

*Pacific
Journal of
Mathematics*

**INVARIANT SUBSPACES AND HARMONIC CONJUGATION ON
COMPACT ABELIAN GROUPS**

NAKHLE HABIB ASMAR, EARL ROBERT BERKSON
AND THOMAS ALASTAIR GILLESPIE

INVARIANT SUBSPACES AND HARMONIC CONJUGATION ON COMPACT ABELIAN GROUPS

NAKHLÉ ASMAR, EARL BERKSON, AND T. A. GILLESPIE

Let Γ be a dense subgroup of the real line \mathbb{R} . Endow Γ with the discrete topology and the order it inherits from \mathbb{R} , and let K be the dual group of Γ . Helson's classic theory of generalized analyticity uses the spectral decomposability of unitary groups to establish a one-to-one correspondence between the cocycles on K and the normalized simply invariant subspaces of $L^2(K)$. This theory has been extended to the invariant subspaces of $L^p(K)$, $1 < p < \infty$, by using recent results concerning the spectral decomposability of uniformly bounded one-parameter groups acting on UMD spaces. We show here that each cocycle A on K can be used to transfer the classical Hilbert transform from $L^1(\mathbb{R})$ to $L^1(K)$ in terms of almost everywhere convergence on K so that in the interesting case (i.e., when A is not a coboundary) the corresponding invariant subspace of $L^p(K)$ is a generalized ergodic Hardy space. This description of the invariant subspaces explicitly identifies the role of the Hilbert transform in generalized analyticity on K . The formulation in terms of almost everywhere convergence on K provides an intrinsic viewpoint which extends to the case $p = 1$.

1. Cocycles and invariant subspaces. Throughout what follows K will be a compact abelian group other than $\{0\}$ or the unit circle \mathbb{T} such that the dual group of K is archimedean ordered. Equivalently, we shall require K to be the dual group of Γ , where Γ arises as a dense subgroup of the additive real line \mathbb{R} , and Γ is then endowed with the natural order of \mathbb{R} and the discrete topology. For each $\lambda \in \Gamma$ we denote by χ_λ the corresponding character on K (evaluation at λ), and for each $t \in \mathbb{R}$ we let e_t be the element of K defined by $e_t(\lambda) = e^{it\lambda}$ for all $\lambda \in \Gamma$. As is well known, $t \rightarrow e_t$ is a continuous isomorphism of \mathbb{R} onto a dense subgroup of K . For $1 \leq p < \infty$ we follow Helson [12] in defining a *simply invariant subspace* of $L^p(K)$ to be a closed subspace M of $L^p(K)$ such that $\chi_\lambda M \subseteq M$ for all $\lambda > 0$, but for some $\alpha < 0$, $\chi_\alpha M$ is not a subset of M . A simply invariant subspace M of $L^p(K)$ is said to be *normalized* provided $M = \bigcap \{\chi_\lambda M : \lambda \in \Gamma, \lambda < 0\}$. The set of all normalized simply invariant subspaces of $L^p(K)$ will be denoted by \mathcal{S}_p . A *cocycle* on

K is a Borel measurable function $A: \mathbb{R} \times K \rightarrow \mathbb{T}$ such that

$$A(t + u, x) = A(t, x)A(u, x + e_t), \quad \text{for } t \in \mathbb{R}, u \in \mathbb{R}, x \in K.$$

After identifying cocycles which are equal almost everywhere (with respect to the Haar measure of $\mathbb{R} \times K$), we denote by \mathcal{C} the collection of all cocycles on K . It is convenient to note that for $A \in \mathcal{C}$, [10, Lemma VII.12.1] shows that $A(t, \cdot)$ moves continuously in $L^p(K)$, $1 \leq p < \infty$, as t runs through \mathbb{R} . A cocycle having the form $A(t, x) = \phi(x)/\phi(x + e_t)$ for some Borel measurable function $\phi: K \rightarrow \mathbb{T}$ is called a *coboundary*. As will be observed in Corollary (3.4) below, the coboundaries determine the Beurling-type elements of \mathcal{S}_p .

For $1 \leq p < \infty$, and $A \in \mathcal{C}$, we define the strongly continuous one-parameter group $\{U_t^{(A,p)}\}$, $t \in \mathbb{R}$, of isometries of $L^p(K)$ by setting

$$(1.1) \quad (U_t^{(A,p)} f)(x) = A(t, x)f(x + e_t),$$

$$\text{for } t \in \mathbb{R}, f \in L^p(K), x \in K.$$

As will be described in detail in §2 below, if $1 < p < \infty$, then the one-parameter group $\{U_t^{(A,p)}\}$ is decomposable by a one-parameter “spectral family” $\mathcal{E}^{(A,p)}$ of projections acting in $L^p(K)$ [4]. With $\mathcal{Q}^{(A,p)}(0)$ denoting the strong limit as $t \rightarrow 0^-$ of $(I - \mathcal{E}^{(A,p)}(t))$, it was shown in [5, Theorem (3.3)] that the mapping

$$\Psi_p: A \rightarrow (\mathcal{Q}^{(A,p)}(0))L^p(K)$$

is a bijection of \mathcal{C} onto \mathcal{S}_p . This result extends Helson’s classic development of Ψ_2 in [11, §3], and can be used ([5, Theorem (5.7)]) to generalize Helson’s characterization of $\Psi_2(A)$ in terms of analyticity in the upper half-plane ([11, §6]). However, the foregoing characterization of \mathcal{S}_p by spectral decomposability can no longer be used when $p = 1$.

The purpose of the present note is to develop, in terms more intrinsic to pointwise operations on K , a unified “Hilbert-transform” approach to the generalized analyticity and cocycle description of \mathcal{S}_p , $1 \leq p < \infty$. More specifically, given $A \in \mathcal{C}$, we describe (in §2) a method for using the one-parameter group $\{U_t^{(A,1)}\}$ to transfer the Hilbert transform from $L^1(\mathbb{R})$ to an operator $\mathcal{H}^{(A)}$ on $L^1(K)$. In fact, denoting (now and henceforth) the normalized Haar measure of K by σ , we obtain $\mathcal{H}^{(A)}f$, for each $f \in L^1(K)$, as the σ -a.e. limit of the transferred truncated Hilbert transforms applied to f . For

$1 < p < \infty$, the restriction $\mathcal{H}^{(A,p)}$ of $\mathcal{H}^{(A)}$ to $L^p(K)$ becomes a bounded linear mapping of $L^p(K)$ into itself, and when $1 < p < \infty$ we use the above-described spectral decomposability considerations to show in Theorem (3.9) that if $\Psi_p(A)$ is not a Beurling-type subspace, then

$$\Psi_p(A) = \{f \in L^p(K) : f = i\mathcal{H}^{(A,p)}f \text{ } \sigma\text{-a.e. on } K\}.$$

Thus for $1 < p < \infty$, the elements of \mathcal{S}_p of interest can be thought of as generalized Hardy spaces. In terms of a suitably defined bijection Ψ_1 of \mathcal{E} onto \mathcal{S}_1 , this result extends to the case $p = 1$.

2. Background and preliminaries. In this section we blend some items from [4], [5], and [3] into a framework suitable for the formulation and treatment of generalized analyticity on K outlined in §1. The relevant facts concerning spectral decomposability will be phrased in terms of the following familiar notion.

DEFINITION. Let $\mathfrak{B}(X)$ be the Banach algebra of all bounded linear mappings of a Banach space X into itself, and denote the identity of $\mathfrak{B}(X)$ by I . A *spectral family of projections in X* is a projection-valued function $E: \mathbb{R} \rightarrow \mathfrak{B}(X)$ such that:

- (i) $\sup\{\|E(t)\| : t \in \mathbb{R}\} < \infty$;
- (ii) $E(s)E(t) = E(t)E(s) = E(s)$, for $-\infty < s \leq t < +\infty$;
- (iii) E is right-continuous on \mathbb{R} in the strong operator topology of $\mathfrak{B}(X)$, and for each $t \in \mathbb{R}$, E has a strong left-hand limit $E(t^-)$ at t ;
- (iv) as $t \rightarrow +\infty$ (respectively, $t \rightarrow -\infty$), $E(t) \rightarrow I$ (respectively, $E(t) \rightarrow 0$) in the strong operator topology of $\mathfrak{B}(X)$.

We shall require some aspects of the integration theory of a spectral family E in X [9, Chapter 17]. For a compact interval $\Delta = [a, b]$ of \mathbb{R} , let $\mathbf{AC}(\Delta)$ be the Banach algebra of all complex-valued, absolutely continuous functions on Δ under the norm $\|\cdot\|_\Delta$ defined by

$$\|f\|_\Delta = |f(b)| + \text{var}(f, \Delta),$$

where “var” denotes total variation. For each $f \in \mathbf{AC}(\Delta)$, the integral $\int_\Delta f(t) dE(t)$ exists as a strong limit of Riemann-Stieltjes sums, and we define $\int_\Delta^\oplus f(t) dE(t)$ by putting

$$\int_\Delta^\oplus f(t) dE(t) = f(a)E(a) + \int_\Delta f(t) dE(t).$$

The mapping $f \rightarrow \int_{\Delta}^{\oplus} f(t) dE(t)$ is an algebra homomorphism of $\mathbf{AC}(\Delta)$ into $\mathfrak{B}(X)$ such that

$$\left\| \int_{\Delta}^{\oplus} f(t) dE(t) \right\| \leq \|f\|_{\Delta} \sup\{\|E(t)\| : t \in \mathbb{R}\}.$$

The Banach spaces X possessing the unconditionality property for martingale differences (written $X \in \text{UMD}$) have been characterized in [7] and [8] as those Banach spaces for which the Hilbert kernel of \mathbb{R} defines a bounded convolution operator on $L^p(\mathbb{R}, X)$ for some, and hence all, p in the range $1 < p < \infty$. When $X \in \text{UMD}$, spectral families of projections in X occur naturally in accordance with the following theorem.

(2.1) **STONE’S THEOREM FOR UMD SPACES** ([4, §5]). *Let $\{V_t\}$, $t \in \mathbb{R}$, be a strongly continuous one-parameter group of operators on a UMD space X such that $\sup\{\|V_t\| : t \in \mathbb{R}\} < \infty$. Then:*

(i) *there is a unique spectral family \mathcal{E} in X (called the Stone-type spectral family of $\{V_t\}$) such that*

$$V_t x = \lim_{u \rightarrow +\infty} \int_{-u}^u e^{its} d\mathcal{E}(s)x, \quad \text{for } t \in \mathbb{R}, \quad x \in X;$$

(ii) *for each $s \in \mathbb{R}$,*

$$(\pi i)^{-1} \int_{\delta \leq |t| \leq \delta^{-1}} t^{-1} e^{ist} V_{-t} dt$$

converges in the strong operator topology of $\mathfrak{B}(X)$, as $\delta \rightarrow 0^+$, to an operator $J_s \in \mathfrak{B}(X)$;

(iii) $J_s = \mathcal{E}(s) + \mathcal{E}(s^-) - I$, for all $s \in \mathbb{R}$;

(iv) $\mathcal{E}(s) = I + 2^{-1}(J_s - J_s^2)$, for all $s \in \mathbb{R}$.

If μ is an arbitrary measure and $1 < p < \infty$, then $L^p(\mu) \in \text{UMD}$. In particular in the setting of §1, if $A \in \mathcal{C}$ and $1 < p < \infty$, then we shall denote the Stone-type spectral family of the one-parameter group $\{U_t^{(A,p)}\}$ in (1.1) by $\mathcal{E}^{(A,p)}$. This implements a bijection of \mathcal{C} onto \mathcal{S}_p , as described in the next theorem.

(2.2) **THEOREM** ([5, §3]). *For $1 < p < \infty$ and $A \in \mathcal{C}$, let*

$$\Psi_p(A) = \{I - \mathcal{E}^{(A,p)}(0^-)\} L^p(K).$$

Then Ψ_p is a one-to-one mapping of \mathcal{C} onto \mathcal{S}_p .

In the sequel we shall cast the relationship between invariant subspaces and generalized analyticity on K in a Hilbert transform setting. In order to do so, we now take up the necessary background material from [3] concerning distributional control, which will later be applied to the one-parameter group $\{U_t^{(A,p)}\}$ in (1.1). Suppose that $1 \leq p < \infty$, (Ω, μ) is an arbitrary measure space, and $u \rightarrow S_u$ is a strongly continuous representation of a locally compact abelian group G in $L^p(\mu)$ such that for some positive real number C_p ,

$$\sup\{\|S_u\|: u \in G\} \leq C_p.$$

Fix a Haar measure ν on G , and for $k \in L^1(G, d\nu)$, let $S_k \in \mathfrak{B}(L^p(\mu))$ denote the *transferred convolution operator* defined by $L^p(\mu)$ -valued Bochner integration as follows:

$$(2.3) \quad S_k f = \int_G k(u) S_{-u} f \, d\nu(u), \quad \text{for all } f \in L^p(\mu).$$

Suppose further that the representation $u \rightarrow S_u$ is also separation-preserving in the sense that whenever $f \in L^p(\mu)$, $g \in L^p(\mu)$, and $f g = 0$ μ -a.e., then $(S_u f)(S_u g) = 0$ μ -a.e. for all $u \in G$. Following [3], we say that $u \rightarrow S_u$ is a *distributionally controlled* representation of G in $L^p(\mu)$ provided that there is a positive real constant C_∞ such that

$$(2.4) \quad \|S_u f\|_\infty \leq C_\infty \|f\|_\infty, \quad \text{for all } f \in L^p(\mu) \cap L^\infty(\mu), \quad u \in G.$$

This terminology is motivated by [3, §2], where it is shown that under the above assumptions on the representation $u \rightarrow S_u$, (2.4) is equivalent to the existence of positive real constants c and α such that S interacts with distribution functions in the following manner:

$$\mu\{\omega \in \Omega: |(S_u f)(\omega)| > y\} \leq c\mu\{\omega \in \Omega: |f(\omega)| > \alpha y\},$$

for all $u \in G$, $f \in L^p(\mu)$, and $y > 0$. The latter characterization permits distributionally controlled representations to transfer weak type bounds of maximal convolution operators, while, apart from (2.4), the separation-preserving requirement permits distributionally controlled representations to transfer strong type maximal bounds (for the case of strong bounds, see [1], [2]). To be more precise, suppose that $\{k_n\}_{n=1}^\infty \subseteq L^1(G, d\nu)$. With $1 \leq p < \infty$, let $N_p(\{k_n\})$ and $N_p^{(w)}(\{k_n\})$ denote the (possibly infinite) strong type (p, p) and weak type (p, p) norms, respectively, of the maximal convolution operator on $L^p(G, d\nu)$ defined by the sequence $\{k_n\}_{n=1}^\infty$. Let $u \rightarrow S_u$ be a distributionally controlled representation of G in $L^p(\mu)$ having

constants C_p and C_∞ as described above. Denote by \mathcal{M} the maximal operator on $L^p(\mu)$ defined by the sequence $\{S_{k_n}\}_{n=1}^\infty$. Under these circumstances we have, in particular, the following two results concerning transference of maximal bounds and transference of almost everywhere convergence ([3, Theorem (2.14), Corollary (2.22)], [1, Théorème 1]).

(2.5) **PROPOSITION.** *With the above notation and assumptions we have:*

(i) for all $f \in L^p(\mu)$,

$$\|\mathcal{M}f\|_p \leq C_p^2 N_p(\{k_n\}) \|f\|_p;$$

(ii) for all $f \in L^p(\mu)$ and all $y > 0$,

$$\mu\{\omega \in \Omega : |(\mathcal{M}f)(\omega)| > y\} \leq (C_p C_\infty)^{2p} [N_p^{(w)}(\{k_n\}) \|f\|_p y^{-1}]^p.$$

(2.6) **PROPOSITION.** *In the foregoing assumptions let $p = 1$, and suppose that the sequence $\{k_n\}_{n=1}^\infty \subseteq L^1(G, d\nu)$ has the following additional properties:*

(i) $N_1^{(w)}(\{k_n\}) < \infty$ and $N_r^{(w)}(\{k_n\}) < \infty$ for some $r \in (1, +\infty)$;

(ii) the corresponding sequence of Fourier transforms $\{\widehat{k_n}\}_{n=1}^\infty$ is uniformly bounded and converges pointwise on the dual group \widehat{G} ;

(iii) for each $s \in G$ and each $\varepsilon > 0$, there is a corresponding relatively compact open neighborhood W of the identity element of G such that

$$\int_{G \setminus W} |k_n(u) - k_n(u+s)| d\nu(u) < \varepsilon,$$

for all sufficiently large n .

Then for each $\varphi \in L^1(G, d\nu)$, the sequence of convolutions $\{k_n * \varphi\}_{n=1}^\infty$ converges ν -a.e. on G , and for each $f \in L^1(\mu)$, the sequence $\{S_{k_n} f\}_{n=1}^\infty$ converges μ -a.e. on Ω .

If κ is a Calderón-Zygmund singular integral kernel of the kind treated in [15, §II.4], then the sequence of truncates of κ , $\{\kappa_n\}_{n=1}^\infty \subseteq L^1(\mathbb{R}^N)$, satisfies the requirements (2.6)–(i), (ii), (iii). Moreover, $N_r(\{\kappa_n\}) < \infty$ for all $r \in (1, +\infty)$. In particular, these comments hold for the sequence $\{h_n\}_{n=1}^\infty$ of truncates of the Hilbert kernel h on \mathbb{R} :

$$(2.7) \quad \begin{aligned} h(t) &= (\pi t)^{-1}, \quad \text{for } t \in \mathbb{R} \setminus \{0\}; \\ h_n(t) &= \begin{cases} h(t) & \text{if } n^{-1} \leq |t| \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Obviously, if $A \in \mathcal{E}$ and $1 \leq p < \infty$, then the one-parameter group $\{U_t^{(A,p)}\}$ in (1.1) is a distributionally controlled representation of \mathbb{R} in $L^p(K, \sigma)$, and we shall now specialize the discussion surrounding Propositions (2.5) and (2.6) to $S = \{U_t^{(A,p)}\}$ and to the sequence of truncated Hilbert kernels $\{h_n\}_{n=1}^\infty$ in (2.7). The Haar measure ν of $G = \mathbb{R}$ will be ordinary Lebesgue measure. Given $k \in L^1(\mathbb{R})$, the corresponding transferred convolution operator $S_k \in \mathfrak{B}(L^p(K, \sigma))$ defined in (2.3) by $L^p(\sigma)$ -valued Bochner integration will be denoted by $U_k^{(A,p)}$ in order to signify the present context explicitly. Applying Fubini's theorem for locally compact spaces ([13, §13]) to (2.3) and (1.1), we easily see that for each $f \in L^p(\sigma)$,

$$(2.8) \quad (U_k^{(A,p)} f)(x) = \int_{\mathbb{R}} k(t) A(-t, x) f(x - e_t) dt,$$

for σ -almost all $x \in K$.

In particular, $U_k^{(A,p)}$ is the restriction to $L^p(\sigma)$ of $U_k^{(A,1)}$. Hence we can conveniently economize on notation by writing $U_k^{(A)}$ to denote the operator $U_k^{(A,1)}$ on $L^1(\sigma)$ when there is no danger of confusion. For each positive integer n , let $\mathcal{H}_n^{(A)}$ be the operator $U_{h_n}^{(A)}$. Thus, applying Proposition (2.6) to the one-parameter group $\{U_t^{(A,1)}\}$ and the sequence $\{h_n\}$, we obtain the *transferred Hilbert transform* $\mathcal{H}^{(A)}$ on $L^1(\sigma)$, which is defined for each $f \in L^1(\sigma)$ by taking $\mathcal{H}^{(A)} f$ to be the limit σ -a.e. on K of the sequence $\{\mathcal{H}_n^{(A)} f\}_{n=1}^\infty$. In view of (2.8), we have for each $f \in L^1(\sigma)$,

$$(2.9) \quad (\mathcal{H}^{(A)} f)(x) = \lim_{n \rightarrow +\infty} \int_{n^{-1} \leq |t| \leq n} (\pi t)^{-1} A(-t, x) f(x - e_t) dt,$$

for σ -almost all $x \in K$.

By either Proposition (2.5)–(i) or by taking $s = 0$ in Theorem (2.1)–(ii), we see that for $1 < p < \infty$, the restriction of $\mathcal{H}^{(A)}$ to $L^p(\sigma)$ (denoted $\mathcal{H}^{(A,p)}$) is a bounded linear mapping of $L^p(\sigma)$ into itself, and that the sequence $\{U_{h_n}^{(A,p)}\}_{n=1}^\infty$ converges in the strong operator topology of $\mathfrak{B}(L^p(\sigma))$ to $\mathcal{H}^{(A,p)}$. Moreover, from the definition of $\mathcal{H}^{(A,p)}$, together with Theorem (2.1)–(ii), (iii), (iv), we obtain the following lemma for use in §3 (in conjunction with Theorem (2.2)).

(2.10) LEMMA. *If $A \in \mathcal{E}$ and $1 < p < \infty$, then the Stone-type spectral family $\mathcal{E}^{(A,p)}$ of $\{U_t^{(A,p)}\}$ satisfies:*

$$(i) \quad -i\mathcal{H}^{(A,p)} = \mathcal{E}^{(A,p)}(0) + \mathcal{E}^{(A,p)}(0^-) - I;$$

$$(ii) \mathcal{E}^{(A,p)}(0) = I + 2^{-1}(\{\mathcal{H}^{(A,p)}\}^2 - i\mathcal{H}^{(A,p)}).$$

Hence

$$(iii) I - \mathcal{E}^{(A,p)}(0^-) = I + 2^{-1}(\{\mathcal{H}^{(A,p)}\}^2 + i\mathcal{H}^{(A,p)}).$$

We close this section by listing in the next proposition some applications of [3, §3] to $\mathcal{H}^{(A)}$ which will be needed for our treatment of generalized analyticity when $p = 1$.

(2.11) **PROPOSITION.** *Given $A \in \mathcal{E}$, the following assertions hold.*

(i) $\mathcal{H}^{(A)}$ is of weak type (1.1) on $L^1(\sigma)$.

(ii) Suppose $f \in L^1(\sigma)$, $\mathcal{H}^{(A)}f \in L^1(\sigma)$, and $g \in L^1(\mathbb{R})$. Then

$$\mathcal{H}^{(A)}(U_g^{(A)}f) = U_g^{(A)}(\mathcal{H}^{(A)}f) \quad \sigma\text{-a.e.}$$

(iii) For $1 \leq p < \infty$, let

$$Y^{(A,p)} = \{f \in L^p(\sigma) : U_t^{(A,p)}f = f \text{ for all } t \in \mathbb{R}\},$$

and let $Z^{(A,p)}$ be the $L^p(\sigma)$ -closure of the linear manifold in $L^p(\sigma)$ spanned by the ranges of the operators $U_g^{(A,p)}$ for all $g \in L^1(\mathbb{R})$ such that $\mathbb{R} \setminus \{0\}$ contains the support of \hat{g} . Then

$$L^p(\sigma) = Y^{(A,p)} \oplus Z^{(A,p)}.$$

(iv) If $f \in L^1(\sigma)$, and $f = i\mathcal{H}^{(A)}f$ σ -a.e., then there is a sequence $\{g_n\}_{n=1}^\infty \subseteq L^\infty(\sigma)$ such that $g_n = i\mathcal{H}^{(A)}g_n$ σ -a.e. for each n , and $\|g_n - f\|_1 \rightarrow 0$, as $n \rightarrow \infty$.

3. Invariant subspaces as Hardy spaces. In this section we shall describe the elements of S_p , $1 \leq p < \infty$, by means of Hardy spaces defined from their corresponding cocycles. In order to include the case $p = 1$ in our considerations, we begin by defining a bijection Ψ_1 of \mathcal{E} onto \mathcal{S}_1 . This is accomplished by recourse to the following scholium, which is readily deduced from [10, Theorem V.6.1] with the aid of the first lemma in [12, §1.6].

(3.1) **SCHOLIUM.** *For each $M \in \mathcal{S}_2$, let $\Theta(M)$ be the $L^1(\sigma)$ -closure of M . Then Θ is a one-to-one mapping of \mathcal{S}_2 onto \mathcal{S}_1 , with inverse mapping given by*

$$\Theta^{-1}(N) = N \cap L^2(\sigma), \quad \text{for all } N \in \mathcal{S}_1.$$

(3.2) **DEFINITION.** We denote by Ψ_1 the one-to-one mapping of \mathcal{E} onto \mathcal{S}_1 given by

$$\Psi_1(A) = \Theta(\Psi_2(A)), \quad \text{for all } A \in \mathcal{E}.$$

The simplest and most obvious elements of \mathcal{S}_p , are the Beurling type subspaces, which are described as follows. For $1 \leq p < \infty$ let $H^p(K)$ be the standard Hardy space defined by $H^p(K) = \{f \in L^p(\sigma) : \hat{f}(\gamma) = 0 \text{ whenever } \gamma \in \Gamma \text{ and } \gamma < 0\}$. A *Beurling type subspace* of $L^p(\sigma)$ is a subspace of the form $\phi H^p(K)$ for some Borel measurable function $\phi : K \rightarrow \mathbb{T}$. Since the multiplication operator on $L^p(K)$ defined by such a function ϕ is a surjective linear isometry which preserves σ -a.e. convergence, the structural features of the Beurling type subspaces mirror those of $H^p(K)$. Consequently, we shall focus on $H^p(K)$ initially in order to elucidate and dispense with the rather special properties of the Beurling type subspaces. This will smooth the way for the subsequent treatment of the main result (Theorem (3.9)).

If A_0 denotes the trivial cocycle, $A_0(x, t) \equiv 1$, then for $1 \leq p < \infty$ and $t \in \mathbb{R}$, the operator $U_t^{(A_0, p)}$ in (1.1) is translation by e_t on $L^p(\sigma)$. Application of (2.9) shows that $\mathcal{H}^{(A_0)} \chi_\lambda = -i \operatorname{sgn}(\lambda) \chi_\lambda$ for each $\lambda \in \Gamma$, where, as usual, the function sgn is defined on Γ by putting $\operatorname{sgn}(\gamma)$ equal to 1, 0, or -1 , according as $\gamma > 0$, $\gamma = 0$, or $\gamma < 0$. It follows that for $1 < p < \infty$, $\mathcal{H}^{(A_0, p)}$ is the translation-invariant operator on $L^p(K)$ corresponding to the $L^p(K)$ -Fourier multiplier $(-i \operatorname{sgn})$ (that is, Bochner's abstract harmonic conjugation operator [6, Theorem 16]), and hence by (2.10)-(iii), $(I - \mathcal{E}^{(A_0, p)}(0^-))$ is the translation-invariant operator on $L^p(K)$ corresponding to the $L^p(K)$ -Fourier multiplier given by the characteristic function on Γ of $\{\lambda \in \Gamma : \lambda \geq 0\}$. This observation leads us to the following scholium (which was observed for the case $p = 2$ in [12, 2.3.(13)]).

(3.3) SCHOLIUM. *Let $A_0 \in \mathcal{C}$ be the trivial cocycle, $A_0(t, x) \equiv 1$. Then for $1 \leq p < \infty$,*

$$\Psi_p(A_0) = H^p(K).$$

Proof. In view of the foregoing and the definition of Ψ_p , for $1 < p < \infty$, in Theorem (2.2), only the case $p = 1$ remains to be considered. By (3.2) and Scholium (3.1), $\Psi_1(A_0)$ is the $L^1(\sigma)$ -closure of $H^2(K)$. Let $\{\tau_\delta\}$ be an approximate identity for $L^1(\sigma)$ consisting of trigonometric polynomials ([14, Theorem (33.12)]). If $f \in H^1(K)$, then the convolution $\tau_\delta * f \in H^2(K)$, for each δ , and we have $\|\tau_\delta * f - f\|_1 \rightarrow 0$. Hence $H^1(K) \subseteq \Psi_1(A_0)$. The reverse inclusion is obvious. \square

(3.4) COROLLARY. *Let $A \in \mathcal{E}$ be a coboundary—that is,*

$$(3.5) \quad A(t, x) = \phi(x)/\phi(x + e_t)$$

for some Borel measurable function $\phi: K \rightarrow \mathbb{T}$.

Then for $1 \leq p < \infty$, $\Psi_p(A) = \phi H^p(K)$.

Proof. For each $t \in \mathbb{R}$, $U_t^{(A,p)} = \phi U_t^{(A_0,p)} \bar{\phi}$. Hence when $1 < p < \infty$, $\mathcal{E}^{(A,p)}(\cdot) = \phi \mathcal{E}^{(A_0,p)}(\cdot) \bar{\phi}$, and the desired conclusion follows in this case from Theorem (2.2), and Scholium (3.3). The desired conclusion for $p = 1$ is immediate from the case $p = 2$, together with (3.2) and Scholium (3.3). \square

Thus for $1 \leq p < \infty$, the Beurling type subspaces are the elements of \mathcal{S}_p corresponding to the coboundaries under Ψ_p , and the study of the Beurling type subspaces from a Hardy space standpoint can be reduced to the cocycle $A_0 \equiv 1$ and the standard Hardy spaces $H^p(K)$. In particular, if A is a coboundary as in Corollary (3.4), then it follows from (2.9) that $\mathcal{H}^{(A)} = \phi \mathcal{H}^{(A_0)} \bar{\phi}$, which renders $\mathcal{H}^{(A,p)}$ isometrically equivalent to Bochner's abstract harmonic conjugation operator when A is a coboundary and $1 < p < \infty$. Since there is no novelty to be gained by studying the Beurling type subspaces from the Hardy space standpoint, we shall be concerned with their complement in \mathcal{S}_p , or, what comes to the same thing, the cocycles which are not coboundaries. In order to expedite these considerations, it will be convenient to characterize the coboundaries A in terms of the existence of nonzero fixed points for the corresponding group $\{U_t^{(A,p)}\}$. This matter is attended to in the following proposition.

(3.6) PROPOSITION. *Let $A \in \mathcal{E}$, and for $1 \leq p < \infty$, let $Y^{(A,p)}$ be as described in Proposition (2.11)–(iii). The following assertions are valid.*

(i) *For $1 < p < \infty$, $Y^{(A,p)} = \{\mathcal{E}^{(A,p)}(0) - \mathcal{E}^{(A,p)}(0^-)\} L^p(\sigma)$.*

(ii) *If $1 \leq p < \infty$, then $Y^{(A,p)} \neq \{0\}$ if and only if A is a coboundary. If A satisfies (3.5), then $Y^{(A,p)}$ is the one-dimensional subspace spanned by ϕ .*

Proof. (i) Since $\mathcal{E}^{(A,p)}$ is the Stone-type spectral family of $\{U_t^{(A,p)}\}$, direct calculation from (2.1)–(i) shows that if

$$f = \{\mathcal{E}^{(A,p)}(0) - \mathcal{E}^{(A,p)}(0^-)\} f,$$

then $f \in Y^{(A,p)}$. Conversely, if $U_t^{(A,p)} f = f$ for all $t \in \mathbb{R}$, then $U_{h_n}^{(A,p)} f = 0$ for all n , and hence $\mathcal{H}^{(A,p)} f = 0$. Using this in (2.10)–(i), (ii), we find that $\mathcal{E}^{(A,p)}(0)f = f$ and $\mathcal{E}^{(A,p)}(0^-)f = 0$. (ii) If $1 \leq p < \infty$, and A has the form (3.5), then direct calculation with (1.1) shows that $\phi \in Y^{(A,p)}$. Conversely, suppose $Y^{(A,p)}$ contains a unit vector ψ . Applying (1.1) to ψ gives, for each $t \in \mathbb{R}$,

$$(3.7) \quad \psi(x) = A(t, x)\psi(x + e_t) \quad \text{for } \sigma\text{-almost all } x \in K.$$

Taking absolute values, we see that for each $t \in \mathbb{R}$, $\rho \equiv |\psi|$ satisfies

$$\rho(x + e_t) = \rho(x) \quad \text{for } \sigma\text{-almost all } x \in K.$$

Taking Fourier transforms on K , we find that $\hat{\rho}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus \{0\}$. So $|\psi| = 1$ σ -a.e. on K . Without loss of generality, we can assume that ψ is Borel measurable and has modulus identically 1 on K . Dividing both sides of (3.7) by $\psi(x + e_t)$ completes the proof of the first assertion in (3.6)–(ii). An obvious variation on the foregoing reasoning with Fourier transforms shows that for a given coboundary A , a unimodular Borel function ϕ satisfying (3.5) for almost all $(t, x) \in \mathbb{R} \times K$ has its equivalence class modulo equality σ -a.e. determined to within a multiplicative unimodular constant. Using this observation together with the foregoing argument also establishes the second assertion in (3.6)–(ii). □

The stage is now set for our main result identifying the non-Beurling type subspaces in \mathcal{S}_p by means of the following generalization of ergodic Hardy space.

(3.8) DEFINITION. For $A \in \mathcal{C}$ and $1 \leq p < \infty$, we define $\mathfrak{H}^p(A)$ by

$$\mathfrak{H}^p(A) = \{f \in L^p(\sigma) : f = i\mathcal{H}^{(A)} f \text{ } \sigma\text{-a.e. on } K\}.$$

Notice that $\mathfrak{H}^p(A)$ is a closed subspace of $L^p(\sigma)$. When $1 < p < \infty$ this observation is an immediate consequence of the fact that $\mathcal{H}^{(A,p)} \in \mathfrak{B}(L^p(\sigma))$, whereas when $p = 1$, it follows from (2.11)–(i).

(3.9) THEOREM. If $A \in \mathcal{C}$, and A is not a coboundary, then for $1 \leq p < \infty$,

$$\Psi_p(A) = \mathfrak{H}^p(A).$$

Proof. Suppose first that $1 < p < \infty$. Since A is not a coboundary, it follows from Proposition (3.6) that $\mathcal{E}^{(A,p)}(0) = \mathcal{E}^{(A,p)}(0^-)$. Using

this in (2.10)–(i), we obtain for all $f \in L^p(\sigma)$:

$$(3.10) \quad (I - \mathcal{E}^{(A,p)}(0^-))f = 2^{-1}(I + i\mathcal{H}^{(A,p)})f.$$

If $f \in \Psi_p(A)$, then the right-hand side of (3.10) equals f , and so $f \in \mathfrak{H}^p(A)$. Similarly, if $f \in \mathfrak{H}^p(A)$, then the left-hand side of (3.10) equals f , and so $f \in \Psi_p(A)$. It remains only to prove the conclusion of the theorem in the case $p = 1$. Suppose first that $f \in \Psi_1(A)$. From the definition of Ψ_1 in (3.2), there is a sequence $\{f_n\}_{n=1}^\infty \subseteq \Psi_2(A)$ such that $\|f_n - f\|_1 \rightarrow 0$. It follows with the aid of (2.11)–(i) that, with respect to σ , $\mathcal{H}^{(A)}f_n \rightarrow \mathcal{H}^{(A)}f$ in measure. Since $f_n = i\mathcal{H}^{(A)}f_n$ for all n , it is clear that $f \in \mathfrak{H}^1(A)$. Conversely if $f \in \mathfrak{H}^1(A)$, then by (2.11)–(iv) and the conclusion already established when $p = 2$, there is a sequence $\{g_n\}_{n=1}^\infty \subseteq \mathfrak{H}^2(A) = \Psi_2(A)$ such that $\|g_n - f\|_1 \rightarrow 0$. Hence $f \in \Psi_1(A)$. \square

REFERENCES

- [1] N. Asmar, E. Berkson, and T. A. Gillespie, *Transfert des opérateurs maximaux par des représentations conservant la séparation*, C. R. Acad. Sci. Paris, **309** (1989), 163–166.
- [2] —, *Transference of strong type maximal inequalities by separation-preserving representations*, Amer. J. Math., **113** (1991), 47–74.
- [3] —, *Distributional control and generalized analyticity*, Integral Equations and Operator Theory, **14** (1991), 311–341.
- [4] E. Berkson, T. A. Gillespie, and P. S. Muhly, *Abstract spectral decompositions guaranteed by the Hilbert transform*, Proc. London Math. Soc., (3) **53** (1986), 489–517.
- [5] —, *Analyticity and spectral decompositions of L^p for compact abelian groups*, Pacific J. Math., **127** (1987), 247–260.
- [6] S. Bochner, *Additive set functions on groups*, Annals of Math., **40** (1939), 769–799.
- [7] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Arkiv för Mat., **21** (1983), 163–168.
- [8] D. L. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, Proceedings of Conference on Harmonic Analysis in Honor of Antoni Zygmund (Chicago, 1981), ed. by W. Beckner et al., Wadsworth Publ., Belmont, California, 1983, pp. 270–286.
- [9] H. R. Dowson, *Spectral Theory of Linear Operators*, London Math. Soc. Monographs, No. 12, Academic Press, New York, 1978.
- [10] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.
- [11] H. Helson, *Compact groups with ordered duals*, Proc. London Math. Soc., (3) **14A** (1965), 144–156.
- [12] —, *Analyticity on compact abelian groups*, in *Algebras in Analysis*, Proceedings of 1973 Birmingham Conference–NATO Advanced Study Institute, Academic Press, London, 1975, pp. 1–62.

- [13] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York, 1963.
- [14] ———, *Abstract Harmonic Analysis II*, Grundlehren der math. Wissenschaften, Band 152, Springer-Verlag, Berlin, 1970.
- [15] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.

Received February 4, 1991. The first author is grateful to the University of Missouri for its Summer Research Fellowship. The work of the second author was supported by a National Science Foundation grant.

UNIVERSITY OF MISSOURI-COLUMBIA
COLUMBIA, MO 65211

UNIVERSITY OF ILLINOIS
URBANA, IL 61801

AND

UNIVERSITY OF EDINBURGH
EDINBURGH EH9 3JZ, SCOTLAND

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

V. S. VARADARAJAN
(Managing Editor)
University of California
Los Angeles, CA 90024-1555
vsv@math.ucla.edu

HERBERT CLEMENS
University of Utah
Salt Lake City, UT 84112
clemens@math.utah.edu

F. MICHAEL CHRIST
University of California
Los Angeles, CA 90024-1555
christ@math.ucla.edu

THOMAS ENRIGHT
University of California, San Diego
La Jolla, CA 92093
tenright@ucsd.edu

NICHOLAS ERCOLANI
University of Arizona
Tucson, AZ 85721
ercolani@math.arizona.edu

R. FINN
Stanford University
Stanford, CA 94305
finn@gauss.stanford.edu

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720
vfr@math.berkeley.edu

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305
spk@gauss.stanford.edu

C. C. MOORE
University of California
Berkeley, CA 94720

MARTIN SCHARLEMANN
University of California
Santa Barbara, CA 93106
mgscharl@henri.ucsb.edu

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH
(1906-1982)

B. H. NEUMANN

F. WOLF
(1904-1989)

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$190.00 a year (10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1992 by Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS

Volume 155 No. 2 October 1992

Invariant subspaces and harmonic conjugation on compact abelian groups	201
NAKHLE HABIB ASMAR, EARL ROBERT BERKSON and THOMAS ALASTAIR GILLESPIE	
Cobcat and singular bordism	215
A. K. DAS and S. S. KHARE	
Poles of Eisenstein series on SL_n induced from maximal parabolics	229
PAUL FEIT	
On the postulation of 0-dimensional subschemes on a smooth quadric	251
SALVATORE GIUFFRIDA, RENATO MAGGIONI and ALFIO RAGUSA	
The Euler class for “piecewise” groups	283
PETER ABRAHAM GREENBERG	
L^p -Fourier transforms on nilpotent Lie groups and solvable Lie groups acting on Siegel domains	295
JUNKO INOUE	
Fuchsian moduli on a Riemann surface—its Poisson structure and Poincaré-Lefschetz duality	319
KATSUNORI IWASAKI	
Interpolation between H^p spaces and noncommutative generalizations. I	341
GILLES PISIER	
The braid index of generalized cables	369
ROBERT FONES WILLIAMS and ROBERT FONES WILLIAMS	
BMO and Hankel operators on Bergman spaces	377
KEHE ZHU	