

*Pacific
Journal of
Mathematics*

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Let G be a compact semisimple Lie group with finite centre. For each positive number s , let μ_{sH} denote the $\text{Ad}(G)$ -invariant probability measure carried on the conjugacy class of $\exp(sH)$ in G . With this one-parameter family of measures, we define the maximal operator \mathcal{M}_H on $\mathcal{E}(G)$. We then estimate the Fourier transform of μ_{sH} and of some derived distributions. Our result leads to the boundedness of \mathcal{M}_H on $L^p(G)$, for all p greater than some index p_0 in $(1, 2)$. This generalizes a recent result of M. Cowling and C. Meaney [2].

Introduction. Let G be a compact semisimple Lie group of rank l with finite centre, and with its Haar measure normalized to have total mass 1. Let \mathfrak{g} denote its Lie algebra, and let \mathfrak{h} be a maximal toral subalgebra of \mathfrak{g} . We denote by Φ the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, and fix $\Delta = \{\alpha_j : j \in I\}$, where $I = \{1, \dots, l\}$, to be a base of Φ (as in [3, §10.1]). With respect to Δ , we write Φ^+ for the set of positive roots, whose members are of the form

$$\alpha = \sum_{j \in I} n_j(\alpha) \alpha_j,$$

with $n_j(\alpha) \in \mathbb{Z}^+ \cup \{0\}$ for all $j \in I$, and Λ^+ for the set of dominant weights, which parametrizes the dual object of G .

We equip the Lie algebra \mathfrak{g} with the positive definite inner product (\cdot, \cdot) derived from the Killing form. For each $\nu \in \mathfrak{h}^*$, we define $H_\nu \in \mathfrak{h}$ by

$$\nu(H) = (H_\nu, H) \quad \forall H \in \mathfrak{h}.$$

We also transfer the inner product to \mathfrak{h}^* via

$$(\nu, \nu') = (H_\nu, H_{\nu'}) \quad \forall \nu, \nu' \in \mathfrak{h}^*.$$

The norm on \mathfrak{h}^* and \mathfrak{h} , induced by these inner products, will then be denoted by $|\cdot|$.

We choose a regular element $H \in \mathfrak{h}$, for which $\alpha(H) \neq 0$ for all $\alpha \in \Phi^+$, and fix $R > 0$ such that $\exp(sH)$ is regular in G for any

$s \in (0, R)$. For a continuous function f on G , the maximal function $\mathcal{M}_H f$ is defined by

$$\mathcal{M}_H f(x) = \sup_{s \in (0, R)} |\mu_{sH} * f(x)| \quad \forall x \in G,$$

where μ_{sH} is the $\text{Ad}(G)$ -invariant probability measure carried on the conjugacy class of $\exp(sH)$ in G . This definition generalizes one in the paper of Cowling and Meaney [2], in which H was a particular regular element of \mathfrak{h} . Our main results are the following.

THEOREM A. *For all $k = 0, 1, 2, \dots$, there exist positive constants $C_k = C_k(H)$ such that*

$$\left| \left(\frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| \leq C_k \frac{(1 + |\lambda|)^k}{(1 + s|\lambda|)^\gamma} \quad \forall s \in (0, R), \lambda \in \Lambda^+,$$

where $\gamma = \min_{j \in I} |\{\alpha \in \Phi^+ : n_j(\alpha) \geq 1\}|$.

It is clear that Theorem A, together with the arguments of [2], imply the boundedness of \mathcal{M}_H on $L^p(G)$ for all $p > 1 + (2\gamma)^{-1}$. So we state

THEOREM B. *For all $p > 1 + (2\gamma)^{-1}$, with γ as above, there exist positive constants $C_p = C_p(H)$ such that*

$$\|\mathcal{M}_H f\|_p \leq C_p \|f\|_p \quad \forall f \in \mathcal{C}(G).$$

We prove Theorem A by handling first the case when G is simple, and then extend the result to the semisimple case. Our method is based on arguments of representation theory, involving formulae for characters and dimensions, a study of root systems, the theory of weights, and properties of the Weyl group, all developed in the first part of this note. The proof of Theorem A will be given in the second part. It is clear that Theorem A is sharp since the explicit expression used in [2] for the particular case in which $H = H_p$ shows no improvement is possible. In the third part of this note, we give an example which shows that Theorem B too is sharp at least in the case where $G = \text{SU}(2)$.

Some related results can be found in M. Christ [1] and C. D. Sogge and E. M. Stein [5].

Throughout this note, the expressions C , C_k , and C_{k_1, k_2, k_3} denote various positive constants which possibly vary from line to line. These constants may depend on G , and some may also depend on the choice

of H . When a constant, C say, depends on H , we write $C(H)$ in place of C .

We are grateful to Professor M. Cowling for his valuable suggestions during the preparation and the writing of this note. In particular, we would like to thank him for helpful discussions concerning the sharpness of the L^p -estimate.

1. Representation theoretic arguments. We shall assume throughout this part that the Lie algebra \mathfrak{g} is simple.

1.1. We start with some formulae for characters and dimensions of representations of G . To each $\lambda \in \Lambda^+$, we associate the representation π_λ , the set of weights ϖ_λ , the character χ_λ , and the dimension $d_\lambda = \chi_\lambda(\mathbf{1})$. For all $\lambda \in \Lambda^+$, we have (see [3, §22])

$$\chi_\lambda(\exp(H)) = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \exp(i\lambda'(H)),$$

where $m_\lambda(\lambda') \in \mathbf{Z}^+$ is the multiplicity of λ' in π_λ . Accordingly,

$$d_\lambda = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda').$$

Let \mathscr{W} be the Weyl group of $(\mathfrak{g}^c, \mathfrak{h}^c)$, generated by the reflections σ_α corresponding to $\alpha \in \Delta$. Introduce the special element $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. For all $\lambda \in \Lambda^+$, the character and dimension formulae of Weyl read (see [3, §24.3])

$$\chi_\lambda(\exp(H)) = \frac{\sum_{\sigma \in \mathscr{W}} \det(\sigma) \exp(i\sigma(\lambda + \rho)(H))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(H)}$$

and

$$d_\lambda = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}.$$

1.2. It is well known that \mathfrak{g}^c has the root space decomposition (see [7, p. 273])

$$\mathfrak{g}^c = \mathfrak{g}^c \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha^c,$$

where \mathfrak{g}_α^c denotes the root subspace of \mathfrak{g}^c corresponding to $\alpha \in \Phi$.

Assuming $l \geq 2$, we choose $j_0 \in I$, and then remove α_{j_0} from Δ to obtain

$$\Delta_0 = \{\alpha_j : j \in I_0\}, \quad \text{where } I_0 = I \setminus \{j_0\}.$$

Set $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$, and put $\Phi_0 = \Phi_0^+ \cup -\Phi_0^+$. Clearly $\Phi_0 = -\Phi_0$ and $\sigma_\alpha \Phi_0 = \Phi_0$ for all σ_α ($\alpha \in \Delta_0$). This shows that

Φ_0 is a root system (see [7, p. 370]). Let \mathfrak{h}_0 be the subspace of \mathfrak{h} spanned by H_α ($\alpha \in \Phi_0$). Then one may verify that

$$\mathfrak{g}_0^c = \mathfrak{h}_0^c \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha^c$$

is a semisimple subalgebra of \mathfrak{g}^c , with maximal toral subalgebra \mathfrak{h}_0^c (see [7, Ex. 30 of Ch. 4]). Evidently Φ_0 is the root system of $(\mathfrak{g}_0^c, \mathfrak{h}_0^c)$, Δ_0 is a base of Φ_0 , and Φ_0^+ is the set of positive roots with respect to Δ_0 .

Write Φ_0 as a disjoint union of irreducible root systems, say

$$\Phi_0 = \Phi_{01} \cup \dots \cup \Phi_{0r}.$$

Let $q \in \{1, \dots, r\}$. Denote by \mathfrak{h}_{0q} the subspace of \mathfrak{h}_0 spanned by H_α ($\alpha \in \Phi_{0q}$). Then we find that

$$\mathfrak{g}_{0q}^c = \mathfrak{h}_{0q}^c \oplus \bigoplus_{\alpha \in \Phi_{0q}} \mathfrak{g}_\alpha^c$$

is a simple ideal of \mathfrak{g}_0^c , with maximal toral subalgebra \mathfrak{h}_{0q}^c . We also note that

$$\mathfrak{h}_0^c = \mathfrak{h}_{01}^c \oplus \dots \oplus \mathfrak{h}_{0r}^c$$

and

$$\mathfrak{g}_0^c = \mathfrak{g}_{01}^c \oplus \dots \oplus \mathfrak{g}_{0r}^c.$$

Now denote by $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_{0q}$ the inner products of \mathfrak{g}_0 and \mathfrak{g}_{0q} respectively. Then we have (see [3, Lemma 5.1])

$$(\cdot, \cdot)_0|_{\mathfrak{g}_{0q} \times \mathfrak{g}_{0q}} = (\cdot, \cdot)_{0q},$$

and so

$$(X, Y)_0 = (X_1, Y_1)_{01} + \dots + (X_r, Y_r)_{0r}$$

for all $X = X_1 + \dots + X_r, Y = Y_1 + \dots + Y_r \in \mathfrak{g}_0$, with $X_q, Y_q \in \mathfrak{g}_{0q}$. Further, since \mathfrak{g} and \mathfrak{g}_{0q} are simple, there exists a positive constant C_q satisfying (see [4, p. 242])

$$(X, Y)_{0q} = C_q(X, Y) \quad \forall X, Y \in \mathfrak{g}_{0q}.$$

We transfer these inner products to the corresponding dual spaces in the usual way.

Let Λ_0^+ denote the set of dominant weights with respect to Δ_0 . We need to determine the set of fundamental dominant weights in Λ_0^{FD} . Suppose $\{\omega_j: j \in I\}$ is the set of fundamental dominant weights in Λ^+ , for which (see [3, §13.1] for definition)

$$2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk} \quad \forall j, k \in I.$$

If we now set

$$\tilde{\omega}_j = \omega_j - \text{proj}_{\omega_{j_0}}(\omega_j) \quad \forall j \in I,$$

then we have the following facts.

Fact 1. For each $j \in I_0$, $\tilde{\omega}_j \in \mathfrak{h}_{0Q}^*$ whenever $\alpha_j \in \mathfrak{h}_{0Q}^*$.

Proof. For all $j, k \in I_0$, we have

$$\begin{aligned} 2 \frac{(\tilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} &= 2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2 \frac{(\text{proj}_{\omega_{j_0}}(\omega_j), \alpha_k)}{(\alpha_k, \alpha_k)} \\ &= 2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2 \frac{(\omega_j, \omega_{j_0}) (\omega_{j_0}, \alpha_k)}{(\omega_{j_0}, \omega_{j_0}) (\alpha_k, \alpha_k)} \\ &= 2 \frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 0 = \delta_{jk}. \end{aligned}$$

Now take $j \in I_0$, and let $Q \in \{1, \dots, r\}$ such that $\alpha_j \in \mathfrak{h}_{0Q}^*$. Clearly

$$\tilde{\omega}_j \perp \mathfrak{h}_{0q}^* \quad \forall q \neq Q.$$

Writing $\tilde{\omega}_j = \tilde{\omega}_{j_1} + \dots + \tilde{\omega}_{j_r}$, with $\tilde{\omega}_{j_q} \in \mathfrak{h}_{0q}^*$ for all $q \in \{1, \dots, r\}$, we find that

$$\tilde{\omega}_{j_q} = 0 \quad \forall q \neq Q.$$

We therefore have

$$\tilde{\omega}_j = \tilde{\omega}_{jQ} \in \mathfrak{h}_{0Q}^*,$$

as stated. □

Fact 2. $\{\tilde{\omega}_j : j \in I_0\}$ is the set of fundamental dominant weights in Λ_0^+ .

Proof. Take $j, k \in I_0$. Suppose $\tilde{\omega}_j \in \mathfrak{h}_{0q}^*$ and $\alpha_k \in \mathfrak{h}_{0q'}^*$ for some $q, q' \in \{1, \dots, r\}$. If $q \neq q'$, then clearly $(\tilde{\omega}_j, \alpha_k)_0 = 0$; otherwise we have

$$2 \frac{(\tilde{\omega}_j, \alpha_k)_0}{(\alpha_k, \alpha_k)_0} = 2 \frac{(\tilde{\omega}_j, \alpha_k)_{0q}}{(\alpha_k, \alpha_k)_{0q}} = 2 \frac{(\tilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk}.$$

Using Fact 1, the assertion follows. □

Fact 3. Suppose $\lambda = \sum_{j \in I} n_j \omega_j \in \Lambda^+$. Then λ can be rewritten as

$$\lambda = \lambda_0 + \lambda_1$$

where $\lambda_0 = \sum_{j \in I_0} n_j \tilde{\omega}_j \in \Lambda_0^+$ (with the same n_j 's) and $\lambda_1 = \text{proj}_{\omega_{j_0}}(\lambda)$.

Proof. Noting that $\tilde{\omega}_{j_0} = 0$, we have

$$\begin{aligned}
 \lambda &= \sum_{j \in I} n_j \omega_j = \sum_{j \in I} n_j \tilde{\omega}_j + \sum_{j \in I} n_j \operatorname{proj}_{\omega_{j_0}}(\omega_j) \\
 &= \sum_{j \in I_0} n_j \tilde{\omega}_j + \sum_{j \in I} n_j \frac{(\omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})} \omega_{j_0} \\
 &= \sum_{j \in I_0} n_j \tilde{\omega}_j + \frac{(\sum_{j \in I} n_j \omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})} \omega_{j_0} \\
 &= \sum_{j \in I_0} n_j \tilde{\omega}_j + \frac{(\lambda, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})} \omega_{j_0} \\
 &= \sum_{j \in I_0} n_j \tilde{\omega}_j + \operatorname{proj}_{\omega_{j_0}}(\lambda) = \lambda_0 + \lambda_1,
 \end{aligned}$$

as claimed. \square

REMARK. It is well known that the special element ρ is a dominant weight in Λ^+ . Indeed, $\rho = \sum_{j \in I} \omega_j$ (see [3, Lemma 13.3A]). By Fact 3, we may rewrite $\rho = \rho_0 + \rho_1$ where $\rho_0 = \sum_{j \in I_0} \tilde{\omega}_j \in \Lambda_0^+$ and $\rho_1 = \operatorname{proj}_{\omega_{j_0}}(\rho)$. But then $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Phi_0^+} \alpha$, giving $\rho_1 = \frac{1}{2} \sum_{\alpha \in \Phi_1^+} \alpha$ where $\Phi_1^+ = \Phi^+ \setminus \Phi_0^+$. As another consequence, we also have $\rho_1 = c \omega_{j_0}$ for some $c > 0$. But we know that $2(\omega_{j_0}, \alpha_{j_0})/(\alpha_{j_0}, \alpha_{j_0}) = 1$, and so we find $c = 2(\rho_1, \alpha_{j_0})/(\alpha_{j_0}, \alpha_{j_0})$. Hence we determine $\omega_{j_0} = \frac{1}{2}((\alpha_{j_0}, \alpha_{j_0})/(\rho_1, \alpha_{j_0}))\rho_1$, with $\rho_1 = \frac{1}{2} \sum_{\alpha \in \Phi_1^+} \alpha$. This offers a method of finding the fundamental dominant weight ω_{j_0} for any given $j_0 \in I$.

Introduce $\mathfrak{h}_1 = \{H \in \mathfrak{h} : \alpha(H) = 0 \ \forall \alpha \in \Delta_0\}$. Obviously \mathfrak{h}_1 is a subalgebra of \mathfrak{h} , which is spanned by H_{ρ_1} (by the above remark). Moreover, we have (like Fact 3)

Fact 4. Every $H \in \mathfrak{h}$ can be written as

$$H = H_0 + H_1$$

where $H_0 \in \mathfrak{h}_0$ and $H_1 \in \mathfrak{h}_1$.

REMARK. $H_0 \in \mathfrak{h}_0$ means that $H_0 = H_{\nu_0}$, where $\nu_0 \in \operatorname{span}(\Delta_0)$, while $H_1 \in \mathfrak{h}_1$ means that $H_1 = H_{\nu_1}$, where $\nu_1 = r\rho_1$ for some $r \in \mathbf{R}$. Thus clearly $\mathfrak{h}_0 \perp \mathfrak{h}_1$, and so Fact 4 actually states that $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$.

Suppose we are in $(\mathfrak{g}_0, \mathfrak{h}_0)$. To each $\lambda_0 \in \Lambda_0^+$, we associate the representation $\tilde{\pi}_{\lambda_0}$, the set of weights $\tilde{\omega}_{\lambda_0}$, the character $\tilde{\chi}_{\lambda_0}$, and the

dimension \tilde{d}_{λ_0} . For all $\lambda_0 \in \Lambda_0^+$ and $H_0 \in \mathfrak{h}_0$, we have

$$\tilde{\chi}_{\lambda_0}(\exp(H_0)) = \sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i\lambda'(H_0))$$

and

$$\tilde{d}_{\lambda_0} = \sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda'),$$

with $\tilde{m}_{\lambda_0}(\lambda') \in \mathbf{Z}^+$ being the multiplicity of λ' in $\tilde{\pi}_{\lambda_0}$.

Let \mathscr{W}_0 (or $\mathscr{W}[\Delta_0]$ if necessary) denote the subgroup of \mathscr{W} generated by σ_α ($\alpha \in \Delta_0$). The Weyl formulae then read

$$\tilde{\chi}_{\lambda_0}(\exp(H_0)) = \frac{\sum_{r \in \mathscr{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(H_0))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_0)}$$

and

$$\tilde{d}_{\lambda_0} = \prod_{\alpha \in \Phi_0^+} \frac{(\lambda_0 + \rho_0, \alpha)}{(\rho_0, \alpha)}.$$

We should note that the inner product in the expression above is really the inner product of \mathfrak{g} . Indeed, we may calculate

$$\begin{aligned} \tilde{d}_{\lambda_0} &= \lim_{s \rightarrow 0} \tilde{\chi}_{\lambda_0}(\exp(sH_{\rho_0})) \\ &= \lim_{s \rightarrow 0} \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(sH_{\rho_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\ &= \lim_{s \rightarrow 0} \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau\rho_0(sH_{\lambda_0 + \rho_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\ &= \lim_{s \rightarrow 0} \frac{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\lambda_0 + \rho_0})}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\ &= \prod_{\alpha \in \Phi_0^+} \frac{\alpha(H_{\lambda_0 + \rho_0})}{\alpha(H_{\rho_0})} = \prod_{\alpha \in \Phi_0^+} \frac{(\lambda_0 + \rho_0, \alpha)}{(\rho_0, \alpha)} \end{aligned}$$

(see [8, p. 106] for clarification).

Allowing \mathscr{W} to act, one may observe that all the above facts still hold for the system constituted by $\sigma\Phi_0$ ($\sigma \in \mathscr{W}$), as well as for that by Φ_0 . Moreover, the two facts below explain the connection between one system and another.

Fact 5. $\sigma\mathscr{W}[\Delta_0]\sigma^{-1} = \mathscr{W}[\sigma\Delta_0]$ for any $\sigma \in \mathscr{W}$.

Proof. Obvious (see [3, Lemma 9.2] for justification). □

Fact 6. $\tilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) = \tilde{\chi}_{\lambda_0}(\exp(H_{\nu_0}))$ for any $\sigma \in \mathscr{W}$.

Proof. For any $\sigma \in \mathscr{W}$, we have (by Fact 5)

$$\begin{aligned} \tilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) &= \frac{\sum_{\tau \in \mathscr{W}[\sigma\Delta_0]} \det(\tau) \exp(i\tau(\sigma\lambda_0 + \sigma\rho_0))(H_{\sigma\nu_0})}{\prod_{\alpha \in \sigma\Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\sigma\nu_0})} \\ &= \frac{\sum_{\tau \in \mathscr{W}[\Delta_0]\sigma^{-1}} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0))(H_{\sigma\nu_0})}{\prod_{\alpha \in \sigma\Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\sigma\nu_0})} \\ &= \frac{\sum_{\tau \in \mathscr{W}[\Delta_0]} \det(\sigma\tau\sigma^{-1}) \exp(i\sigma\tau(\lambda_0 + \rho_0))(H_{\sigma\nu_0})}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\sigma\alpha(H_{\sigma\nu_0})} \\ &= \frac{\sum_{\tau \in \mathscr{W}[\Delta_0]} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0))(H_{\nu_0})}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\nu_0})} \\ &= \tilde{\chi}_{\lambda_0}(\exp(H_{\nu_0})), \end{aligned}$$

as stated. □

2. The proof of the theorem. The outline of the proof is as follows. We first look for an estimate for all $s \in (0, R)$, then examine the decay for large s , and finally combine the results. The result obtained is valid under the assumption that G is simple, but then it extends to every semisimple Lie group G .

2.1. For all $s \in (0, R)$, $\lambda \in \Lambda^+$, we have (see [2, p. 813])

$$\hat{\mu}_{sH}(\lambda) = \frac{\chi_\lambda(\exp(sH))}{d\lambda}.$$

Using the multiplicity formulae, we write

$$\hat{\mu}_{sH}(\lambda) = \frac{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \exp(i\lambda'(sH))}{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda')}.$$

Hence, we have

$$\begin{aligned} \left| \left(\frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| &\leq \frac{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \left| \left(\frac{\partial}{\partial s} \right)^k \exp(i\lambda'(sH)) \right|}{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda')} \\ &\leq |H|^k |\lambda|^k = C_k(H) |\lambda|^k, \end{aligned}$$

for all $k = 0, 1, 2, \dots$

2.2. By the Weyl formulae, for all $s \in (0, R)$, $\lambda \in \Lambda^+$, we have

$$\hat{\mu}_{sH}(\lambda) = \frac{\sum_{\sigma \in \mathscr{W}} \det(\sigma) \exp(i(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(sH)} \prod_{\alpha \in \Phi^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)}.$$

In the case $l = 1$, one can easily obtain

$$\left| \left(\frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| \leq C_k(H) \frac{|\lambda + \rho|^k}{s^{|\lambda + \rho|}},$$

for all $k = 0, 1, 2, \dots$. So assume, hereafter, that $l \geq 2$.

For each $\lambda \in \Lambda^+$, choose $j_0 \in I$ for which $(\lambda + \rho, \alpha_{j_0})$ is maximal. As before, we write $\Delta_0 = \Delta \setminus \{\alpha_{j_0}\}$, $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$, and $\Phi_1^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) \geq 1\}$. (Note that $\Phi_1^+ = \Phi^+ \setminus \Phi_0^+$, and that Φ_1^+ depends on the choice of j_0 , and so depends on λ .) Clearly, if $\alpha \in \Phi_0^+$, then

$$(\lambda + \rho, \alpha) \geq (\rho, \alpha) \geq C,$$

and if $\alpha \in \Phi_1^+$, then (by the choice of j_0)

$$(\lambda + \rho, \alpha) \geq n_{j_0}(\alpha)(\lambda + \rho, \alpha_{j_0}) \geq C|\lambda + \rho|.$$

Moreover,

$$\gamma = \min_{j \in I} |\{\alpha \in \Phi^+ : n_j(\alpha) \geq 1\}| \leq |\Phi_1^+|.$$

Recall that \mathscr{W}_0 is the subgroup of \mathscr{W} generated by σ_α ($\alpha \in \Delta_0$). For an appropriate $\mathscr{S} \subset \mathscr{W}$, we write $\mathscr{W} = \bigcup_{\sigma \in \mathscr{S}} \sigma \mathscr{W}_0$ (disjoint union). We then obtain

$$\hat{\mu}_{sH}(\lambda) = \sum_{\sigma \in \mathscr{S}} \left(\frac{\sum_{\tau \in \mathscr{W}_0} \det(\sigma\tau) \exp(i\sigma\tau(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(sH)} \prod_{\alpha \in \Phi^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right).$$

For each reflection $\sigma_\alpha \in \mathscr{W}$, we know that $\det(\sigma_\alpha) = -1$, $\sigma_\alpha \alpha = -\alpha$, and $\sigma_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$ (see [3, Lemma 10.2B]). Thus, for any $\sigma \in \mathscr{W}$, we have

$$\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(sH) = \det(\sigma) \prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\sigma\alpha(sH).$$

It follows that

$$\begin{aligned} \hat{\mu}_{sH}(\lambda) &= \sum_{\sigma \in \mathscr{S}} \left(\frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\sigma\tau(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\sigma\alpha(sH)} \prod_{\alpha \in \Phi_0^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right) \\ &\quad \times \left(\prod_{\alpha \in \Phi_1^+} \frac{1}{2i \sin \frac{1}{2}\sigma\alpha(sH)} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right). \end{aligned}$$

Now fix $\sigma \in \mathcal{S}$. We write $H = H_{\sigma\nu}$, with $\nu = \nu_0 + \nu_1$, where $\nu_0 \in \text{span}(\Delta_0)$ and $\nu_1 = r\rho_1$ for some $r \in \mathbf{R}$. Then put $H_0 = H_{\nu_0}$ and $H_1 = H_{\nu_1}$. Next recall that Λ_0^+ is the set of dominant weights corresponding to Φ_0^+ . For each $\lambda \in \Lambda^+$, we write $\lambda = \lambda_0 + \lambda_1$, where $\lambda_0 \in \Lambda_0^+$ and $\lambda_1 = c\rho_1$ for some $c \in \mathbf{R}^+$. Hence, for all $\alpha \in \Phi_0^+$, we have $(\rho, \alpha) = (\rho_0, \alpha)$ and $(\lambda + \rho, \alpha) = (\lambda_0 + \rho_0, \alpha)$. Further, for all $\alpha \in \Phi_0^+$,

$$\begin{aligned} \sigma\alpha(H) &= (\sigma\alpha, \sigma\nu) = (\alpha, \nu) \\ &= (\alpha, \nu_0 + \nu_1) = (\alpha, \nu_0) \quad (\text{as } \nu_1 \perp \alpha) \\ &= \alpha(H_{\nu_0}) = \alpha(H_0), \end{aligned}$$

and whenever $\tau \in \mathcal{W}_0$,

$$\begin{aligned} \sigma\tau(\lambda + \rho)(H) &= (\sigma\tau(\lambda + \rho), \sigma\nu) = (\tau(\lambda + \rho), \nu) \\ &= (\tau(\lambda_0 + \rho_0) + \tau(\lambda_1 + \rho_1), \nu_0 + \nu_1) \\ &= (\tau(\lambda_0 + \rho_0) + (\lambda_1 + \rho_1), \nu_0 + \nu_1) \quad (\text{as } \tau \in \mathcal{W}_0) \\ &= (\tau(\lambda_0 + \rho_0), \nu_0) + (\lambda_1 + \rho_1, \nu_1) \quad (\text{by orthogonality}) \\ &= \tau(\lambda_0 + \rho_0)(H_{\nu_0}) + (\lambda_1 + \rho_1)(H_{\nu_1}) \\ &= \tau(\lambda_0 + \rho_0)(H_0) + (\lambda_1 + \rho_1)(H_1). \end{aligned}$$

It turns out that

$$\begin{aligned} &\frac{\sum_{\tau \in \mathcal{W}_0} \det(\tau) \exp(i\sigma\tau(\lambda + \rho)(sH))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\sigma\alpha(sH)} \prod_{\alpha \in \Phi_0^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \\ &= \exp(i(\lambda_1 + \rho_1)(sH_1)) \frac{\sum_{\tau \in \mathcal{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(sH_0))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_0)} \\ &\quad \times \prod_{\alpha \in \Phi_0^+} \frac{(\rho_0, \alpha)}{(\lambda_0 + \rho_0, \alpha)} \\ &= \exp(i(\lambda_1 + \rho_1)(sH_1)) \frac{\check{\chi}_{\lambda_0}(\exp(sH_0))}{\check{d}_{\lambda_0}} \\ &= \exp(i(\lambda_1 + \rho_1)(sH_1)) \frac{\sum_{\lambda' \in \check{\omega}_{\lambda_0}} \check{m}_{\lambda_0}(\lambda') \exp(i\lambda'(sH_0))}{\sum_{\lambda' \in \check{\omega}_{\lambda_0}} \check{m}_{\lambda_0}(\lambda')} \\ &= \frac{\sum_{\lambda' \in \check{\omega}_{\lambda_0}} \check{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda' \in \check{\omega}_{\lambda_0}} \check{m}_{\lambda_0}(\lambda')} \quad (\text{by orthogonality}). \end{aligned}$$

So we have

$$\hat{\mu}_{sH}(\lambda) = \sum_{\sigma \in \mathcal{S}} \left(\frac{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \right) \times \left(\prod_{\alpha \in \Phi_1^+} \frac{1}{\sigma\alpha(sH)} \frac{\sigma\alpha(sH)}{2i \sin \frac{1}{2}\sigma\alpha(sH)} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right).$$

For all $k = 0, 1, 2, \dots$, we have the estimates

$$(1) \quad \left| \left(\frac{\partial}{\partial s} \right)^k \frac{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda' \in \tilde{\omega}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \right| \leq |H|^k |\lambda + \rho|^k,$$

$$(2) \quad \left| \left(\frac{\partial}{\partial s} \right)^k \prod_{\alpha \in \Phi_1^+} \sigma\alpha(sH)^{-1} \right| \leq C_k(H) s^{-k - |\Phi_1^+|},$$

$$(3) \quad \left| \left(\frac{\partial}{\partial s} \right)^k \prod_{\alpha \in \Phi_1^+} \frac{\sigma\alpha(sH)}{2i \sin \frac{1}{2}\sigma\alpha(sH)} \right| \leq C_k \quad (\text{by Leibniz' rule}),$$

$$(4) \quad \left| \prod_{\alpha \in \Phi_1^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)} \right| \leq C |\lambda + \rho|^{-|\Phi_1^+|}$$

(as $(\lambda + \rho, \alpha) \geq C |\lambda + \rho|$ for all $\alpha \in \Phi_1^+$).

Therefore, by Leibniz' rule for the derivatives of products, we obtain

$$\begin{aligned} & \left| \left(\frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| \\ & \leq \sum_{\sigma \in \mathcal{S}} \sum_{k_1+k_2+k_3=k} C_{k_1, k_2, k_3} \left| \left(\frac{\partial}{\partial s} \right)^{k_1} (\text{1st term}) \right| \\ & \quad \times \left| \left(\frac{\partial}{\partial s} \right)^{k_2} (\text{2nd term}) \right| \left| \left(\frac{\partial}{\partial s} \right)^{k_3} (\text{3rd term}) \right| |\text{4th term}| \\ & \leq \sum_{\sigma \in \mathcal{S}} \sum_{k_1+k_2+k_3=k} C_{k_1, k_2, k_3}(H) |H|^{k_1} |\lambda + \rho|^{k_1} s^{-k_2} (s|\lambda + \rho|)^{-|\Phi_1^+|} \\ & \leq C_k(H) (1 + |H|)^k \frac{|\lambda + \rho|^k}{(s|\lambda + \rho|)^{|\Phi_1^+|}} \quad (\text{provided } s|\lambda + \rho| > 1) \\ & \leq C_k(H) \frac{|\lambda + \rho|^k}{(s|\lambda + \rho|)^\gamma} \quad (\text{as } \gamma \leq |\Phi_1^+|), \end{aligned}$$

for all $k = 0, 1, 2, \dots$, as desired.

Combining this with the previous estimate, we obtain the result.

2.3. We shall now extend our result to every semisimple Lie group G . The key is to prove that Fact 2 in §1.2 is still valid.

Let us write Φ as a disjoint union of irreducible root systems

$$\Phi = \Phi^{(1)} \cup \dots \cup \Phi^{(n)},$$

and split Δ into

$$\Delta = \Delta^{(1)} \cup \dots \cup \Delta^{(n)},$$

with $\Delta^{(m)} = \Delta \cap \Phi^{(m)}$ being a base of $\Phi^{(m)}$ for each $m \in \{1, \dots, n\}$.

The Lie algebra \mathfrak{g}^c is now a direct sum of simple ideals

$$\mathfrak{g}^c = \mathfrak{g}^{(1)c} \oplus \dots \oplus \mathfrak{g}^{(n)c}.$$

As before, we choose $j_0 \in I$ and remove α_{j_0} from Δ to obtain

$$\Delta_0 = \Delta \setminus \{\alpha_{j_0}\}.$$

But $\alpha_{j_0} \in \Delta^{(M)}$ for some $M \in \{1, \dots, n\}$, and so

$$\Delta_0 = \Delta^{(1)} \cup \dots \cup \Delta_0^{(M)} \cup \dots \cup \Delta^{(n)},$$

with $\Delta_0^{(M)} = \Delta^{(M)} \setminus \{\alpha_{j_0}\}$. The Lie algebra \mathfrak{g}_0 (as in §1.2) then decomposes into

$$\mathfrak{g}_0^c = \mathfrak{g}^{(1)c} \oplus \dots \oplus \mathfrak{g}_0^{(M)c} \oplus \dots \oplus \mathfrak{g}^{(n)c},$$

where $\mathfrak{g}_0^{(M)c}$ is the Lie subalgebra corresponding to $\Delta_0^{(M)}$. Now let $K, K_0, K^{(m)},$ and $K_0^{(M)}$ denote the Killing forms of $\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}^{(m)},$ and $\mathfrak{g}_0^{(M)}$ respectively. Then, for each $m \in \{1, \dots, n\}, m \neq M,$ we have

$$K_0|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}} = K^{(m)} = K|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}};$$

while for $m = M,$ the connection between $K^{(M)}$ and $K_0^{(M)}$ is explained in §1.2. We therefore find that Fact 2 still holds, and thus the extension is clear.

3. An example: The sharpness of the estimate. We shall here consider an example concerning the sharpness of the L^p -estimate.

Let $G = \text{SU}(2),$ the Lie group consisting of 2×2 complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1.$ Its Lie algebra \mathfrak{g} then contains all matrices of the form

$$\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix}$$

with $a \in \mathbf{R}$, $b \in \mathbf{C}$. Here $\gamma = 1$ and the special element is

$$H_\rho = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In \mathfrak{g} , one may define the norm $|\cdot|$ by

$$\left| \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \right| = (a^2 + |b|^2)^{1/2} \quad \forall a \in \mathbf{R}, b \in \mathbf{C}.$$

For any $y \in G$, $X \in \mathfrak{g}$, one may observe that

$$X' = yXy^{-1} \in \mathfrak{g},$$

with $|X'| = |X|$. Conversely, for any $X, X' \in \mathfrak{g}$ with $|X| = |X'|$, one can find $y \in G$ such that $X' = yXy^{-1}$.

Denote by $B_0(\pi)$ the ball in \mathfrak{g} which has centre 0 and radius π . It is then evident that the map $\exp: B_0(\pi) \rightarrow G$ is injective. Indeed, for each $x \in G$, there exists a unique $X \in B_0(\pi)$ for which $x = \exp(X)$. Diagonalizing such an X , one has

$$x = y \exp(\omega H_\rho) y^{-1}, \quad \text{where } \omega = |X|,$$

for some $y \in G$. It is seen here that $\text{trace}(x) = 2 \cos \omega$.

As suggested in [6], let us consider the function $f: G \rightarrow \mathbf{R}^+$ given by

$$f(\exp(X)) = \begin{cases} \frac{|X|^{-2}}{\log |X|^{-1}}, & \text{if } 0 < |X| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

One may observe that $f \in L^p(G)$, whenever $1 \leq p \leq \frac{3}{2}$. On the other hand, regarding the maximal function $\mathcal{M}f = \mathcal{M}_{H_\rho} f$, we claim that $\mathcal{M}f(x) = \infty$ for all $x \in G$.

Before verifying our claim, we remark that

$$f(-\exp(X)) = f(\exp(X'))$$

where $|X'| = \pi - |X|$. Moreover, $f(yxy^{-1}) = f(x)$ for all $x, y \in G$. In fact, for all $x, y \in G$, we have

$$\begin{aligned} f(yxy^{-1}) &= f(y \exp(X) y^{-1}) \quad (\text{for some } X \in \mathfrak{g}) \\ &= f(\exp(yXy^{-1})) \\ &= f(\exp(X')) \quad (\text{where } |X'| = |X|) \\ &= f(\exp(X)) = f(x). \end{aligned}$$

Similarly, we observe that $\mathcal{M} f(yxy^{-1}) = \mathcal{M} f(x)$ for all $x, y \in G$. To be precise, for all $x, y \in G$, we have

$$\begin{aligned} \mathcal{M} f(yxy^{-1}) &= \sup_{s \in (0, \pi)} \int_G f(y \times y^{-1} g \exp(sH_\rho) g^{-1}) dg \\ &= \sup_{s \in (0, \pi)} \int_G f(xy^{-1} g \exp(sH_\rho) g^{-1} y) dg \\ &= \sup_{s \in (0, \pi)} \int_G f(xg' \exp(sH_\rho) g'^{-1}) dg' = \mathcal{M} f(x). \end{aligned}$$

We shall now verify our claim. First, for $x = \pm 1$, we have

$$\begin{aligned} \mathcal{M} f(\pm 1) &= \sup_{s \in (0, \pi)} \int_G f(\pm g \exp(sH_\rho) g^{-1}) dg \\ &= \sup_{s \in (0, \pi)} \int_G f(\pm \exp(sH_\rho)) dg = \sup_{s \in (0, \pi)} f(\pm \exp(sH_\rho)) \\ &= \sup_{s \in (0, \frac{1}{2})} \frac{s^{-2}}{\log s^{-1}} = \infty. \end{aligned}$$

Next, for $x \neq \pm 1$, we may assume that $x = \exp(\frac{t}{2}H_\rho)$ for some $0 < t < 2\pi$, and hence

$$\begin{aligned} \mathcal{M} f(x) &= \mathcal{M} f(\exp(\frac{t}{2}H_\rho)) \\ &= \sup_{s \in (0, \pi)} \int_G f(\exp(\frac{t}{2}H_\rho) g \exp(sH_\rho) g^{-1}) dg \\ &\geq \int_G f(\exp(\frac{t}{2}H_\rho) g \exp(\frac{t}{2}H_\rho) g^{-1}) dg. \end{aligned}$$

Writing each $g \in G$ as $g = h_\theta k_\phi h_{\theta'}$, where $h_\theta = \exp(\frac{\theta}{2}H_\rho)$ and k_ϕ is the matrix of rotation with angle $\frac{\phi}{2}$, we have (see [9, pp. 99–100])

$$\begin{aligned} \mathcal{M} f(x) &\geq \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi f(h_t h_\theta k_\phi h_{\theta'} h_t h_{-\theta'} k_{-\phi} h_{-\theta}) \sin \phi d\phi d\theta d\theta' \\ &= \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta}) \sin \phi d\phi d\theta d\theta' \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta}) \sin \phi d\phi d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(h_{-\theta} h_t h_\theta k_\phi h_t k_{-\phi}) \sin \phi d\phi d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(h_t k_\phi h_t k_{-\phi}) \sin \phi d\phi d\theta \\ &= \frac{1}{2} \int_0^\pi f(h_t k_\phi h_t k_{-\phi}) \sin \phi d\phi. \end{aligned}$$

Let us now investigate the integrand. Multiplying out, we get

$$h_t k_\phi h_t k_{-\phi} = \begin{pmatrix} e^{it} \cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} & \cos \frac{\phi}{2} \sin \frac{\phi}{2} (1 - e^{it}) \\ -\cos \frac{\phi}{2} \sin \frac{\phi}{2} (1 - e^{-it}) & e^{-it} \cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} \end{pmatrix}.$$

As seen before, this matrix is similar to $\exp(\omega H_\rho)$, where

$$\omega = \cos^{-1} \left(\sin^2 \frac{\phi}{2} + \cos^2 \frac{\phi}{2} \cos t \right).$$

By observation (thanks to John Cornwall for making it easier), there exists a constant $C = C_t \in (0, 1)$ such that

$$\cos(\pi - \phi) \leq \sin^2 \frac{\phi}{2} + \cos^2 \frac{\phi}{2} \cos t \leq \cos C(\pi - \phi) \quad \forall \phi \in (\pi - \frac{1}{2}, \pi),$$

and accordingly

$$0 < C(\pi - \phi) \leq \omega \leq \pi - \phi < \frac{1}{2} \quad \forall \phi \in (\pi - \frac{1}{2}, \pi).$$

Hence we find that

$$f(h_t k_\phi h_t k_{-\phi}) = f(\exp(\omega H_\rho)) = \frac{\omega^{-2}}{\log \omega^{-1}} \geq \frac{(\pi - \phi)^{-2}}{\log\{C(\pi - \phi)\}^{-1}},$$

for all $\phi \in (\pi - \frac{1}{2}, \pi)$. It therefore follows that

$$\begin{aligned} \mathcal{M} f(x) &\geq \frac{1}{2} \int_{\pi-1/2}^\pi \frac{(\pi - \phi)^{-2}}{\log\{C(\pi - \phi)\}^{-1}} \sin \phi \, d\phi \\ &\geq \frac{1}{4} \int_{\pi-1/2}^\pi \frac{(\pi - \phi)^{-1}}{\log\{C(\pi - \phi)\}^{-1}} \, d\phi \\ &= \frac{1}{4} \int_0^{C/2} \frac{\varphi^{-1}}{\log \varphi^{-1}} \, d\varphi = \frac{1}{4} \int_{\log(2/C)}^\infty \frac{d\psi}{\psi} = \infty, \end{aligned}$$

as claimed.

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Received March 21, 1991 and in revised form October 16, 1991.

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PACIFIC JOURNAL OF MATHEMATICS

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