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INTERPOLATION BETWEEN H^p SPACES AND NON-COMMUTATIVE GENERALIZATIONS. I

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We give an elementary proof that the H^p spaces over the unit disc (or the upper half plane) are the interpolation spaces for the real method of interpolation between H^1 and H^∞. This was originally proved by Peter Jones. The proof uses only the boundedness of the Hilbert transform and the classical factorisation of a function in H^p as a product of two functions in H^q and H^r with $1/q + 1/r = 1/p$. This proof extends without any real extra difficulty to the non-commutative setting and to several Banach space valued extensions of H^p spaces. In particular, this proof easily extends to the couple $H^{p_0}(l_{q_0})$, $H^{p_1}(l_{q_1})$, with $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. In that situation, we prove that the real interpolation spaces and the K-functional are induced (up to equivalence of norms) by the same objects for the couple $L_{p_0}(l_{q_0})$, $L_{p_1}(l_{q_1})$. In another direction, let us denote by C_p the space of all compact operators x on Hilbert space such that $\mathrm{tr}(|x|^p) < \infty$. Let T_p be the subspace of all upper triangular matrices relative to the canonical basis. If $p = \infty$, C_p is just the space of all compact operators. Our proof allows us to show for instance that the space $H^p(C_p)$ (resp. T_p) is the interpolation space of parameter $(1/p, p)$ between $H^1(C_1)$ (resp. T_1) and $H^\infty(C_\infty)$ (resp. T_∞). We also prove a similar result for the complex interpolation method. Moreover, extending a recent result of Kaftal-Larson and Weiss, we prove that the distance to the subspace of upper triangular matrices in C_1 and C_∞ can be essentially realized simultaneously by the same element.

Introduction. Let $0 < p \leq \infty$. We will denote simply by L_p the L_p-space relative to the circle group $\mathbf{T}$ equipped with its normalised Haar measure denoted by m. We will denote by H^p the classical Hardy space of analytic functions (on the unit disc D of the complex plane). It is well known that this space can be identified with a closed subspace of L_p, namely the closure in L_p (for $p = \infty$ we must take the weak*-closure) of the linear span of the functions $\{e^{\mathrm{int}} | n \geq 0\}$. We refer e.g. to **[G]** or **[GR]** for more information on H^p-spaces.

Let us recall here the definitions of the K_t and J_t functionals which are fundamental in the real interpolation method. Let A_0, A_1 be a compatible couple of Banach (or quasi-Banach) spaces. This just means that A_0, A_1 are continuously included into a larger topological vector space (most of the time left implicit), so that we can consider unambiguously the sets $A_0 + A_1$ and $A_0 \cap A_1$.

For all $x \in A_0 + A_1$ and for all $t > 0$, we let

$$K_t(x, A_0, A_1) = \inf(\|x_0\|_{A_0} + t\|x_1\|_{A_1} | x = x_0 + x_1, \ x_0 \in A_0, \ x_1 \in A_1).$$

For all $x \in A_0 \cap A_1$ and for all $t > 0$, we let

$$J_t(x, A_0, A_1) = \max(\|x_0\|_{A_0}, t\|x_1\|_{A_1}).$$

Recall that the (real interpolation) space $(A_0, A_1)_{\theta,p}$ is defined as the space of all x in $A_0 + A_1$ such that $\|x\|_{\theta,p} < \infty$ where

$$\|x\|_{\theta,p} = \left(\int (t^{-\theta} K_t(x, A_0, A_1))^p \, dt/t\right)^{1/p}.$$

We also recall that there is a parallel definition of $(A_0, A_1)_{\theta,p}$ using the J_t functional which leads to the same Banach space with an equivalent norm. For example, if $1 \le p_0, p_1, q \le \infty$ and $0 < \theta < 1$, we have

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p,q}$$

where $L_{p,q}$ is the classical Lorentz space, identical to L_p if $p = q$. We refer to **[BL]** for more details.

In §1, we give a new proof of the following interpolation theorem of Peter Jones **[J1]**, as reformulated by Sharpley (cf. **[BS]** p. 414):

There is a constant C such that

$$\forall f \in H^1 + H^\infty, \quad \forall t > 0, \quad K_t(f, H^1, H^\infty) \le C K_t(f, L^1, L^\infty).$$

We should recall that

$$K_t(f, L^1, L^\infty) = \int_0^t f^* \, ds = \sup\left\{\int_E |f| \, dm | E \subset \mathbf{T}, m(E) = t\right\},$$

where we have denoted by f^* the non-increasing rearrangement of the function $|f|$. The difficulty of Jones' theorem lies in the fact that the optimal decomposition which realizes $K_t(f, L^1, L^\infty)$ is obtained by truncating the function f. If f is analytic, this operation clearly spoils the analyticity, and the problem is to find a substitute, something like a truncation but which preserves analyticity.

We should mention that a different proof of Jones' results (including some results which cannot be obtained by our method) has already been obtained a few months ago by Quanhua Xu. However, Xu's argument does not seem to extend to the non-commutative case.

We now describe the (very simple) method of proof we use throughout this paper, we call it the "square/dual/square" argument.

Let us say that the couple (H^p, H^q) is K-closed if there is a constant C such that

$$\forall f \in H^p + H^q, \quad \forall t > 0, \quad K_t(f, H^p, H^q) \leq C K_t(f, L^p, L^q).$$

The first step consists in showing the following "squaring" property:

(0.1) If (H^{2p}, H^{2q}) is K-closed and if the pointwise product defines a bounded bilinear map from $H^{2p} \times H^{2q}$ into the interpolation space $(H^p, H^q)_{1/2,\infty}$, then (H^p, H^q) is K-closed.

The next step is a dualisation (this seems to be the point that has been overlooked by previous researchers).

(0.2) The couple (H^p, H^q) is K-closed iff $(H^{p'}, H^{q'})$ is also K-closed $(1 \leq p, q \leq \infty,\ 1/p + 1/p' = 1/q + 1/q' = 1)$.

We can then sketch our "square/dual/square"-proof of the fact that (H^1, H^∞) is K-closed as follows:

By (0.1), it suffices to show that (H^2, H^∞) is K-closed, then by (0.2) it suffices to show that (H^2, H^1) is K-closed, but then by (0.1) again, it suffices to show that (H^4, H^2) is K-closed, and this is an obvious and well-known consequence of Marcel Riesz's theorem on the simultaneous boundedness of the Hilbert transform on L_p for all $1 < p < \infty$.

Our proof emphasizes the existence of a "simultaneous good approximation" to H^1 and H^∞. More precisely, we have

(0.3) There is a constant C, such that for all $f \in L_\infty$, there is a function $h \in H^\infty$ such that we have simultaneously

$$\|f - h\|_\infty \leq C \operatorname{dist}_{L_\infty}(f, H^\infty) \quad \text{and} \quad \|f - h\|_1 \leq C \operatorname{dist}_{L_1}(f, H^1).$$

As far as we know at the time of this writing, these results are known only in dimension 1, and are open in higher dimension either for the ball or the polydisc. We refer the reader to **[J2]** for a survey of what is known in the latter case.

In §2, we prove a non-commutative analogue of Peter Jones' theorem, where the space L^p is replaced by the space C_p of all compact operators x on l_2 such that $\operatorname{tr}|x|^p < \infty$, and H^p is replaced by the subspace T_p of all upper triangular matrices.

This result, which was motivated by and which improves a result of **[KLW]**, says again that the K_t-functional for the couple (T_1, T_∞) is induced (up to a constant independent of t) by the K_t-functional for the couple (C_1, C_∞). As a corollary, we identify the real interpolation spaces for the couple (T_1, T_∞).

In that case also, there is a simultaneous good approximation to T_1 and T_∞, as in (0.3) above.

In §3, we discuss the case of Banach space valued H^p-spaces. In particular, we show that Jones' theorem is also true for the couple of operator valued H^p-spaces $(H^1(C_1), H^\infty(C_\infty))$, i.e. the K_t-functional is induced (up to a constant independent of t) by the K_t-functional for the couple $(L_1(C_1), L_\infty(C_\infty))$. (See Theorem 3.3 and its corollaries for more precision.) As a consequence, we can again identify the real interpolation spaces. This result is closely related to the result in §2. (In fact, one can deduce from it the above result on (T_1, T_∞).) More generally, we obtain similar results for the couples $(H^{p_0}(l_{q_0}), H^{p_1}(l_{q_1}))$, and $(H^{p_0}(C_{q_0}), H^{p_1}(C_{q_1}))$, with $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. We doubt that Jones' proof can be adapted to all these cases. In the case of $(H^\infty(l_2), H^\infty(l_\infty))$, our argument leads to a new proof of a theorem of Bourgain **[B]**, but we chose to write this separately, we refer to **[P]**.

In §4, we consider similar problems for the complex interpolation method. Peter Jones **[J1]** proved that $H^p = (H^1, H^\infty)_\theta$, with $1/p = 1-\theta$. Using what seems to be a new idea in this context, we show that this result can be deduced from a slightly extended version of the real case. Our argument extends to the non-commutative case and gives $T_p = (T_1, T_\infty)_\theta$.

Although we state and prove our results on the unit disc, there is no problem to extend them to the case of the upper half plane. We leave this to the reader.

We now introduce a specific notation needed to treat the Banach space valued case. Let T be the circle group equipped with its normalized Haar measure m. Let $0 < p \leq \infty$. When B is a Banach space, we denote by $L_p(B)$ the usual space of Bochner-p-integrable B-valued functions on $(\mathbf{T}, m)$, so that when $p < \infty$, $L_p \otimes B$ is dense in $L_p(B)$. We denote by $\widetilde{H}^p(B)$ the closure in $L_p(B)$ of all the finite sums of the form $\sum_{0 \leq k < n} x_k e^{ikt}$ with $x_k \in B$. In other words, if we denote by $\mathscr{T}$ the space of all analytic trigonometric polynomials, $\widetilde{H}^p(B)$ is the closure in $L_p(B)$ of $\mathscr{T} \otimes B$. We reserve the notation $H^p(B)$ (and simply H^p if B is one dimensional) for the Hardy space of B-valued analytic functions f such that

$$\sup_{r<1} \left(\int \|f(re^{it})\|^p \, dm(t) \right)^{1/p} < \infty .$$

Again, see **[G, GR]** for more information.

When B is reflexive, is a separable dual or is an L_1-space (in particular if B is finite dimensional), then it is well known that $\widetilde{H}^p(B) = H^p(B)$ for all $p < \infty$, and $H^p(B)$ can be identified with a subspace of $L_p(B)$ for all $p \leq \infty$. We refer to **[BuD, E, HP]** for more information on this property, called the analytic Radon-Nikodym property.

The next proposition although very simple will be essential in the sequel. We suspect that the importance of the equivalence (i) $\Leftrightarrow$ (ii) has been overlooked although its proof is routine. We should emphasize that the exponents p, q in (i) and (ii) are the same, they are *not* conjugate to each other.

PROPOSITION 0.1. *Let $1 \leq p \leq q \leq \infty$. Consider an interpolation couple of Banach spaces (A_0, A_1), the following are equivalent:*

(i) *There is a constant C' such that*

$$\forall f \in \widetilde{H}^p(A_0) + \widetilde{H}^q(A_1), \forall t > 0,$$
$$K_t(f, \widetilde{H}^p(A_0), \widetilde{H}^q(A_1)) \leq C' K_t(f, L^p(A_0), L^q(A_1)).$$

(ii) *There is a constant C such that*

$$\forall f \in [L^p(A_0)/\widetilde{H}^p(A_0)] \cap [L^q(A_1)/\widetilde{H}^q(A_1)],$$
$$\forall t > 0, \quad \exists \hat{f} \in L^p(A_0) \cap L^q(A_1)$$

satisfying

$$J_t(\hat{f}, L^p(A_0), L^q(A_1)) \leq C J_t(f, L^p(A_0)/\widetilde{H}^p(A_0), L^q(A_1)/\widetilde{H}^q(A_1)).$$

(iii) *There is a constant C such that*

$$\forall f \in [L^p(A_0)/\widetilde{H}^p(A_0)] \cap [L^q(A_1)/\widetilde{H}^q(A_1)], \quad \exists \hat{f} \in L^p(A_0) \cap L^q(A_1)$$

satisfying

$$\|\hat{f}\|_{L^p(A_0)} \leq C\|f\|_{L^p(A_0)/\widetilde{H}^p(A_0)},$$
$$\|\hat{f}\|_{L^q(A_1)} \leq C\|f\|_{L^q(A_1)/\widetilde{H}^q(A_1)}.$$

In the above statement we regard the spaces $L^p(A_0)/\widetilde{H}^p(A_0)$ and $L^q(A_1)/\widetilde{H}^q(A_1)$ as included via the Fourier transform $f \to (\hat{f}(-1), \hat{f}(-2), \hat{f}(-3), \ldots)$ in the space of all sequences in $A_0 + A_1$. In this way, we may view these quotient spaces as forming a compatible couple for interpolation. (For the subspaces $\widetilde{H}^p(A_0)$, $\widetilde{H}^q(A_1)$, there is no problem, we may clearly consider them as a compatible couple in the obvious way.)

Proof. For brevity, we will denote simply $L^p/H^p(A_0)$ instead of $L^p(A_0)/\widetilde{H}^p(A_0)$, we will also write L^p, H^p, ... instead of $L^p(A_0)$, $\widetilde{H}^p(A_0)$, ... no confusion should arise. The proof is routine. We indicate first the argument for (i) $\Rightarrow$ (ii) which is the one we use below.

Assume (i). Let f be as above such that

$$J_t(f, L^p/H^p(A_0), L^q/H^q(A_1)) < 1.$$

Then let $g_p \in L^p(A_0)$ and $g_q \in L^q(A_1)$ be such that

$$\|g_p\|_{L^p} < 1, \quad \|g_q\|_{L^q} < t^{-1}, \quad f = g_p + H^p(A_0), \quad f = g_q + H^q(A_1).$$

Therefore, $g_p - g_q$ must be in $H^p + H^q$ and

$$K_t(g_p - g_q, L^p(A_0), L^q(A_1)) \leq \|g_p\|_{L^p} + t\|g_q\|_{L^q} < 2.$$

By (i), we have $K_t(g_p - g_q, H^p, H^q) < 2C'$, and hence there are $h_p \in H^p(A_0)$ and $h_q \in H^q(A_1)$ such that $g_p - g_q = h_p - h_q$ and $\|h_p\|_{H^p} + t\|h_q\|_{H^q} < 2C'$. Now if we let $\hat{f} = g_p - h_p = g_q - h_q$, then we have that $\hat{f} \in L^p(A_0) \cap L^q(A_1)$, $f = \hat{f} + H^p(A_0)$ in the space $L^p/H^p(A_0)$ and $f = \hat{f} + H^q(A_1)$ in the space $L^q/H^q(A_1)$ and moreover

$$J_t(\hat{f}, L^p, L^q) \leq \max(\|\hat{f}\|_{L^p}, t\|\hat{f}\|_{L^q}) \leq 1 + 2C'.$$

By homogeneity this completes the proof of (i) $\Rightarrow$ (ii) with $C \leq 1 + 2C'$. The converse is similar, we skip the details. The implication (ii) $\Rightarrow$ (iii) is easy; just take

$$t = (\|f\|_{L^p(A_0)/H^p(A_0)})(\|f\|_{L^q(A_1)/H^q(A_1)})^{-1}.$$

The converse (iii) $\Rightarrow$ (ii) is trivial. □

REMARK. The preceding statement would also remain valid if we had defined $\widetilde{H}^\infty(B)$ as the subspace of $L_\infty(B)$ formed by the functions with a Fourier transform vanishing on the negative integers. See the end of this section for a more general viewpoint.

We recall the following basic fact: If $1 < p_0 < p_1 < \infty$ then there is a constant C such that for all $t > 0$ we have:

$$\text{(0.4)} \qquad \forall f \in H^{p_0} + H^{p_1}, \quad \forall t > 0,$$
$$K_t(f, H^{p_0}, H^{p_1}) \leq C K_t(f, L^{p_0}, L^{p_1}).$$

This is an obvious consequence of the simultaneous boundedness of the orthogonal projection $P: L^2 \to H^2$ on all the L^p spaces (or equivalently of the same for the Hilbert transform). This "simultaneous" boundedness of P obviously also implies that if $1 < p_0 < p < p_1 < \infty$ and if $1/p = (1-\theta)/p_0 + \theta/p_1$, we have

$$H^p = (H^{p_0}, H^{p_1})_{\theta,p}$$

or more generally, if we define $H^{p,q}$ as the space of analytic functions in the disc with boundary values in the Lorentz space $L^{p,q}$ (on the circle), then, for any $1 \le q \le \infty$, we have

$$H^{p,q} = (H^{p_0}, H^{p_1})_{\theta,q},$$

and in particular

$$H^p \subset H^{p,\infty} = (H^{p_0}, H^{p_1})_{\theta,\infty}. \tag{0.5}$$

1. The proof of Peter Jones' theorem. We prove the theorem in several steps: starting from (0.4) restricted to p_0, p_1 both finite and more than 1, we will progressively extend the set of couples (p_0, p_1) for which (0.4) is valid until we eventually have eliminated all restrictions on p_0, p_1.

PROPOSITION 1.1. *For all $1 < p < q < \infty$ we have*

$$H^p \subset (H^1, H^q)_{\theta,\infty}, \tag{1.1}$$

with norm bounded by some constant $K(p,q)$, where $0 < \theta < 1$ satisfies $1/p = 1 - \theta + \theta/q$.

Proof. Choose any number $r > q$, and define r', s and t by the relations

$$1/r + 1/r' = 1, \quad 1/r + 1/s = 1/p, \quad 1/r + 1/t = 1/q.$$

Observe that $1/s = (1-\theta)/r' + \theta/t$. Let f be in the unit ball of H^p, and write $f = gh$ with g and h respectively in the unit balls of H^r and H^s. By the above basic fact (0.4) we have

$$H^s \subset (H^{r'}, H^t)_{\theta,\infty}$$

and this inclusion has norm less than (say) C. Observe that the operation of multiplication by g maps (by Hölder's inequality) the unit ball of $H^{r'}$ (resp. H^t) into that of H^1 (resp. H^q); hence it maps the unit ball of $(H^{r'}, H^t)_{\theta,\infty}$ into that of $(H^1, H^q)_{\theta,\infty}$. Therefore the norm of $f = gh$ in the space $(H^1, H^q)_{\theta,\infty}$ is less than C, which completes the proof. (This statement is also immediate using the complex interpolation method). □

The proof of the next proposition, although very simple, is important in the sequel.

Proposition 1.2. *For each $1 < q < \infty$, there is a constant C' such that*

$$\forall t > 0, \quad \forall f \in H^1 + H^q, \quad K_t(f, H^1, H^q) \leq C' K_t(f, L^1, L^q).$$

Proof. Let f be analytic in the disc and such that $K_t(f, L^1, L^q) < 1$. We factorize f as $f = BF^2$ with F non-vanishing and B a Blaschke product. Then since $|F| = |f|^{1/2}$ on the unit circle, we clearly have $K_{t^{1/2}}(F, L^2, L^{2q}) < 2^{1/2}$; hence by (0.1),

$$K_{t^{1/2}}(F, H^2, H^{2q}) < 2^{1/2} C.$$

Therefore, there are *analytic* functions g_0 and g_1 such that

$$F = g_0 + g_1, \qquad \|g_0\|_2 + t^{1/2}\|g_1\|_{2q} < 2^{1/2} C. \tag{1.2}$$

Now we can write $f = B(g_0 + g_1)^2 = B(g_0^2 + g_1^2 + 2g_0 g_1)$; hence

$$K_t(f, H^1, H^q) \leq K_t(g_0^2 + g_1^2, H^1, H^q) + K_t(2g_0 g_1, H^1, H^q). \tag{1.3}$$

By (1.3) we have

$$K_t(g_0^2 + g_1^2, H^1, H^q) \leq 2C^2, \tag{1.4}$$

and on the other hand by Hölder $\|2g_0 g_1\|_p \leq 2C^2 t^{-1/2}$ where $1/p = 1/2 + 1/2q$. Note that $1 < p < q$, and that $1/p = 1 - \theta + \theta/q$ with $\theta = 1/2$, so that by Proposition 1.1 for some constant K we have

$$\|2g_0 g_1\|_{(H^1, H^q)_{\theta,\infty}} \leq K 2C^2 t^{-1/2}.$$

Hence, in particular, $t^{-\theta} K_t(2g_0 g_1, H^1, H^q) \leq K2C^2 t^{-1/2}$, so that

$$K_t(2g_0 g_1, H^1, H^q) \leq K2C^2. \tag{1.5}$$

Returning to (1.3), we see that (1.4) and (1.5) imply

$$K_t(f, H^1, H^q) \leq 2C^2 + K2C^2. \qquad \square$$

Remark. At this point, we can easily check (0.1) by a minor modification of the preceding proof. We will refer to (0.1) in the sequel as "the squaring argument."

The special nature of the K and J functionals on one hand and of H^p and $(H^p)^\perp$ on the other hand imply that Proposition 1.2 has the following consequence.

Proposition 1.3. *For each $1 < q < \infty$, there is a constant C_q such that*

$$\forall t > 0, \quad \forall f \in L^1/H^1 \cap L^q/H^q,$$

$$\exists \hat{f} \in L^1 \cap L^q \text{ satisfying } J_t(\hat{f}, L^1, L^q) \le C_q J_t(f, L^1/H^1, L^q/H^q).$$

Equivalently, $\forall f \in L^1/H^1 \cap L^q/H^q$,

(1.6) $\exists \hat{f} \in L^1 \cap L^q$ *satisfying*

$$\|\hat{f}\|_{L^1} \le C_q \|f\|_{L^1/H^1}, \ \|\hat{f}\|_{L^q} \le C_q \|f\|_{L^q/H^q}.$$

Proof. By Proposition 0.1, this follows from Proposition 1.2. □

Up to now we have not used the duality between the K_t and J_t functionals, we now do so. We record below the dual versions of the preceding two propositions.

Proposition 1.2*. *For each $1 < p < \infty$, there is a constant C'_p such that*

$$\forall t > 0, \quad \forall f \in L^\infty/H^\infty \cap L^p/H^p,$$
$$\exists \hat{f} \in L^\infty \cap L^p \text{ satisfying}$$
$$J_t(\hat{f}, L^\infty, L^p) \le C'_p J_t(f, L^\infty/H^\infty, L^p/H^p).$$

Proposition 1.3*. *For each $1 < p < \infty$, there is a constant C_p such that*

$$\forall t > 0, \quad \forall f \in H^\infty + H^p, \quad K_t(f, H^\infty, H^p) \le C_p K_t(f, L^\infty, L^p).$$

The proof is obvious, we just recall that if p and q are conjugate then the dual of the space $K_t(L^1/H^1, L^p/H^p)$ (resp. $J_t(L^1/H^1, L^p/H^p)$) is isometrically identifiable with the space $J_{t^{-1}}(H^\infty, H^q)$ (resp. $K_{t^{-1}}(H^\infty, H^q)$), and that an injection is an isomorphic embedding iff its adjoint is onto (the relevant constants being the same).

Let us record here an immediate consequence of Proposition 1.3 and Proposition 1.2*.

Corollary 1.4. *Assume $1 < p < q < \infty$ with $1/p = (1-\beta)/1 + \beta/q$ and with $1/q = (1-\gamma)/\infty + \gamma/p$. Then for all $f \in L^1/H^1 \cap L^q/H^q$*

$$\|f\|_{L^p/H^p} \le C_q (\|f\|_{L^1/H^1})^{1-\beta} \cdot (\|f\|_{L^q/H^q})^\beta. \tag{1.7}$$

And for all $f \in L^p/H^p \cap L^\infty/H^\infty$

$$\|f\|_{L^q/H^q} \le C'_p (\|f\|_{L^p/H^p})^\gamma \cdot (\|f\|_{L^\infty/H^\infty})^{1-\gamma}. \tag{1.8}$$

Proof. (1.7) follows immediately from (1.6), and (1.8) can be proved similarly using Proposition 1.2* instead of Proposition 1.3. □

We now use a simple "extrapolation" trick to obtain

PROPOSITION 1.5. *For each* $1 < p < \infty$, *there is a constant* K_p *such that*

$$\forall f \in L^\infty/H^\infty \cap L^1/H^1 \quad \|f\|_{L^p/H^p} \leq K_p(\|f\|_{L^1/H^1})^{1-\theta} \cdot (\|f\|_{L^\infty/H^\infty})^\theta,$$

where $1/p = (1-\theta)/1 + \theta/\infty$.

Proof. Combining (1.7) and (1.8), we find,

$$\|f\|_{L^p/H^p} \leq C_q(\|f\|_{L^1/H^1})^{1-\beta} \cdot (C'_p\|f\|^\gamma_{L^p/H^p} \cdot \|f\|^{1-\gamma}_{L^\infty/H^\infty})^\beta.$$

Hence,

$$\|f\|^{1-\beta\gamma}_{L^p/H^p} \leq C_q(C'_p)^\beta \|f\|^{1-\beta}_{L^1/H^1} \|f\|^{(1-\gamma)\beta}_{L^\infty/H^\infty}.$$

This yields the desired inequality with

$$K_p = (C_q(C'_p)^\beta)^{1/1-\beta\gamma},$$

since $1 - \theta = (1-\beta)(1-\beta\gamma)^{-1}$. □

We can now complete our proof of Peter Jones' theorem. (A different proof has already been given a few months ago by Quanhua Xu **[X3]**.)

THEOREM 1.6. *There is a constant* C *such that for all* $t > 0$ *we have*:

$$\forall f \in H^1 + H^\infty, \quad \forall t > 0, \quad K_t(f, H^1, H^\infty) \leq CK_t(f, L^1, L^\infty).$$

Proof. We simply reproduce the proof of Proposition 1.2, but this time we can take $q = \infty$ because of Proposition 1.3* (applied to the case $p = 2$). Moreover, Proposition 1.5 allows us to complete that same proof because by duality Proposition 1.5 is equivalent to the assertion

$$H^{p'} \subset (H^1, H^\infty)_{1-\theta,\infty},$$

with norm bounded by some constant K. In other words, Proposition 1.1 remains valid for $q = \infty$. It is then easy to complete the proof by the squaring argument of Proposition 1.2. □

COROLLARY 1.7. *For all* $0 < \theta < 1$, $1 \le q \le \infty$ *we have*

$$H^{pq} = (H^1, H^\infty)_{\theta q}$$

where $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty}$.

REMARK 1.8. To prove Proposition 1.5, we can invoke Tom Wolff's interpolation theorem [W]. Indeed, Proposition 1.3 gives us the "right answer" for the interpolation spaces $(L_1/H^1, L_q/H^q)$ for $q < \infty$ and Proposition 1.2 gives us the case $(L_p/H^p, L_\infty/H^\infty)$ for $p > 1$. Actually, Corollary 1.7 can be deduced directly from Propositions 1.2 and 1.3* using Wolff's results in [W].

REMARK 1.9. It is easy to extend Theorem 1.6 to the case of H^r with $0 < r < 1$. First, we can check

$$H^1 \subset (H^r, H^\infty)_{\alpha,\theta} \tag{1.9}$$

with $\frac{1}{1} = \frac{1-\alpha}{r} + \frac{\alpha}{\infty}$. Indeed, we easily prove $H^1 \subset (H^r, H^q)_{\alpha,\infty}$ with $\frac{1}{1} = \frac{1-\alpha}{r} + \frac{\alpha}{q}$ for $q < \infty$ by the same method as above. Then we can obtain (1.9) from Wolff's theorem [W]. Using (1.9), it is immediate to adapt the preceding arguments to prove Theorem 1.6 with H^r and L^r $(0 < r < 1)$ instead of H^1 and L^1.

REMARK 1.10. (i) The same method will prove that the couple of quasi-normed spaces $(H^{1,\infty}, H^\infty)$ is K-closed relative to $(L^{1,\infty}, L^\infty)$. The same argument works. Note however that we already know a priori from Corollary 1.7 that the real interpolation spaces between $(H^{1,\infty}, H^\infty)$ are the same (by reiteration) as those between (H^1, H^∞). Indeed, the inclusion $(H^{1,\infty}, H^\infty)_{\theta,q} \subset H^{p,q}$ is the trivial direction, and Corollary 1.7 provides the converse. A fortiori the same is true for the interpolation spaces between $(L_1/\overline{H}_0^1, H^\infty)$.

(ii) By Holmstedt's formula (cf. [BL], pp. 52–53), it follows from Jones' theorem that all the couples (H^p, H^q) are K-closed, for any $0 < p$, $q \le \infty$, and similarly for couples of Lorentz spaces $(H^{p_0,q_0}, H^{p_1,q_1})$ with $p_0 \neq p_1$.

Let us recapitulate and at the same time formalize the preceding argument.

Consider a compatible couple (A_0, A_1) of Banach (or quasi-Banach) spaces. Assume given a closed subspace $S \subset A_0 + A_1$ and let

$$S_0 = S \cap A_0, \quad S_1 = S \cap A_1.$$

Let $Q_0 = A_0/S_0$ and $Q_1 = A_1/S_1$ be the associated quotient spaces. Clearly (Q_0, Q_1) form a compatible couple since there are natural inclusion maps

$$Q_0 \to (A_0 + A_1)/S \quad \text{and} \quad Q_1 \to (A_0 + A_1)/S.$$

We will say that the couple (S_0, S_1) is K-closed (relative to (A_0, A_1)) if there is a constant C such that

$$\forall t > 0 \quad \forall x \in S_0 + S_1$$
$$K_t(x, S_0, S_1) \leq CK_t(x, A_0, A_1).$$

We will say that (Q_0, Q_1) is J-closed if for some constant C we have

$$\forall t > 0 \quad \forall x \in Q_0 \cap Q_1 \quad \exists \hat{x} \in A_0 \cap A_1 \text{ such that}$$
$$J_t(\hat{x}, A_0, A_1) \leq J_t(x, Q_0, Q_1).$$

By the same argument as in Proposition 0.1 above one can show that this is equivalent to the following "simultaneous lifting property:" $\forall x \in Q_0 \cap Q_1 \quad \exists \hat{x} \in A_0 \cap A_1$ such that

$$x = \hat{x} + S_0 \text{ in } Q_0, \qquad x = \hat{x} + S_1 \text{ in } Q_1$$

and

$$\|\hat{x}\|_{A_0} \leq C\|x\|_{Q_0}, \qquad \|\hat{x}\|_{A_1} \leq C\|x\|_{Q_1}.$$

Our terminology is motivated by the fact that, roughly speaking, (S_0, S_1) is K-closed iff $S_0 + S_1$ is closed in $A_0 + A_1$ with a uniformity over t, while (Q_0, Q_1) is J-closed iff $Q_0 \cap Q_1$ is closed in $(A_0 + A_1)/S$ with a uniformity over t. Then our key observation in the preceding proof can be reformulated more "abstractly" as follows:

Proposition 1.11. *(S_0, S_1) is K-closed iff (Q_0, Q_1) is J-closed.*

We leave the routine proof to the reader.

Remark 1.12. Let us denote $A_{\theta,p} = (A_0, A_1)_{\theta,p}$ and $S_{\theta,p} = (S_0, S_1)_{\theta,p}$. Assume that (S_0, S_1) is K-closed (relative to (A_0, A_1)). Then $S_{\theta,p}$ can obviously be identified with a subspace of $A_{\theta,p}$ and the norm induced by $A_{\theta,p}$ on $S_{\theta,p}$ is equivalent to the norm of $S_{\theta,p}$. Moreover, the Holmstedt reiteration formula (cf. **[BL]**, pp. 52–53) for the K-functional shows that if $0 < \theta_0 \neq \theta_1 < 1$, and if $1 \leq p_0, p_1 \leq \infty$, then the couple $(S_{\theta_0,p_0}, S_{\theta_1,p_1})$ is a fortiori K-closed relative to $(A_{\theta_0,p_0}, A_{\theta_1,p_1})$, and also the couples (S_0, S_{θ_1,p_1}) and (S_{θ_0,p_0}, S_1) are K-closed relative to respectively (A_0, A_{θ_1,p_1}) and (A_{θ_0,p_0}, A_1).

We can reformulate Proposition 1.11 using duality. Assume $A_0 \cap A_1$ dense in A_0 and in A_1 and also assume that there is a subspace $s \subset A_0 \cap A_1$ which is dense in S_0 with respect to A_0, and in S_1 with respect to A_1. Then (S_0, S_1) is K-closed in (A_0, A_1) iff $(S_0^\perp, S_1^\perp)$ is K-closed in (A_0^*, A_1^*).

REMARK 1.13. An even more abstract fact is behind the preceding statement. Indeed, Proposition 1.10 can be viewed as a consequence of the following statement: Let X_1, X_2 be two closed subspaces of a Banach space X. Let $Q_i \colon X \to X/X_i$ be the quotient map $(i = 1, 2)$. Then $Q_1(X_2)$ is closed iff $Q_2(X_1)$ is closed. This can also be made more quantitative. Let us say that a surjective operator is a λ-surjection if the image of the open ball with center 0 and radius λ contains the open unit ball with center 0. Now in the above situation, if $Q_1|_{X_2}$ is a λ-surjection onto its image $Q_1(X_2)$, then $Q_2|_{X_1}$ is a $(\lambda+1)$-surjection onto its image $Q_2(X_1)$. (To see the connection with Proposition 1.10, consider the case $X = A_0 \times A_1$, $X_1 = S_0 \times S_1$ and $X_2 = \{(x, -x) | x \in A_0 \cap A_1\}$.)

2. The non commutative case. Let H be a separable Hilbert space. Let us denote by $C_p(H)$ or simply by C_p the Schatten ideal formed by all the compact operators T on H such that $\operatorname{tr}|T|^p < \infty$ and equipped with the norm $\|T\|_p = (\operatorname{tr}|T|^p)^{1/p}$. Here $1 \le p < \infty$. If $p = \infty$, we denote by C_∞ the space of all compact operators on H. In the above, we have taken $|T|$ defined as $(T^*T)^{1/2}$, but actually this choice is unimportant here since (as is well known) $\operatorname{tr}(T^*T)^{p/2} = \operatorname{tr}(TT^*)^{p/2}$, and hence $\|T\|_{C_p} = \|T^*\|_{C_p}$.

Assume H separable (possibly finite dimensional) and let (e_n) be a fixed orthonormal basis. Let $E_k = \operatorname{span}(e_i, i \le k)$ for all $k \ge 1$. We will simply say that a bounded operator $T \colon H \to H$ is triangular if $T(E_k) \subset E_k$ for all k. This definition can be extended formally to the case when the indexing set for the orthonormal basis is any countable totally ordered set in the place of the set of all positive integers.

We will denote by $T_p(H)$ or simply by T_p the subspace of $C_p(H)$ formed by all the triangular operators. We will show the following non-commutative version of P. Jones' theorem proved in the preceding section. The first point was proved recently, with C_2 and T_2 in the place of C_1 and T_1, in **[KLW]**. It is this result from **[KLW]** which motivated the present paper.

THEOREM 2.1. (i) *There is a constant* K *such that for any* x *in* C_1, *there is an operator* $\hat{x}$ *in* T_1 *such that we have simultaneously*

$$\|x - \hat{x}\| \le K d_1(x, T_1),$$
$$\|x - \hat{x}\|_\infty \le K d_\infty(x, T_\infty)$$

where we have denoted

$$d_p(x, T_p) = \inf\{\|x - y\|_p, y \in T_p\}.$$

(ii) *There is a constant* C *such that for any* x *in* $T_1 + T_\infty$, *we have*

$$\forall t > 0 \quad K_t(x, T_1, T_\infty) \le C K_t(x, C_1, C_\infty).$$

(iii) *If* $0 < \theta < 1$ *and* $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty}$, *we have*

$$(T_1, T_\infty)_{\theta p} = T_p$$

with equivalent norms (and similarly for the Lorentz space case).

The proof of this theorem is entirely similar to the proof given in §1, so that we will only briefly review the main ingredients one by one.

First we recall the following well-known fact (due to Mačaev, cf. e.g. **[GK]**).

LEMMA 2.2. *The orthogonal projection* $P: C_2 \to T_2$ *is bounded simultaneously on* C_p *for all* $1 < p < \infty$. *Therefore, in particular* (0.4) *extends to the present non-commutative setting as follows: If* $1 < p_0 < p_1 < \infty$ *then there is a constant* C *such that for all* $t > 0$ *we have*

$$\forall x \in T_{p_0} + T_{p_1}, \quad \forall t > 0, \tag{2.1}$$
$$K_t(x, T_{p_0}, T_{p_1}) \le C K_t(x, C_{p_0}, C_{p_1}).$$

The following fact is also well known.

LEMMA 2.3. *Let* $1 \le p, q, r \le \infty$ *with* $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. *Assume (for simplicity) that* H *is finite dimensional. Then every invertible* x *in* T_p *can be written as* $x = ab$ *with* $a \in T_r$, $b \in T_q$ *and* $\|a\|_r \|b\|_q = \|x\|_p$.

Proof. Note that either $p/q \le 1/2$ or $p/r \le 1/2$. Assume $p/q \le 1/2$. Also, assume $\|x\|_p = 1$. By the Cholesky factorization, we have $|x|^{2p/q} = b^*b$ for some b triangular. Moreover, b is necessarily invertible and $|b|^q = |x|^p$, so that $\|b\|_q = 1$. Let $\alpha = 1 - 2p/q$. Note $\alpha \ge 0$. We have $x = U|x|$ with U unitary. Hence, $x = U|x|^\alpha b^* b$. Then, let $a = xb^{-1} = U|x|^\alpha b^*$. Clearly, a is triangular (since a and b^{-1} are so, and triangular operators form an algebra) and moreover,

$$aa^* = U|x|^\alpha b^* b |x|^\alpha U^* = U|x|^{2(1-p/q)} U^*$$
$$= U|x|^{2p/r} U^*.$$

Hence $(aa^*)^{1/2} = U|x|^{p/r}U^*$ so that $\|a\|_r = 1$. This completes the proof if $\frac{p}{q} \le \frac{1}{2}$.

In case we have instead $\frac{p}{r} \le \frac{1}{2}$, an entirely similar argument works. We first write $x = (xx^*)^{1/2}V$ then $(xx^*)^{p/r} = aa^*$ with a triangular and $b = a^{-1}x = a^*(xx^*)^{1/2-p/r}V$. Then the rest is the same. □

It is now easy to complete the proof of the following non-commutative version of Proposition 1.1, we skip the proof.

PROPOSITION 2.4. *For all* $1 < p < q < \infty$ *we have*

$$T_p \subset (T_1, T_q)_{\theta,\infty} \tag{2.2}$$

with norms bounded by some constant $K(p, q)$ *where* $0 < \theta < 1$ *satisfies* $\frac{1}{p} = 1 - \theta + \frac{\theta}{q}$.

To extend the squaring argument (0.1) (cf. the proof of Proposition 1.2), we need a non-commutative analogue of the "scaling" that we used heavily for an analytic function without zeros. In the present setting, this is somewhat easier. Indeed, let us denote by $\lambda(|x|) = (\lambda_n(|x|))_{n\ge0}$ the sequence of the eigenvalues of $|x|$ (arranged in non-increasing order, and repeated as usual according to their multiplicity). Observe that for all $\alpha > 0$

$$\lambda_n(|x|^\alpha) = (\lambda_n(|x|))^\alpha. \tag{2.3}$$

PROPOSITION 2.5. *If* $1 \le p_0 \le p_1 \le \infty$ *then for all* x *in* $C_{p_0} + C_{p_1}$ *we have for all* $t > 0$

$$K_t(x, C_{p_0}, C_{p_1}) = K_t(|x|, C_{p_0}, C_{p_1}) = K_t(\lambda(|x|), l_{p_0}, l_{p_1}).$$

Moreover, a similar double identity holds for the J*-functional. If we allow* $0 < p_0 \le p_1 \le \infty$, *then the first identity still holds and the second one becomes there is a constant* C *such that for all* $t > 0$

$$K_t(x, C_{p_0}, C_{p_1}) \le K_t(\lambda(|x|), l_{p_0}, l_{p_1}) \le CK_t(x, C_{p_0}, C_{p_1}).$$

The first equality is easy to check using the unitary invariance of the spaces C_p. The second equality follows from the existence of a projection simultaneously bounded on all C_p's ($p \ge 1$) onto the elements which are diagonal on the same basis as $|x|$. In the quasi-normed case, the last assertion can be checked as follows. Let $a_n(x)$ be the distance (in the operator norm) of x to the set of all operators of rank $< n$. It is well known that $a_n(x) = \lambda_n(|x|)$, and also that $a_{2n}(x_0 + x_1) \le a_n(x_0) + a_n(x_1)$. Using this it is an easy exercise to check the last assertion. In any case, we refer the reader to **[PT]** for more details on such results.

Using Proposition 2.5, it is easy to extend Proposition 1.2 to the non-commutative case, as follows.

PROPOSITION 2.6. *For each* $1 < q < \infty$, *there is a constant* C' (*depending only on* q) *such that*

$$\forall t > 0 \quad \forall x \in T_1 + T_q \quad K_t(x, T_1, T_q) \leq C' K_t(x, C_1, C_q).$$

Proof. Assume w.l.o.g. that H is finite dimensional and x invertible. Let T be triangular such that

$$|x| = (x^*x)^{1/2} = b^*b = |b|^2.$$

We have $\lambda(|b|) = \lambda(|x|^{1/2})$. Hence by Proposition 2.5 and (2.3) we have

$$K_{t^{1/2}}(b, C_{2p_0}, C_{2p_1}) \leq (2K_t(x, C_{p_0}, C_{p_1}))^{1/2}.$$

Assume for simplicity $2K_t(x, C_{p_0}, C_{p_1}) < 1$. Then by (2.1), there are g_0, g_1 triangular such that $b = g_0 + g_1$ and $\|g_0\|_2 + t^{1/2}\|g_1\|_{2q} < C$. Note that $x = U|x| = Ub^*b = ab$ where $a = xb^{-1}$ is triangular. Since $a = Ub^*$ (and $\|x\|_p = \|x\|_p$) we have obviously

$$K_{t^{1/2}}(a, C_{2p_0}, C_{2p_1}) = K_{t^{1/2}}(b, C_{2p_0}, C_{2p_1}).$$

Hence by (2.1) again, $a = h_0 + h_1$ with h_0, h_1 triangular such that $\|h_0\|_2 + t^{1/2}\|h_1\|_{2q} < C$. Finally, $x = ab = (h_0 + h_1)(g_0 + g_1)$ can be estimated as in the (commutative) proof of Proposition 1.2. □

REMARK 2.7. The analogue of Proposition 1.3 is clearly valid in the case of T_p with the same proof. The same comment applies to Proposition 1.2* and Proposition 1.3*. Moreover, Proposition 1.5 clearly also extends to the non-commutative case, so that the proof of Theorem 2.1 can be completed exactly as in §1.

3. The Banach space valued case. We first remark that Jones' theorem remains valid for a couple $(H^1(B), H^\infty(B))$ for an arbitrary Banach space B. Indeed, if f is $H^1(B)$, using an elementary outer function argument, one can factor f as $F\phi$, where F is analytic, scalar valued and such that $|F| = \|f\|_B$, while ϕ is bounded analytic B-valued and such that $\|\phi\| = 1$ a.s. on $\mathbf{T}$. Moreover, if f is in the unit ball of $\widetilde{H}^1(B)$, then f can be approximated in $H^1(B)$

by products $F\phi$ with F in the unit ball of H^1 and ϕ in the unit ball of $\widetilde{H}^\infty(B)$. This reduces the problem to the scalar case, since it easy to verify that $K_t(f, H^1(B), H^\infty(B)) \le K_t(F, H^1, H^\infty)$ and $K_t(F, L_1, L_\infty) \le K_t(f, L_1(B), L_\infty(B))$. Similarly, for any $1 \le p$, $q \le \infty$, the couples $(H^p(B), H^q(B))$ and $(\widetilde{H}^p(B), \widetilde{H}^q(B))$ are K-closed, by reduction to the scalar case.

However, the general case of a compatible couple (A_0, A_1) of two different Banach spaces, is more delicate. We refer the reader to **[BX]** for more information and for a counterexample showing that Jones' theorem does not extend in that degree of generality (cf. also **[X1]**, **[X2]**). Nevertheless, we show below that in a number of nice cases, it does extend. There seems to be no counterexample known at the time of this writing within couples of Banach lattices.

Using the same method as in §1, we can prove

THEOREM 3.1. *Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$. *(Moreover, the space* l_∞ *must be replaced by* c_0 *wherever it appears.) Then the couple* $(\widetilde{H}^{p_0}(l_{q_0}), \widetilde{H}^{p_1}(l_{q_1}))$ *is K-closed with respect to* $(L_{p_0}(l_{q_0}), L_{p_1}(l_{q_1}))$.

REMARK. One can derive from Jones' proof the case $p_0 = q_0 = 1$, $p_1 = q_1 = \infty$, but probably not the other cases. However, some other cases can be derived from **[B]**. More precisely, Bourgain states explicitly in **[B]** a theorem which in our terminology means that the couple $(\widetilde{H}^1(l_1), \widetilde{H}^1(l_\infty))$ is K-closed. By a rather simple factorisation argument (such as Theorem 2.7 in **[HP]**), one can show that a couple $(\widetilde{H}^p(A_0), \widetilde{H}^p(A_1))$ is K-closed for some $1 \le p \le \infty$ iff it is K-closed for all $1 \le p \le \infty$. Therefore, Bourgain's theorem does imply certain cases of Theorem 3.1. But actually, it is interesting that one can go conversely: in **[P]** we indeed do recover most of the results of **[B]** by the methods of the present paper.

We will denote simply by $g \cdot h$ the pointwise and coordinatewise product of two sequences $g = (g_n)$ and $h = (h_n)$ of scalar analytic functions. We first observe that in the situation of Theorem 3.1, if

$$\frac{1}{p} = \frac{1}{2p_0} + \frac{1}{2p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1}{2q_0} + \frac{1}{2q_1}, \tag{3.1}$$

then the unit ball of $H^p(l_q)$ coincides with the set of all products $g \cdot h$ with g and h in the unit balls respectively of $H^{2p_0}(l_{2q_0})$ and $H^{2p_1}(l_{2q_1})$. Indeed, this is easy to check using outer functions. Then, the squaring argument (0.1) suitably generalized, becomes

LEMMA 3.2. *In the same situation as in Theorem* 3.1 (*replacing* l_∞ *by* c_0 *wherever it appears*), *if* $(\widetilde{H}^{2p_0}(l_{2q_0}), \widetilde{H}^{2p_1}(l_{2q_1}))$ *is K-closed and if, with* p *and* q *as in* (3.1) *above,*

$$\widetilde{H}^p(l_q) \subset (\widetilde{H}^{p_0}(l_{q_0}), \widetilde{H}^{p_1}(l_{q_1}))_{1/2,\infty}, \tag{3.2}$$

then $(\widetilde{H}^{p_0}(l_{q_0}), \widetilde{H}^{p_1}(l_{q_1}))$ *is K-closed.*

Proof. The assumptions allow to reduce to the case of finite dimensional l_p-spaces with constants independent of the dimension. Then, the distinction between $H^p(l_q)$ and $\widetilde{H}^p(l_q)$ becomes irrelevant and we can argue exactly as in Proposition 1.2. □

Proof of Theorem 3.1. Let us record here the preliminary observation that if we a priori know that $(\widetilde{H}^{2p_0}(l_{2q_0}), \widetilde{H}^{2p_1}(l_{2q_1}))$ is K-closed, then (3.2) holds iff the couple $(\widetilde{H}^{p_0}(l_{q_0}), \widetilde{H}^{p_1}(l_{q_1}))$ is K-closed. Indeed, Lemma 3.2 gives the only if part, and the converse is clear since $\widetilde{H}^p(l_q)$ is included into $L^p(l_q)$, hence into the complex interpolation space $(L^{p_0}(l_{q_0}), L^{p_1}(l_{q_1}))_{1/2}$, and a fortiori into $(L^{p_0}(l_{q_0}), L^{p_1}(l_{q_1}))_{1/2,\infty}$, but, if we assume K-closedness, the latter space induces on the subspace of analytic functions a norm equivalent to that of $(\widetilde{H}^{p_0}(l_{q_0}), \widetilde{H}^{p_1}(l_{q_1}))_{1/2,\infty}$, which proves the converse part.

Using the observations preceding Lemma 3.2, when all the indices are finite it is easy to extend Proposition 1.1 to the present setting with essentially the same proof; more precisely, we have an inclusion

$$\widetilde{H}^{p_\theta}(l_{q_\theta}) \subset (\widetilde{H}^{p_0}(l_{q_0}), \widetilde{H}^{p_1}(l_{q_1}))_{\theta,\infty},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$, and all the indices are finite. Indeed, this can be checked using the well known fact (apparently going as far back as **[BB]**) that the Hilbert transform is bounded simultaneously on $L_p(l_q)$ for all $1 < p$, $q < \infty$, a fact which provides us with a substitute for (0.4), and proves the preceding inclusion when all indices are strictly between 1 and ∞ (this can also be seen, perhaps more easily, using complex interpolation). Then, the factorisation argument as in Proposition 1.1 yields the preceding inclusion assuming only that all indices are finite. This extension and Lemma 3.2 give us Theorem 3.1 in case all the four indices p_0, p_1, q_0, q_1 are finite. We now dualize. To avoid irrelevant complications let us assume for the moment that, everywhere until said otherwise, l_p is the finite dimensional space $\mathbf{C}^n$ equipped with the l_p-norm. Then, by Proposition 0.1 and dualisation, we obtain Theorem 3.1 and (3.2) in case all the four indices are more than 1. At this stage, by the preliminary remark recorded at the start of the proof, to conclude it suffices to check that (3.2) holds in full generality. To do so, we note that if q_0, q_1 are both finite and p_0, p_1 are both more than 1, but possibly infinite, then we still have (3.2) and hence K-closedness because we can apply the argument for Proposition 1.1 to the second indices only. More precisely, choosing r large enough, we can write any f in the unit ball of $\widetilde{H}^p(l_q)$ as a product gh with g in the unit ball of $H^\infty(l_r)$ and h in the unit ball of $\widetilde{H}^p(l_s)$ with $1/r + 1/s = 1/q$, and this "translation" by $1/r$ reduces the problem to the case of all indices more than 1, which has already been settled. By duality (or by a similar argument applied to the first indices), if q_0, q_1 are both more than 1 and p_0, p_1 both finite, we also have K-closedness. Let us now check that all the other cases follow. To describe the argument, it is convenient to denote $x_0 = (1/p_0, 1/q_0)$, $x_1 = (1/p_1, 1/q_1)$ and to view these two points as the extremities on a line segment lying in the unit square of $\mathbf{R}^2$. Then using the same argument as in Proposition 1.5 (or invoking Wolff's theorem [W]) we can obtain (3.2) for the segment (x_0, x_1) everytime we know it for two subsegments (x_0, y) and (z, x_1) which intersect in a non-empty open subsegment (z, y). In this way, it is then an entirely elementary matter to check all the remaining cases, using the already settled ones. This concludes the proof in the case of finite dimensional l_p-spaces, with constants independent of the dimension. By a density argument, (note that the presence of the tildes and the substitution of c_0 for l_∞ allows the reduction to the finite dimensional case) it is easy to deduce the general case from the finite dimensional one. □

THEOREM 3.3. *In the same situation as in Theorem* 3.1, *the couple* $(\widetilde{H}^{p_0}(C_{q_0}), \widetilde{H}^{p_1}(C_{q_1}))$ *is K-closed with respect to* $(L_{p_0}(C_{q_0}), L_{p_1}(C_{q_1}))$.

Proof. This is entirely analogous to the preceding argument for Theorem 3.1, but of course we must use suitable matrix-valued extensions of the classical factorization theorems used in §1. Sarason's paper [S] contains all that is needed here, but actually, by density we need only prove the matrix valued case, with constants independent of the size of the matrix. In that case, if H is finite dimensional, it can be deduced from classical results of Wiener-Masani-Helson-Lowdenslager (see [H]) that if $1 \le p_1, s_1, r_1 \le \infty$ and $1/p_1 = 1/s_1 + 1/r_1$, every f in the unit ball of $H^\infty(C_{p_1}(H))$ can be written as a product $f = gh$ with g in the unit ball of $H^\infty(C_{s_1}(H))$ and h in the unit ball of $H^\infty(C_{r_1}(H))$. Here of course the product means the pointwise product of operator valued functions. This can be checked following the same argument as for Lemma 2.3 but using the fact that any positive matrix valued function W, such that (say) $W \ge \varepsilon I$ for some $\varepsilon > 0$, can be written as b^*b for some invertible (actually outer) matrix valued analytic function b, cf. [H]. One can get rid of ε a posteriori by a weak*-compactness limiting argument. Taking into account the remarks at the beginning of this section, we find that, if $1 \le p_0, s_0, r_0 \le \infty$ and $1/p_0 = 1/s_0 + 1/r_0$, any f in the unit ball of $H^{p_0}(C_{p_1}(H))$ can be written as a product $f = gh$ with g in the unit ball of $H^{s_0}(C_{s_1}(H))$ and h in the unit ball of $H^{r_0}(C_{r_1}(H))$. The proof of Theorem 3.3 can then be computed easily following the same line of reasoning as in §2. Let us indicate here an "economic" route for the inexperienced reader. We assume H finite dimensional, but all our constants will be independent of its dimension. Let us explain more technically what the "squaring" argument becomes in the non-commutative case. We will show that if $(\widetilde{H}^{2p_0}(C_{2q_0}), \widetilde{H}^{2p_1}(C_{2q_1}))$ is K-closed and if (3.2) holds, then $(\widetilde{H}^{p_0}(C_{q_0}), \widetilde{H}^{p_1}(C_{q_1}))$ is K-closed. To prove that consider f in $\widetilde{H}^{p_0}(C_{q_0}) + \widetilde{H}^{p_1}(C_{q_1})$ such that $K_t(f, L_{p_0}(C_{q_0}), L_{p_1}(C_{q_1})) < 1$, this means there are $f_0 \in L_{p_0}(C_{q_0})$ and $f_1 \in L_{p_1}(C_{q_1})$ such that

$$f = f_0 + f_1, \quad \|f_0\|_{L_{p_0}(C_{q_0})} + t\|f_1\|_{L_{p_1}(C_{q_1})} < 1. \tag{3.3}$$

Fix $\varepsilon > 0$. Let F be an analytic matrix valued function such that

$$F^*F = |f| + \varepsilon I \tag{3.4}$$

a.e. on $\mathbf{T}$, and such that $z \to F(z)^{-1}$ is analytic. This is possible by choosing F outer, cf. [H]. Let us denote by $\mathscr{U}$ the set of all matrix valued (measurable) functions V such that $\|V(t)\| \le 1$ a.e. on $\mathbf{T}$. Note that $|f|^{1/2} = VF$ for some $V \in \mathscr{U}$. By polar decomposition, $f = U|f| = U|f|^{1/2}VF$ for some $U \in \mathscr{U}$. Therefore, $f = GF$ with $G = U|f|^{1/2}V$. But G must be analytic since $G = fF^{-1}$. Now we claim that for some $\delta > 0$ which can be made arbitrarily small by letting ε tend to zero, we have

$$\begin{aligned} &K_{t^{1/2}}(F, L_{2p_0}(C_{2q_0}) + L_{2p_1}(C_{2q_1})) < 2^{1/2} + \delta, \\ &K_{t^{1/2}}(G, L_{2p_0}(C_{2q_0}) + L_{2p_1}(C_{2q_1})) < 2^{1/2}. \end{aligned} \tag{3.5}$$

Let us justify this. Going back to our assumption on f, we have $|f| = U^*f = U^*f_0 + U^*f_1$. Using the diagonal projection mentioned after Proposition 2.5 (more precisely, the inequality $\|\sum a_{ii}e_i \otimes e_i\|_p \leq \|(a_{ij})\|_p$ valid for any $p \geq 1$ and any orthonormal basis (e_i)), we can project the last decomposition and we obtain $|f| = g_0 + g_1$ where g_0 and g_1 are diagonal for the same basis as $|f|$ and by (3.3) they satisfy $\|g_0\|_{L_{p_0}(C_{q_0})} + t\|g_1\|_{L_{p_1}(C_{q_1})} < 1$. Now since $|f|$, g_0, g_1 all commute we can write $|f|^{1/2} \leq |g_0|^{1/2} + |g_1|^{1/2}$ from which it follows exactly as in the commutative case that

$$K_t(|f|^{1/2}, L^{2p_0}(C_{2q_0}), L^{2p_1}(C_{2q_1})) < 2^{1/2},$$

which, recalling (3.4) and the value of G, obviously implies the above claim (3.5). The rest of the proof is then clear: since we assume K-closedness for the doubled indices, we can write, for some constant C,

$$F = F_0 + F_1, \quad G = G_0 + G_1$$

with

$$\|F_0\|_{\widetilde{H}^{2p_0}(C_{2q_0})} + t^{1/2}\|F_1\|_{\widetilde{H}^{2p_1}(C_{2q_1})} < C(2^{1/2} + \delta),$$
$$\|G_0\|_{\widetilde{H}^{2p_0}(C_{2q_0})} + t^{1/2}\|G_1\|_{\widetilde{H}^{2p_1}(C_{2q_1})} < C2^{1/2}.$$

Finally, we have

$$f = GF = (G_0F_0 + G_1F_1) + (G_0F_1 + G_1F_0)$$

and we can conclude exactly as we did for Proposition 1.2. This concludes the proof of the squaring argument. With this, it is now easy to complete the proof exactly as in Theorem 3.1. □

COROLLARY 3.4. *Let H be a separable Hilbert space. The couple $(H^1(C_1(H)), H^\infty(B(H)))$ is K-closed relative to $(L_1(C_1(H)), L_\infty^w(B(H)))$, where we have denoted by $L_\infty^w(B(H))$ the space of all essentially bounded weak*-measurable functions with values in $B(H)$ (if we view the $B(H)$-valued functions as matrix valued, weak*-measurability simply means here that all the entries are measurable).*

Proof. Consider $f \in H^1(C_1(H)) + H^\infty(B(H))$ with

$$K_t(f, L_1(C_1(H)), L_\infty^w(B(H))) < 1.$$

We view f as a doubly infinite matrix valued function. Let f_n be the function which has the same entries as f on the upper left $n \times n$

square and zero elsewhere. By Theorem 3.3, for any $t > 0$, we can write $f_n = g_n + h_n$, with

$$\|g_n\|_{H^1(C_1(H))} \leq C \quad \text{and} \quad \|h_n\|_{H^\infty(B(H))} \leq C/t. \tag{3.6}$$

By compactness, we can assume w.l.o.g. that g_n and h_n converge in the weak operator topology, uniformly on compact subsets of the unit disc D to g and h. Clearly (3.6) remains valid in the limit and $f = g+h$, so that we conclude $K_t(f, H^1(C_1(H)), H^\infty(B(H))) \leq 2C$. By homogeneity, this completes the proof. □

By well-known results on the interpolation of L_p-spaces (cf. **[BL]** p. 130, note 5.8.6, and **[PT]**) the preceding results immediately imply

COROLLARY 3.5. *If* $1/p = 1 - \theta$, *then*

$$(H^1(l_1), H^\infty(l_\infty))_{\theta,p} = H^p(l_p),$$

and

$$(H^1(C_1(H)), H^\infty(B(H)))_{\theta,p} = H^p(C_p(H))$$

for any separable Hilbert space H. *Moreover, a similar result holds for the* $\widetilde{H}^p$*-spaces.*

4. Complex interpolation. In this section, we deduce the complex version of Peter Jones' theorem from the real one. Somehow, we feel that the idea in the proof of this deduction is of some (theoretical) interest even for couples of L_p spaces.

Let us denote by $H_0^{p,q}$ the subspace of $H^{p,q}$ formed by the boundary values of the analytic functions vanishing at 0. We will denote by $\overline{H}_0^{p,q}$ the subspace of $L_{p,q}$ formed by all the antianalytic functions whose complex conjugates lie in $H_0^{p,q}$. When $p = q$, as usual we denote these spaces by H_0^p, $\overline{H}_0^p$. We wish to prove that if $1 < p < \infty$, and $1/p = 1 - \theta$, then $H^p = (H^1, H^\infty)_\theta$. By standard methods, it suffices to show that

$$(L_1/\overline{H}_0^1, L_\infty/\overline{H}_0^\infty)_{1-\theta} \subset H^q \tag{4.1}$$

where $1/q = \theta$.

Let us denote by dn the counting measure on the integers. We will denote simply by $\Lambda_{q,\infty}$ the space $L_{q,\infty}(dm \otimes dn)$ and by $h^{q,\infty}$ (resp. $\overline{h}_0^{q,\infty}$) the subspace formed by the elements $f(t, n)$ such that for each n, the function $f(\cdot, n)$ is in $H^{q,\infty}$ (resp. $\overline{H}_0^{q,\infty}$). When $q = \infty$ we denote similarly by Λ_∞ and h^∞ the corresponding spaces. By the method of the preceding section, it is easy to show that the couple

$(\overline{h}_0^{1,\infty}, \overline{h}_0^{\infty})$ (which, of course is equivalent to the couple $(h^{1,\infty}, h^{\infty})$) is K-closed with respect to $(\Lambda_{1,\infty}, \Lambda_\infty)$. Indeed, by Theorem 3.1 and reiteration (cf. Remark 1.12) we know that the couple $(h^{2,\infty}, h^\infty)$ is K-closed, so that by the squaring argument, it suffices to check that

$$h^{2,\infty} \subset (h^{1,\infty}, h^\infty)_{1/2,\infty}.$$

The latter inclusion is clear since by Theorem 3.1, we have $h^{2,\infty} = (h^1, h^\infty)_{1/2,\infty}$ and $h^1 \subset h^{1,\infty}$. Thus, we have checked the K-closedness of the couple $(\overline{h}_0^{1,\infty}, \overline{h}_0^{\infty})$ with respect to $(\Lambda_{1,\infty}, \Lambda_\infty)$.

In particular, by Proposition 0.1, this implies we have a simultaneous good lifting for the quotient spaces $(\Lambda_{1,\infty}/\overline{h}_0^{1,\infty}, L_\infty(l_\infty)/\overline{h}_0^{\infty})$. From this, it is easy to deduce using the J-method that the space $(\Lambda_{1,\infty}/\overline{h}_0^{1,\infty}, L_\infty(l_\infty)/\overline{h}_0^{\infty})_{1-\theta,\infty}$ can be identified with the space $\Lambda_{q,\infty}/\overline{h}_0^{q,\infty}$ where $1/q = \theta$.

By interpolation and reiteration, the natural (i.e. orthogonal) projection is bounded from $\Lambda_{q,\infty}$ into $h_{q,\infty}$ if $1 < q < \infty$. Therefore, $\Lambda_{q,\infty}/\overline{h}_0^{q,\infty}$ can simply be identified with $h_{q,\infty}$. The result of this discussion is the following

LEMMA 4.1. *If* $1 < q < \infty$, *and* $1/q = \theta$, *then there is a bounded natural inclusion*

$$(\Lambda_{1,\infty}/\overline{h}_0^{1,\infty}, L_\infty(l_\infty)/\overline{h}_0^{\infty})_{1-\theta,\infty} \subset h_{q,\infty}.$$

We will now introduce a mapping J_q from $L_q/\overline{H}_0^q$ into $\Lambda_{q,\infty}/\overline{h}_0^{q,\infty}$ for all $1 \le q \le \infty$, as follows. We start by defining a mapping $K_q: L_q \to \Lambda_{q,\infty}$ by setting

$$\forall F \in L_q \quad K_q(F)(t, n) = n^{-1/q} F(t).$$

For any positive real x, we denote by $[x]$ the largest integer n which is less than x. Then, we have

$$\sum_{n>0} t^q m\{n^{-1/q}|F| > t\} = t^q \int \left[\frac{|F|^q}{t^q}\right] dm \le \int |F|^q \, dm. \tag{4.2}$$

Moreover, the supremum of the left side over all $t > 0$ is equal to the right-hand side (to check this, simply let t tend to 0). Hence $\|K_q(F)\| = \|F\|$, so that K_q has norm 1. Note that K_q obviously maps $\overline{H}_0^q$ into $\overline{h}_0^{q,\infty}$, and therefore we may define $J_q: L_q/\overline{H}_0^q \to \Lambda_{q,\infty}/\overline{h}_0^{q,\infty}$ as the mapping canonically associated to K_q. For $f \in L_q/\overline{H}_0^q$, if $F \in L_q$ is a representant of the equivalence class of f, then $(n^{-1/q}F)$ is a representant of $J_q(f)$. The next result is a key observation allowing us to deduce the complex interpolation theorem from the real one.

LEMMA 4.2. *If* $1 < q < \infty$, $1/q = \theta$, *the operator* J_q *defines a bounded mapping from* $(L_1/\overline{H}_0^1, L_\infty/\overline{H}_0^\infty)_{1-\theta}$ *into* $h_{q,\infty}$.

Proof. For any z with $0 < \mathrm{Re}(z) < 1$, let J^z be the operator defined exactly as J_q but with n^{z-1} in the place of $n^{-1/q}$. Then, by (4.2), if $\mathrm{Re}(z) = 0$, J^z is clearly a contraction from $L_1/\overline{H}_0^1$ into $\Lambda_{1,\infty}/\overline{h}_0^{1,\infty}$, and if $\mathrm{Re}(z) = 1$, it is a contraction from $L_\infty/\overline{H}_0^\infty$ into $L_\infty(l_\infty)/\overline{h}_0^\infty$. Hence, by complex interpolation (namely Stein's interpolation theorem for analytic families of operators), $J_q = J^{1/q}$ is a contraction from $(L_1/\overline{H}_0^1, L_\infty/\overline{H}_0^\infty)_{1-\theta}$ into $(\Lambda_{1,\infty}/\overline{h}_0^{1,\infty}, L_\infty(l_\infty)/\overline{h}_0^\infty)_{1-\theta}$, hence a fortiori (cf. e.g. **[BL]** p. 102, see also the following remark for a technical precision), it is bounded from $(L_1/\overline{H}_0^1, L_\infty/\overline{H}_0^\infty)_{1-\theta}$ into $(\Lambda_{1,\infty}/\overline{h}_0^{1,\infty}, L_\infty(l_\infty)/\overline{h}_0^\infty)_{1-\theta,\infty}$, so that we can conclude the proof by Lemma 4.1. □

REMARK. In the preceding argument, there is a slight problem because $\Lambda_{1,\infty}$ is not normable, and the complex interpolation method is usually developed in the locally convex setting (see however **[JJ]**). This difficulty can be circumvented easily. Indeed, let us denote simply $Q_1 = L_\infty(l_\infty)/\overline{h}_0^\infty$. Let B_0 be the Banach space of all sequences of measurable functions (x_n) such that $\int \sup(n|x_n|)\, dm < \infty$, equipped with the norm $\|(x_n)\| = \int \sup(n|x_n|)\, dm$. We will denote by S_0 the subspace of B_0 formed by the sequences (x_n) such that $x_n \in \overline{H}_0^1$ for all n. Finally, we set $Q_0 = B_0/S_0$. We will use the observation that $B_0 \subset \Lambda_{1,\infty}$ and this inclusion has norm one, so that we also have $Q_0 \subset \Lambda_{1,\infty}/\overline{h}_0^{1,\infty}$ with norm one. Then, the preceding argument shows that J_q is a contraction from $(L_1/\overline{H}_0^1, L_\infty/\overline{H}_0^\infty)_{1-\theta}$ into $(Q_0, Q_1)_{1-\theta}$, hence a fortiori it is bounded into $(Q_0, Q_1)_{1-\theta,\infty}$, and finally by the preceding observation, into $(\Lambda_{1,\infty}/\overline{h}_0^{1,\infty}, L_\infty(l_\infty)\overline{h}_0^\infty)_{1-\theta,\infty}$. In this manner, we have managed to remain with Banach spaces.

We can now obtain the complex case of Peter Jones' theorem as a consequence of the real case.

THEOREM 4.3. *If* $1 < p < \infty$, *and* $1/p = 1 - \theta$, *then*

$$H^p = (H^1, H^\infty)_\theta .$$

Proof. Let p be the conjugate of q, so that $1/p + 1/q = 1$. The inclusion $(H^1, H^\infty)_\theta \subset H^p$ is obvious. To prove the converse we dualize. Hence we have to prove that

$$(L_1/\overline{H}_0^1, L_\infty/\overline{H}_0^\infty)_{1-\theta} \subset H^q$$

where $1/q = \theta$. By Lemma 4.2, it suffices to show

$$\forall x \in H^\infty \quad \|x\|_{H^q} = \|(n^{-1/q}x)\|_{h_{q,\infty}}.$$

But this follows from the simple identity

$$\int |x|^q\, dm = \sup_{t>0}\left\{t^q \sum m(|x| > tn^{1/q})\right\}. \tag{4.3}$$

This concludes the proof. (Note that (4.3) means that K_q is an isometric embedding of L_q into $\Lambda^{q,\infty}$.) □

Corollary 4.4. *For any Banach space* B, *in the same situation as in Theorem* 4.3, *we have*

$$\widetilde{H}^p(B) = (\widetilde{H}^1(B), \widetilde{H}^\infty(B))_\theta, \tag{4.4}$$

and

$$H^p(B) = (H^1(B), H^\infty(B))_\theta. \tag{4.5}$$

Proof. The obvious inclusion $(\widetilde{H}^1(B), \widetilde{H}^\infty(B))_\theta \subset (L_1(B), L_\infty(B))_\theta = L_p(B)$ implies

$$(\widetilde{H}^1(B), \widetilde{H}^\infty(B))_\theta \subset \widetilde{H}^p(B).$$

Moreover, we recall that for any f in $H^p(B)$, and any $r < 1$, the function f_r defined by $f_r(z) = f(rz)$ is clearly in $\widetilde{H}^p(B)$, and $\|f\|_{H^p(B)} = \sup_{0<r<1} \|f_r\|_{\widetilde{H}^p(B)}$. Using this, we obtain similarly

$$(H^1(B), H^\infty(B))_\theta \subset H^p(B).$$

To check the converse, by the factorisation argument mentioned at the beginning of §3, we can write any f in $H^p(B)$ as a product of $f = gh$, with $g \in H^p$ and $h \in H^\infty(B)$. By Theorem 4.3, g belongs to $(H^1, H^\infty)_\theta$; hence, by interpolation, since the multiplication by h maps H^1 into $H^1(B)$ and H^∞ into $H^\infty(B)$, the function gh belongs to $(H^1(B), H^\infty(B))_\theta$. This completes the proof of (4.5). We leave the rest of the proof to the reader.

The proof of Theorem 4.3 extends with almost no change to the non-commutative case (as was pointed out to me by QuanHua Xu), as follows

THEOREM 4.5. *If* $1 < p < \infty$, *and* $1/p = 1 - \theta$, *then*

$$T_p = (T_1, T_\infty)_\theta \tag{4.6}$$

and

$$(H^1(C_1(H)), H^\infty(B(H)))_\theta = H^p(C_p(H))$$

for any separable Hilbert space H. *Moreover, a similar result holds for the* $\widetilde{H}^p$*-spaces.*

Proof. The argument is entirely similar to the above. Let us indicate how (4.6) can be checked. First, the inclusion $(T_1, T_\infty)_\theta \subset T_p$ is obvious, so that it suffices to prove the converse one. Let us denote $S_q(H) = C_q(H) \cap T_1^\perp$ and $S_{q,\infty}(H) = C_{q,\infty}(H) \cap T_1^\perp$. To abbreviate, we will sometimes write simply S_q instead of $S_q(H)$. By duality, it suffices to show, in analogy with (4.1) that

$$(T_1/S_1, T_\infty/S_\infty)_{1-\theta} \subset T_q$$

where $1/q = \theta = 1 - 1/p$. Let $H \otimes l_2$ be the Hilbert space which is the Hilbertian tensor product of H and l_2. We define a mapping K_q from $C_q(H)$ into $C_{q,\infty}(H \otimes l_2)$ by letting

$$K_q(x) = \sum n^{-1/q} x \otimes \delta_n \otimes \delta_n$$

where we have denoted by (δ_n) the canonical basis of l_2. It is easy to check that K_q is an isometric embedding from $C_q(H)$ into $C_{q,\infty}(H \otimes l_2)$. Let us denote $\widehat{H} = H \otimes l_2$. Since we assume given an orthonormal basis (e_n) in H, we can order the basis $(e_n \otimes \delta_k)$ using the lexicographic order, so that we can define as usual the notion of a "triangular" operator on $\widehat{H}$. Then obviously, K_q maps S_q into $S_{q,\infty}(\widehat{H})$, and hence it induces a mapping J_q from C_q/S_q into $C_{q,\infty}(\widehat{H})/S_{q,\infty}(\widehat{H})$. We again denote by J^z the same mapping but with n^{1-z} in the place of $n^{-1/q}$. By reasoning exactly as in Lemma 4.2, we can show that J_q is bounded from $(C_1/S_1, C_\infty/S_\infty)_{1-\theta}$ into $T_{q,\infty}(\widehat{H})$. Let us denote by $(a_k(x))_{k\geq 0}$ the sequence of singular numbers of an operator x, (i.e. with the notation of §2, we have $a_k(x) = \lambda_k(|x|))$. We observe that the sequence $(a_n(K_q(x)))$ coincides with the non-increasing rearrangement of the collection $\{n^{-1/q}a_k(x) \mid n \geq 1, k \geq 0\}$. This implies that $\|K_q(x)\|_{q,\infty} = \|x\|_q$. Hence, we can argue exactly as for Theorem 4.3 above, and we obtain (4.6). We leave the rest of the proof to the reader.

REMARK. While we were completing the present paper, we received a copy of a preprint by Paul Müller [M] which contains a strikingly simple proof of Peter Jones' theorem, or at least of Corollary 1.7 above, by an extremely simple probabilistic stopping time argument. It seems unlikely however that his idea will yield the non-commutative case. More recently (in Oberwolfach in September 91) S. Kisliakov showed me an extremely simple classical proof of Jones' theorem (Theorem 1.6) which only uses the boundedness of the Hilbert transform in L_2. This proof uses the same idea as in **[K]** (where the more delicate H^p-spaces with weights are considered), but is much simpler. See also **[KX]** for more results related to the above §3.

FINAL REMARK. The research for this paper was motivated by a preprint of Kaftal, Larson and Weiss, where Proposition 1.2* and its non-commutative analogue for nest algebras are proved for $p = 2$ using an operator algebraic method related to Arveson's distance formula. The author is most grateful to David Larson for showing him a copy of that paper and for stimulating conversations. I am also grateful to Svante Janson for observing some point that needed a correction in the preprint version.

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