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A general free boundary problem is investigated and the qualitative behavior of the fixed boundary is compared with that of the fixed boundary. As an illustration, consider the following situation. Let Γ^* be a given Jordan curve in $\mathfrak{R}^2$. For each Jordan curve Γ in $\mathfrak{R}^2$ which surrounds Γ^*, we let $\Omega = \Omega(\Gamma^*, \Gamma)$ be the region between Γ^* and Γ. Let Q be the second-order elliptic operator given by

$$Qu \equiv au_{xx} + 2bu_{xy} + cu_{yy} \quad \text{in } \Omega$$

where a, b, c depend on x, y, u_x, and u_y and $ac - b^2 > 0$. Consider the free boundary problem of finding a curve Γ and a function $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma) \cap C^0(\overline{\Omega})$ such that

$$\begin{aligned} Qu &= 0 \quad \text{in } \Omega \\ u &= 1 \quad \text{on } \Gamma^* \end{aligned}$$

and, for a fixed $\lambda > 0$,

$$u = 0, \quad |\nabla u| = \lambda \quad \text{on } \Gamma,$$

where $\Omega = \Omega(\Gamma^*, \Gamma)$. Suppose Γ and u constitute a solution of this free boundary problem. Using curves of constant gradient direction, the geometry of the free boundary Γ is compared to the geometry of the fixed boundary Γ^*. In particular, Γ is shown to have a "simpler" geometry than does Γ^*.

0. Introduction. Let $a, b, c \in C^0(\mathfrak{R}^4)$ with $ac - b^2 > 0$ in $\mathfrak{R}^4$ and define Q to be the quasilinear, elliptic, second-order partial differential operator given by

$$Qu = au_{xx} + 2bu_{xy} + cu_{yy} \tag{1}$$

for $u \in C^2$, where $a = a(x, y, p, q)$, $b = b(x, y, p, q)$, $c = c(x, y, p, q)$ and $p = u_x(x, y)$, $q = u_y(x, y)$. We are interested in the following free boundary problem.

Quasilinear free boundary problem. Given Γ^* a Jordan curve in $\mathfrak{R}^2$ or a finite collection of pairwise disjoint Jordan curves in $\mathfrak{R}^2$ and a number $\lambda > 0$, find a bounded domain $\Omega \subset \mathfrak{R}^2$, a finite collection Γ

of pairwise disjoint Jordan curves in $\mathfrak{R}^2$, and a function $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma) \cap C^0(\overline{\Omega})$ such that $\Gamma \cap \Gamma^* = \varnothing$, $\partial\Omega = \Gamma \cup \Gamma^*$, and

$$Qu = 0 \quad \text{in } \Omega, \tag{2a}$$

$$u = 1 \quad \text{on } \Gamma^*, \tag{2b}$$

$$u = 0 \quad \text{on } \Gamma, \tag{2c}$$

$$|\nabla u| = \lambda \quad \text{on } \Gamma. \tag{2d}$$

We are also interested in the following related free boundary problem. Let $F \in C^0(\mathfrak{R}^2 \times \mathfrak{R}^2 \times \mathfrak{R}^{2\times 2})$ satisfy:

$F(x, y, P, R)$ is locally uniformly Lipschitz with respect to the $P \in \mathfrak{R}^2$ and $R \in \mathfrak{R}^{2\times 2}$ variables;

F is elliptic;

$|F_P|$ is locally bounded;

$F(x, y, P, 0) = 0$ for all $(x, y) \in \mathfrak{R}^2$ and $P = (p, q) \in \mathfrak{R}^2$, where $\mathfrak{R}^{2\times 2}$ denotes the 3-dimensional space of real, symmetric 2×2 matrices (see [**12**], pp. 441–446). Let $\mathfrak{J}$ be the elliptic, fully nonlinear partial differential operator of second-order depending on x, y, Du, D^2u given by

$$\mathfrak{J}u = F(x, y, P, R) \tag{1'}$$

with $P = Du$ and $R = D^2u \in \mathfrak{R}^{2\times 2}$.

Fully nonlinear free boundary problem. Given Γ^* a Jordan curve in $\mathfrak{R}^2$ or a finite collection of pairwise disjoint Jordan curves in $\mathfrak{R}^2$ and a number $\lambda > 0$, find a bounded domain $\Omega \subset \mathfrak{R}^2$, a finite collection Γ of pairwise disjoint Jordan curves in $\mathfrak{R}^2$, and a function $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma) \cup C^0(\overline{\Omega})$ such that $\Gamma \cap \Gamma^* = \varnothing$, $\partial\Omega = \Gamma \cup \Gamma^*$, and

$$\mathfrak{J}u = 0 \quad \text{in } \Omega, \tag{2a'}$$

$$u = 1 \quad \text{on } \Gamma^*, \tag{2b'}$$

$$u = 0 \quad \text{on } \Gamma, \tag{2c'}$$

$$|\nabla u| = \lambda \quad \text{on } \Gamma. \tag{2d'}$$

We will call Γ^* the *fixed boundary* and Γ the *free boundary* of this problem. We should note that the requirement that Ω be bounded can be relaxed.

A number of authors have considered questions about certain qualitative aspects of the free boundary. For example, D. Tepper ([**20**], [**21**]; see also [**8**], pp. 432–443) proved that if $Q = \Delta$, Ω is an annular domain, and Γ^* is convex or starlike, then Γ is also convex or starlike. For an inhomogeneous free boundary problem, T. Vogel ([**22**]) obtained the convexity or starlikeness of Γ when Γ^* is convex or starlike. For parabolic free boundary problems, K. Nickel ([**14**], [**15**]) considered profiles of solutions of the heat equation and A. Friedman and R. Jensen ([**10**]) obtained the convexity of the free boundary in Stefen and dam problems.

In 1983 (approximately), A. Acker ([**1**]) and A. Friedman-T. Vogel ([**11**]) independently considered the case $Q = \Delta$ (i.e. u is harmonic in Ω) and used curves of constant gradient direction to obtain qualitative information about the geometry of the free boundary in terms of the geometry of the fixed boundary when Ω is (equivalent to) a doubly connected or annular domain (i.e. Γ and Γ^* are Jordan curves with one of the curves lying inside the other); these curves of constant gradient direction are related to "nodal lines" ([**9**]; see also [**10**], [**14**], [**17**]). Vogel ([**22**]) also used these curves to examine a galvanization problem. Subsequently, Acker showed that curves of constant gradient direction can be a powerful tool for investigating the geometry of free boundaries by obtaining more detailed qualitative information about the free boundary in terms of information about the fixed boundary ([**2**], [**3**]) and obtaining qualitative results without the assumption that Ω is doubly connected ([**4**]). Further, Acker, together with the author, used this method to study free boundary problems for parametric minimal surfaces in $\mathfrak{R}^3$ ([**6**]) and the one-dimensional heat equation ([**7**]). Acker ([**5**]) has also found examples in $\mathfrak{R}^n$, $n > 2$, which show that the qualitative results above are generally false in three or more dimensions.

We will show that the "method of curves of constant direction" can be used to investigate the quasilinear and fully nonlinear free boundary problems for any operator Q or $\mathfrak{J}$ as given by (1) or (1′) respectively. We will prove that each component of Γ has a "simpler" geometry than does Γ^*. For example, when Ω is an annular domain, we will prove that the total curvature, the number of local maxima (minima) with respect to a prescribed direction $\vec{\nu}$, and the number of inflection points of Γ are less than or equal to the total curvature, the number of local maxima (minima) with respect to $\vec{\nu}$, and the number of inflection points of Γ^* respectively. When we do not assume Ω has a particular topological structure, we see that total "positive curvature" and the number of $\vec{\nu}$-minima of Γ are less than or equal to the total "positive curvature" and the number of $\vec{\nu}$-minima of Γ^* respectively. We note that all of the qualitative results of Acker for the harmonic free boundary problem obtained using curves of constant gradient direction remain valid for our free boundary problems. Since, in addition to the free boundary problems (2) and (2′), this method has proven useful for inhomogeneous ([22], [3]), parabolic ([7]), and axial-symmetric ([5]) free boundary problems, we suspect that curves of constant gradient direction will eventually prove useful for investigating a number of additional elliptic and parabolic free boundary problems.

1. Main results. We will adopt the notation of [2] and [4]. Suppose Γ^* and λ are given and Ω, Γ, u constitute a solution of the free boundary problem (2). We assume Γ^* is a C^1 curve or union of curves and has bounded curvature. We will orient $\partial\Omega$ so that the forward direction on $\partial\Omega$ is such that Ω lies locally to the left of Ω and locally to the right of Γ^* (e.g. Figure 1, [4]). We let $\vec{n}(x, y)$ denote the unit normal vector to $\partial\Omega$ at $(x, y) \in \partial\Omega$ which points to the left; hence $\nabla u(x, y) = |\nabla u(x, y)|\vec{n}(x, y)$ for $(x, y) \in \partial\Omega$. We will assume that for each unit vector $\vec{e}$, the curve Γ^* contains at most finitely many maximal segments (including isolated points) on which $\vec{n}(x, y) = \vec{e}$.

DEFINITION. Given a unit vector $\vec{\nu}$, we call $(x_0, y_0) \in \Gamma$ a *$\vec{\nu}$-minimum* (*$\vec{\nu}$-maximum*) of Γ if $\vec{n}(x_0, y_0) = \vec{\nu}$ and (x_0, y_0) is a strict local minimum (maximum) relative to Γ of $f(x, y) = \vec{\nu}\cdot(x, y)$ (see, for example, Figures 2 and 3, [4]).

DEFINITION. Given a unit vector $\vec{\nu}$, we call $(x_0, y_0) \in \Gamma^*$ a *$\vec{\nu}$-minimum* (*$\vec{\nu}$-maximum*) of Γ^* if $\vec{n}(x_0, y_0) = \vec{\nu}$ and either (x_0, y_0) is a strict local minimum (maximum) relative to Γ of $f(x, y) = \vec{\nu} \cdot (x, y)$ or there is a closed line segment $\gamma^* \subset \Gamma^*$ such that $(x_0, y_0) \in \gamma^*$ and $\vec{\nu} \cdot (x, y) > (<)\ \vec{\nu} \cdot (x_0, y_0)$ for $(x, y) \in \Gamma^* \setminus \gamma^*$ near γ^*. Here γ^* is considered as a single local extremum.

DEFINITION. Given $\sigma \subset \Gamma$, we say that γ has *positive* (*negative*) *curvature* on σ if and only if Γ has nonvanishing curvature on σ and for each $(x, y) \in \sigma$, there exists $r > 0$ such that the set of points which are within r of (x, y) and lie to the left (right) of $\partial\Omega$ is a convex set. We define the notation of positive and negative curvature of Γ^* similarly.

DEFINITION. We call $(x, y) \in \partial\Omega$ a *positive* (*negative*) *inflection point* of $\partial\Omega$ if and only if $\partial\Omega$ has negative (positive) curvature locally before (x, y) and has positive (negative) curvature locally after

(x, y) or there is a line segment $\sigma \subset \Gamma^*$ with $(x, y) \in \sigma$ such that Γ^* has negative (positive) curvature locally before σ and positive (negative) curvature locally after σ.

THEOREM 1. *Let Γ^* be a given C^2 Jordan curve and $\lambda > 0$. Suppose Ω, Γ, u is a solution of either free boundary problem with the following properties:*

(i) *Ω is a bounded, C^2 annular domain.*

(ii) *$u \in C^2(\overline{\Omega})$.*

Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a positively ordered set of distinct points on Γ (i.e. $(x_1, y_1) < (x_2, y_2) < \cdots < (x_n, y_n) < (x_1, y_1)$ in the natural ordering on Γ) such that for each i, the points (x_i, y_i) has one of the following properties:

(a) *For given unit vector $\vec{\nu}_i$, $\vec{n}(x_i, y_i) = \vec{\nu}_i$.*

(b) *$\vec{n}(x_i, y_i) = \vec{\nu}_i$ and (x_i, y_i) is a $\vec{\nu}$-maximum ($\vec{\nu}$-minimum) of Γ.*

(c) *(x_i, y_i) is a positive (negative) inflection point of Γ.*

Then there is a positively ordered set of distinct (possibly degenerate) line segments $\sigma_1, \ldots, \sigma_n$ on Γ^* such that for each i, each point (x, y) in the segment σ_i has the same property relative to Γ^* that (x_i, y_i) has relative to Γ (e.g. $\vec{n}(x, y) = \vec{\nu}_i$ for each $(x, y) \in \sigma_i$.) Further, the total positive (negative) curvature of Γ is less than or equal to that of Γ^*.

When Γ, Γ^*, and u are real-analytic and Ω is an annular domain, we obtain:

THEOREM 2. *Let Γ^* be a given analytic Jordan curve and $\lambda > 0$. Suppose Ω, Γ, u is a solution of either free boundary problem with the following properties:*

(i) *Ω is a bounded, analytic annular domain.*

(ii) *u is real-analytic on $\overline{\Omega}$.*

Let $\vec{\nu}$ be a unit vector and let $(x_1, y_1), \ldots, (x_n, y_n)$ be the distinct $\vec{\nu}$-minima ($\vec{\nu}$-maxima) of Γ. Then each point (x_i, y_i) is joined by a simple, piecewise-analytic directed curve γ_i to a point $(x_i^*, y_i^*) \in \Gamma^*$ such that

(a) *$|\nabla u|$ is strictly increasing (decreasing) on γ_i.*

(b) $\arg(\nabla u)$ *is constant on γ_i.*

(c) *The points $(x_1^*, y_1^*), \ldots, (x_n^*, y_n^*)$ are distinct $\vec{\nu}$-minima ($\vec{\nu}$-maxima) of Γ^*.*

(d) $|\nabla u(x_i^*, y_i^*)| > (<)\ \lambda$ *for each* i.

(e) *For* $i \neq j$, *the directed curves* γ_i *and* γ_j *do not cross or coalesce. Further, the total positive (negative) curvature of* Γ *is less than or equal to that of* Γ^*.

When Ω is a bounded domain which is not annular, we obtain the following two results.

THEOREM 3. *Let* Γ^* *be a given finite union of pairwise disjoint,* C^2 *Jordan curves and* $\lambda > 0$. *Suppose* Ω, Γ, u *is a solution of either free boundary problem with the following properties:*

(i) Ω *is a bounded* C^2 *domain.*

(ii) $u \in C^2(\overline{\Omega})$.

Let $(x_1, y_1), \ldots, (x_n, y_n)$ *be a positively ordered set of distinct* $\vec{\nu}$-*minima of* Γ. *Then there is a positively ordered set of distinct (possibly degenerate) line segments* $\sigma_1, \ldots, \sigma_n$ *on* Γ^* *such that for each* i, *each point* (x, y) *in the segment* σ_i *is a* $\vec{\nu}$-*minimum of* Γ^*. *Further, the total positive curvature of* Γ *is less than or equal to that of* Γ^*.

THEOREM 4. *Let* Γ^* *be a given finite union of pairwise disjoint, analytic Jordan curves and* $\lambda > 0$. *Suppose* Ω, Γ, u *is a solution of either free boundary problem with the following properties:*

(i) Ω *is a bounded, analytic domain.*

(ii) u *is real-analytic on* $\overline{\Omega}$.

Let $\vec{\nu}$ *be a unit vector and let* $(x_1, y_1), \ldots, (x_n, y_n)$ *be the distinct* $\vec{\nu}$-*minima of* Γ. *Then each point* (x_i, y_i) *is joined by a simple, piecewise-analytic directed curve* γ_i *to a point* $(x_i^*, y_i^*) \in \Gamma^*$ *such that*

(a) $|\nabla u|$ *is strictly increasing on* γ_i.

(b) $\arg(\nabla u)$ *is constant on* γ_i.

(c) *The points* $(x_1^*, y_1^*), \ldots, (x_n^*, y_n^*)$ *are distinct* $\vec{\nu}$-*minima of* Γ^*.

(d) $|\nabla u(x_i^*, y_i^*)| > \lambda$ *for each* i.

(e) *For* $i \neq j$, *the directed curves* γ_i *and* γ_j *do not cross or coalesce. Further, the total positive curvature of* Γ *is less than or equal to that of* Γ^*.

2. Preliminary results. Let us define $e(x, y) \equiv u_{xx}(x, y)u_{yy}(x, y) - u_{xy}(x, y)^2$, $E = \{(x, y) \in \overline{\Omega} : e(x, y) < 0\}$, and $Z = \{(x, y) \in \overline{\Omega} : |\nabla u(x, y)| > 0\}$. Further, set

$$\phi(x, y) = |\nabla u(x, y)|^2, \qquad (x, y) \in \overline{\Omega}, \tag{3}$$

and

$$\psi(x, y) = \arg(\nabla u(x, y)), \qquad (x, y) \in \overline{\Omega} \setminus Z. \tag{4}$$

Notice that ψ is a multiple-valued function. If $\alpha \in \mathfrak{R}$, $\vec{\eta} = (\sin(\alpha), -\cos(\alpha))$, and $\vec{\eta}^{\perp} = (\cos(\alpha), \sin(\alpha))$, then $\{(x, y) \in \overline{\Omega}\backslash Z : \psi(x, y) = \alpha\} = \{(x, y) \in \overline{\Omega}: \vec{\eta} \cdot \nabla u(x, y) = 0, \vec{\eta}^{\perp} \cdot \nabla u(x, y) > 0\}$.

The following lemma indicates the behavior of ϕ on level sets of ψ when the graph of u is a saddle surface.

LEMMA 1. *Suppose* $u \in C^2(\Omega)$ *satisfies* $u_{xx}u_{yy} - u_{xy}^2 \leq 0$ *on* Ω. *For* $\alpha \in \mathfrak{R}$, *define*

$$S_{\alpha} = \{(x, y) \in \Omega: \psi(x, y) = \alpha\}. \tag{5}$$

Suppose $(x_0, y_0) \in S_{\alpha} \cap E \cap Z$. *Then locally near* (x_0, y_0), *the set* S_{α} *is a simple,* C^1 *curve* σ *which divides its complement into two connected components on which* $\psi - \alpha$ *has opposite signs. Further,* ϕ *is strictly increasing on* σ *if we choose the forward direction such that* $\psi > \alpha$ *locally to the right of* σ *(or* $\psi < \alpha$ *locally to the left of* σ*).*

Proof. Notice that $\nabla\psi = |\nabla u|^{-2}(u_x u_{xy} - u_y u_{xx}, u_x u_{yy} - u_y u_{xy})$ and $\nabla\phi = 2(u_x u_{xx} + u_y u_{xy}, u_x u_{xy} + u_y u_{yy})$. Now $|\nabla\psi(x, y)| = 0$ iff $e(x, y) = 0$ iff $(x, y) \notin E$. Since $(x_0, y_0) \in E$, $|\nabla\psi(x_0, y_0)| \neq 0$. The first part now follows from the implicit function theorem. Let us now orient σ so that $\psi > \alpha$ locally to the right of σ. Let us set $\nabla\psi^{\perp} = (-\psi_y, \psi_x)$. Notice that $\nabla\psi$ is orthogonal to σ and points to the right of σ. Also, $\nabla\psi^{\perp}$ is a (forward) tangent vector to σ. Let us write

$$\nabla\phi(x, y) = \beta_1(x, y)\nabla\psi(x, y) + \beta_2(x, y)\nabla\psi^{\perp}(x, y), \tag{6}$$

where $\beta_2(x, y) = \nabla\phi(x, y) \cdot \nabla\psi^{\perp}(x, y)/|\nabla\psi(x, y)|^2$. We claim that $\beta(x, y) > 0$ for all $(x, y) \in \sigma$. In fact, a direct computation yields

$$\nabla\phi \cdot \nabla\psi^{\perp} = 2(u_{xy}^2 - u_{xx}u_{yy}) > 0$$

on σ and our claim follows. If we parametrize σ by $(x(t), y(t))$, then

$$\nabla\phi \cdot (x', y') = \beta_2(\nabla\psi^{\perp} \cdot (x', y')) > 0 \tag{7}$$

and so $\phi(x(t), y(t))$ is strictly increasing in t. □

COROLLARY. *Suppose* $u \in C^2(\overline{\Omega})$ *satisfies* $u_{xx}u_{yy} - u_{xy}^2 \leq 0$ *on* Ω, (2b), *and* (2c) *and suppose* σ *is an open line segment with* $\sigma \subseteq \partial\Omega$ *such that* $\sigma \cap E$ *is dense in* σ. *Then* ϕ *is strictly decreasing on* σ *and* $\psi \leq \alpha$ *locally to the right of* σ.

Here we orient σ as a subset of $\partial\Omega$; that is, Ω is locally to the left of σ when $\sigma \subseteq \Gamma$ and locally to the right of σ when $\sigma \subset \Gamma^*$.

Proof. On σ, notice that $\nabla\psi$ is orthogonal to σ and $\nabla\psi^{\perp}$ points backwards (or to the left) along σ. The fact that ϕ is strictly decreasing on σ follows from the proof of Lemma 1. Now if $\psi > \alpha$ locally to the right of a point z of σ, then Lemma 1 would imply that ϕ is strictly increasing on σ near z, a contradiction. Thus $\psi \leq \alpha$ in a neighborhood in Ω of σ. □

REMARK. When $u \in C^2(\Omega)$ is a solution of (2a′), we may regard u as a solution of the quasilinear equation (2a) for some Q as in **[12**, p. 444]. If we set

$$F_{ij}(x, y, p, q, r_{11}, r_{12}, r_{21}, r_{22}) = \frac{\partial F}{\partial r_{ij}}(x, y, p, q, r_{11}, r_{12}, r_{21}, r_{22}), \tag{8}$$

for $i, j = 1, 2$ (with $r_{12} = r_{21}$), then we see that u is a solution of (2a) when $a = a^{11}$, $b = a^{12}$, and $c = a^{22}$ and

$$a^{ij}(x, y, p, q) = \int_0^1 F_{ij}(x, y, p, q, \theta D^2u(x, y))\, d\theta. \tag{9}$$

Thus, in the proofs throughout this note we will only consider solutions of (2a) (e.g. **[12**, §17.1]).

LEMMA 2. *Suppose* $u \in C^2(\Omega)$ *satisfies either* (2a) *or* (2a′). *Then* $u_{xx}u_{yy} - u_{xy}^2 \leq 0$ *in* Ω *and* $u_{xx}u_{yy} - u_{xy}^2 = 0$ *at a point if and only if* $u_{xx} = u_{xy} = u_{yy} = 0$ *at the point. Further, if* $D^2u \not\equiv 0$, *then* $E \cap \Omega$ *and* $Z \cap \Omega$ *are open, dense subsets of* Ω *and if* $\eta_1, \eta_2 \in \mathfrak{R}$ *with* $\eta_1^2 + \eta_2^2 > 0$ *and* $\sigma = \{(x, y) \in \Omega: \eta_1 u_x(x, y) + \eta_2 u_y(x, y) = 0\}$, *then* $\sigma \cap E$ *is dense in* σ. *If* $u \in C^2(\overline{\Omega})$ *and* $\sigma = \{(x, y) \in \overline{\Omega}: \eta_1 u_x(x, y) + \eta_2 u_y(x, y) = 0\}$, *then* $\sigma \cap E$ *is also dense in* σ.

Proof. We may assume u satisfies (2a). If we set $r = u_{xx}$, $s = u_{xy}$, and $t = u_{yy}$, we see that $0 = r(ar + 2bs + ct)$ and so $rt - s^2 = -\frac{1}{c}(ar^2 2brs + cs^2)$. Since Q is elliptic, $\alpha\xi_1^2 + 2b\xi_1\xi_2 + c\xi_2^2 > 0$ if and only if $\vec{\xi} = (\xi_1, \xi_2) \neq 0$. Thus $u_{xx}u_{yy} - u_{xy}^2 \leq 0$. The fact that E and Z are dense follows from the strong maximum principle and the density of $\sigma \cap E$ in σ follows from the proof of Lemma 1 of **[13]** after we rotate Ω so that $\vec{\eta}$ becomes $(0, 1)$. □

REMARK. Suppose Ω is a C^2 domain and $u \in C^2(\Omega \cup \Gamma) \cap C^1(\overline{\Omega})$. Then $|\nabla u| \neq 0$ on $\partial\Omega$ and Γ cannot contain any line segments. Notice that $|\nabla u|$ is bounded in $\overline{\Omega}$ and so Q or $\mathfrak{J}$ is uniformly elliptic for u. The first claim follows from the Hopf boundary point lemma (Lemma 3.4, **[12]**) and the second follows from the corollary to Lemma 1.

A. *Annular domains.* Here we will assume the following:

(i) Γ^* is a given C^2 Jordan curve and $\lambda > 0$.

(ii) Ω, Γ, u constitutes a solution of one of the free boundary problems.

(iii) Ω is an annular domain and Γ is a C^2 Jordan curve.

(iv) $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

LEMMA 3. *For each* $(x, y) \in \overline{\Omega}$, $|\nabla u(x, y)| > 0$ *and* $Z = \overline{\Omega}$.

Proof. Let $(x_0, y_0) \in E$ and define $\tilde{u}$ by

$$\tilde{u}(x, y) = z_0 + p_0(x - x_0) + q_0(y - y_0), \tag{10}$$

where $z_0 = u(x_0, y_0) > 0$, $p_0 = u_x(x_0, y_0)$, and $q_0 = u_y(x_0, y_0)$. As in Lemma 2, **[13]**, we see that $\tilde{u} \equiv u$ or there are two curves σ_1, σ_2 which meet at (x_0, y_0) and divide a neighborhood of (x_0, y_0) into four open "sectors" ω_1, ω_2, ω_3, ω_4 such that $u > \tilde{u}$ in $\omega_2 \cup \omega_3$ and $u < \tilde{u}$ in $\omega_2 \cup \omega_4$. If the plane $z = \tilde{u}$ does not intersect $\Sigma_0 \cup \Sigma_1$, where $\Sigma_0 = \Gamma \times \{0\}$ and $\Sigma_1 = \Gamma^* \times \{1\}$, then $\tilde{u} > 0 = u$ on Γ and $\tilde{u} < 1 = u$ on Γ^*. Since Ω is an annular domain, the previous two statements are in contradiction; this follows, for example, from the maximum principle and the Jordan curve theorem (e.g. the proof in §373, **[16]**). Hence, the tangent plane to the graph of u at (x_0, y_0) must intersect $\Sigma_0 \cup \Sigma_1$. This implies that

$$\begin{aligned} &|\nabla u(x_0, y_0)| \\ &\quad \geq \min\{u(x_0, y_0)d(x_0, y_0), (1 - u(x_0, y_0))d^*(x_0, y_0)\} > 0 \end{aligned} \tag{11}$$

for every $(x_0, y_0) \in E$, where $d(x, y) = \inf\{|(x, y) - (s, t)|^{-1} : (s, t) \in \Gamma\}$ and $d^*(x, y) = \inf\{|(x, y) - (s, t)|^{-1} : (s, t) \in \Gamma^*\}$. Since

$u \in C^2(\Omega)$, (4) holds for all $(x_0, y_0) \in \Omega$. Since we already know $|\nabla u| > 0$ on $\partial\Omega$, the lemma follows. $\square$

LEMMA 4. *Let $(x_0, y_0) \in \Gamma$ and let $\vec{\eta} = (\eta_1, \eta_2)$ with $|\eta| = 1$ and $\vec{\eta} \cdot \nabla u(x_0, y_0) \neq 0$. Let ω denote the connected component of $\{(x, y) \in \overline{\Omega}: \vec{\eta} \cdot \nabla u(x, y) \neq 0\}$ which contains (x_0, y_0). Let $\alpha \in \mathfrak{R}$ such that $\vec{\eta} = (\sin(\alpha), -\cos(\alpha))$. Then:*

(a) *ω is relatively open in $\overline{\Omega}$ and $\omega \cap \Gamma^*$ is relatively open in Γ^*.*

(b) *ω is simply connected and $\omega \cap \Gamma$ is connected.*

(c) *$\vec{\eta} \cdot \nabla u = 0$ on $\Omega \cap \partial\omega$ and ψ is constant ($= \alpha$ or $\alpha + \pi$ (mod 2π)) on each component of $\Omega \cap \partial\omega$.*

(d) *$\omega \cap \Gamma^* \neq \varnothing$.*

(e) *$\partial\omega$ is a simple, C^1 curve in a neighborhood of each point of $\partial\omega \cap E$.*

(f) *If a component γ of $\Omega \cap \partial\omega$ is oriented so that $\vec{\eta} \cdot \nabla u < 0$ locally to the right or $\vec{\eta} \cdot \nabla u > 0$ locally to the left, then ϕ is strictly increasing on $\overline{\gamma}$.*

Proof. Notice that (a), (c), and (e) are clear. If γ is a component of $\partial\omega \cap \Omega$, then $\gamma \cap E$ is dense in γ and $\psi = \theta$ with $\theta = \alpha$ or $\theta = \alpha + \pi \bmod 2\pi$. Suppose γ is oriented as in (f). Then $\psi > \theta$ locally to the right or $\psi < \theta$ locally to the left and so ϕ is strictly increasing on $\gamma \cap E$ by Lemma 1. Since $\phi \in C^0(\overline{\Omega})$ and $\gamma \cap E$ is relatively open and dense in γ, (f) follows. Suppose ω is not simply connected. Then there is a component γ of $\partial\omega$ and a bounded component U of $\mathfrak{R} \setminus \omega$ such that $\gamma = \partial U$. If $\gamma \subseteq \Omega$, then the strict monotonicity of ϕ yields a contradiction unless U is a single point, in which case the strong maximum principle implies $\vec{\eta} \cdot \nabla u \neq 0$ in U and so $U \subset \omega$, a contradiction. Thus $\gamma \cap \partial\Omega \neq \varnothing$. For convenience, let us assume $\vec{\eta} \cdot \nabla u > 0$ in ω. The monotonicity of ϕ implies $\gamma \cap \Gamma$ contains no more than one point and U must contain a component Γ_0^* of Γ^*. Let $(x_0, y_0) \in \Gamma$ with $\vec{\eta} \cdot \nabla u(x_0, y_0) < 0$ and let ω_0 be the component of W which contains (x_0, y_0). Then $\omega \cap \omega_0 = \varnothing$ and so $\omega_0 \cap \Gamma_0^* = \varnothing$. If Γ^* contains only one component, this contradicts (d). Suppose next that $\omega \cap \Gamma$ is not connected. Then there is a component γ of $\partial\omega \cap \Omega$ which joins two points of Γ. Since $\vec{\eta} \cdot \nabla u = 0$ on γ, ϕ is strictly monotonic on γ. However, this contradicts the fact that $\phi = \lambda^2$ on Γ; hence (b) holds. Finally suppose $\omega \cap \Gamma^* = \varnothing$. Then $\partial\omega$ is a Jordan curve in $\Omega \cup \Gamma$. Now $\vec{\eta} \cdot \nabla u = 0$ on $\partial\omega \cap \Omega$ and so ϕ is strictly monotonic on $\partial\omega \cap \Omega$. Once again, this contradicts $\phi = \lambda^2$ on Γ and so (d) follows. $\square$

LEMMA 5. *Let ω_1, ω_2 be connected components of $\{(x, y) \in \overline{\Omega}: \vec{\eta} \cdot \nabla u(x, y) \neq 0\}$ for some $\vec{\eta}$ with $|\vec{\eta}| = 1$. If $(\omega_1 \cup \omega_2) \cap \Gamma$ is not connected, then $\omega_1 \cap \omega_2 = \varnothing$.*

Proof. If $\omega_1 \cap \omega_2 \neq \varnothing$, then $\omega_1 = \omega_2$ and so $(\omega_1 \cup \omega_2) \cap \Gamma = \omega_1 \cap \Gamma$ is connected by (b) of Lemma 3. □

B. *General domains.* Here we will assume the following:

(i) Γ^* is a given finite union of disjoint C^2 Jordan curves and $\lambda > 0$.

(ii) Ω, Γ, u constitutes a solution of one of the free boundary problems.

(iii) Ω is a bounded domain and Γ is a finite union of disjoint C^2 Jordan curves.

(iv) $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

In this case, we expect $|\nabla u| = 0$ at some points.

LEMMA 6. *Let $(x_0, y_0) \in \Gamma$ and let $\vec{\eta} = (\eta_1, \eta_2)$ with $|\eta| = 1$ and $\vec{\eta} \cdot \nabla u(x_0, y_0) \neq 0$. Let ω denote the connected component of $\{(x, y) \in \overline{\Omega}: \vec{\eta} \cdot \nabla u(x, y) \neq 0\}$ which contains (x_0, y_0). Let $\alpha \in \Re$ such that $\vec{\eta} = (\sin(\alpha), -\cos(\alpha))$. Then:*

(a) *ω is relatively open in $\overline{\Omega}$ and $\omega \cap \Gamma^*$ is relatively open in Γ^*.*

(b) *$\omega \cap \Gamma$ is connected and if ω is not simply connected and if U is a bounded component of $\Re^2 \setminus \omega$, then U contains a component Γ_0^* of Γ^*.*

(c) *$\overline{\eta} \cdot \nabla u = 0$ on $\Omega \cap \partial\omega$ and ψ is constant ($= \alpha$ or $\alpha + \pi$ (mod 2π)) on each component of $Z \cap \partial\omega$.*

(d) *$\omega \cap \Gamma^* \neq \varnothing$.*

(e) *$\partial\omega$ is a simple, C^1 curve in a neighborhood of each point of $\partial\omega \cap E \cap Z$.*

(f) *If a component γ of $Z \cap \partial\omega$ is oriented so that $\vec{\eta} \cdot \nabla u < 0$ locally to the right or $\vec{\eta} \cdot \nabla u > 0$ locally to the left, then ϕ is strictly increasing on $\overline{\gamma}$.*

The proof is essentially the same as the proof of Lemma 4. Notice that $|\nabla u| \neq 0$ in ω.

LEMMA 7. *Let ω_1, ω_2 be connected components of $\{(x, y) \in \overline{\Omega}: \vec{\eta} \cdot \nabla u(x, y) \neq 0\}$ for some $\vec{\eta}$ with $|\vec{\eta}| = 1$. If $(\omega_1 \cup \omega_2) \cap \Gamma$ is not connected, then $\omega_1 \cap \omega_2 = \varnothing$.*

The proof is the same as the proof of Lemma 5.

C. *The real analytic case.* Here we will assume the following:

(i) Γ^* is a given finite union of disjoint real-analytic Jordan curves and $\lambda > 0$.

(ii) Ω is a bounded domain and Γ is a finite union of disjoint real-analytic Jordan curves.

(iii) u is real-analytic on $\overline{\Omega}$ and either Q is real-analytic on $\mathfrak{R}^2 \times \mathfrak{R}^2$ or $\mathfrak{J}$ is real-analytic on $\mathfrak{R}^2 \times \mathfrak{R}^2 \times \mathfrak{R}^{2\times 2}$.

For each unit vector $\vec{\nu} = v_1\vec{i} + \nu_2\vec{j}$, let $H_{\vec{\nu}}$ denote the set of piecewise analytic, directed curves γ in $\overline{\Omega}$ on which $|\nabla u|$ is strictly increasing and $\nabla u(x, y)$ points in the $\vec{\nu}$-direction at each point (x, y) of γ. Since u is real-analytic on $\overline{\Omega}$, we wish to regard u as a real-analytic solution of (2a) in a neighborhood of $\overline{\Omega}$. This will simplify the statement of certain results (e.g. Lemma 9).

LEMMA 8. *Suppose* $(x_0, y_0) \in \Omega$. *Set*

$$\tilde{u}(x, y) = z_0 + p_0(x - x_0) + q_0(y - y_0) \tag{12}$$

where $z_0 = u(x_0, y_0) \in (0, 1)$, $p_0 = u_x(x_0, y_0)$, *and* $q_0 = u_y(x_0, y_0)$. *Then either* $u \equiv \tilde{u}$ *or*

$$u(x, y) = \tilde{u}(x, y) + H(\overline{x}, \overline{y}) + O(r^{n+1}) \quad \text{as } r \to 0, \tag{13}$$

where $n \geq 2$ *is an integer,* $r = \operatorname{dist}((x, y), (x_0, y_0))$, H *is a harmonic, homogeneous polynomial of degree* n *in* $(\overline{x}, \overline{y})$, *and* $\overline{x}$, $\overline{y}$ *are linearly independent, linear functions of* x, y.

Let $\eta_1, \eta_2 \in \mathfrak{R}$ *with* $\eta_1^2 + \eta_2^2 = 1$. *Suppose* $u \not\equiv \tilde{u}$, $\eta_1 u_x(x_0, y_0) + \eta_2 u_y(x_0, y_0) = 0$, *and* (x_0, y_0) *is a branch point of* u *(i.e.* $u_{xx}(x_0, y_0) = u_{xy}(x_0, y_0) = u_{yy}(x_0, y_0) = 0$). *Then* $n \geq 3$ *and*

$$\eta_1 u_x(x, y) + \eta_2 u_y(x, y) = G(\overline{x}, \overline{y}) + O(r^n) \quad \text{as } r \to 0, \tag{14}$$

where G *is a harmonic, homogeneous polynomial of degree* $n - 1$. *Hence the zeros of* $\eta_1 u_x + \eta_2 u_y$ *in a neighborhood of* (x_0, y_0) *lie on* $n - 1 \geq 2$ *analytic curves which intersect at* (x_0, y_0) *and divide a neighborhood of* (x_0, y_0) *into* $2(n-1)$ *disjoint open "sectors"* $\omega_1, \dots, \omega_{2n-2}$ *such that* $\eta_1 u_x + \eta_2 u_y < 0$ *in* $\omega_1, \dots, \omega_{2n-3}$ *and* $\eta_1 u_x + \eta_2 u_y > 0$ *in* $\omega_2, \dots, \omega_{2n-2}$.

Proof. The proof of the first part is similar to [**19**, p. 380]; also see [**13**]. Let us consider the second part. Suppose $\overline{x} = \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$ and $\overline{y} = \delta_1(x - x_0) + \delta_2(y - y_0)$; here, $\varepsilon_1\delta - \varepsilon_2\delta_1 \neq 0$. Now $\frac{\partial H}{\partial x}(\overline{x}, \overline{y}) = \varepsilon_1 H_{\overline{x}}(\overline{x}, \overline{y}) + \delta_1 H_{\overline{y}}(\overline{x}, \overline{y})$ and $\frac{\partial H}{\partial y}(\overline{x}, \overline{y}) = \varepsilon_2 H_{\overline{x}}(\overline{x}, \overline{y}) +$

$\delta_2 H_{\bar{y}}(\bar{x}, \bar{y})$. Since $\eta_1 p_0 + \eta_2 q_0 = 0$, we obtain (14) when $G(\bar{x}, \bar{y}) = (\eta_1\varepsilon_1 + \eta_2\varepsilon_2)H_{\bar{x}}(\bar{x}, \bar{y}) + (\eta_1\delta_1 + \eta_2\delta_2)H_{\bar{y}}(\bar{x}, \bar{y})$. Since $H_{\bar{x}}$ and $H_{\bar{y}}$ are harmonic, homogeneous polynomials of degree $n-1$, either $G \equiv 0$ or G is as claimed. Since $u \not\equiv \tilde{u}$, $H_{\bar{x}}$ and $H_{\bar{y}}$ are linearly independent; it then follows that $G \not\equiv 0$. The remainder of the proof follows using standard facts about harmonic functions (e.g. **[16]**, §373). □

LEMMA 9. *For $\alpha \in \mathfrak{R}$, define*

$$S_\alpha = \{(x, y) \in Z \cup \partial\Omega : \psi(x, y) = \alpha\}.$$

Then

(a) *If $|\nabla\psi(x_0, y_0)| \neq 0$ at a point $(x_0, y_0) \in S_\alpha$, then locally at (x_0, y_0) the set S_α is an analytic simple curve γ which divides its complement into two connected components on which $\psi - \alpha$ has opposite signs. Further, ϕ is strictly increasing on γ if we choose the forward direction such that $\psi > \alpha$ locally to the right of γ.*

(b) *Suppose $\nabla\psi$ has a zero of order n at $(x_0, y_0) \in S_\alpha$, for some integer $n \geq 1$. Then locally at (x_0, y_0), the set S_α is swept out by $2n+2$ directed, analytic arcs $C_1, C_2, \ldots, C_{2n+2}$ which emanate from (x_0, y_0) and divide its complement into $2n+2$ open sectors*

$$\omega_1, \omega_2, \ldots, \omega_{2n+2}$$

on which $\psi - \alpha$ alternates in sign. We may choose our notation so that $C_i \cup C_{i+1}$ is the local boundary of ω_i (with $C_{2n+3} = C_1$), $(-1)^i(\psi - \alpha)$ is negative in ω_i, and $(-1)^i\phi$ is strictly decreasing on C_i.

Proof. Notice that (a) follows Lemma 1. If we set $\vec{\eta} = (\sin(\alpha), -\cos(\alpha))$ and $\vec{\eta}^\perp = (\cos(\alpha), \sin(\alpha))$, then $S_\alpha = \{(x, y) \in \overline{\Omega} : \vec{\eta} \cdot \nabla u(x, y) = 0,\ \vec{\eta}^\perp \cdot \nabla u(x, y) > 0\}$. Suppose $(x_0, y_0) \in S_\alpha$ such that $|\nabla u(x_0, y_0)| = 0$. Notice that $(x_0, y_0) \notin E$. According to the second part of Lemma 8, we see that near (x_0, y_0) the set $\vec{\eta} \cdot \nabla u = 0$ consists of $n-1 \geq 2$ analytic curves through (x_0, y_0). Since S_α is contained in the relatively open set $\{(x, y) \in \overline{\Omega} : \vec{\eta}^\perp \cdot \nabla u(x, y) > 0\}$, we see that the set $\psi = \alpha$ consists of the same $n-1$ analytic curves and the remainder of the lemma follows from Lemma 1 and Lemma 8. □

LEMMA 10. *Let $\vec{\nu}$ be a unit vector and let Γ have a $\vec{\nu}$-minimum at (x_0, y_0). Then:*

(a) *At least one directed curve $\tilde{\gamma} \in H_{\vec{\nu}}$, which exists locally at (x_0, y_0), emanates into Ω from (x_0, y_0).*

(b) *Let* $\gamma \in H_{\vec{\nu}}$ *be any maximal continuation of* $\tilde{\gamma}$ *in* $H_{\vec{\nu}}$. *Then* γ *remains uniformly bounded away from any zero of* $|\nabla u|$ *and any closed, connected set in* Γ^* *on which* $\vec{n}(x, y) \neq \vec{\nu}$ *and* $\gamma \setminus \tilde{\gamma}$ *remains bounded away from* Γ.

(c) *The curve* γ *must terminate at a point* $(x_0^*, y_0^*) \in \Gamma^*$ *such that*

$$\vec{n}(x_0^*, y_0^*) = \vec{\nu} \tag{15}$$

and

$$|\nabla u(x_0^*, y_0^*)| > \lambda. \tag{16}$$

REMARK. If Ω is an annular domain, then Lemma 3 implies $|\nabla u| > 0$ on $\overline{\Omega}$ and the conclusions of Lemma 10 hold when Γ has a $\vec{\nu}$-maximum at (x_0, y_0) with the modifications that $\tilde{\gamma} \in H_{\vec{\nu}}$ terminates at (x_0, y_0), γ begins at (x_0^*, y_0^*), and $|\nabla u(x_0^*, y_0^*)| < \lambda$.

The proof of Lemma 10 follows from Lemma 9 as in the proof of Lemma 2, [**4**].

3. Proof of main results.

Proof of Theorem 1. Suppose first that $(x_0, y_0) \in \Gamma$ and $\vec{n}(x_0, y_0) = \vec{\nu} = (\nu_1, \nu_2)$. Set $\vec{\eta} = -\vec{\nu}^{\perp} = (\nu_2, -\nu_1)$ and $W = \{(x, y) \in \Omega : \vec{\eta} \cdot \nabla u(x, y) \neq 0\}$. Since Γ contains no line segments and $\nabla u \in C^0(\overline{\Omega})$, $(x_0, y_0) \in \partial W$. Let ω be a component of W such that $(x_0, y_0) \in \partial\omega$ and let γ be a component of $\partial\omega \cap \Omega$ with $(x_0, y_0) \in \overline{\gamma}$. From Lemma 4, we see that $\overline{\gamma} \cap \Gamma^* \neq \varnothing$ and $\vec{\eta} \cdot \nabla u = 0$ on $\overline{\gamma}$. Also $\vec{\eta}^{\perp} \cdot \nabla u > 0$ on $\overline{\gamma}$ since $\vec{\eta}^{\perp} \cdot \nabla u(x_0, y_0) = \vec{\nu} \cdot \nabla u(x_0, y_0) > 0$ and $|\nabla u| \neq 0$ on $\overline{\Omega}$. Thus if $(x, y) \in \overline{\gamma} \cap \Gamma^*$, $\vec{n}(x, y) = \vec{\nu}$. If σ_0 is the maximal line segment on Γ^* which contains $\overline{\gamma} \cap \Gamma^*$, then $\vec{n}(x, y) = \vec{\nu}$ for all $(x, y) \in \sigma_0$.

Suppose next that $(x_0, y_0) \in \Gamma$ with $\vec{n}(x_0, y_0) = \vec{\nu}$ such that (x_0, y_0) is a $\vec{\nu}$-minimum of Γ. Let us use the notation of the previous paragraph. Then $\vec{\nu} \cdot (x, y) > \vec{\nu} \cdot (x_0, y_0)$ for $(x, y) \in \Gamma \setminus \{(x_0, y_0)\}$ near (x_0, y_0). Since Γ is the 0-level curve of u, this implies $\vec{\eta} \cdot \nabla u > 0$ locally before (x_0, y_0) on Γ and $\vec{\eta} \cdot \nabla u < 0$ locally after (x_0, y_0) on Γ. Now let $\omega_{\pm}$ be the components of W such that (x_0, y_0) is the left (or initial) endpoint of $\overline{\omega_+} \cap \Gamma$ and is the right (or terminal) endpoint of $\overline{\omega_-} \cap \Gamma$. Then $\vec{\eta} \cdot \nabla u < 0$ in ω_+ and $\vec{\eta} \cdot \nabla u > 0$ in ω_-. Let γ_+ be the component of $\partial\omega_+ \cap \Omega$ whose closure contains (x_0, y_0) and let γ_- be the component of $\partial\omega_- \cap \Omega$ whose closure contains (x_0, y_0). As above, $\overline{\gamma_{\pm}} \cap \Gamma^* \neq \varnothing$ and if $(x, y) \in \overline{\gamma_{\pm}} \cap \Gamma^*$, then $\vec{n}(x, y) = \vec{\nu}$. Notice that $\overline{\gamma_-} \cap \Gamma^*$ lies to the left (or before)

$\overline{\gamma_\pm} \cap \Gamma^*$. If $\sigma_\pm$ are the (possibly degenerate) closed line segments in Γ^* such that $\overline{\gamma_+} \cap \Gamma^* \subset \sigma_+$ and $\overline{\gamma_-} \cap \Gamma^* \subset \sigma_-$ and if z_-, $z_+ \in \Gamma^*$ are the left endpoint of σ_- and the right endpoint of σ_+ respectively, then $\vec{n} \cdot \nabla u > 0$ and $\psi < \alpha$ locally to the left of z_- on Γ^* and $\vec{\eta} \cdot \nabla u < 0$ and $\psi > \alpha$ locally to the right of z_+ on Γ^*, where $\alpha \in \mathfrak{R}$ with $(\cos(\alpha), \sin(\alpha)) = \vec{\nu}$. The fact that Γ^* has a $\vec{\nu}$-minimum between z_- and z_+ follows using an argument similar to that of **[1]** (in particular, see Lemma 1 and the proof of Theorem 1 in **[1]**). The case where (x_0, y_0) is a $\vec{\nu}$-maximum of Γ follows similarly.

Now suppose $(x_0, y_0) \in \Gamma$ with $\vec{n}(x_0, y_0) = \vec{\nu}$ and (x_0, y_0) is a positive inflection point of Γ. Then if $\vec{\eta} = -\vec{\nu}^\perp$ as above, $\vec{\nu} \cdot \nabla u < 0$ locally before and locally after (x_0, y_0) on Γ. Let $W = \{(x, y) \in \Omega\colon \vec{\eta} \cdot \nabla u(x, y) < 0\}$ and let $\omega_\pm$ be the components of W and $\gamma_\pm$ be the components of $\partial\omega_\pm \cap \Omega$ as in the previous paragraph. Then $\vec{\eta} \cdot \nabla u = 0$ on $\gamma_\pm$ and ϕ is strictly increasing (decreasing) on $\overline{\gamma_+}$ ($\overline{\gamma_-}$) as $(x, y) \in \overline{\gamma_\pm}$ leaves (x_0, y_0). This implies $\overline{\gamma_-} \cap \overline{\gamma_+} = \{(x_0, y_0)\}$. Let z_-, $z_+ \in \Gamma^*$ be the leftmost point of $\overline{\gamma_-} \cap \Gamma^*$ and the rightmost point of $\overline{\gamma_+} \cap \Gamma^*$ respectively. Let V be the open subset of Ω bounded by $\gamma_- \cup \gamma_+$ and that portion of Γ^* between z_- and z_+. Let us suppose that $\vec{\eta} \cdot \nabla u \leq 0$ in V; this will lead to a contradiction. Let $z \in \partial V \cap E \subset \gamma_- \cup \gamma_+$ and pick $\varepsilon > 0$ so that $\vec{\eta} \cdot \nabla u \leq 0$ in $B(z, \varepsilon) \subset \Omega$. Now $\partial B(z, \varepsilon) \cap (\omega_- \cup \omega_+) \neq \varnothing$ and so $\vec{\eta} \cdot \nabla u < 0$ on a portion of the boundary of $B(z, \varepsilon)$. If we rotate Ω so that $\vec{\eta} = (0, 1)$ and so $\vec{\eta} \cdot \nabla u = u_y$ and if we notice that u_y is the generalized solution of a linear, elliptic, homogeneous equation (e.g. **[12]**, §13.2), we see that the strong maximum principle (e.g. **[12]**, Theorem 8.19) implies $\vec{\eta} \cdot \nabla u < 0$ in $B(z, \varepsilon)$. However, $\vec{\eta} \cdot \nabla u = 0$ on $\gamma_- \cup \gamma_+$ and $z \in \gamma_- \cup \gamma_+$. This contradiction implies $\vec{\eta} \cdot \nabla u > 0$ for some points in V. Now let D be a component of $\{(x, y) \in \Omega\colon \vec{\eta} \cdot \nabla u(x, y) > 0\}$ such that $D \subset V$. Since ϕ is strictly monotonic on $\partial D \cap \Omega$, we see that $\overline{D} \cap \Gamma^* \neq \varnothing$ and $\overline{D} \cap \Gamma^*$ must contain a point z_0 at which $\vec{\eta} \cdot \nabla u > 0$. Since $\vec{\eta} \cdot \nabla u < 0$ locally to the left of (or before) z_- and locally to the right of (or after) z_+ and z_0 lies between z_- and z_+, Γ^* must have a positive inflection point between z_- and z_+.

Notice that the final conclusion of the theorem concerning the total positive and negative curvature follows from (a) in a manner similar to the proof of Theorem 2 of **[4]** for the positive curvature case. To complete the proof, we need only observe that if $\vec{n}(x_1, y_1) = \vec{n}(x_2, y_2) = \vec{\nu}$ for (x_1, y_1), $(x_2, y_2) \in \Gamma$ with $(x_1, y_1) \neq (x_2, y_2)$ and if γ_i is γ, γ_-, or γ_+ as above with $(x_0, y_0) = (x_i, y_i)$, then

$\overline{\gamma_1} \cap \overline{\gamma_2} = \varnothing$. To see this, let $(x^0, y^0) \in \Gamma$ lie between (x_1, y_1) and (x_2, y_2) such that $\vec{n}(x^0, y^0) = \vec{e} \neq \vec{\nu}$. Set $\vec{\delta} = -\vec{e}^{\perp}$ and $W^0 = \{(x, y) \in \Omega: \vec{\delta} \cdot \nabla u(x, y) \neq 0\}$. Now let ω^0 be a component of W^0 with $(x^0, y^0) \in \overline{\omega^0}$ and let γ^0 be a component of $\partial\omega^0 \cap \Omega$ with $(x^0, y^0) \in \overline{\gamma^0}$. Then γ^0 is a curve from (x^0, y^0) with $\overline{\gamma^0} \cap \Gamma^* \neq \varnothing$ and $\vec{\delta} \cdot \nabla u = 0$ on γ^0. Since $|\nabla u| \neq 0$ in $\overline{\Omega}$, $\overline{\gamma^0}$ (strictly) separates $\overline{\gamma_1}$ and $\overline{\gamma_2}$. □

Proof of Theorem 2. When the (x_i, y_i) are $\vec{\nu}$-minima of Γ, the proof of the existence of curves $\gamma_i \in H_{\vec{\nu}}$ and $\vec{\nu}$-minima $(x_i^*, y_i^*) \in \Gamma^*$ as indicated in the theorem is essentially the same as the proof of Theorem 5 of **[4]** with Lemmas 9 and 10 taking the places of Lemmas 1 and 2 of **[4]**. Since $|\nabla u| > 0$ in $\overline{\Omega}$ (Lemma 3), we may modify the proof in **[4]** when the (x_i, y_i) are $\vec{\nu}$-maxima of Γ. In fact, if we modify the rules in **[4]** for continuation of curves $\psi = \alpha$ so as to keep $\psi < \alpha$ locally to the right except at negative inflection points of Γ^*, where we require $\psi > \alpha$ locally to the left, then the proof of the existence of curves γ_i on which ϕ is decreasing and which begin at (x_i, y_i) and end at $\vec{\nu}$-maxima (x_i^*, y_i^*) of Γ^* as in the theorem is similar to the proof in the $\vec{\nu}$-minima case. The last conclusion of Theorem 2 follows from Theorem 1. □

Proof of Theorem 3. In **[4]**, continuation rules for curves of constant gradient direction were developed using the analyticity of u and Γ^* and these curves, which began at $\vec{\nu}$-minima of Γ, were shown to terminate at $\vec{\nu}$-minima of Γ^*. Since we do not know the behavior of u in a neighborhood of a point z of Γ^* at which $e(z) = 0$, a considerable portion of this proof will involve technical details required to allow us to continue curves of constant gradient direction which have reached Γ^*.

Suppose $(x_0, y_0) \in \Gamma$ with $\vec{n}(x_0, y_0) = \vec{\nu}$ and (x_0, y_0) is a $\vec{\nu}$-minimum of Γ and set $\vec{\eta} = -\vec{\nu}^{\perp}$. Let $W_+ = \{(x, y) \in \Omega: \vec{\eta} \cdot \nabla u(x, y) < 0\}$ and $W_- = \{(x, y) \in \Omega: \vec{\eta} \cdot \nabla u(x, y) > 0\}$. Notice that on Γ, $\vec{\eta} \cdot \nabla u < 0$ locally to the right (or after) and $\vec{\eta} \cdot \nabla u > 0$ locally to the left (or before) (x_0, y_0); hence $(x_0, y_0) \in \overline{W_+} \cap \overline{W_-}$. Let ω_1 be a component of W_+ whose closure contains (x_0, y_0) and let γ_1 be the component of $\partial\omega_1 \cap \Omega$ whose closure contains (x_0, y_0). Let us orient γ_1 so that ω_1 lies to the right of γ_1; that is, so that (x_0, y_0) is the initial point of γ_1. By Lemma 6, ϕ is strictly increasing on γ_1, $\overline{\gamma_1} \cap \Gamma^* \neq \varnothing$, and $\vec{\eta} \cdot \nabla u = 0$ on γ_1. Notice that $\gamma_1 \subset \partial W_-$, since otherwise we obtain a contradiction of the strong maximum principle (e.g. the third paragraph of the proof of Theorem 1).

Let $(x_1, y_1) \in \overline{\gamma_1} \cap \Gamma^*$ and notice that $\vec{n}(x_1, y_1) = \vec{\nu}$. If (x_1, y_1) is a $\vec{\nu}$-minimum of Γ^* and $\overline{\omega_1}$ contains a portion of Γ^* to the right of (x_1, y_1), we stop. Otherwise we continue γ_1 beyond (x_1, y_1). If (x_1, y_1) is not a positive inflection point of Γ^*, then either $\vec{\eta} \cdot \nabla u > 0$ locally to the right of (x_1, y_1) on Γ^* or there is a line segment $\sigma \subset \Gamma^*$ with $(x_1, y_1) \in \sigma$ and $\vec{\eta} \cdot \nabla u > 0$ locally to the right of σ on Γ^*. If (x_1, y_1) is a positive inflection point of Γ^*, then $\vec{\eta} \cdot \nabla u < 0$ locally to the left of (x_1, y_1) or locally to the left of a line segment $\sigma \subset \Gamma^*$ with $(x_1, y_1) \in \sigma$. We will describe a rule for obtaining a set γ with $\gamma_1 \subset \gamma$ such that γ begins at (x_0, y_0) and ends at a $\vec{\nu}$-minimum of Γ^*.

Suppose (x_1, y_1) is not a positive inflection point of Γ^*. Let σ be the (possibly degenerate) maximal closed line segment such that $(x_1, y_1) \in \sigma \subset \Gamma^*$ and let z_1, z_2 be the initial and terminal endpoints of σ respectively (with $z_1 = z_2$ if σ is degenerate). By the corollary to Lemma 1, we see that $\overline{\omega_1} \cap \sigma \subseteq \{z_1, z_2\}$ and if $z_1 \neq z_2$, then $\phi(z_1) > \phi(z_2)$. If $\overline{\omega} \cap \sigma = \{z_1, z_2\}$, $z_1 \neq z_2$, and ξ is the component of W_- whose closure contains σ, then $\gamma = \partial\xi \cap \Omega$ satisfies $\overline{\gamma} \cap \partial\Omega = \{z_1, z_2\}$ and $\vec{\eta} \cdot \nabla u = 0$ on γ; since ξ lies to the left of γ as $(x, y) \in \gamma$ moves from z_1 to z_2, $\phi(z_1) < \phi(z_2)$, which is a contradiction. Thus $\overline{\omega_1} \cap \sigma = \{z_0\}$, where z_0 is z_1 or z_2, and $\overline{\omega_1}$ does not intersect the portion of Γ^* locally to the right of z_0. Let ω_ε be the component of $\omega_1 \cap B(z_0, \varepsilon)$ which satisfies $\partial\omega_\varepsilon \cap \gamma_1 \cap \Omega \neq \varnothing$, for each $\varepsilon > 0$. Let γ_2 be the component of $\partial\omega \cap \Omega$ which is disjoint from γ_1 and satisfies $\gamma_2 \cap \partial\omega_\varepsilon \neq \varnothing$ for each $\varepsilon > 0$. Notice that γ_2 is the component of $\partial\omega_1 \cap \Omega$ "immediately to the right" of γ_1. From Lemma 6, we see that $\phi > \lambda^2$ on $\overline{\gamma_2}$ and so $\overline{\gamma_2} \cap \Gamma = \varnothing$. Let us orient γ_2 so that ω_1 is to the right of γ_2. Then as $(x, y) \in \gamma_2$ leaves z_0, $\phi(x, y)$ increases (strictly). Notice that there is no open set $\tilde{\omega} \subset \Omega$ with $\partial\tilde{\omega} \subset \overline{\gamma_2}$ (i.e. γ_2 cannot cross itself) because of the strict monotonicity of ϕ. Therefore $\Gamma^* \cap \overline{\gamma_2} \setminus \sigma \neq \varnothing$.

Suppose (x_1, y_1) is a positive inflection point of Γ^*. Let σ be the (possibly degenerate) maximal closed line segment satisfying $(x_1, y_1) \in \sigma \subset \Gamma^*$ and let z_1 and z_2 be the initial and terminal endpoints of σ respectively. As above, $\overline{\omega_1} \cap \sigma = \{z_0\}$, where $z_0 = (x_1, y_1)$ is either z_1 or z_2, and $\phi(z_1) > \phi(z_2)$ if $z_1 \neq z_2$. Since $\vec{\eta} \cdot \nabla u < 0$ locally to the left of z_1, $\overline{\omega_1}$ does not intersect the portion of Γ^* locally to the left of z_1.

Suppose $z_1 \neq z_2$ and let ω_0 be the component of W_- whose closure contains σ. Let λ_1 and λ_2 be the components of $\partial\omega_0 \cap \Omega$ with $z_2 \in \overline{\lambda_1}$ and $z_1 \in \overline{\lambda_2}$ respectively such that $\partial\omega_\varepsilon \cap \partial\omega_0 \subseteq \lambda_1 \cup \sigma \cup \lambda_2$ for all sufficiently small $\varepsilon > 0$, where ω_ε is the component of $\{(x, y) \in \omega_0 \colon \operatorname{dist}((x, y), \sigma) < \varepsilon\}$ whose closure contains σ. Suppose also $z_0 = z_2$ and $\gamma_1 \cap \gamma_1 \neq \varnothing$. Then $\gamma_1 \cap B(z_2, \varepsilon) \subseteq \lambda_1$ for some $\varepsilon > 0$. Let us define $\gamma_2 = \sigma \cup \lambda_2$ and orient γ_2 beginning at z_2 and going toward (and beyond) z_1.

Suppose now $z_0 = z_1$ or $z_0 \neq z_1$ and $\gamma_1 \cap \lambda_1 = \varnothing$. Let $y^0 \in \gamma_1 \cap E$; by Lemma 1, $\vec{\eta} \cdot \nabla u < 0$ to the right of γ_1 near y^0 and $\vec{\eta} \cdot \nabla u > 0$ to left of γ_1 near y^0. Let (y^n) be a collection of points after y^0 on γ_1, given in increasing order on γ_1, at which the condition $\vec{\eta} \cdot \nabla u > 0$ to the left of γ_1 near y^n is not satisfied. By Lemma 2, for each n, there exists $z^n \in \gamma_1 \cap E$ between y^{n-1} and y^n and so $\vec{\eta} \cdot \nabla u > 0$ to the left of γ_1 near z^n. Then there is a component w^n of W_+ with $y^n \in \partial w^n$ such that w^n lies to the left of γ_1 near y^n for each n. Let λ^n be a component of $\partial w^n \setminus (\gamma_1 \cup \partial\Omega)$, oriented to begin at y^n, such that w^n lies to the right of λ^n and so ϕ is strictly increasing on λ^n. Now $\overline{\lambda^n} \cap \Gamma = \varnothing$ and so $\overline{\lambda^n}$ must interest Γ^*. Suppose $\overline{\lambda^m}$ and $\overline{\lambda^n}$ either intersect one another or intersect the same component of $\Gamma^*_{\vec{\eta}} = \{(x, y) \in \Gamma^* \colon \vec{\eta} \cdot \nabla(x, y) = 0\}$. Then the component U_{mn} of $\Omega \setminus (\lambda^m \cup \lambda^n \cup \gamma_1)$ whose closure contains $\lambda^m \cup \lambda^n$ must enclose a component of Γ^* because otherwise the strong maximum principle yields a contradiction. Since Γ^* has only a finite number of components, only a finite number of pairs (m, n) can satisfy the last supposition. Since $\Gamma^*_{\vec{\nu}}$ has a finite number of components, there can only be a finite number of points (y^n) as supposed and hence $\vec{\nu} \cdot \nabla u > 0$ locally to the left of γ_1 in a neighborhood of $\gamma_1 \cap B(z_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. Let ω_2 be the component of W_- whose closure contains $\overline{\gamma_1} \cap B(z_0, \varepsilon_0)$. Notice that if $z_0 = z_2 \neq z_1$, then $\omega_2 \cap \omega_0 = \varnothing$ (at least in a neighborhood of σ). Now let γ_2 be the component of $\partial\omega_2 \cap \Omega$ whose closure contains z_0 and which lies immediately to the left of γ_1 on $\partial\omega_2$. Let us orient γ_2 to begin at z_0.

Let us review our procedure. If (x_1, y_1) is a $\vec{\nu}$-minimum of Γ^* and the component of $\partial\omega_1 \cap \Omega$ immediately to the right of γ_1 is a portion of Γ^*, we terminate our procedure. Otherwise, we see that there exists a set γ_2 beginning at (x_1, y_1) on which $\vec{\eta} \cdot \nabla u = 0$ and ϕ is strictly increasing. In particular, γ_2 is the component of $\partial\omega_1 \cap \Omega$ immediately to the right of γ_1 if (x_1, y_1) is not a positive inflection point of Γ^* and $\gamma_2 \setminus \sigma$ is the component of $\partial\omega_2 \cap \Omega$ immediately to the left of γ_1 if (x_1, y_1) is a positive inflection point of Γ^*, where ω_2 is the component of W_- immediately to the left of γ_1 near (x_1, y_1). We may continue this process to obtain points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ and a set $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ such that ϕ is strictly increasing on γ and $\vec{\eta} \cdot \nabla u = 0$ on γ. Due to the monotonicity of ϕ and the maximum principle, γ cannot cross itself and (x_i, y_i) and (x_j, y_j) can be elements of the same component σ of $\Gamma^*_{\vec{\nu}}$ only if σ is a nondegenerate line segment, (x_i, y_i) is the terminal endpoint of σ, (x_j, y_j) is the initial endpoint of σ, and $\gamma_{i+1} \cup \cdots \cup \gamma_j$ surrounds a component of $\partial\Omega$. Since $\Gamma^*_{\vec{\nu}}$ and $\partial\Omega$ each have a finite number, say M and N, of components, this process must terminate after a finite number $(\leq M + N)$ of steps. On the other hand, the process cannot terminate except at a $\vec{\nu}$-minimum of Γ^*, since we can continue γ past (x_n, y_n) if (x_n, y_n) is not a $\vec{\nu}$-minimum of Γ^*. Hence γ terminates at a $\vec{\nu}$-minimum of Γ^*. From the rules used for continuation, we see that two curves γ and $\tilde{\gamma}$ cannot terminate at the same point unless $\gamma_i = \tilde{\gamma}_j$ for some i and j. However, this cannot occur unless γ and $\tilde{\gamma}$ begin at the same point. The fact that the total positive curvature of Γ is bounded by the total positive curvature of Γ^* follows as in the proof of Theorem 2 of **[4]**. □

The proof of Theorem 4 follows from the proof of Theorem 3 above or from the proof of Theorem 5 of **[4]**.

REMARK. In **[11]**, Friedman and Vogel examined two-dimensional ideal fluid flows with a cavity behind an obstacle in an infinite channel with an oscillatory wall. One aspect of their work involved determining some geometric aspects of the free boundary and used curves of constant gradient direction. Using the ideas of this paper, especially Lemma 1, geometric properties of two-dimensional, compressible, irrotational, inviscid, subsonic cavitation flows (e.g. [**18**, p. 109]) past an obstacle in an infinite, oscillatory channel could be determined. When Ω is assumed bounded, the results of this section apply directly to our free boundary problems when, for example, (2a) or (2a′) is the minimal surface equation, the p-Laplace equation, or Pucci's equation ([**12**]). When Ω is not assumed bounded, we can still obtain qualitative information about the free boundary in the same manner as for compressible flows.

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