COBCAT AND SINGULAR BORDISM

A. K. DAS AND S. S. KHARE

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Dold proved that a homomorphism $\phi\colon H^n(\mathrm{BO}) \to \mathbb{Z}_2$ corresponds to a manifold M^n if and only if $\phi(\mathrm{Sq}^p u + v_p \cdot u) = 0$, $\forall p \geq 0$ and $\forall u \in H^{n-p}(\mathrm{BO})$, v_p being the Wu class. The object of the present work is to have a singular analogue of this result and to study the bordism classification of singular manifolds in BO.

1. **Introduction.** Singh [1] has developed the notion of cobcat for a manifold M^n and has classified, upto bordism, all manifolds M^n with $\mathrm{cobcat}(M^n) \leq 3$. Cobcat$(M^n)$ was defined to be the smallest positive integer k such that the number $\langle W_{i_1} \cdots W_{i_p}, [M^n] \rangle = 0$ for all partitions $i_1 + \cdots + i_p = n$ with $k \leq p \leq n$.

Here we develop the notion of cobcat for a singular manifold (M^n, f) in a space X and discuss the bordism classification of all singular manifolds (M^n, f) in BO with $\mathrm{cobcat}(M^n, f) \leq 3$, $n = 2^r$.

Here all the manifolds are to be unoriented, smooth and closed, and all the homology and cohomology coefficients are to be in $\mathbb{Z}_2$. The space X is such that for each n, $H_n(X)$ and hence $H^n(X)$ is a finite dimensional vector space over $\mathbb{Z}_2$.

2. **Preliminaries.** Consider the set $N_n(X)$ of bordism classes of n-dimensional singular manifolds (M^n, f) in X, $f\colon M^n \to X$ being a continuous map. We know that $N_n(X)$ is an abelian group under the operation "disjoint union"

$$[M_1^n, f_1] + [M_2^n, f_2] = [M_1^n \sqcup M_2^n, f_1 \sqcup f_2],$$

where $f_1 \sqcup f_2\colon M_1^n \sqcup M_2^n \to X$ is given by

$$f_1 \sqcup f_2(x) = \begin{cases} f_1(x) & \text{if } x \in M_1^n, \\ f_2(x) & \text{if } x \in M_2^n. \end{cases}$$

Further, we have

$$N_*(X) = \bigoplus_{n \geq 0} N_n(X).$$

We know that for a point, $N_*(pt) = N_*$, the unoriented bordism ring, and there is a N_*-module structure in $N_*(X)$ given by

$$[M^n, f] \times [N^m] = [M^n \times N^m, f\pi],$$

where $\pi: M^n \times N^m \to M^n$ is the projection.

For a singular manifold (M^n, f) in X let $\tau: M^n \to$ BO be the classifying map of the tangent bundle over M^n. Then there is defined a homomorphism $t: H^n(\text{BO} \times X) \to \mathbb{Z}_2$ given by

$$t(w \otimes x) = \langle (\tau, f)^*(w \otimes x), [M^n] \rangle = \langle \tau^*(w) f^*(x), [M^n] \rangle,$$

where $w \otimes x \in H^n(\text{BO} \times X) = \bigoplus_{i=0}^{n} H^{n-i}(\text{BO}) \otimes H^i(X)$ and (τ, f): $M^n \to \text{BO} \times X$ is given by $(\tau, f)(z) = (\tau(z), f(z))$.

The number $\langle W_{i_1} \cdots W_{i_r} f^*(x_{n-p}), [M^n] \rangle$ is called the Stiefel-Whitney number of (M^n, f) associated to the cohomology class $x_{n-p} \in H^{n-p}(X)$ corresponding to the partition $i_1 + \cdots + i_r = p$. Moreover, this number is as usual bordism invariant [2].

Analogous to [1], given a singular manifold (M^n, f) in X there is associated a Poincaré algebra P^* given as follows:

Let $J = \{z \in H^*(\text{BO} \times X)$: either $\dim z > n$, or for all $z' \in H^{n-\dim z}(\text{BO} \times X)$, $\langle (\tau, f)^* z (\tau, f)^* z', [M^n] \rangle = 0\}$.

It is easy to see that J is an ideal of the graded algebra $H^*(\text{BO} \times X)$. Set

$$P^* = \frac{H^*(\text{BO} \times X)}{J}, \quad \text{the quotient algebra.}$$

Let $q: H^*(\text{BO} \times X) \to P^*$ be the quotient map. Clearly, $P^* = 0$ if and only if (M^n, f) bounds. Let $z \in H^*(\text{BO} \times X)$; we say, " $z = 0$ in P^* " if $q(z) = 0$.

As in [1], we have the following proposition, whose verification is a routine matter.

PROPOSITION 2.1. *If* (M^n, f) *is not a boundary then*
(a) P^* *is an n-dimensional graded algebra with Poincaré duality,*
(b) *the Steenrod algebra acts on P^* with the action given by*

$$\text{Sq}^i(q(z)) = q(\text{Sq}^i(z)),$$

(c) *if* $z \in H^n(\text{BO} \times X)$ *then* $q(z) = 0$ *if and only if*

$$t(z) = \langle (\tau, f)^* z, [M^n] \rangle = 0. \qquad \square$$

It is easy to see that for all $p \geq 0$, and for all $z \in H^{n-p}(\text{BO} \times X)$,

$$t(\text{Sq}^p(z) + (v_p \otimes 1)z) = 0,$$

where $v_p \in H^p(\text{BO})$ is the Wu class. So, in view of the above proposition, we have $\text{Sq}^p(z) = (v_p \otimes 1)z$ in P^*.

3. Extension of Dold's and Milnor's results to singular case. Dold [3] has proved the following

Result 3.1. For each integer $n \geq 0$, if $\phi\colon H^n(\mathrm{BO}) \to \mathbb{Z}_2$ is a homomorphism then there is an n-dimensional closed manifold M^n with $\phi(w) = \langle \tau(w), [M^n] \rangle$ for all $w \in H^n(\mathrm{BO})$ if and only if $\phi(\mathrm{Sq}^p(u) + v_p \cdot u) = 0$ for all $u \in H^{n-p}(\mathrm{BO})$ and for all $p \geq 0$, $v_p \in H^p(\mathrm{BO})$ being the Wu class. $\qquad\square$

Here we shall extend this result to the singular case as follows:

THEOREM 3.2. *For each* $n \geq 0$, *if* $h\colon H^n(\mathrm{BO} \times X) \to \mathbb{Z}_2$ *is a homomorphism then there is an* n-*dimensional singular manifold* (M^n, f) *with* $h(w \otimes x) = \langle \tau^*(w) f^*(x), [M^n] \rangle$ *for all* $w \otimes x \in H^n(\mathrm{BO} \times X)$ *if and only if* $h(\mathrm{Sq}^p(u \otimes y) + (v_p \cdot u) \otimes y) = 0$ *for all* $u \otimes y \in H^{n-p}(\mathrm{BO} \times X)$ *and for all* $p \geq 0$, $v_p \in H^p(\mathrm{BO})$ *being the Wu class.*

Proof. It is easy to see that the condition is necessary. We prove that the condition is sufficient also. Let $\{c_{m,i}\}_{i \in I_m}$ denote a basis for the vector space $H_m(X)$ over $\mathbb{Z}_2$, $m \geq 0$. Let $c^{m,i} \in H^m(X)$ be the cohomology class dual to $c_{m,i}$ i.e. $\langle c^{m,i}, c_{m,j} \rangle = \delta_{ij}$. Note that $\{c^{m,i}\}$ forms a basis for $H^m(X)$. Now, for each $c_{m,j}$ we can choose a singular manifold (M_j^m, f_j^m) with $(f_j^m)_*([M_j^m]) = c_{m,j}$, [2]. Thus, we have

$$\langle (f_j^m)^* c^{m,i}, [M_j^m] \rangle = \delta_{ij}.$$

Now,

$$H^n(\mathrm{BO} \times X) = \bigoplus_{l=0}^{n} H^l(\mathrm{BO}) \otimes H^{n-l}(X).$$

Define,

$$h_i^0\colon H^0(\mathrm{BO}) \to \mathbb{Z}_2$$

by

$$h_i^0(1) = h(1 \otimes c^{n,i}).$$

Clearly, h_i^0 satisfies the condition given in Result (3.1) and so there exists a manifold V_i^0 such that

$$h_i^0(1) = \langle \tau^*(1), [V_i^0] \rangle = \langle 1, [V_i^0] \rangle,$$

for each $i \in I_n$.

Define $h_i^1 \colon H^1(\mathrm{BO}) \to \mathbb{Z}_2$ by

$$
\begin{aligned}
h_i^1(w) &= h(w \otimes c^{n-1,i}) \\
&\quad + \sum_{j \in I_n} \langle \tau^*(w)(f_j^n \pi)^*(c^{n-1,i}), [M_j^n \times V_j^0] \rangle \\
&= h(w \otimes c^{n-1,i}) \\
&\quad + \sum_j \langle \tau^* w \cdot ((f_j^n)^* c^{n-1,i} \otimes 1), [M_j^n \times V_j^0] \rangle .
\end{aligned}
$$

Now,

$$
\begin{aligned}
h_i^1(&\mathrm{Sq}^1(1) + v_1 \cdot 1) \\
&= h(v_1 \otimes c^{n-1,i}) + \sum_j \langle \tau^* v_1 \cdot ((f_j^n)^* c^{n-1,i} \otimes 1), [M_j^n \times V_j^0] \rangle \\
&= h(\mathrm{Sq}^1(1 \otimes c^{n-1,i})) + \sum_j \langle \mathrm{Sq}^1((f_j^n)^* c^{n-1,i} \otimes 1), [M_j^n \times V_j^0] \rangle \\
&= h(1 \otimes \mathrm{Sq}^1 c^{n-1,i}) + \sum_j \langle (f_j^n)^* \mathrm{Sq}^1 c^{n-1,i}, [M_j^n] \rangle \langle 1, [V_j^0] \rangle .
\end{aligned}
$$

Since, $\mathrm{Sq}^1 c^{n-1,i} \in H^n(X)$, there is a subset $K_n \subset I_n$ such that

$$
\mathrm{Sq}^1 c^{n-1,i} = \sum_{k \in K_n} c^{n,k} .
$$

Therefore,

$$
\begin{aligned}
h_i^1(&\mathrm{Sq}^1(1) + v_1 \cdot 1) \\
&= \sum_{k \in K_n} h(1 \otimes c^{n,k}) + \sum_j \sum_{k \in K_n} \langle (f_j^n)^* c^{n,k}, [M_j^n] \rangle h_j^0(1) \\
&= \sum_k h_k^0(1) + \sum_k h_k^0(1), \quad \text{since } \langle (f_j^n)^* c^{n,k}, [M_j^n] \rangle = \delta_{kj} \\
&= 0 .
\end{aligned}
$$

So, by Result (3.1), there exists a manifold V_i^1 such that

$$
h_i^1(w) = \langle \tau^* w, [V_i^1] \rangle ,
$$

for each $i \in I_{n-1}$.

Now, using induction, we define $h_i^l \colon H^l(\mathrm{BO}) \to \mathbb{Z}_2$ by

$$
\begin{aligned}
h_i^l(w) &= h(w \otimes c^{n-l,i}) \\
&\quad + \sum_{\substack{j \in I_{n-m} \\ 0 \le m < l}} \langle \tau^* w \cdot ((f_j^{n-m})^* c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m] \rangle
\end{aligned}
$$

where $1 \le l \le n$, and each V_j^m is given by h_j^m $(m < l)$.

Now, it is enough to show that h_i^l satisfies the condition given in (3.1). For if it is so, then there exists a family $\{V_i^l\}_{i \in I_{n-l}}$ of manifolds such that for each $i \in I_{n-l}$,

$$h_i^l(w) = \langle \tau^* w, [V_i^l] \rangle \qquad (1 \leq l \leq n).$$

Also, we already have a family $\{V_i^0\}_{i \in I_n}$ of manifolds such that

$$h_i^0(1) = \langle 1, [V_i^0] \rangle,$$

for each $i \in I_n$. It is then easy to see that the given homomorphism $h: H^n(\mathrm{BO} \times X) \to \mathbb{Z}_2$ corresponds to the singular manifold (M^n, f) given by

$$M^n = \bigsqcup M_i^{n-l} \times V_i^l \quad \text{and} \quad f = \bigsqcup (f_i^{n-l} \pi),$$

where the disjoint union $\bigsqcup$ is taken over all $i \in I_{n-l}$ and all $0 \leq l \leq n$, and $\pi: M_i^{n-l} \times V_i^l \to M_i^{n-l}$ denotes the projection map.

Note that, for each $p \geq 0$ and for each $u \in H^{l-p}(\mathrm{BO})$,

$$h_i^l(\mathrm{Sq}^p u + v_p \cdot u) = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$
\begin{aligned}
\mathrm{I} &= h((\mathrm{Sq}^p u + v_p \cdot u) \otimes c^{n-l, i}) \\
&= h(\mathrm{Sq}^p u \otimes c^{n-l, i} + (v_p \cdot u) \otimes c^{n-l, i}) \\
&= h\left(\sum_{\substack{r+s=p \\ r \neq p}} \mathrm{Sq}^r u \otimes \mathrm{Sq}^s c^{n-l, i} \right), \qquad \text{by hypothesis on } h, \\
&= \sum_{\substack{r+s=p \\ r \neq p}} h(\mathrm{Sq}^r u \otimes \mathrm{Sq}^s c^{n-l, i}),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{II} &= \sum_{\substack{j \in I_{n-m} \\ 0 \leq m < l}} \langle \tau^*(\mathrm{Sq}^p u)((f_j^{n-m})^* c^{n-l, i} \otimes 1), [M_j^{n-m} \times V_j^m] \rangle \\
&= \sum_{j, m} \langle \mathrm{Sq}^p(\tau^* u)((f_j^{n-m})^* c^{n-l, i} \otimes 1), [M_j^{n-m} \times V_j^m] \rangle, \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{III} &= \sum_{j, m} \langle \tau^*(v_p \cdot u)((f_j^{n-m})^* c^{n-l, i} \otimes 1), [M_j^{n-m} \times V_j^m] \rangle \\
&= \sum_{j, m} \langle \mathrm{Sq}^p(\tau^* u((f_j^{n-m})^* c^{n-l, i} \otimes 1)), [M_j^{n-m} \times V_j^m] \rangle.
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathrm{II} + \mathrm{III} &= \sum_{j,m} \sum_{\substack{r+s=p \\ r\neq p}} \langle \mathrm{Sq}^r(\tau^* u)(\mathrm{Sq}^s(f_j^{n-m})^* c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle \\
&= \sum_{j,m} \sum_{\substack{r+s=p \\ r\neq p}} \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-m})^* \mathrm{Sq}^s c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle .
\end{aligned}
$$

Now, since $\mathrm{Sq}^s c^{n-l,i} \in H^{n-l+s}(X)$, there is a subset K_{n-l+s} of I_{n-l+s} such that

$$
\mathrm{Sq}^s c^{n-l,i} = \sum_{k \in K_{n-l+s}} c^{n-l+s,k}, \qquad (1 \le s \le p).
$$

So,

$$
\begin{aligned}
(1) \quad & h(\mathrm{Sq}^r u \otimes \mathrm{Sq}^s c^{n-l,i}) \\
&= \sum_{k \in K_{n-l+s}} h(\mathrm{Sq}^r u \otimes c^{n-l+s,k}) \\
&= \sum_k \left\{ \sum_{\substack{j,m \\ (m<l-s)}} \langle \tau^*(\mathrm{Sq}^r u)((f_j^{n-m})^* c^{n-l+s,k} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle \right. \\
& \hspace{7cm} \left. + \langle \tau^*(\mathrm{Sq}^r u), [V_k^{l-s}]\rangle \right\},
\end{aligned}
$$

$$
\text{noting that } l - s < l
$$

$$
\begin{aligned}
&= \sum_{\substack{j,m \\ (m<l-s)}} \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-m})^* \mathrm{Sq}^s c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle \\
& \quad + \sum_k \langle \mathrm{Sq}^r(\tau^* u), [V_k^{l-s}]\rangle .
\end{aligned}
$$

Also,

$$
\begin{aligned}
(*) \quad & \sum_{\substack{j,m \\ (l-s\le m<l)}} \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-m})^* \mathrm{Sq}^s c^{n-l,i} \otimes 1), [M_j^{n-m} \times V_j^m]\rangle \\
&= \sum_j \langle \mathrm{Sq}^r(\tau^* u)((f_j^{n-l+s})^* \mathrm{Sq}^s c^{n-l,i} \otimes 1), [M_j^{n-l+s} \times V_j^{l-s}]\rangle ,
\end{aligned}
$$

by dimensional consideration, since $m \ge l-s$. Further, we note that u is a polynomial in Stiefel-Whitney classes of BO, so that $\mathrm{Sq}^r(\tau^* u)$ is

a polynomial in Stiefel-Whitney classes of $M_j^{n-l+s} \times V_j^{l-s}$. Therefore the above expression $(*)$ becomes equal to

$$(2) \quad \sum_j \langle (f_j^{n-l+s})^* \operatorname{Sq}^s c^{n-l,i}, [M_j^{n-l+s}] \rangle \langle \operatorname{Sq}^r(\tau^* u), [V_j^{l-s}] \rangle$$

$$= \sum_j \sum_{k \in K_{n-l+s}} \langle (f_j^{n-l+s})^* c^{n-l+s,k}, [M_j^{n-l+s}] \rangle \langle \operatorname{Sq}^r(\tau^* u), [V_j^{l-s}] \rangle$$

$$= \sum_k \langle \operatorname{Sq}^r(\tau^* u), [V_k^{l-s}] \rangle.$$

Hence, combining I with (1) and II + III with (2), it follows that

$$h_i^l(\operatorname{Sq}^p u + v_p \cdot u) = \mathrm{I} + \mathrm{II} + \mathrm{III} = 0.$$

That is, h_i^l satisfies the condition given in (3.1). $\qquad\qquad\square$

Now, consider the universal bundle $\gamma \colon EO \to BO$ and the cartesian product $\gamma \times \gamma$ over $BO \times BO$. Let $\mu \colon BO \times BO \to BO$ be the classifying map of $\gamma \times \gamma$. μ has the property that

$$\mu^*(W_i) = \sum_{k=0}^{i} W_k \otimes W_{i-k}.$$

The product of two singular manifolds (M^m, f) and (M^n, g) in BO is given by $(M^m \times M^n, \mu \circ (f \times g))$, and this product induces a multiplication in $N_*(BO)$ given by

$$[M^m, f] \times [M^n, g] = [M^m \times M^n, \mu \circ (f \times g)],$$

which makes $N_*(BO)$ an algebra over $\mathbb{Z}_2$.

Analogous to [4], we have

LEMMA 3.3. *The Stiefel-Whitney numbers*

$$\langle W_{2i_1} \cdots W_{2i_r} (\mu \circ (g \times g))^*(W_{2i_{r+1}} \cdots W_{2i_{r+s}}), [N \times N] \rangle$$

of the product $(N, g) \times (N, g)$ *in* BO *are equal to*

$$\langle W_{i_1} \cdots W_{i_r} g^*(W_{i_{r+1}} \cdots W_{i_{r+s}}), [N] \rangle,$$

while the numbers

$$\langle W_{j_1} \cdots W_{j_p} (\mu \circ (g \times g))^*(W_{j_{p+1}} \cdots W_{j_{p+q}}), [N \times N] \rangle$$

are zero if some j_h *is odd.*

Proof. Routine verification. $\qquad\qquad\square$

THEOREM 3.4. *Let (M^{2n}, f) be a singular manifold in* BO, *such that*

$$\langle W_{j_1} \cdots W_{j_p} f^*(W_{j_{p+1}} \cdots W_{j_{p+q}}), [M^{2n}] \rangle = 0,$$

whenever some j_h is odd. Then

$$[M^{2n}, f] = [(N^n, g) \times (N^n, g)] \quad in \ N_{2n}(\mathrm{BO}).$$

Proof. We shall construct a singular manifold (N^n, g) in BO whose Stiefel-Whitney numbers

$$\langle W_{i_1} \cdots W_{i_r} g^*(W_{i_{r+1}} \cdots W_{i_{r+s}}), [N^n] \rangle$$

are equal to

$$\langle W_{2i_1} \cdots W_{2i_r} f^*(W_{2i_{r+1}} \cdots W_{2i_{r+s}}), [M^{2n}] \rangle .$$

This will imply that (M^{2n}, f) is cobordant to $(N^n, g) \times (N^n, g)$, by (3.3).

Let $R^n \subset H^n(\mathrm{BO} \times \mathrm{BO})$ be the vector space generated by all elements of the form $\mathrm{Sq}^p(x \otimes y) + (v_p \cdot x) \otimes y$. The Stiefel-Whitney numbers of each manifold (N^n, g) determine a homomorphism

$$h_N \colon H^n(\mathrm{BO} \times \mathrm{BO}) \to \mathbb{Z}_2$$

given by $h_N(x \otimes y) = \langle (\tau, g)^*(x \otimes y), [N] \rangle$, and by Theorem (3.2) we know that a given homomorphism $H^n(\mathrm{BO} \times \mathrm{BO}) \to \mathbb{Z}_2$ corresponds to a singular manifold in BO if and only if it annihilates R^n.

Define the "doubling homomorphism"

$$d \colon H^*(\mathrm{BO} \times \mathrm{BO}) \to H^*(\mathrm{BO} \times \mathrm{BO})$$

by

$$d(W_i \otimes W_j) = W_{2i} \otimes W_{2j} .$$

Let (M^{2n}, f) satisfy the hypothesis of Theorem (3.4). Then we shall show that $h_M \circ d \colon H^n(\mathrm{BO} \times \mathrm{BO}) \to \mathbb{Z}_2$ annihilates R^n. This will prove the existence of the required manifold (N^n, g).

Let $I \subset H^*(\mathrm{BO} \times \mathrm{BO})$ denote the ideal generated by the family $\{W_i \otimes 1, 1 \otimes W_i\}_{i \ \mathrm{odd}}$. Note that

$$\mathrm{Sq}^{2i} d(W_j \otimes 1) = (\mathrm{Sq}^{2i} W_{2j}) \otimes 1$$

$$= \left(\sum_{k=0}^{2i} \binom{2j - 2i + k - 1}{k} W_{2i-k} W_{2j+k} \right) \otimes 1,$$

where $\binom{p}{q}$ denotes the binomial coefficients reduced modulo 2. Therefore using the fact that

$$\binom{2j - 2i + 2l - 1}{2l} = \binom{j - i + l - 1}{l}$$

we get

$$\text{Sq}^{2i}\, d(W_j \otimes 1) \equiv \left(\sum_{l=0}^{i} \binom{j - i + l - 1}{l} W_{2i-2l} W_{2j+2l} \right) \otimes 1 \quad \text{mod } I$$

$$\equiv d(\text{Sq}^i(W_j \otimes 1)) \quad \text{mod } I.$$

Similarly, $\text{Sq}^{2i}\, d(1 \otimes W_j) \equiv d(\text{Sq}^i(1 \otimes W_j))$ mod I. Further, if

$$\text{Sq}^{2i}\, d(x \otimes y) \equiv d(\text{Sq}^i(x \otimes y)) \text{ mod } I \quad \text{and}$$
$$\text{Sq}^{2i}\, d(x \otimes y) \equiv d(\text{Sq}^i(x' \otimes y')) \text{ mod } I,$$

then

$$\text{Sq}^{2i}\, d((x \otimes y)(x' \otimes y'))$$

$$\equiv \sum_{p+q=i} (\text{Sq}^{2p}\, d(x \otimes y))(\text{Sq}^{2q}\, d(x' \otimes y')) \quad \text{mod } I$$

$$\equiv \sum_{p+q=i} (d(\text{Sq}^p(x \otimes y)))(d(\text{Sq}^q(x' \otimes y'))) \quad \text{mod } I$$

$$\equiv d(\text{Sq}^i((x \otimes y)(x' \otimes y'))) \quad \text{mod } I.$$

Hence, by induction, it follows that

$$\text{Sq}^{2i}\, d(x \otimes y) \equiv d(\text{Sq}^i(x \otimes y)) \quad \text{mod } I,$$

for each $x \otimes y \in H^*(\text{BO} \times \text{BO})$.

It is simple to verify that I is closed under Steenrod squaring operation. Applying induction on p, one gets

$$d(v_p \otimes 1) \equiv (v_{2p} \otimes 1) \quad \text{mod } I.$$

Now, consider the manifold (M^{2n}, f). By the hypothesis on (M^{2n}, f) we have

$$h_M(I^{2n}) = 0, \quad \text{where } I^{2n} = I \cap H^{2n}(\text{BO} \times \text{BO}).$$

Therefore, for any generator $\text{Sq}^p(x \otimes y) + (v_p \otimes 1)(x \otimes y)$ of R^n we have, using the congruences established above,

$$(h_M \circ d)(\text{Sq}^p(x \otimes y) + (v_p \otimes 1)(x \otimes y))$$
$$= h_M(\text{Sq}^{2p}(d(x \otimes y)) + (v_{2p} \otimes 1)\, d(x \otimes y) + (\text{terms in } I^{2n}))$$
$$= 0.$$

That is, $h_M \circ d$ annihilates R^n and so by Theorem (3.2) there exists a singular manifold (N^n, g) in BO such that $h_M \circ d = h_N$. Hence, the theorem follows. $\qquad \square$

4. Cobcat and singular bordism in BO. Analogous to [1], we define the cobcat for a singular manifold (M^n, f) in X as follows

DEFINITION. Cobcat(M^n, f) is the smallest positive integer k such that for each m, $0 \leq m \leq n$, the number

$$\langle W_{i_1} \cdots W_{i_p} f^*(x_{j_1} \cdots x_{j_q}), [M^n] \rangle = 0$$

for all partitions $i_1 + \cdots + i_p$ of m and for all partitions $j_1 + \cdots + j_q$ of $n - m$, with $k \leq p + q \leq n$ $(x_{j_h} \in H^{j_h}(X)$ for all $j_h)$. If no such k exists define cobcat$(M^n, f) = n + 1$.

REMARK 4.1. (a) Cobcat$(M^n) \leq$ cobcat(M^n, f),
(b) cobcat$(M^n, f) = 1$ if and only if (M^n, f) bounds,
(c) cobcat$(M^n, f) \leq$ nil$(\text{Im}(\tau, f)^*) \leq$ cat$(\tau, f) \leq$ cat(M^n),
(d)

$$\text{cobcat}(M_1^n \sqcup M_2^n, f_1 \sqcup f_2) \leq \max\{\text{cobcat}(M_1^n, f_1), \ \text{cobcat}(M_2^n, f_2)\}.$$

Now we shall discuss the singular version of some results proved in [1]. Let P^* be the Poincaré algebra associated to the singular manifold (M^n, f) in X. As in [1], an element z of any graded algebra A^* will be called k-decomposable if it is zero or is the sum of the products $z_1 \cdot z_2 \cdot \ \cdots \ \cdot z_p$ where $z_i \in A^*$ with dim $z_i > 0$ for each i, and $p \geq k$.

PROPOSITION 4.2. Let cobcat$(M^n, f) \leq k$.
(a) If $z \in P^*$ is k-decomposable, then z is zero.
(b) If $z \in P^*$ is $(k-1)$-decomposable and dim $z < n$ then z is zero.

Proof. Note that any k-decomposable element z of $H^*(\text{BO} \times X) \cong H^*(\text{BO}) \otimes H^*(X)$ can be written as a sum of the products $z_1 \cdot z_2 \cdot \ \cdots \ \cdot z_p$, where each z_i is of the type $W_j \otimes 1$ or $1 \otimes x_j$, and $p \geq k$. Also for any $z' \in H^{n-\dim z}(\text{BO} \times X)$ we have

$$\langle (\tau, f)^* z_1 (\tau, f)^* z_2 \cdots (\tau, f)^* z_p (\tau, f)^* z', [M^n] \rangle = 0,$$

since cobcat$(M^n, f) \leq k$ and $p \geq k$. Hence (a) follows, using the fact that any k-decomposable element of P^* can be obtained from a k-decomposable element of $H^*(\text{BO} \times X)$.

As in ([1], 1.2), (b) follows from (a). $\qquad \square$

From now on, the ambient space X will be taken to be the universal base space BO and (M^n, f) will denote a singular manifold in BO with $\mathrm{cobcat}(M^n, f) \leq 3$.

LEMMA 4.3. (a) *If* $z(W_i \otimes 1) = 0$ *and* $z(1 \otimes W_i) = 0$ *in* P^*, *where* $z \in H^{n-i}(\mathrm{BO} \times \mathrm{BO})$ *and* $0 < i < n$, *then* $z = 0$ *in* P^*.
   (b) *For* $j > 0$,

$$W_{2j+1} \otimes 1 = \begin{cases} \mathrm{Sq}^1(W_{2j} \otimes 1) & \textit{if } 2j+1 < n, \\ 0 & \textit{if } 2j+1 = n, \end{cases}$$

and

$$1 \otimes W_{2j+1} = \mathrm{Sq}^1(1 \otimes W_{2j}) \quad \textit{if } 2j+1 < n$$

in P^*.

Proof. (a) By the hypothesis, the last proposition and the fact that $\mathrm{cobcat}(M^n, f) \leq 3$, we have

$$\langle (\tau, f)^* z (\tau, f)^* z', [M^n] \rangle = \langle (\tau, f)^*(z \cdot z'), [M^n] \rangle = 0$$

for all $z' \in H^i(\mathrm{BO} \times \mathrm{BO})$. So $z = 0$ in P^*.
   (b) Note that

$$\mathrm{Sq}^1(W_{2j} \otimes 1) = (W_1 \otimes 1)(W_{2j} \otimes 1) + W_{2j+1} \otimes 1.$$

If $2j+1 < n$ then $(W_1 \otimes 1)(W_{2j} \otimes 1)$, being decomposable, is zero in P^*. If $2j+1 = n$, then $(W_1 \otimes 1)(W_{2j} \otimes 1) = \mathrm{Sq}^1(W_{2j} \otimes 1)$ in P^*. For the last part of (b) one has

$$\mathrm{Sq}^1(1 \otimes W_{2j}) = (1 \otimes W_1)(1 \otimes W_{2j}) + 1 \otimes W_{2j+1}. \qquad \square$$

LEMMA 4.4. *If n is even and $n > 2$, then $W_i \otimes 1 = 0$ and $1 \otimes W_i = 0$ in P^* for all odd i.*

Proof. By Lemma (4.3), it is enough to show that
(a) $(W_i \otimes 1)(W_{n-i} \otimes 1) = 0$,
(b) $(W_i \otimes 1)(1 \otimes W_{n-i}) = 0$,
(c) $(1 \otimes W_i)(1 \otimes W_{n-i}) = 0$ in P^* for all odd i.
For (a), let $i = 1$; then

$$(W_1 \otimes 1)(W_{n-1} \otimes 1) = \mathrm{Sq}^1(W_{n-1} \otimes 1) = \mathrm{Sq}^1 \mathrm{Sq}^1(W_{n-2} \otimes 1) = 0$$

in P^*, using (4.3) and the fact that $\mathrm{Sq}^1 \mathrm{Sq}^1 = 0$. Now let $i = 2j+1$,

$j > 0$ and $n - i = 2k + 1$, $k > 0$; then by (4.3)

$$(W_i \otimes 1)(W_{n-i} \otimes 1) = \mathrm{Sq}^1(W_{2j} \otimes 1)\,\mathrm{Sq}^1(W_{2k} \otimes 1)$$
$$= \mathrm{Sq}^1((W_{2j} \otimes 1)\,\mathrm{Sq}^1(W_{2k} \otimes 1))$$
$$= (W_1 \otimes 1)(W_{2j} \otimes 1)\,\mathrm{Sq}^1(W_{2k} \otimes 1) = 0$$

in P^*, as it is 3-decomposable. Hence (a) follows. For (b), using the same technique as in (a), we have $(W_i \otimes 1)(1 \otimes W_{n-i}) = 0$ in P^* for all odd $i < n - 1$. However, for $i = n - 1$, we have by (4.3)

$$(W_{n-1} \otimes 1)(1 \otimes W_1) = \mathrm{Sq}^1(W_{n-2} \otimes 1)(1 \otimes W_1)$$
$$= \mathrm{Sq}^1((W_{n-2} \otimes 1)(1 \otimes W_1)) + (W_{n-2} \otimes 1)\,\mathrm{Sq}^1(1 \otimes W_1)$$
$$= (W_1 \otimes 1)(W_{n-2} \otimes 1)(1 \otimes W_1)$$
$$+ (W_{n-2} \otimes 1)(1 \otimes W_1)(1 \otimes W_1) = 0$$

in P^*, as it is 3-decomposable. Thus (b) follows. Now, (c) can be proved by the same technique used in (a) and (b) above.   □

PROPOSITION 4.5. *If (M^n, f) is a non-bounding n-dimensional singular manifold in* BO *with* cobcat$(M^n, f) \le 3$, *where n is even and $n > 2$, then (M^n, f) is cobordant to a product $(N, g) \times (N, g)$ in* BO, *where (N, g) is also non-bounding and* cobcat$(N, g) \le 3$.

Proof. By Theorem (3.4) and Lemma (4.4), there exists a singular manifold (N, g) is BO of dimension $n/2$ such that (M^n, f) is cobordant to the product $(N, g) \times (N, g)$. Also,

$$\langle W_{i_1} \cdots W_{i_p} g^*(W_{i_{p+1}} \cdots W_{i_{p+q}}), [N] \rangle$$
$$= \langle W_{2i_1} \cdots W_{2i_p} f^*(W_{2i_{p+1}} \cdots W_{2i_{p+q}}), [M] \rangle,$$

where $i_1 + \cdots + i_{p+q} = n/2$ is a partition of $n/2$. Hence the proposition follows.   □

In the above proposition one can observe that if the underlying manifold M^n in (M^n, f) were a boundary then N in (N, g) would also be a boundary. Further, using induction, one can easily get the following

COROLLARY 4.6. *Let (M^n, f) be an n-dimensional singular manifold in* BO *with* cobcat$(M^n, f) \le 3$. *Let $n = 2^r \cdot m$ where either m is odd and $m \ge 3$, or $m = 2$. Then either (M^n, f) is a boundary or else (M^n, f) is cobordant to $(N, g)^{2^r}$, where (N, g) is a non-bounding m-dimensional singular manifold in* BO *with* cobcat$(N, g) \le 3$.   □

Finally, in view of the above results and the fact that $N_2(BO)$ is generated by the bordism classes $[(RP^1, \tau_1)^2]$, $[RP^2, \tau_1]$ and $[RP^2, c]$, where $\tau_1 \colon RP^i \to BO$ is the classifying map of the canonical line bundle over RP^i $(i = 1, 2)$ and $c \colon RP^2 \to BO$ is the constant map, we can make the following remarks.

REMARK 4.7. Let (M^n, f) be as in (4.6) and $n = 2^r$, $r \geq 1$. Then either (M^n, f) is a boundary or else (M^n, f) is cobordant to $(N, g)^{2^{r-1}}$, where (N, g) is a 2-dimensional singular manifold generated by $(RP^1, \tau_l)^2$, (RP^2, τ_l) and/or (RP^2, c). $\qquad\Box$

REMARK 4.8. In Remark (4.7) if, in addition, the underlying manifold M^n in (M^n, f) were a boundary then (N, g) would be equal to exactly one of the following

  (i) $(RP^1, \tau_l)^2$,
  (ii) $(RP^2, c) \sqcup (RP^2, \tau_l)$, or
  (iii) $(RP^1, \tau_l)^2 \sqcup (RP^2, c) \sqcup (RP^2, \tau_l)$,

where $(M_1, f_1) \sqcup (M_2, f_2) = (M_1 \sqcup M_2, f_1 \sqcup f_2)$. $\qquad\Box$

REFERENCES

[1]   H. Singh, *Lusternik-Schnirelmann category and cobordism*, Proc. Amer. Math. Soc., **102** (1) (1988), 183–190.
[2]   P. E. Conner, *Differentiable Periodic Maps*, Lecture Notes in Mathematics, vol. 738, Springer-Verlag, Berlin-Heidelberg-New York 1978.
[3]   R. E. Stong, *Notes on Cobordism Theory*, Math. Notes, Princeton Univ. Press, Princeton, N.J., 1968.
[4]   J. W. Milnor, *On the Stiefel-Whitney numbers of complex manifolds and spin manifolds*, Topology, **3** (1965), 223–230.

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