

Pacific Journal of Mathematics

**ENVELOPING ALGEBRAS OF LIE GROUPS WITH DISCRETE
SERIES**

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In this article we shall prove that the enveloping algebra of the Lie algebra of a class of unimodular Lie groups having discrete series, when localized at some element of the center, is isomorphic to the tensor product of a Weyl algebra over the ring of Laurent polynomials of one variable and the enveloping algebra of some reductive algebra. In particular, it will be proved that the Lie algebra of a unimodular solvable Lie group having discrete series satisfies the Gelfand-Kirillov conjecture.

1. Introduction. Let G be a real connected Lie group with center Z , \mathcal{G} and \mathcal{Z} the Lie algebras of G and Z respectively. Let \mathcal{G}^* be the linear dual of \mathcal{G} . Then G is said to be an H -group if there exists a linear functional $l \in \mathcal{G}^*$ such that the co-adjoint orbit of l in \mathcal{G}^* is the hyperplane $l + \mathcal{Z}^\perp$ where $\mathcal{Z}^\perp = \{f \in \mathcal{G}^*; f(\mathcal{Z}) = 0\}$ (see Definition 2.1 of [2]).

In [2] it was proved that a connected Lie group G with center Z is an H -group if and only if G is unimodular and there exists $l \in \mathcal{G}^*$ such that $B_l(\cdot, \cdot) = l([\cdot, \cdot])$ is a non-degenerate skew-symmetric bilinear form on \mathcal{G}/\mathcal{Z} .

The class of H -groups plays the key role in the problem of classifying unimodular Lie groups with discrete series. Let us recall that a Lie algebra \mathcal{H} is called an H -algebra if it is the Lie algebra of an H -group. The main results of [1] and [2] may be stated in another form as follows:

A Lie algebra \mathcal{G} is the Lie algebra of some connected unimodular Lie group with discrete series iff \mathcal{G} may be written as the semi direct product of an H -algebra \mathcal{H} with center \mathcal{Z} and a reductive Lie algebra S acting trivially on \mathcal{Z} such that:

- *the maximal semisimple subalgebra of \mathcal{S} has a compact Cartan subalgebra.*
- *the center of $\text{ad}_{\mathcal{H}}(\mathcal{S})$ is the Lie subalgebra of $\text{gl}(\mathcal{H})$ corresponding to a compact torus in $\text{GL}(\mathcal{H})$*

Such an \mathcal{S} clearly acts in a completely reducible manner on \mathcal{H} . In the following we shall consider a slightly more general situation:

namely \mathcal{G} is the semidirect product of an H -algebra \mathcal{H} with center \mathcal{Z} and a subalgebra \mathcal{S} acting trivially on \mathcal{Z} such that \mathcal{H} contains an \mathcal{S} -invariant subspace $\overline{\mathcal{H}}$ complementing \mathcal{Z} . Our aim is to determine the enveloping algebra of such a semidirect product and apply this result to compute the characters of discrete series representations later. In the present article we treat only the case $\dim(\mathcal{Z}) = 1$. Although the case $\dim(\mathcal{Z}) > 1$ is not much different from this, its proof requires one to extend the ground field to an arbitrary field of characteristic 0 and will be treated in another paper.

The main result may be stated as follows:

THEOREM 1. *Let $\mathcal{G} = \mathcal{H} \odot \mathcal{S}$ and \mathcal{Z} be as above.¹ Then for any $\zeta \neq 0$ in \mathcal{Z} , the localized ring $\mathbf{A} = \mathbf{U}(\mathcal{H})_\zeta$ is isomorphic to a Weyl algebra $A_n \otimes k[\zeta, \zeta^{-1}]$, where $n = \frac{1}{2} \dim(\mathcal{H}/\mathcal{Z})$. Moreover there exists a Lie algebra homomorphism $X \mapsto a_X$ from \mathcal{S} into \mathbf{A} such that $[X, u] = [a_X, u]$, $\forall u \in \mathbf{A}$. In particular $\mathbf{U}(\mathcal{G})_\zeta$ is isomorphic to $A_n \otimes k[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathcal{S})$.*

In fact, the above isomorphism will be described in more detail for later applications (see Theorem 4.3).

The authors would like to express their gratitude to the Department of Mathematics at the University of HoChiMinh City. The first author would also like to express his gratitude to the International Center for Theoretical Physics at Trieste, Italy for its hospitality during his stay as a Visiting Scientist.

2. Notation. $\mathbf{N}, \mathbf{R}, \mathbf{C}$ always stand for the natural integers, the real and complex numbers. Recall that if \mathcal{G} is a Lie algebra with one-dimensional center $\mathcal{Z} = \mathbf{R}\zeta$, then the localized enveloping algebra $\mathbf{U}(\mathcal{G})_\zeta$ is defined to be the set of all elements of the form $\zeta^{-n}u$, $n \in \mathbf{N}$, $u \in \mathbf{U}(\mathcal{G})$ with the multiplication: $(\zeta^{-n}u)(\zeta^{-m}v) = \zeta^{-(n+m)}uv$. Let τ be the principal anti-automorphism of $\mathbf{U}(\mathcal{G})$ so that:

$$\tau(X_1 X_2 \cdots X_n) = (-1)^n X_n X_{n-1} \cdots X_1, \quad \forall X_1, \dots, X_n \in \mathcal{G}.$$

Then it is clear that τ may be extended to an anti-automorphism of $\mathbf{U}(\mathcal{G})_\zeta$ by defining: $\tau(\zeta^{-n}u) = (-1)^n \zeta^{-n} \tau(u)$. An element $u \in \mathbf{U}(\mathcal{G})_\zeta$ is said to be symmetric (resp. skew-symmetric) if $\tau(u) = u$ (resp. $\tau(u) = -u$).

¹ \odot denotes the semidirect product.

Let R be an algebra over \mathbf{R} , $n \in \mathbf{N}$; then the *Weyl algebra* $A_n(R)$ is the algebra over R generated by the set $\mathscr{W} = \{\bar{p}_1, \dots, \bar{p}_n, \bar{q}_1, \dots, \bar{q}_n\}$ with relations:

$$\bar{p}_i \bar{q}_j - \bar{q}_j \bar{p}_i = \delta_{ij}, \quad 1 \leq i \leq n$$

where δ_{ij} is the Kronecker symbol. We also say that \mathscr{W} is a *Gelfand-Kirillov basis* of $A_n(R)$. More generally, let $A(R)$ be any algebra over R ; then a generating subset $\mathscr{W} = \{p_1, \dots, p_n, q_1, \dots, q_n\}$ is said to be a *Gelfand-Kirillov basis* of $A(R)$ if the mapping: $p_i \mapsto \bar{p}_i$, $q_i \mapsto \bar{q}_i$, $1 \leq i \leq n$ may be extended to an algebra isomorphism between $A(R)$ and $A_n(R)$. We often identify $A(R)$ with $A_n(R)$ and p_i with \bar{p}_i , q_i with \bar{q}_i , $1 \leq i \leq n$.

Let \mathscr{H}_n be the $(2n + 1)$ -dimensional Heisenberg algebra with the standard basis ζ , ξ_i , η_i , $1 \leq i \leq n$ such that the only nonzero Lie brackets among the elements of this basis are:

$$[\xi_i, \eta_i] = \zeta, \quad 1 \leq i \leq n.$$

It is clear that $U(\mathscr{H}_n)_\zeta$ is a Weyl algebra over $\mathbf{R}[\zeta, \zeta^{-1}]$ with Gelfand-Kirillov basis $p_i = \xi_i$, $q_i = \zeta^{-1} \eta_i$, $1 \leq i \leq n$. Let τ be the principal anti-automorphism of $U(\mathscr{H}_n)_\zeta$. Then we have:

$$\tau(p_i) = -p_i, \quad \tau(q_i) = q_i, \quad 1 \leq i \leq n$$

and

$$\tau(\zeta) = -\zeta, \quad \tau(\zeta^{-1}) = -\zeta^{-1}$$

Such an anti-automorphism of the Weyl algebra $A_n = A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}]$ is also called the *principal anti-automorphism* of A_n .

3. The nilpotent case. Let $\mathscr{H} = \mathscr{H}_n$ be the Heisenberg algebra with standard basis ζ , ξ_i , η_i , $1 \leq i \leq n$ as above. Let $\overline{\mathscr{H}} = \sum_{i=1}^n (\mathbf{R}\xi_i + \mathbf{R}\eta_i)$. Then there is a natural symplectic form on $\overline{\mathscr{H}}$ with the canonical symplectic basis ξ_i , η_i , $1 \leq i \leq n$. The matrix of any $X \in \text{sp}(\overline{\mathscr{H}})$ with respect to this basis has the form:

$$\begin{pmatrix} \mathbf{a}^X & \mathbf{b}^X \\ \mathbf{c}^X & -{}^t\mathbf{a}^X \end{pmatrix}$$

where \mathbf{a}^X , \mathbf{b}^X , \mathbf{c}^X are $n \times n$ -real matrices such that \mathbf{b}^X and \mathbf{c}^X are

symmetric, and ${}^t\mathbf{a}^X$ is the transpose of \mathbf{a}^X . Put

$$\begin{aligned} S_X &= -\frac{1}{2}\zeta^{-1} \sum_{i,j=1}^n \mathbf{a}_{ij}^X (\xi_i \eta_j + \eta_j \xi_i) + \frac{1}{2}\zeta^{-1} \sum_{i,j=1}^n (\mathbf{b}_{ij}^X \xi_i \xi_j - \mathbf{c}_{ij}^X \eta_i \eta_j) \\ &= -\frac{1}{2} \sum_{i,j=1}^n \mathbf{a}_{ij}^X (p_i q_j + q_j p_i) + \frac{1}{2}\zeta^{-1} \sum_{i,j=1}^n \mathbf{b}_{ij}^X p_i p_j \\ &\quad - \frac{1}{2}\zeta \sum_{i,j=1}^n \mathbf{c}_{ij}^X q_i q_j. \end{aligned}$$

LEMMA 3.1. $X \mapsto S_X$ is a Lie algebra homomorphism from $\mathfrak{sp}(\overline{\mathcal{H}})$ into $\mathbf{U}(\mathcal{H})_\zeta$ such that

$$[X, u] = [S_X, u], \quad \forall X \in \mathfrak{sp}(\overline{\mathcal{H}}), \quad \forall u \in \mathbf{U}(\mathcal{H})_\zeta.$$

Proof. For $1 \leq i_0 \leq n$ we have:

$$[S_X, \xi_{i_0}] = \sum_{i=1}^n \mathbf{a}_{i i_0}^X \xi_i + \sum_{i=1}^n \mathbf{c}_{i i_0}^X \eta_i = [X, \xi_{i_0}].$$

Similarly, we have:

$$[S_X, \eta_{i_0}] = [X, \eta_{i_0}].$$

Hence it follows that:

$$[S_X, u] = [X, u], \quad \forall u \in \mathbf{U}(\mathcal{H})_\zeta.$$

Finally by using the commutation relations:

$$\begin{aligned} [p_i q_j, p_k q_l] &= \delta_{il} p_k q_j - \delta_{jk} p_i q_l, \\ [p_i q_j, p_k p_l] &= -\delta_{jk} p_i p_l - \delta_{jl} p_i p_k, \\ [p_i q_j, q_k q_l] &= \delta_{ik} q_l q_j + \delta_{il} q_k q_j, \\ [p_i p_j, q_k q_l] &= \delta_{ik} q_l p_j + \delta_{il} q_k p_j + \delta_{jk} p_k q_l + \delta_{jl} p_i p_k, \end{aligned}$$

we see that

$$[S_X, S_Y] = S_{[X, Y]}, \quad \forall X, Y \in \mathfrak{sp}(\mathcal{H}). \quad \square$$

REMARK. The above expression of S_X is just the expression of D_n in [3] rewritten in the terminology of enveloping algebras instead of that of symmetric algebras as in [3].

Now let $\mathcal{G} = \mathcal{H} \odot \mathcal{S}$ where \mathcal{H} is an H -algebra with one-dimensional center $\mathcal{Z} = \mathbf{R}\zeta$. Assume that \mathcal{Z} centralizes \mathcal{S} and that \mathcal{H} contains an \mathcal{S} -invariant subspace $\overline{\mathcal{H}}$ complementing \mathcal{Z} . Let \mathcal{N} be

the greatest nilpotent ideal of \mathcal{H} . Assume also that the center of \mathcal{N} is equal to \mathcal{Z} and that there exists an abelian ideal \mathcal{K} of \mathcal{G} contained in \mathcal{H} such that \mathcal{K}/\mathcal{Z} is central in \mathcal{N}/\mathcal{Z} . Put $\overline{\mathcal{H}} = \mathcal{H} \cap \overline{\mathcal{H}}$, $\overline{\mathcal{N}} = \mathcal{N} \cap \overline{\mathcal{H}}$. Let η_1, \dots, η_m be a basis of $\overline{\mathcal{H}}$. Put:

$$\mathcal{H}_0\{X \in \mathcal{H}; [X, \overline{\mathcal{H}}] \subset \overline{\mathcal{H}}\}.$$

Then $\mathcal{N}_0 = \mathcal{H}_0 \cap \mathcal{N}$ is precisely the centralizer of \mathcal{K} in \mathcal{N} . Let $\overline{\mathcal{H}}_0 = \mathcal{H}_0 \cap \overline{\mathcal{H}}$ and $\overline{\mathcal{N}}_0 = \mathcal{N}_0 \cap \overline{\mathcal{H}}$. Let $m = \dim \overline{\mathcal{H}}$ and $n = \frac{1}{2} \dim(\mathcal{H}/\mathcal{Z})$. Then we have

PROPOSITION 3.2. *Let the notation be as above. Let X_1, \dots, X_{2n} be any basis of \mathcal{H} . Then there exist a Weyl subalgebra A_m of $\mathbf{U}(\mathcal{G})_\zeta$ with Gelfand-Kirillov basis $\mathcal{W} = \{p_i, q_i; 1 \leq i \leq m\}$ and a Lie algebra homomorphism χ from $\mathcal{H}_0 \odot \mathcal{S}$ onto a Lie subalgebra $\tilde{\mathcal{G}}$ of $\mathbf{U}(\mathcal{G})_\zeta$ satisfying the following properties:*

1. $\mathbf{U}(\tilde{\mathcal{G}})_\zeta$ can be identified with a subalgebra of $\mathbf{U}(\mathcal{G})_\zeta$ commuting with A_m such that

$$\mathbf{U}(\mathcal{G})_\zeta \simeq \mathbf{U}(\tilde{\mathcal{G}})_\zeta \otimes A_m.$$

Moreover the restriction of the principal anti-automorphism τ of $\mathbf{U}(\mathcal{G})_\zeta$ to $\mathbf{U}(\tilde{\mathcal{G}})_\zeta$ coincides with the principal anti-automorphism of the latter, and:

$$\tau(p_i) = -p_i, \quad \tau(q_i) = q_i, \quad 1 \leq i \leq m.$$

2. Let $\tilde{\mathcal{H}} = \chi(\mathcal{H}_0)$; then χ induces an isomorphism from $\mathcal{H}_0/\overline{\mathcal{H}}$ onto $\tilde{\mathcal{H}}$. Moreover there exists a basis $\tilde{X}_1, \dots, \tilde{X}_{2n-2m}$ of $\tilde{\mathcal{H}}$ such that each X_i may be expressed as:

$$(1) \quad X_i = \zeta G_i(q, \zeta^{-1}\tilde{X}, \zeta^{-1}p), \quad 1 \leq i \leq 2n$$

where $q = (q_1, \dots, q_m)$, $\zeta^{-1}\tilde{X} = (\zeta^{-1}\tilde{X}_1, \dots, \zeta^{-1}\tilde{X}_{2n-2m})$, $\zeta^{-1}p = (\zeta^{-1}p_1, \dots, \zeta^{-1}p_m)$ and each G_i is a polynomial of $2n$ indeterminates $\theta = (\theta_1, \dots, \theta_m)$, $\psi = (\psi_1, \dots, \psi_{2n-2m})$, $\omega = (\omega_1, \dots, \omega_m)$ which are in fact linear combinations of $1, \psi, \omega$ with coefficients in $\mathbf{R}[\theta]$ such that the mapping $(\theta, \psi, \omega) \mapsto (G_i(\theta, \psi, \omega))_{1 \leq i \leq 2n}$ is an automorphism of the polynomial ring $\mathbf{R}[\theta, \psi, \omega]$ with Jacobian 1.

3. χ is, in fact, an isomorphism from \mathcal{S} onto $\tilde{\mathcal{S}} = \chi(\mathcal{S})$ and the action of $\tilde{\mathcal{S}}$ on $\tilde{\mathcal{H}}$ is induced from that of \mathcal{S} on $\mathcal{H}_0/\overline{\mathcal{H}}$. Moreover for each $Y \in \mathcal{S}$, $\chi(Y)$ can be expressed as:

$$(2) \quad \chi(Y) = Y - \zeta S_Y(q, \zeta^{-1}\tilde{X}, \zeta^{-1}p),$$

where the polynomial $S_Y(\theta, \psi, \omega)$ is a linear combination of $1, \psi, \omega$ with coefficients in $\mathbf{R}[\theta]$.

Proof. By making a preliminary change of basis if necessary, we may assume that the basis has the form: $\{\eta_1, \dots, \eta_m, X_1, \dots, X_{2n-2m}, \xi_1, \dots, \xi_m\}$ where:

- η_1, \dots, η_m is a basis of $\overline{\mathcal{H}}$,
- X_1, \dots, X_r is a basis of $\overline{\mathcal{N}}_0 \bmod \overline{\mathcal{H}}$,
- $X_{r+1}, \dots, X_{2n-2m}$ is a basis of $\overline{\mathcal{H}}_0 \bmod \overline{\mathcal{N}}_0$,
- ξ_1, \dots, ξ_m is a basis of $\overline{\mathcal{N}} \bmod \overline{\mathcal{N}}_0$.

Moreover it follows from Proposition 3.1 of [2] (see also Proposition 4.2 of [1]) that ξ_1, \dots, ξ_m may be chosen so that:

$$[\xi_i, \eta_j] = \delta_{ij} \zeta, \quad 1 \leq i \leq m.$$

Put $q_i = \zeta^{-1} \eta_i$, $1 \leq i \leq m$. Now for every $X \in \mathcal{H}_0 \odot \mathcal{S}$ there exists a real $m \times m$ -matrix S^X such that:

$$[X, \eta_i] = - \sum_{j=1}^m S_{ij}^X \eta_j, \quad 1 \leq i \leq m.$$

Note that $S^X = 0$ if $X \in \mathcal{N}_0$. Let l be the linear form on \mathcal{H} such that $l(\zeta) = 1$, $l(\overline{\mathcal{H}}) = 0$, and let B_l be the associated skew-symmetric bilinear form on \mathcal{H} . For $X \in \mathcal{H}_0 \odot \mathcal{S}$ and $1 \leq i, j \leq m$ we have:

$$B_l([X, \xi_i], \eta_j) + B_l(\xi_i, [X, \eta_j]) = l([X, [\xi_i, \eta_j]]) = 0.$$

Hence

$$[X, \xi_j] = \sum_{i=1}^m S_{ij}^X \xi_i \pmod{\mathcal{N}_0}.$$

Put

$$S_X = -\frac{1}{2} \sum_{i,j=1}^m S_{ij}^X (\xi_i q_j + q_j \xi_i).$$

Then $X - S_X$ commutes with the q_i 's. Moreover for $1 \leq i \leq m$ we have:

$$(3) \quad [X - S_X, \xi_i] = 0 \pmod{\mathcal{N}_0 + \sum_{j=1}^m q_j \mathcal{N}_0}.$$

It follows that for $X, Y \in \mathcal{H}_0 \odot \mathcal{S}$ we have:

$$[X, S_Y] = [S_X, Y] = [S_X, S_Y] \pmod{\sum_{i=1}^m q_i \mathcal{N}_0 + \sum_{i,j=1}^m q_i q_j \mathcal{N}_0}.$$

Hence

$$\begin{aligned} [X - S_X, Y - S_Y] &= [X, Y] - [X, S_Y] - [S_X, Y] + [S_X, S_Y] \\ &= [X, Y] - [S_X, S_Y] \left(\text{mod } \sum_{i=1}^m q_i \mathcal{N}_0 + \sum_{i,j=1}^m q_i q_j \mathcal{N}_0 \right). \end{aligned}$$

On the other hand, by a similar computation as in the proof of Lemma 3.1 we see that:

$$[S_X, S_Y] = S_{[X, Y]} \left(\text{mod } \sum_{i,j=1}^m q_i q_j \mathcal{N}_0 \right).$$

Hence

$$\begin{aligned} (4) \quad [X - S_X, Y - S_Y] &= [X, Y] - S_{[X, Y]} \left(\text{mod } \sum_{i=1}^m q_i \mathcal{N}_0 + \sum_{i,j=1}^m q_i q_j \mathcal{N}_0 \right). \end{aligned}$$

Let Y_1, \dots, Y_t be a basis of \mathcal{S} . Then it follows from (3) that

$$\begin{aligned} \mathbf{U}(\mathcal{G})_\zeta &= \mathbf{U}(\mathcal{N}_0)_\zeta [\xi_1, \dots, \xi_m] [X_{r+1} - S_{X_{r+1}}, \dots, X_{2n-2m} - S_{X_{2n-2m}}] \\ &\quad \cdot [Y_1 - S_{Y_1}, \dots, Y_t - S_{Y_t}] \\ &= \mathbf{U}(\mathcal{N}_0)_\zeta [X_{r+1} - S_{X_{r+1}}, \dots, X_{2n-2m} - S_{X_{2n-2m}}] \\ &\quad \cdot [Y_1 - S_{Y_1}, \dots, Y_t - S_{Y_t}] [\xi_1, \dots, \xi_m] \\ &= \mathbf{A}[\xi_1, \dots, \xi_m] \end{aligned}$$

where

$$\mathbf{A} = \mathbf{U}(\mathcal{N}_0)_\zeta [X_{r+1} - S_{X_{r+1}}, \dots, X_{2n-2m} - S_{X_{2n-2m}}] [Y_1 - S_{Y_1}, \dots, Y_t - S_{Y_t}].$$

Put $p_1 = \xi_1$, and for $1 \leq i \leq m-1$ put

$$p_{i+1} = \sum_{j_1, \dots, j_i} \frac{(-1)^{j_1 + \dots + j_i}}{j_1! \dots j_i!} (\text{ad } \xi_1)^{j_1} \dots (\text{ad } \xi_i)^{j_i} (\xi_{i+1}) q_1^{j_1} \dots q_i^{j_i}.$$

On the other hand for $Y \in \mathbf{A}$ put

$$\nu(Y) = \sum_{j_1, \dots, j_m} \frac{(-1)^{j_1 + \dots + j_m}}{j_1! \dots j_m!} (\text{ad } \xi_1)^{j_1} \dots (\text{ad } \xi_m)^{j_m} (Y) q_1^{j_1} \dots q_m^{j_m}.$$

Now by applying successively Lemma 4.7.6 of [5] we see that ν is a homomorphism from \mathbf{A} onto a subalgebra $\tilde{\mathbf{A}}$ of $\mathbf{U}(\mathcal{G})_\zeta$ commuting with the p_i 's and q_i 's so that

$$\mathbf{U}(\mathcal{G})_\zeta \simeq \tilde{\mathbf{A}} \otimes A_m.$$

Note that it follows also from Lemma 4.7.5 of [5] that ν induces an isomorphism from $\mathbf{A}/\overline{\mathcal{H}}\mathbf{A}$ onto $\tilde{\mathbf{A}}$. On the other hand it follows from (4) that $\{X - S_X + \overline{\mathcal{H}}\mathbf{A}; X \in \mathcal{H}_0\}$ (resp. $\{Y - S_Y + \overline{\mathcal{H}}\mathbf{A}; Y \in \mathcal{S}\}$) is a Lie subalgebra of $\mathbf{A}/\overline{\mathcal{H}}\mathbf{A}$ isomorphic to $\mathcal{H}_0/\overline{\mathcal{H}}$ (resp. \mathcal{S}). Thus $X \mapsto \chi(X) = \nu(X - S_X)$ is a Lie algebra homomorphism from $\mathcal{H}_0 \odot \mathcal{S}$ onto a Lie subalgebra $\tilde{\mathcal{G}}$ of $\tilde{\mathbf{A}}$ which induces an isomorphism from $(\mathcal{H}_0/\overline{\mathcal{H}}) \odot \mathcal{S}$ onto $\tilde{\mathcal{G}}$. Note that $\tilde{\mathbf{A}}$ can be identified with $\mathbf{U}(\tilde{\mathcal{G}})_{\zeta}$. Moreover let $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{S}}$ be the images of \mathcal{H}_0 and \mathcal{S} respectively; then the action of \mathcal{S} on $\mathcal{H}_0/\overline{\mathcal{H}}$ is transformed into the action of $\tilde{\mathcal{S}}$ on $\tilde{\mathcal{H}}$.

Now it is clear that $\tilde{X}_j = \chi(X_j)$ ($1 \leq j \leq 2n - 2m$) and p_j ($1 \leq j \leq m$) may be expressed in the form

$$(5) \quad \begin{aligned} \tilde{X}_j &= \sum_{i=1}^r \mathbf{a}_{ij} X_i + e_j \zeta, & 1 \leq j \leq r, \\ \tilde{X}_{r+j} &= X_{r+j} + \sum_{i=1}^r \mathbf{b}_{ij} X_i + \sum_{i=1}^m \mathbf{c}_{ij} \xi_i + e_{r+j} \zeta, \\ & & 1 \leq j \leq 2n - 2m - r, \\ p_j &= \xi_j + \sum_{i=1}^r \mathbf{d}_{ij} X_i + f_j \zeta, & 1 \leq j \leq m, \end{aligned}$$

where $e_1, \dots, e_{2n-2m}, f_1, \dots, f_m \in \mathbf{R}[q]$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are matrices with coefficients in $\mathbf{R}[q]$ of dimension $r \times r$, $r \times (2n - 2m - r)$, $m \times (2n - 2m - r)$ and $r \times m$ respectively. Moreover since \mathcal{N} is nilpotent, we may choose X_i , $1 \leq i \leq r$ so that \mathbf{a} is a unipotent matrix and hence $\det(\mathbf{a}) = 1$. For an arbitrary basis $\{X_i, 1 \leq i \leq r\}$ of $\overline{\mathcal{N}}_0$ we can make a change of basis for $\{\tilde{X}_i\}$ with real matrix coefficients which preserves $\det(\mathbf{a})$. Therefore \mathbf{a}^{-1} is also a matrix with coefficients in $\mathbf{R}[q]$. Hence it follows that the η_i 's, X_i 's and ξ_i 's may be expressed in the form (1) with

- $G_i(\theta, \psi, \omega) = \theta_i$, $1 \leq i \leq m$,
- for $1 \leq i \leq r$, $G_{m+i}(\theta, \psi, \omega)$ is a linear combination of $1, \psi_1, \dots, \psi_r$ with coefficients in $\mathbf{R}[\theta]$,
- for $1 \leq i \leq 2n - 2m - r$, $G_{m+r+i}(\theta, \psi, \omega) - \psi_{r+i}$ is a linear combination of $1, \psi_1, \dots, \psi_r, \omega$ with coefficients in $\mathbf{R}[\theta]$,
- for $1 \leq i \leq m$, $G_{2n-m+i}(\theta, \psi, \omega) - \omega_i$ is a linear combination of $1, \psi_1, \dots, \psi_r$ with coefficients in $\mathbf{R}[\theta]$.

Hence it is clear that the polynomial map defined by the G_i 's is an automorphism of the polynomial ring $\mathbf{R}[\theta, \psi, \omega]$ with Jacobian 1.

Finally (2) follows immediately from the definition of χ and a similar computation as above. Note that X_1, \dots, X_r commute with the q_i 's so that

$$\tilde{X}_j = \sum_{i=1}^r X_i \mathbf{a}_{ij} + e_j \zeta, \quad 1 \leq j \leq r.$$

Therefore

$$\tau(\tilde{X}_j) = \sum_{i=1}^r \tau(\mathbf{a}_{ij})\tau(X_i) + \tau(e_j \zeta) = - \sum_{i=1}^r \mathbf{a}_{ij} X_i - e_j \zeta = -\tilde{X}_j.$$

This together with (5) imply that the restriction of τ to $\mathbf{U}(\tilde{\mathcal{G}})_\zeta$ is precisely the principal anti-automorphism of $\mathbf{U}(\tilde{\mathcal{G}})_\zeta$. \square

THEOREM 3.3. *Let $\mathcal{G} = \mathcal{H} \odot \mathcal{S}$ where \mathcal{H} is a nilpotent H -algebra with one-dimensional center $\mathcal{Z} = \mathbf{R}_\zeta$. Assume that \mathcal{Z} centralizes \mathcal{S} and that \mathcal{H} contains an \mathcal{S} -invariant subspace $\overline{\mathcal{H}}$ complementing \mathcal{Z} . Let $n = \frac{1}{2} \dim(\mathcal{H}/\mathcal{Z})$.*

1. *Under these conditions, for an arbitrary basis X_1, \dots, X_{2n} of $\overline{\mathcal{H}}$, there exists a Gelfand-Kirillov basis $\mathcal{W} = \{p_i, q_i; 1 \leq i \leq n\}$ of $\mathbf{U}(\mathcal{H})_\zeta$ such that*

(i) $\tau(p_i) = -p_i$, $\tau(q_i) = q_i$, $1 \leq i \leq n$ where τ is the principal anti-automorphism of $\mathbf{U}(\mathcal{H})_\zeta$;

(ii) for $1 \leq i \leq 2n$, $\zeta^{-1}X_i$ is a linear combination of 1 , $\zeta^{-1}p_1, \dots, \zeta^{-1}p_n$ with coefficients in $\mathbf{R}[q]$ and the corresponding polynomials of $2n$ indeterminates $\theta_1, \dots, \theta_n$, $\omega_1, \dots, \omega_n$ define an automorphism of the polynomial ring $\mathbf{R}[\theta, \omega]$ with Jacobian 1.

2. *For each $Y \in \mathcal{S}$ there exists a polynomial $a_Y(\theta, \omega)$ which is a polynomial of degree ≤ 2 in $\omega_1, \dots, \omega_n$ with coefficients in $\mathbf{R}[\theta]$ such that:*

(i) $Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from \mathcal{S} into $\mathbf{U}(\mathcal{H})_\zeta$;

(ii) $a_Y(q, \zeta^{-1}p)$ is symmetric and

$$[Y, u] = [\zeta a_Y(q, \zeta^{-1}p), u], \quad \forall u \in \mathbf{U}(\mathcal{H})_\zeta;$$

(iii) *the mapping $Y \mapsto Y - \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra isomorphism from \mathcal{S} onto a Lie subalgebra \mathcal{S}' of $\mathbf{U}(\mathcal{G})_\zeta$ so that*

$$\begin{aligned} \mathbf{U}(\mathcal{G})_\zeta &\simeq \mathbf{U}(\mathcal{H})_\zeta \otimes \mathbf{U}(\mathcal{S}') \\ &\simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathcal{S}). \end{aligned}$$

Proof. The proof is carried out by induction on $\dim(\mathcal{H})$. If \mathcal{H} is isomorphic to a Heisenberg algebra with center \mathcal{Z} then the theorem follows from Lemma 3.1. Otherwise there is always an abelian ideal \mathcal{K} of \mathcal{G} contained in \mathcal{H} satisfying the conditions of Proposition 3.2 (see Proposition 2.3 of [1]). By making a preliminary change of basis if necessary we may assume that

$$\begin{aligned} X_i &= \eta_i, & 1 \leq i \leq m, \\ X_{2n-m+i} &= \xi_i, & 1 \leq i \leq m, \end{aligned}$$

where $m = \dim(\mathcal{H}/\mathcal{Z})$. Hence it follows from Proposition 3.2 that there exist a Lie algebra homomorphism χ from $\mathcal{H}_0 \odot \mathcal{S}$ onto a Lie subalgebra $\tilde{\mathcal{G}}$ of $\mathbf{U}(\mathcal{G})_\zeta$ and elements p_i , $1 \leq i \leq m$ of $\mathbf{U}(\mathcal{H})_\zeta$ satisfying the following properties.

- \mathbf{A}_m be the subalgebra generated by $\mathcal{W}_1 = \{p_i, q_i; 1 \leq i \leq m\}$ which is in fact a Weyl algebra with Gelfand-Kirillov basis \mathcal{W}_1 . Then

$$\mathbf{U}(\mathcal{G})_\zeta \simeq \mathbf{U}(\tilde{\mathcal{G}})_\zeta \otimes \mathbf{A}_m.$$

- Let τ be the principal anti-automorphism of $\mathbf{U}(\mathcal{G})_\zeta$. Then the restriction of τ to $\mathbf{U}(\tilde{\mathcal{G}})_\zeta$ coincides with the principal anti-automorphism of the latter, and furthermore

$$\tau(p_i) = -p_i, \quad \tau(q_i) = q_i \quad 1 \leq i \leq m,$$

- χ induces an isomorphism from $\mathcal{H}_0/\overline{\mathcal{H}}$ onto $\tilde{\mathcal{H}} = \chi(\mathcal{H}_0)$ and

$$\mathbf{U}(\mathcal{H})_\zeta \simeq \mathbf{U}(\tilde{\mathcal{H}})_\zeta \otimes \mathbf{A}_m.$$

- For $m+1 \leq i \leq 2n$, $\zeta^{-1}X_i$ may be expressed as a linear combination of $1, \zeta^{-1}\tilde{X}_1, \dots, \zeta^{-1}\tilde{X}_{2n-2m}, \zeta^{-1}p_1, \dots, \zeta^{-1}p_m$ with coefficients in $\mathbf{R}[q_1, \dots, q_m]$, where $\tilde{X}_1, \dots, \tilde{X}_{2n-2m}$ is a basis of $\tilde{\mathcal{H}}$ as described in Proposition 3.2. Let $G_i(\theta, \psi, \omega)$ be the corresponding real polynomials. Then the mapping

$$(\theta, \psi, \omega) \mapsto (\theta, G_{m+1}(\theta, \psi, \omega), \dots, G_{2n}(\theta, \psi, \omega))$$

is an automorphism of $\mathbf{R}[\theta, \psi, \omega]$ with Jacobian 1.

- χ is in fact an isomorphism from \mathcal{S} onto $\tilde{\mathcal{S}}$ such that

$$\chi(Y) = Y - \zeta \hat{s}_Y(\hat{q}, \zeta^{-1}\tilde{X}, \zeta^{-1}\hat{p}), \quad \forall Y \in \mathcal{S}$$

where $\hat{q} = (q_1, \dots, q_m)$, $\hat{p} = (p_1, \dots, p_m)$, $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_{2n-2m})$, and $\hat{s}_Y(\theta, \psi, \omega)$ is a linear combination of $1, \psi_1, \dots, \psi_{2n-2m}, \omega_1, \dots, \omega_m$ with coefficients in $\mathbf{R}[\theta]$.

Now by the induction hypothesis $\mathbf{U}(\widetilde{\mathcal{H}})_\zeta$ is isomorphic to a Weyl algebra with Gelfand-Kirillov basis $\widetilde{\mathcal{W}} = \{\tilde{p}_i, \tilde{q}_i; 1 \leq i \leq n - m\}$ where the following hold

$$(\alpha) \quad \tau(\tilde{p}_i) = -\tilde{p}_i, \tau(\tilde{q}_i) = \tilde{q}_i, \quad 1 \leq i \leq n - m,$$

(β) For $1 \leq i \leq 2n - 2m$, $\zeta^{-1}\tilde{X}_i$ is a linear combination of $1, \zeta^{-1}\tilde{p}_1, \dots, \zeta^{-1}\tilde{p}_{n-m}$ with coefficients in $\mathbf{R}[\tilde{q}]$ such that the corresponding polynomials $\tilde{F}_i(\tilde{\theta}, \tilde{\omega})$, $1 \leq i \leq 2n - 2m$ of $2n - 2m$ indeterminates $(\tilde{\theta}, \tilde{\omega}) \equiv (\tilde{\theta}_1, \dots, \tilde{\theta}_{n-m}, \tilde{\omega}_1, \dots, \tilde{\omega}_{n-m})$ determine an automorphism \tilde{F} of $\mathbf{R}[\tilde{\theta}, \tilde{\omega}]$ with Jacobian 1. Put

$$F_i(\theta, \tilde{\theta}, \tilde{\omega}, \omega) = \begin{cases} \theta_i, & 1 \leq i \leq m, \\ G_i(\theta, \tilde{F}(\tilde{\theta}, \tilde{\omega}), \omega), & m + 1 \leq i \leq 2n. \end{cases}$$

Then the mapping

$$(\theta, \tilde{\theta}, \tilde{\omega}, \omega) \mapsto (F_1(\theta, \tilde{\theta}, \tilde{\omega}, \omega), \dots, F_{2n}(\theta, \tilde{\theta}, \tilde{\omega}, \omega))$$

is an automorphism of $\mathbf{R}[\theta, \tilde{\theta}, \tilde{\omega}, \omega]$ with Jacobian 1. Moreover we have

$$X_i = \zeta F_i(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}), \quad 1 \leq i \leq 2n.$$

On the other hand it follows also from the induction hypothesis that for each $\tilde{Y} \in \widetilde{\mathcal{S}}$ there exists a polynomial $\tilde{a}_{\tilde{Y}}(\tilde{\theta}, \tilde{\omega})$ which is in fact a polynomial of degree ≤ 2 in $\tilde{\omega}_1, \dots, \tilde{\omega}_{n-m}$ with coefficients in $\mathbf{R}[\tilde{\theta}]$ such that $\tilde{Y} \mapsto \zeta \tilde{a}_{\tilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p})$ is a Lie algebra homomorphism from $\widetilde{\mathcal{S}}$ into $\mathbf{U}(\widetilde{\mathcal{H}})_\zeta$ and moreover

$$(6) \quad [\tilde{Y}, \tilde{u}] = [\zeta \tilde{a}_{\tilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p}), \tilde{u}], \quad \forall \tilde{u} \in \mathbf{U}(\widetilde{\mathcal{H}})_\zeta.$$

Put

$$a_Y(\theta, \tilde{\theta}, \tilde{\omega}, \omega) = \tilde{a}_{\chi(Y)}(\tilde{\theta}, \tilde{\omega}) + s_Y(\theta, \tilde{\theta}, \tilde{\omega}, \omega), \quad \forall Y \in \mathcal{S}$$

where $S_Y(\theta, \tilde{\theta}, \tilde{\omega}, \omega) = \hat{s}_Y(\theta, \tilde{F}(\tilde{\theta}, \tilde{\omega}), \omega)$. Then for $\tilde{Y} = \chi(Y)$ we have

$$\begin{aligned} Y - \zeta a_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}) \\ &= Y - \zeta \tilde{a}_{\tilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p}) - \zeta s_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}) \\ &= \tilde{Y} - \zeta \tilde{a}_{\tilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p}). \end{aligned}$$

Hence

$$[Y, \tilde{u}] - [\zeta a_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}), \tilde{u}] = 0, \quad \forall \tilde{u} \in \mathbf{U}(\widetilde{\mathcal{H}})_\zeta.$$

On the other hand since \tilde{Y} and $\zeta \tilde{a}_{\tilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p})$ commute with $\{p_i, q_i; 1 \leq i \leq m\}$ we have

$$(7) \quad [Y, u] = [\zeta a_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}), u], \quad \forall u \in \mathbf{U}(\mathcal{H})_\zeta.$$

Now it follows from (6) that

$$\begin{aligned} [\zeta \tilde{a}_{\tilde{Y}_1}(\tilde{q}, \zeta^{-1}\tilde{p}), \zeta \tilde{a}_{\tilde{Y}_2}(\tilde{q}, \zeta^{-1}\tilde{p})] &= [\tilde{Y}_1, \zeta \tilde{a}_{\tilde{Y}_2}(\tilde{q}, \zeta^{-1}\tilde{p})] \\ &= [\zeta \tilde{a}_{\tilde{Y}_1}(\tilde{q}, \zeta^{-1}\tilde{p}), \tilde{Y}_2]. \end{aligned}$$

Hence

$$\begin{aligned} &[\tilde{Y}_1 - \zeta \tilde{a}_{\tilde{Y}_1}(\tilde{q}, \zeta^{-1}\tilde{p}), \tilde{Y}_2 - \zeta \tilde{a}_{\tilde{Y}_2}(\tilde{q}, \zeta^{-1}\tilde{p})] \\ &= [\tilde{Y}_1, \tilde{Y}_2] - \zeta \tilde{a}_{[\tilde{Y}_1, \tilde{Y}_2]}(\tilde{q}, \zeta^{-1}\tilde{p}) \quad \forall \tilde{Y}_1, \tilde{Y}_2 \in \tilde{\mathcal{S}}. \end{aligned}$$

Put $p = (\hat{p}, \tilde{p})$, $q = (\hat{q}, \tilde{q})$. Then for $Y_1, Y_2 \in \mathcal{S}$ and $\tilde{Y}_i = \chi(Y_i)$, $i = 1, 2$, we have

$$\begin{aligned} &[Y_1 - \zeta a_{Y_1}(q, \zeta^{-1}p), Y_2 - \zeta a_{Y_2}(q, \zeta^{-1}p)] \\ &= [\tilde{Y}_1, \tilde{Y}_2] - \zeta \tilde{a}_{[\tilde{Y}_1, \tilde{Y}_2]}(\tilde{q}, \zeta^{-1}\tilde{p}) \\ &= \chi([Y_1, Y_2]) - \zeta \tilde{a}_{\chi([Y_1, Y_2])}(\tilde{q}, \zeta^{-1}\tilde{p}) \\ &= [Y_1, Y_2] - \zeta a_{[Y_1, Y_2]}(q, \zeta^{-1}p) \end{aligned}$$

i.e. $Y \mapsto Y - \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism which is in fact an isomorphism. Let \mathcal{S}' be the image of \mathcal{S} by this isomorphism. Then \mathcal{S}' commutes with $\mathbf{U}(\mathcal{H})_\zeta$ and hence

$$\mathbf{U}(\mathcal{G})_\zeta \simeq \mathbf{U}(\mathcal{H})_\zeta \otimes \mathbf{U}(\mathcal{S}') \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathcal{S}).$$

Finally (7) implies that

$$\begin{aligned} &[\zeta a_{Y_1}(q, \zeta^{-1}p), \zeta a_{Y_2}(q, \zeta^{-1}p)] \\ &= [Y_1, \zeta a_{Y_2}(q, \zeta^{-1}p)] \\ &= [\zeta a_{Y_1}(q, \zeta^{-1}p), Y_2], \quad Y_1, Y_2 \in \mathcal{S}. \end{aligned}$$

Hence

$$\begin{aligned} &[Y_1, Y_2] - \zeta a_{[Y_1, Y_2]}(q, \zeta^{-1}p) \\ &= [Y_1, Y_2] - [\zeta a_{Y_1}(q, \zeta^{-1}p), \zeta a_{Y_2}(q, \zeta^{-1}p)] \end{aligned}$$

i.e. $Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from \mathcal{S} into $\mathbf{U}(\mathcal{H})_\zeta$. \square

REMARK. This theorem contains Lemma 3.2 and Theorem 3.5 of [6] as special cases.

4. The general case. Let $\mathcal{G} = \mathcal{H} \odot \mathcal{S}$ as in Theorem 1 of the Introduction. Assume also that there exists a nilpotent ideal \mathcal{N} of \mathcal{G} contained in \mathcal{H} such that

- \mathcal{N} is an H -algebra with center \mathcal{Z} ,
- the action of \mathcal{S} on \mathcal{H}/\mathcal{N} is trivial.

Then it follows from Theorem 2.9 and Lemma 2.3 of [2] that there exists a Heisenberg subalgebra \mathcal{H}_1 of \mathcal{H} with center \mathcal{Z} such that $\mathcal{H} = \mathcal{N} + \mathcal{H}_1$, $\mathcal{N} \cap \mathcal{H}_1 = \mathcal{Z}$ and $[\mathcal{H}_1, \overline{\mathcal{N}}] \subset \overline{\mathcal{N}}$. Moreover \mathcal{H}_1 commutes with \mathcal{S} .

Now by applying Theorem 3.3 for $\mathcal{G}_1 = \mathcal{N} \odot (\text{ad}_{\overline{\mathcal{N}}}(\mathcal{H}_1) \times \mathcal{S})$ we see that for any basis X_1, \dots, X_{2m} of \mathcal{N} there exists a Gelfand-Kirillov basis $\mathcal{W}_1 = \{p_i, q_i; 1 \leq i \leq m\}$ of $\mathbf{U}(\mathcal{N})_\zeta$ satisfying the following properties.

- (i) For $1 \leq i \leq m$, p_i is skew-symmetric (resp q_i is symmetric).
- (ii) For $1 \leq i \leq 2m$, $\zeta^{-1}X_i$ is a linear combination of $1, \zeta^{-1}p_1, \dots, \zeta^{-1}p_m$ with coefficients in $\mathbf{R}[q]$. Furthermore the corresponding polynomials F_i , $1 \leq i \leq 2m$, of $2m$ indeterminates (θ, ω) define an automorphism of $\mathbf{R}[\theta, \omega]$ with Jacobian 1.
- (iii) For every $Y \in \mathcal{S}_1 = \text{ad}_{\overline{\mathcal{N}}}(\mathcal{H}_1) \times \mathcal{S}$ there exists a polynomial $a_Y(\theta, \omega)$ which may be expressed as a polynomial of $\deg \leq 2$ in ω with coefficients in $\mathbf{R}[\theta]$ such that $a_Y(q, \zeta^{-1}p)$ is symmetric and:

- $Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from \mathcal{S}_1 into $\mathbf{U}(\mathcal{N})_\zeta$.
- $[Y, u] = [\zeta a_Y(q, \zeta^{-1}p), u]$, $\forall u \in \mathbf{U}(\mathcal{N})_\zeta$.
- $Y \mapsto Y - \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra isomorphism from \mathcal{S}_1 into $\mathbf{U}(\mathcal{G}_1)_\zeta$.

Let ζ, ξ_i, η_i , $1 \leq i \leq n - m$, be the standard Heisenberg basis of \mathcal{H}_1 , i.e.

$$[\xi_i, \eta_j] = \delta_{ij}\zeta, \quad 1 \leq i, j \leq n - m.$$

For $1 \leq i \leq n - m$ put

$$\begin{aligned} \tilde{p}_i &= \xi_i - \zeta a_{\text{ad } \xi_i}(q, \zeta^{-1}p), \\ \tilde{q}_i &= \zeta^{-1}\eta_i - a_{\text{ad } \eta_i}(q, \zeta^{-1}p). \end{aligned}$$

Note that

$$[\zeta a_{\text{ad } \xi_i}(q, \zeta^{-1}p), \zeta a_{\text{ad } \eta_j}(q, \zeta^{-1}p)] = \zeta a_{\text{ad}[\xi_i, \eta_j]}(q, \zeta^{-1}p) = 0.$$

Hence

$$[\tilde{p}_i, \tilde{q}_j] = [\xi_i, \zeta^{-1}\eta_j] = \delta_{ij}.$$

On the other hand for all $\tilde{u} \in \mathbf{U}(\mathcal{N})_\zeta$ we have

$$[\zeta a_{\text{ad } \xi_i}(q, \zeta^{-1}p), \tilde{u}] = \text{ad } \xi_i(\tilde{u})$$

i.e.

$$[\tilde{p}_i, \tilde{u}] = 0, \quad 1 \leq i \leq n - m.$$

Similarly, we have:

$$[\tilde{q}_i, \tilde{u}] = 0, \quad 1 \leq i \leq n - m.$$

In particular $\mathcal{W} = \{p_i, \tilde{p}_j, q_i, \tilde{q}_j; 1 \leq i \leq m, 1 \leq j \leq n - m\}$ is a Gelfand-Kirillov basis for $\mathbf{U}(\mathcal{H})_\zeta$. Furthermore it is clear that the \tilde{p}_i 's are skew symmetric (resp. the \tilde{q}_i 's are symmetric). It follows that for an arbitrary basis X_{2m+1}, \dots, X_{2n} of \mathcal{H} complementing \mathcal{N} there exist polynomials F_i , $2m + 1 \leq i \leq 2n$, of $2n$ indeterminates $(\theta, \tilde{\theta}, \tilde{\omega}, \omega)$ of $\deg \leq 1$ in $\tilde{\omega}$ (resp. of $\deg \leq 2$ in ω) with coefficients in $\mathbf{R}[\theta, \tilde{\theta}]$ such that

$$X_i = \zeta F_i(q, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}p), \quad 2m + 1 \leq i \leq 2n.$$

Moreover the mapping $(\theta, \tilde{\theta}, \tilde{\omega}, \omega) \mapsto (F(\theta, \omega), \tilde{F}(\theta, \tilde{\theta}, \tilde{\omega}, \omega))$ is an automorphism of $\mathbf{R}[\theta, \tilde{\theta}, \tilde{\omega}, \omega]$ with Jacobian 1, where $F = (F_i)_{1 \leq i \leq 2m}$, $\tilde{F} = (F_i)_{2m+1 \leq i \leq 2n}$. Finally $Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from \mathcal{S} into $\mathbf{U}(\mathcal{N})_\zeta$ such that $a_Y(q, \zeta^{-1}p)$ is symmetric and $\mathcal{S}' = \{Y - \zeta a_Y(q, \zeta^{-1}p); Y \in \mathcal{S}\}$ is a Lie subalgebra of $\mathbf{U}(\mathcal{G})_\zeta$ isomorphic to \mathcal{S} and commuting with the elements of \mathcal{W} so that

$$\mathbf{U}(\mathcal{G})_\zeta \simeq \mathbf{U}(\mathcal{H})_\zeta \otimes \mathbf{U}(\mathcal{S}') \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathcal{S}).$$

Thus by changing slightly the notation we obtain the following

PROPOSITION 4.1. *Let $\mathcal{G}, \mathcal{H}, \mathcal{S}, \mathcal{N}$ be as above. Then for any basis X_1, \dots, X_{2n} of \mathcal{H} such that X_1, \dots, X_{2m} is a basis of $\mathcal{N} = \mathcal{H} \cap \mathcal{N}$, we may choose a Gelfand-Kirillov basis $\mathcal{W} = \{p_i, q_i; 1 \leq i \leq n\}$ of $\mathbf{U}(\mathcal{H})_\zeta$ such that $\mathcal{W}_1 = \{p_i, q_i; 1 \leq i \leq m\}$ is a Gelfand-Kirillov basis for $\mathbf{U}(\mathcal{N})_\zeta$ with skew-symmetric p_i (resp. symmetric q_i), $1 \leq i \leq n$. Moreover there exist polynomials F_i , $1 \leq i \leq 2n$ and a_Y , $Y \in \mathcal{S}$ of $2n$ indeterminates (θ, ω) satisfying the same properties as those in Theorem 3.3 with the only exceptions:*

- (i) for $1 \leq i \leq 2m$, F_i is a polynomial of $\deg \leq 1$ in $\omega_1, \dots, \omega_m$ with coefficients in $\mathbf{R}[\theta_1, \dots, \theta_m]$
- (ii) for $2m + 1 \leq i \leq 2n$, F_i is a polynomial of $\deg \leq 2$ in $\omega_1, \dots, \omega_m$ (resp. of $\deg \leq 1$ in $\omega_{m+1}, \dots, \omega_n$) with coefficients in $\mathbf{R}[\theta_1, \dots, \theta_n]$

(iii) a_Y depends only on $(\theta_1, \dots, \theta_m, \omega_1, \dots, \omega_m)$. In particular

$$\mathbf{U}(\mathcal{G})_\zeta \simeq \mathbf{U}(\mathcal{H})_\zeta \otimes \mathbf{U}(\mathcal{S}') \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathcal{S})$$

where $\mathcal{S}' = \{Y - \zeta a_Y(q_1, \dots, q_m, \zeta^{-1}p_1, \dots, \zeta^{-1}p_m); Y \in \mathcal{S}\}$ is a Lie subalgebra of $\mathbf{U}(\mathcal{G})_\zeta$ isomorphic to \mathcal{S} and commuting with \mathcal{W}

REMARK. The following lemma, which follows immediately from Proposition V.2.5 of [4], shows that the assumptions of Proposition 4.1 certainly hold if the greatest nilpotent ideal of \mathcal{H} is an H -algebra with center \mathcal{Z} .

LEMMA 4.2. *Let \mathcal{G} be a Lie algebra over a field of characteristic 0. Let \mathcal{H} be a solvable ideal of \mathcal{G} and \mathcal{N} the greatest nilpotent ideal of \mathcal{H} . Then $[\mathcal{G}, \mathcal{H}] \subset \mathcal{N}$.*

We are now ready to state the

THEOREM 4.3. *Let $\mathcal{G} = \mathcal{H} \odot \mathcal{S}$ and assume that there exists an \mathcal{S} -invariant subspace $\overline{\mathcal{H}}$ as usual. Then*

1. *For any basis X_1, \dots, X_{2n} of \mathcal{H} we may choose a Gelfand-Kirillov basis $\mathcal{W} = \{p_i, q_i; 1 \leq i \leq n\}$ of $\mathbf{U}(\mathcal{H})_\zeta$ with skew-symmetric p_i (resp. symmetric q_i), $1 \leq i \leq n$ and polynomials F_i , $1 \leq i \leq 2n$ of $2n$ indeterminates (θ, ω) satisfying the following properties:*

(i) *for $1 \leq i \leq 2n$, F_i is in fact a polynomial of $\deg \leq 2$ in ω with coefficients in $\mathbf{R}[\theta]$;*

(ii) *the mapping $(\theta, \omega) \mapsto (F_i(\theta, \omega))_{1 \leq i \leq 2n}$ is an automorphism of $\mathbf{R}[\theta, \omega]$ with Jacobian 1;*

(iii) $X_i = \zeta F_i(q, \zeta^{-1}p)$, $1 \leq i \leq 2n$;

2. *for each $Y \in \mathcal{S}$ there exists a polynomial $a_Y(\theta, \omega)$ which is in fact of $\deg \leq 2$ in ω with coefficients in $\mathbf{R}[\theta]$ such that*

(i) $a_Y(q, \zeta^{-1}p)$ is symmetric;

(ii) $Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from \mathcal{S} into $\mathbf{U}(\mathcal{H})_\zeta$;

(iii) $\mathcal{S}' = \{Y - \zeta a_Y(q, \zeta^{-1}p); Y \in \mathcal{S}\}$ is Lie subalgebra of $\mathbf{U}(\mathcal{G})_\zeta$ isomorphic to \mathcal{S} and commuting with \mathcal{W} so that

$$\mathbf{U}(\mathcal{G})_\zeta \simeq \mathbf{U}(\mathcal{H})_\zeta \otimes \mathbf{U}(\mathcal{S}') \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathcal{S}).$$

Proof. The proof is carried out by induction on $\dim \mathcal{H}$. Let \mathcal{N} be the greatest nilpotent ideal of \mathcal{H} .

(α) If \mathcal{N} is isomorphic to a Heisenberg algebra with center \mathcal{Z} then the theorem follows from Proposition 4.1.

(β) Thus assume that \mathcal{N} is not isomorphic to any Heisenberg algebra with center \mathcal{Z} but the center of \mathcal{N} is still \mathcal{Z} . In this case the proof is carried out exactly as in Theorem 3.3. The only difference is when applying the induction hypothesis we obtain polynomials $\tilde{F}_i(\tilde{\theta}, \tilde{\omega})$, $1 \leq i \leq 2n - 2m$ which have $\deg \leq 2$ (instead of $\deg \leq 1$) in $\tilde{\omega}$ with coefficients in $\mathbf{R}[\tilde{\theta}]$. Thus in the final results the polynomials F_i are of $\deg \leq 2$ in ω with coefficients in $\mathbf{R}[\theta]$ as stated in (1.i)

(γ) Finally assume that the center of \mathcal{N} contains \mathcal{Z} strictly. Let \mathcal{H} be a minimal abelian ideal of \mathcal{G} contained in the center of \mathcal{N} such that $\mathcal{H} \neq \mathcal{Z}$. Since the action of \mathcal{S} on \mathcal{H}/\mathcal{N} is trivial by Lemma 4.2, by contragredient the action of \mathcal{S} on \mathcal{H}/\mathcal{Z} is also trivial. Therefore it follows from the proof of Theorem 2.9 of [2] that $\dim(\mathcal{H}/\mathcal{Z}) = 1$. Thus there exist $\xi \in \mathcal{H} \setminus \mathcal{N}$ and $\eta \in \mathcal{H} = \overline{\mathcal{H}} \cap \mathcal{H}$ such that $[\xi, \eta] = \zeta$ and $[\mathcal{S}, \xi] = [\mathcal{S}, \eta] = \{0\}$. Put $q_1 = \zeta^{-1}\eta$ and

$$\mathcal{H}_0\{X \in \mathcal{H}; [X, \overline{\mathcal{H}}] \subset \overline{\mathcal{H}}\} = \text{Cent}_{\mathcal{H}}(\eta).$$

Let D_1 and D_2 be the nilpotent and semisimple parts of the derivation $\text{ad } \xi$ so that D_1 may be extended to a locally nilpotent derivation of $\mathbf{U}(\mathcal{H}_0 \odot \mathcal{S})_\zeta$ such that $D_1 q_1 = 1$, $D_1(\mathcal{S}) = 0$. Now the action of $\mathbf{R}D_2 \times \mathcal{S}$ on \mathcal{H}_0 defines a semidirect product $\mathcal{G}_0 = \mathcal{H}_0 \odot (\mathbf{R}D_2 \times \mathcal{S})$ which contains $H_0 \odot \mathcal{S}$ as an ideal. Moreover by modifying \mathcal{H} outside of the subspace generated by $[\mathcal{S}, \mathcal{H}]$ if necessary we may assume that $\overline{\mathcal{H}}$ is also invariant under the action of D_2 . For $X \in \mathbf{U}(\mathcal{G}_0)_\zeta$ put

$$\chi(X) = \sum_i \frac{(-1)^i}{i!} D_1^i(X) q_1^i.$$

Then it follows from Lemma 4.7.5 of [5] that χ is a homomorphism from $\mathbf{U}(\mathcal{G}_0)_\zeta$ onto a subalgebra \mathbf{A} of $\mathbf{U}(\mathcal{G})_\zeta$ commuting with q_1 such that the action of D_1 on \mathbf{A} is trivial. Moreover since D_1 commutes with $\mathbf{R}D_2 \times \mathcal{S}$ it is clear that the action of $\chi(\mathbf{R}D_2 \times \mathcal{S})$ on $\chi(\mathcal{H}_0) = \widetilde{\mathcal{H}}_0$ is induced from the action of $\mathbf{R}D_2 \times \mathcal{S}$ on \mathcal{H}_0 . Note that $\widetilde{\mathcal{H}}_0$ is a Lie subalgebra of \mathbf{A} isomorphic to $\mathcal{H}_0/\mathbf{R}\eta$. Again by some preliminary change of basis we may assume that $X_1 = \eta$, $X_{2n} = \xi$. Hence by using an argument similar to that in the proof of Theorem 3.3 and the induction hypothesis, we see that there exist a Gelfand-Kirillov basis $\mathcal{W}_1 = \{p_i, q_i; 2 \leq i \leq n\}$ of \mathbf{A} with skew-symmetric p_i (resp. sym-

metric q_i) $2 \leq i \leq n$, and polynomials F_i , $2 \leq i \leq 2n-1$ of $\deg \leq 2$ in $\hat{\omega} \equiv (\omega_2, \dots, \omega_n)$ with coefficients in $\mathbf{R}[\hat{\theta}] \equiv \mathbf{R}[\theta_2, \dots, \theta_n]$ such that

- $X_i = \zeta F_i(\hat{q}, \zeta^{-1}\hat{p})$, $2 \leq i \leq 2n-1$,
- $(\hat{\theta}, \hat{\omega}) \mapsto (F_i(\hat{\theta}, \hat{\omega}))_{2 \leq i \leq 2n-1}$ is an automorphism of $\mathbf{R}[\hat{\theta}, \hat{\omega}]$ with Jacobian 1.

Moreover there exist polynomials d , a_Y ($Y \in \mathcal{S}$) of $\deg \leq 2$ in $\hat{\omega}$ with coefficients in $\mathbf{R}[\hat{\theta}]$ such that

$$(tD_2, Y) \mapsto t\zeta d(\hat{q}, \zeta^{-1}\hat{p}) + \zeta a_Y(\hat{q}, \zeta^{-1}\hat{p})$$

is a Lie algebra homomorphism from $\mathbf{R}D_2 \times \mathcal{S}$ into \mathbf{A} and

$$\begin{cases} [D_2, u] = [\zeta d(\hat{q}, \zeta^{-1}\hat{p}), u], \\ [Y, u] = [\zeta a_Y(\hat{q}, \zeta^{-1}\hat{p}), u], \end{cases} \quad \forall u \in \mathbf{A}.$$

In particular

$$[D_2 - \zeta d(\hat{q}, \zeta^{-1}\hat{p}), u] = 0, \quad \forall u \in \mathbf{A}.$$

This shows that $p_1 \equiv \xi - \zeta d(\hat{q}, \zeta^{-1}\hat{p})$ commutes with \mathbf{A} and

$$[p_1, q_1] = D_1(q_1) = 1,$$

i.e. $\mathbf{U}(\mathcal{G})_\zeta$ is isomorphic to a Weyl algebra with Gelfand-Kirillov basis $\mathcal{W} = \{p_i, q_i; 1 \leq i \leq n\}$. Finally by putting

$$F_i(\theta_1, \dots, \theta_n, \omega_1, \dots, \omega_n) = \begin{cases} \theta_1 & \text{if } i = 1, \\ F_i(\hat{\theta}, \hat{\omega}) & \text{if } 2 \leq i \leq 2n-1, \\ \omega_1 + d(\hat{\theta}, \hat{\omega}) & \text{if } i = 2n, \end{cases}$$

and

$$a_Y(\theta_1, \hat{\theta}, \omega_1, \hat{\omega}) = a_Y(\hat{\theta}, \hat{\omega})$$

we see that F_i , $1 \leq i \leq 2n$, and a_Y , $Y \in \mathcal{S}$, satisfy the statements (1) and (2) of the theorem. \square

COROLLARY 4.4. *Let $\mathcal{G} = \mathcal{H} \odot \mathcal{S}$ as in Theorem 4.3. Let $\mathbf{Z}(\mathcal{G}_c)$ be the center of $\mathbf{U}(\mathcal{G}_c)$, where \mathcal{G}_c is the complexification of \mathcal{G} . Then $\mathbf{Z}(\mathcal{G}_c)_\zeta$ is isomorphic to the localized polynomial ring $\mathbf{C}[Y_1, \dots, Y_r, \zeta, \zeta^{-1}]$ where Y_1, \dots, Y_r is a basis of some Cartan subalgebra of \mathcal{S}_c .*

REMARK. This corollary gives a generalization of Theorem 2 in [3].

COROLLARY 4.5. *The Gelfand-Kirillov conjecture holds for the Lie algebras of connected unimodular solvable Lie groups having discrete series with one-dimensional center. In particular it also holds for H -algebras with one-dimensional center.*

Proof. We can apply the theorem with \mathcal{S} abelian and get

$$U(\mathcal{G})_{\zeta} \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes S(\mathcal{S})$$

where $S(\mathcal{S})$ is the symmetric algebra of \mathcal{S} . From this it follows that the (skew) field of quotients of $U(\mathcal{G})$ is isomorphic to the (skew) field of quotients of the Weyl algebra A_n over the polynomial ring $\mathbf{R}[Y_1, \dots, Y_t, \zeta_a]$ where Y_1, \dots, Y_t is a basis of \mathcal{S} . \square

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Received December 3, 1990 and in revised form October 22, 1991.

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Volume 156 No. 1 November 1992

Enveloping algebras of Lie groups with discrete series	1
NGUYEN HUU ANH and VUONG MANH SON	
Asymptotic behavior of eigenvalues for a class of pseudodifferential operators on \mathbf{R}^n	19
JUNICHI ARAMAKI	
A hybrid of theorems of Vinogradov and Piatetski-Shapiro	45
ANTAL BALOG and JOHN BENJAMIN FRIEDLANDER	
Chaos in terms of the map $x \rightarrow \omega(x, f)$	63
ANDREW MICHAEL BRUCKNER and JACK GARY CEDER	
Local real analytic boundary regularity of an integral solution operator of the $\bar{\partial}$ -equation on convex domains	97
ZHENHUA CHEN	
On some properties of exhaustion maps between bounded domains	107
CHI KEUNG CHEUNG	
A generalization of maximal functions on compact semisimple Lie groups	119
HENDRA GUNAWAN	
Stability of nonsingular group orbits	135
CLARK DEAN HORTON	
Bordism and regular homotopy of low-dimensional immersions	155
JOHN FORBES HUGHES	
On six-connected finite H -spaces	185
JAMES PEICHENG LIN and FRANK WILLIAMS	