, Z_v its center) for all v such that the product of the volumes $|K_v/Z_v \cap K_v|$ converges; K_v is a maximal compact subgroup of G_v , chosen to be $K_v = \underline{G}(R_v)$ at the finite places. Then $dg = \otimes dg_v$ is a measure on \mathbb{G}/\mathbb{Z} . The convolution operator $r(\varphi) = \int_{\mathbb{G}/\mathbb{Z}} \varphi(g)r(g)dg$ is an integral operator on $L(G)$ with the kernel $K_\varphi(x, y) = \sum \varphi(x^{-1}\gamma y)$ ($\gamma \in G/Z$). In this paper we work only with discrete functions φ .

DEFINITION. The function φ is called *discrete* if for every $x \in \mathbb{G}$ and $\gamma \in G$ we have $\varphi(x^{-1}\gamma x) = 0$ unless γ is elliptic regular.

Recall that γ is called *regular* if its centralizer $Z_\gamma(\mathbb{G})$ is a torus, and *elliptic* if it is semi-simple and $Z_\gamma(\mathbb{G})/Z_\gamma(G)\mathbb{Z}$ has finite volume. The centralizer $Z_\gamma(G)$ of an elliptic regular $\gamma \in G$ is the multiplicative group of a field extension E of F of degree n . For a general elliptic γ , we have that $Z_\gamma(G)$ is $GL(m, F')$ with $n = m[F' : F]$.

The proof of Theorem 1 is based on integrating the kernel $K_\varphi(x, y)$ on $x = y$ against an Eisenstein series, as in [Z] and [JZ].

Identify $GL(n-1)$ with a subgroup of $GL(n)$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Let U be the unipotent radical of the upper triangular parabolic subgroup of type $(n-1, 1)$. Put $Q = GL(n-1)U$. Given a local field F ,

let $S(F^n)$ be the space of smooth and rapidly decreasing (if F is archimedean), or locally constant compactly supported (if F is non-archimedean) complex valued functions on F^n . Denote by Φ^0 the characteristic function of R^n in F^n if F is non-archimedean. For a global field F let $S(\mathbb{A}^n)$ be the linear span of the functions $\Phi = \otimes \Phi_v$, $\Phi_v \in S(F_v^n)$ for all v , Φ_v is Φ_v^0 for almost all v . Put $\underline{\varepsilon} = (0, \dots, 0, 1) \in \mathbb{A}^n$. The integral of

$$(1.1) \quad f(g, s) = \omega(\det g) |\det g|^s \int_{\mathbb{A}^\times} \Phi(a\underline{\varepsilon}g) |a|^{ns} \omega^n(a) d^\times a$$

converges absolutely, uniformly in compact subsets of $\text{Re } s \geq \frac{1}{n}$. The absolute value is normalized as usual, and ω is a character of $\mathbb{A}^\times/F^\times$.

It follows from Lemmas (11.5), (11.6) of [GoJ] that the Eisenstein series

$$E(g, \Phi, \omega, s) = \sum f(\gamma g, s) \quad (\gamma \in ZQ \backslash G)$$

converges absolutely in $\text{Re } s > 1$. In [JS], (4.2), p. 545, and [JS2], (3.5), p. 7, it is shown (with a slight modification caused by the presence of ω here) that $E(g, \Phi, \omega, s)$ extends to a meromorphic function on $\text{Re } s > 0$, in fact to the entire complex s -plane with a functional equation $E(g, \Phi, \omega, s) = E({}^t g^{-1}, \hat{\Phi}, \omega^{-1}, 1-s)$; here ${}^t g$ is the transpose of g and $\hat{\Phi}$ is the Fourier transform of Φ . Moreover, $E(g, \Phi, \omega, s)$ is slowly increasing in $g \in G \backslash \mathbb{G}$, and it is holomorphic except for a possible simple pole at $s = 1$ and 0 . Note that $f(g)$ and $E(g, s)$ are \mathbb{Z} -invariant.

2. PROPOSITION. *For any character ω of $\mathbb{A}^\times/F^\times$, Schwartz function Φ in $S(\mathbb{A}^n)$, and discrete function φ on \mathbb{G} , for each extension E of degree n of F there is an entire holomorphic function $A(\Phi, \varphi, \omega, E, s)$ in s such that*

$$(2.1) \quad \int_{\mathbb{Z}G \backslash \mathbb{G}} K_\varphi(x, x) E(x, \Phi, \omega, s) dx = \sum_E A(\Phi, \varphi, \omega, E, s) L(s, \omega|E)$$

on $\text{Re } s > 1$. The sum over E ranges over a finite set depending on (the support of) φ .

Proof. Since the function φ is discrete the sum in $K_\varphi(x, x) = \sum \varphi(x^{-1}\gamma x)$ ranges only over the elliptic regular elements γ in G/Z .

It can be expressed as

$$(2.2) \quad K_\varphi(x, x) = \sum_T [W(T)]^{-1} \sum_{\gamma \in T/Z} \sum_{\delta \in G/T} \varphi(x^{-1}\delta^{-1}\gamma\delta x).$$

Here T ranges over a set of representatives for the conjugacy classes in G of elliptic tori (T is isomorphic over F to the multiplicative group of a field extension E of degree n of F ; T is uniquely determined by such E , and each such E is so obtained). The cardinality of the Weyl group (normalizer/centralizer) $W(T)$ of T in G is denoted by $[W(T)]$. It is easy to check that for any elliptic T we have $G = TQ$, and $T \cap Q = \{1\}$. Hence the sum over δ can be taken to range over Q .

The left side of (2.1) is equal, in the domain of absolute convergence of the series which defines the Eisenstein series, to

$$\int_{\mathbb{Z}G \backslash G} K_\varphi(x, x) \sum_{\gamma \in \mathbb{Z}Q \backslash G} f(\gamma x, s) dx = \int_{\mathbb{Z}Q \backslash G} K_\varphi(x, x) f(x, s) dx,$$

since $x \mapsto K_\varphi(x, x)$ is left G -invariant. Substituting (2.2) this is equal to

$$\begin{aligned} & \int_{\mathbb{Z}Q \backslash G} \sum_T [W(T)]^{-1} \sum_{\gamma \in T/Z} \sum_{\delta \in Q} \varphi(x^{-1}\delta^{-1}\gamma\delta x) f(x, s) dx \\ & = \sum_T [W(T)]^{-1} \sum_{\gamma \in T/Z} \int_{\mathbb{Z} \backslash G} \varphi(x^{-1}\gamma x) f(x, s) dx; \end{aligned}$$

note that $x \mapsto f(x, s)$ is left Q -invariant.

To justify the change of summation and integration note that given φ , the sums over T and γ are finite. Indeed, the coefficients of the characteristic polynomial of γ are rational, and lie in a compact set depending on the support of φ (and a discrete subset of a compact is finite). This explains also the finiteness assertion at the end of the proposition.

Substituting now the expression (1.1) for $f(x, s)$ we obtain a sum over T and γ of

$$\begin{aligned} & \int_{\mathbb{Z} \backslash G} \varphi(x^{-1}\gamma x) f(x, s) dx = \int_G \varphi(x^{-1}\gamma x) \omega(\det x) |\det x|^s \Phi(\underline{\epsilon}x) dx \\ & = \int_{\mathbb{T} \backslash G} \varphi(x^{-1}\gamma x) \int_{\mathbb{T}} \Phi(\underline{\epsilon}tx) \omega(\det tx) |\det tx|^s dt dx. \end{aligned}$$

Here $\mathbb{T} = \underline{T}(\mathbb{A}) \simeq \mathbb{A}_E^\times$, where \underline{T} is the centralizer of γ in \underline{G} , and $\underline{T}(F) = T$. The inner integral, over \mathbb{T} , is a ‘‘Tate integral’’ for

$L(s, \omega|E)$; it is a multiple of $L(s, \omega|E)$ by a function which is holomorphic in s in \mathbb{C} and smooth in x , depending on Φ, ω and E . The integral over x ranges over a compact in $\mathbb{T} \backslash \mathbb{G}$, since φ is compactly supported modulo \mathbb{Z} . The proposition follows.

We now turn to the spectral expression for the kernel $K_\varphi(x, y)$.

DEFINITION. The function φ on \mathbb{G} is called *cuspidal* if for every x, y in \mathbb{G} and every proper F -parabolic subgroup \underline{P} of \underline{G} , we have $\int_{\mathbb{N}} \varphi(xny)dn = 0$, where $\mathbb{N} = \underline{N}(\mathbb{A})$ is the unipotent radical of $\mathbb{P} = \underline{P}(\mathbb{A})$.

When φ is cuspidal, the convolution operator $r(\varphi)$ factorizes through the projection on $L_0(G)$. Then $r(\varphi)$ is an integral operator whose kernel has the form

$$K_\varphi(x, y) = \sum_{\pi} K_{\varphi}^{\pi}(x, y), \quad \text{where } K_{\varphi}^{\pi}(x, y) = \sum_{\phi^{\pi}} (r(\varphi)\phi^{\pi})(x)\overline{\phi^{\pi}}(y).$$

The sum over π ranges over all cuspidal \mathbb{G} -modules in $L_0(G)$. The ϕ^{π} range over an orthonormal basis consisting of $\mathbb{K} = \prod_v K_v$ -finite vectors in π . The ϕ^{π} are rapidly decreasing functions and the sum over ϕ^{π} is finite for each φ (uniformly in x and y) since φ is \mathbb{K} -finite. The sum over π converges in L^2 , and hence also in a space of rapidly decreasing functions. Hence $K_{\varphi}(x, y)$ is rapidly decreasing in x and y , and the product of $K_{\varphi}(x, x)$ with the slowly increasing functions $E(x, \Phi, \omega, s)$, is integrable over $\mathbb{ZG} \backslash \mathbb{G}$. The resulting integral, which is equal to (2.1), can also be expressed then in the form

$$\sum_{\pi} \sum_{\phi^{\pi}} \int_{\mathbb{ZG} \backslash \mathbb{G}} (r(\varphi)\phi^{\pi})(x)\overline{\phi^{\pi}}(x)E(x, \Phi, \omega, s) dx.$$

To prove Theorem 1 we now assume that $L(s, \omega)$ is zero at $s = s_0$. It is well known then that $|\operatorname{Re} s_0 - \frac{1}{2}| < \frac{1}{2}$, hence $s_0 \neq 0, 1$. If s_0 is a zero of order m of $L(s, \omega)$, then by (Ass; E, ω) the function $L(s, \omega|E)$ vanishes at s_0 to the order m . Making this assumption for every separable field extension E of degree n of F we conclude that (2.1) vanishes at $s = s_0$ to the order m , and that for all j ($0 \leq j \leq m$) we have

$$(2.3)_j \quad \sum_{\pi} \sum_{\phi^{\pi}} \int_{\mathbb{ZG} \backslash \mathbb{G}} (\pi(\varphi)\phi^{\pi})(x)\overline{\phi^{\pi}}(x)E^{(j)}(x, \Phi, \omega, s_0) dx = 0.$$

Here $E^{(j)}(*, s_0) = \frac{d^j}{ds^j} E(*, s)|_{s=s_0}$.

At our disposal we have all cuspidal discrete functions φ on \mathbb{G} , and our aim is to show the vanishing of some summands in the last

double sum over π and ϕ^π . In fact, fix a π for which Theorem 1 will now be proven. Let V be a finite set of F -primes, containing the archimedean primes and those where π or ω ramify. Consider $\varphi = \otimes_v \varphi_v$ (product over all F -places v) where each φ_v is a smooth compactly supported modulo Z_v function on G_v which transforms under Z_v via ε_v^{-1} . For almost all v the function φ_v is the unit element φ_v^0 in the Hecke algebra \mathbb{H}_v of K_v -biinvariant (compactly supported modulo Z_v transforming under Z_v via ε_v^{-1}) functions on G_v . For all $v \notin V$ the component φ_v is taken to be spherical, namely in \mathbb{H}_v .

Each of the operators $\pi_v(\varphi_v)$ for $v \notin V$ factorizes through the projection on the subspace $\pi_v^{K_v}$ of K_v -fixed vectors in π_v . This subspace is zero unless π_v is unramified, in which case $\pi_v^{K_v}$ is one-dimensional. On this K_v -fixed vector, the operator $\pi_v(\varphi_v)$ acts as the scalar $\varphi_v^\vee(t(\pi_v))$, where φ_v^\vee denotes the Satake transform of φ_v . Put $\varphi^\vee(t(\pi^V))$ for the product over $v \notin V$ of $\varphi_v^\vee(t(\pi_v))$, and $\pi_V(\varphi_V) = \otimes_{v \in V} \pi_v(\varphi_v)$. Then (2.3)_j takes the form

$$(2.4)_j \quad \sum_{\{\pi; \pi^{K,V} \neq 0\}} \varphi^\vee(t(\pi^V)) a(\pi, \varphi_V, j, \Phi, \omega, s_0) = 0,$$

where

$$(2.5)_j \quad a(\pi, \varphi_V, j, \Phi, \omega, s) = \sum_{\phi^\pi} \int_{\mathbb{Z}G \backslash G} (\pi_V(\varphi_V)\phi^\pi)(x) \overline{\phi^\pi}(x) E^{(j)}(x, \Phi, \omega, s) dx.$$

The sum over π ranges over the cuspidal \mathbb{G} -modules $\pi = \otimes \pi_v$ with $\pi_v^{K_v} \neq \{0\}$ for all $v \notin V$; $\pi^{K,V}$ denotes the space of $\prod_{v \notin V} K_v$ -fixed vectors in π . The sum over ϕ^π ranges over those elements in the orthonormal basis of π which appears in (2.3)_j, which, for any $v \notin V$, as functions in $x \in G_v$, are K_v -invariant and eigenfunctions of $\pi_v(\varphi_v)$, $\varphi_v \in \mathbb{H}_v$, with eigenvalues $t(\pi_v)$. In particular $\phi^\pi(x) = \phi_V^\pi(x_V) \prod_{v \notin V} \phi_v^\pi(x_v)$, for such ϕ_v^π ($v \notin V$).

A standard argument (see, e.g., Theorem 2 in [FK] in a more elaborate situation), based on the absolute convergence of the sum over π in (2.4)_j, standard estimates on the Hecke parameter $t(\pi_v)$ of the unitary unramified π_v ($v \notin V$), and the Stone-Weierstrass theorem, implies the following.

3. PROPOSITION. *Let π be a cuspidal \mathbb{G} -module which has a supercuspidal component. Let ω be a character of \mathbb{Z}/Z . Suppose that*

$L(s, \omega|E)$ vanishes at $s = s_0$ to the order m for every separable extension E of F of degree n . Then for any Φ and a function φ_V such that φ is cuspidal and discrete with any choice of $\otimes \varphi_v$ ($v \notin V$), we have that $a(\pi, \varphi_V, j, \Phi, \omega, s_0)$ is zero.

We shall now recall the relation between the summands in (2.5) _{j} and the L -function $L(s, \pi \otimes \omega \times \tilde{\pi})$. Let ψ be an additive non-trivial character of \mathbb{A} modulo F (into the unit circle in \mathbb{C}), and denote by ψ_v its component at v . An irreducible admissible G_v -module π_v is called *generic* if $\text{Hom}_{N_v}(\pi_v, \psi_v) \neq \{0\}$. By [GK], or Corollary 5.17 of [BZ], such π_v embeds in the G_v -module $\text{Ind}(\psi_v; G_v, N_v)$ induced from the character $n = (n_{ij}) \mapsto \psi(n) = \psi(\sum_{1 \leq i < n} n_{i, i+1})$ of the unipotent upper triangular subgroup N_v of G_v . Moreover, this embedding is unique, equivalently the dimension of $\text{Hom}_{N_v}(\pi_v, \psi_v)$ is at most one. The embedding is given by $\pi_v \ni \xi \mapsto W_\xi$, where $W_\xi(g) = \lambda(\pi(g)\xi)$ ($g \in G$) and $\lambda \neq 0$ is a fixed element in $\text{Hom}_{N_v}(\pi_v, \psi_v)$. Since π_v is admissible, each of the functions W_ξ is smooth (under right action by G_v). If π_v is generic, denote by $W(\pi_v)$ its realization in $\text{Ind}(\psi_v)$; $W(\pi_v)$ is called the *Whittaker model* of π_v . It is well-known that any component of a cuspidal G -module is generic.

Given π , consider $W'_v \neq 0$ in $W(\pi_v)$ for all v , such that W'_v is the normalized unramified vector $W'_v{}^0$ (it is K_v -invariant and $W'_v{}^0(1) = 1$) for all $v \notin V$. The function $\phi'(x) = \sum_{p \in N \setminus Q} W'(px)$, where $W'(x) = \prod_v W'_v(x_v)$, is a cuspidal function in the space of $\pi \subset L_0(G)$. Substituting the series definition of $E(x, \Phi, \omega, s) = \sum_{ZQ \setminus G} f(\gamma x, s)$ in

$$\int_{ZG \setminus G} \phi''(x) \overline{\phi}'(x) E(x, \Phi, \omega, s) dx \quad (\phi'' \in \pi \subset L_0(G))$$

one obtains

$$\int_{ZQ \setminus G} \phi''(x) \overline{\phi}'(x) f(x, s) dx = \int_{ZN \setminus G} \phi''(x) \overline{W}'(x) f(x, s) dx.$$

Since $W'(nx) = \psi(n)W'(x)$, and $\int_{N \setminus N} \phi''(nx) \overline{\psi}(n) dn = W_{\phi''}(x)$ is the Whittaker function associated to the cusp form ϕ'' , the integral is equal to

$$\begin{aligned} & \int_{ZN \setminus G} W_{\phi''}(x) \overline{W}'(x) f(x, s) dx \\ &= \int_{N \setminus G} W_{\phi''}(x) \overline{W}'(x) \Phi(\underline{x}) \omega(\det x) |\det x|^s dx. \end{aligned}$$

If ϕ'' is also of the form $\phi''(x) = \sum_{p \in N \setminus \mathcal{Q}} W''(px)$, where $W''(x) = \prod_v W''_v(x_v)$ is factorizable, then $W_{\phi''} = W''$ and the integral factorizes as a product over all v of the local integrals

$$(3.1) \quad \int_{N_v \setminus G_v} W''_v(x) \overline{W''_v}(x) \Phi_v(\underline{\varepsilon}x) \omega_v(\det x) |\det x|_v^s dx,$$

provided that $\Phi(x) = \prod_v \Phi_v(x_v)$.

When $W'_v = W''_v = W''_v$, and Φ_v is the characteristic function Φ_v^0 of R_v^n (and $v \notin V$), the integral (3.1) is easily seen (on using Schur function computations; see [F3], p. 305) to be equal to $L(s, \pi_v \otimes \omega_v \times \tilde{\pi}_v)$. For a non-archimedean $v \in V$ the L -factor is defined in [JPS], Theorem 2.7, as a “g.c.d” of the integrals (3.1) for all $W_{1v}, W_{2v} \in \mathcal{W}(\pi_v)$ and Φ_v . In the archimedean case the L -factor is defined in [JS1], Theorem 5.1. It is shown in [JPS] and [JS1] that the L -factor lies in the span of the integrals (3.1). The product of the L -factors, as well as the various manipulations above, converges absolutely for s in some right half plane.

4. LEMMA. *The functions $W'_v \in \mathcal{W}(\pi_v)$ (and so $\phi' \in \pi$) can be chosen to have the property that ϕ' factorizes as $\otimes_v \phi'_v$.*

Proof. Since W'_v is K_v -invariant for $v \notin V$, so is ϕ' , and we have

$$\phi'(x) = \phi'_V(x_v) \prod_{v \notin V} \phi_v^0(x_v),$$

where ϕ_v^0 is the K_v -invariant function on G_v which takes the value 1 at 1 and is the eigenfunction of the operators $\pi_v(\varphi_v)$, $\varphi_v \in \mathbb{H}_v$, with the eigenvalue $t(\pi_v)$.

The space $\pi \subset L_0(G)$ is spanned by factorizable functions, namely ϕ' is a finite sum over j ($1 \leq j \leq J$) of products $\otimes_v \phi'_{jv}$ of functions ϕ'_{jv} on G_v (which are smooth, compactly supported modulo Z_v , transform under Z_v via ε_v), with $\phi'_{jv} = \phi_v^0$ for all $v \notin V$. Each of the functions ϕ'_{1v} ($v \in V$) is (right) invariant under a congruence subgroup K'_v of the standard compact subgroup K_v of G_v . Namely ϕ'_{1v} is a non-zero vector in the finite dimensional space $\pi_v^{K'_v}$ of K'_v -fixed vectors in π_v . The Hecke algebra $\mathbb{H}(K'_v)$ of K'_v -biinvariant compactly supported modulo Z_v functions on G_v which transform under Z_v via ε_v^{-1} generate the algebra of endomorphisms of the finite dimensional space $\pi_v^{K'_v}$. Consider $\varphi_v \in \mathbb{H}(K'_v)$ such that $\pi_v(\varphi_v)$ acts

as an orthogonal projection on ϕ'_{1v} . Then $(\bigotimes_{v \in V} \pi_v(\varphi_v))\phi'$ lies in π , is of the form $\bigotimes_v \phi'_{1v}$, and is defined by the Whittaker functions $\pi_v(\varphi_v)W'_v$, as required.

Proof of Theorem 1. For π as in the theorem, and s_0 as in (2.3)_j, we shall choose $W'_v \in W(\pi_v)$ with factorizable $\phi'(x) = \bigotimes_v \phi'_v(x_v) = \sum_{p \in N \setminus Q} W'(px)$ and proceed to show the vanishing of the corresponding summand in (2.5)_j. Recall that by the assumption of Theorem 1 there is an F -place v_2 such that π_{v_2} is supercuspidal. Let v_1 be another F -place in V , say where π and ω are unramified. Put $V'' = V - \{v_2\}$ and V' for $V'' - \{v_1\}$.

Consider the matrix coefficient $\phi'_{v_2}(x) = \langle \pi_{v_2}(x^{-1})\phi'_{v_2}, \phi'_{v_2} \rangle$ of the supercuspidal G_{v_2} -module π_{v_2} . Note that ϕ'_{v_2} is a C_c^∞ -function on G_{v_2} modulo Z_{v_2} , and $\langle \cdot, \cdot \rangle$ denotes the natural inner product. The function ϕ'_{v_2} is smooth and compactly supported on G_{v_2} modulo Z_{v_2} , and it is a supercusp form ($\int \phi'_{v_2}(xny) dn = 0, n \in N_{v_2} =$ unipotent radical of any parabolic subgroup of G_{v_2}). It is well-known that a function $\varphi = \bigotimes \varphi_v$ whose component at v_2 is a supercusp form is cuspidal. By the Schur orthogonality relations, the convolution operator $\pi_{v_2}(\phi'_{v_2})$ acts as an orthogonal projection on the subspace generated by ϕ'_{v_2} . Working with $\varphi = \bigotimes \varphi_v$ whose component at v_2 is ϕ'_{v_2} we then have that φ is cuspidal and that the sum in (2.5)_j ranges only over the $\phi (= \phi^\pi)$ whose component at v_2 is ϕ'_{v_2} (up to a scalar multiple).

As in the proof of Lemma 4, for each $v \in V'$ we may choose ϕ'_v in $\mathbb{H}(K'_v)$ such that $\pi_v(\phi'_v)$ acts as an orthogonal projection to the subspace of π'_v spanned by ϕ'_v . Choosing the components φ_v of φ at $v \in V'$ to be of the form $\varphi''_v * \phi'_v$, with any φ''_v , the sum in (2.5)_j for our π extends only over those ϕ in the orthonormal basis of the chosen $\pi \subset L_0(G)$ whose component at $v \neq v_1$ is ϕ'_v . But ϕ is left G -invariant, being a cusp form, and $\mathbb{G} = G \prod_{v \neq v_1} G_v$. Hence the only ϕ which contributes to the sum in (2.5)_j is ϕ' , whatever φ_{v_1} is.

We still need to choose φ_{v_1} such that $\varphi = \bigotimes \varphi_v$ be discrete. It suffices to choose φ_{v_1} to be supported on the regular elliptic set in G_{v_1} . Moreover, since ϕ'_{v_1} is right invariant under a compact open subgroup K'_{v_1} of $K_{v_1} \subset G_{v_1}$, we can choose the support of φ_{v_1} to be contained in $Z_{v_1}K'_{v_1}$. Then $\pi_{v_1}(\varphi_{v_1})$ acts as a scalar on ϕ'_1 , and we normalize φ_{v_1} so that this scalar be one.

In conclusion, for any choice of $W'_v \in W(\pi_v)$ for all v , with $W'_v =$

W_v^0 for $v \notin V$, and any choice of φ_v ($v \in V'$), we have that

$$\begin{aligned} & \int_{\mathbb{Z}G \backslash \mathbb{G}} (\pi_{V'}(\varphi_{V'})\phi')(x)\bar{\phi}'(x)E(x, \Phi, \omega, s) dx \\ &= \prod_{v \in V'} \int_{N_v \backslash G_v} (\pi_v(\varphi_v)W'_v)(x)\bar{W}'_v(x)\Phi_v(\underline{x})\omega_v(\det x)|\det x|_v^s dx \\ & \cdot \prod_{v \notin V'} L(s, \pi_v \otimes \omega_v \times \check{\pi}_v) \end{aligned}$$

vanishes at s_0 to the order m . Here $\pi_{v_1}(\varphi_{v_1})W'_{v_1} = W'_{v_1}$. In fact we may choose W'_{v_1} to be $W_{v_1}^0 \in \mathcal{W}(\pi_{v_1})$, and Φ_{v_1} to be $\Phi_{v_1}^0$. Since π_{v_1} and ω_{v_1} are unramified, the corresponding integral is then equal to the L -factor, so v_1 can be deleted from the set V .

To complete the proof of Theorem 1, note that the L -function $L(s, \pi_v \otimes \omega_v \times \check{\pi}_v)$ lies in the span of the integrals (3.1). Hence the assumption for every separable extension E of F of degree n that $L(s, \omega|E)$ vanishes at $s = s_0$ to the order m , implies the vanishing of $\prod L(s, \pi_v \otimes \omega_v \times \check{\pi}_v)$ to the order m . This completes the proof of Theorem 1.

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