Pacific Journal of Mathematics

ON SIX-CONNECTED FINITE H-SPACES

JAMES PEICHENG LIN AND FRANK WILLIAMS

Volume 156 No. 1 November 1992

PACIFIC JOURNAL OF MATHEMATICS
Vol. 156, No. 1, 1992

ON SIX-CONNECTED FINITE H-SPACES

JAMES P. LIN AND FRANK WILLIAMS

In this note we shall prove the following theorem.

MAIN THEOREM. ***Let X be a 6-connected finite H-space with associative* mod 2 *homology. Further, suppose that* $\mathrm{Sq}^4 H^7(X; Z_2) = 0$ *and* $\mathrm{Sq}^{15} H^{15}(X; Z_2) = 0$. *Then X is either contractible or has the homotopy type of a product of seven-spheres.***

0. Introduction. It should be noted that there are several results related to this theorem. Lin showed that any finite H-space with associative mod 2 homology has its first nonvanishing homotopy in degrees 1, 3, 7, or 15 (or is contractible). A seven-sphere is an H-space, but not a mod 2 homotopy-associative one [**4**, **10**]. Further work of Hubbuck [**5**], Sigrist and Suter [**12**], and others has shown that spaces whose mod 2 cohomology has the form

$$\Lambda(x_7, x_{11}) \quad \text{or} \quad \Lambda(x_7, x_{11}, x_{13})$$

are not realizable as H-spaces. (Here x_i denotes an element of degree i.) One is led to conjecture that

Conjecture 1. Every two-torsion-free 6-connected finite H-space is homotopy equivalent to a product of seven-spheres (or is acyclic).

Conjecture 2. Every two-torsion-free homotopy-associative 6-connected finite H-space is acyclic.

Conjecture 1 implies Conjecture 2 by [**4**, **11**].

Henceforth, X will denote an H-space that satisfies the hypotheses of the Main Theorem, and $H^*(X)$ will denote $H^*(X; Z_2)$. The proof of the Main Theorem will be accomplished in a series of steps, which we record here. Our goal is to show that under the hypotheses, X has mod 2 cohomology an exterior algebra on 7-dimensional generators. This relies heavily on the following theorem.

Steenrod Connections [**8**]. Let X be a finite simply-connected H-space with associative mod 2 homology. Then for $r \geq 0$, $k > 0$,

$$QH^{2^r+2^{r+1}k-1}(X; Z_2) = \mathrm{Sq}^{2^r k} QH^{2^r+2^r k-1}(X; Z_2), \text{ and}$$
$$\mathrm{Sq}^{2^r} QH^{2^r+2^{r+1}k-1}(X; Z_2) = 0.$$

(Here QH^* denotes the indecomposable quotient.)

In §1 we shall use a relation in the Steenrod algebra and the methods of [**1**] to produce a new factorization of Sq^{16}. We then apply this factorization to show that $H^{23}(X) = 0$. This implies that $H^*(X)$ is an exterior algebra on generators in degrees of the form $2^d - 1$, $d \geq 3$, with trivial action of the Steenrod algebra. In §2 we use the Cartan formula for secondary operations, [7], and a particular factorization of the cube of a certain 8-dimensional cohomology class, [**10**], to show that $H^{15}(X) = 0$. In §3 we turn to the c-invariant, [**14**], to complete our calculations by showing that no algebra generators for $H^*(X)$ exist in degrees greater than seven. Once it is shown that the mod 2 cohomology is exterior on 7-dimensional generators, it follows by the Bockstein spectral sequence that the rational cohomology has the same form. But since the rational cohomology is isomorphic to the E_∞ term of the mod p Bockstein spectral for any prime p, it follows by [2] that $H^*(X; Z)$ has no odd torsion. Thus $H^*(X; Z)$ is torsion-free, and we may use the Hurewicz map together with the multiplication in X to obtain a homotopy equivalence

$$S^7 \times \cdots \times S^7 \to X.$$

1. $H^{23}(X)$. In this section we prove that there are no 23-dimensional generators in $H^*(X)$. We will also show that $H^*(X)$ is an exterior algebra with trivial action of the Steenrod algebra. We shall use the notation $\mathrm{Sq}^{i,j}$ to denote $\mathrm{Sq}^i\mathrm{Sq}^j$.

THEOREM 1.1. *Let Y be a space and $x \in H^k(Y)$ be the reduction of an integral class. If x is in the intersection of the kernels of* Sq^2, Sq^7, Sq^8, *and* $\mathrm{Sq}^{8,4}$, *then there exist classes $v_i \in H^{k+i}(Y)$, $i = 3, 7, 8, 10, 12, 13, 14, 15$, such that*

$$\begin{aligned}(1.1)\qquad \mathrm{Sq}^{16}x = {} & \mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})v_7 \\ & + (\mathrm{Sq}^8 + \mathrm{Sq}^{6,2})v_8 + \mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4 v_{12} \\ & + \mathrm{Sq}^3 v_{13} + \mathrm{Sq}^2 v_{14} + \mathrm{Sq}^1 v_{15}.\end{aligned}$$

Proof. Consider the following matrix of relations:

$$(1.2)\qquad \begin{matrix} v_3 \\ v_7 \\ v_8 \\ v_{10} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{matrix} \begin{pmatrix} \mathrm{Sq}^2 & 0 & 0 & 0 \\ 0 & \mathrm{Sq}^1 & 0 & 0 \\ 0 & \mathrm{Sq}^2 & \mathrm{Sq}^1 & 0 \\ \mathrm{Sq}^9 & \mathrm{Sq}^4 & \mathrm{Sq}^3 & 0 \\ \mathrm{Sq}^{8,2,1} & 0 & 0 & \mathrm{Sq}^1 \\ \mathrm{Sq}^{12} & 0 & \mathrm{Sq}^{4,2} & 0 \\ \mathrm{Sq}^{13}+\mathrm{Sq}^{12,1} & \mathrm{Sq}^8 & 0 & \mathrm{Sq}^3 \\ \mathrm{Sq}^{14} & 0 & \mathrm{Sq}^8 & \mathrm{Sq}^4 \end{pmatrix} \begin{pmatrix} \mathrm{Sq}^2 \\ \mathrm{Sq}^7 \\ \mathrm{Sq}^8 \\ \mathrm{Sq}^{8,4} \end{pmatrix} = 0.$$

Let

$$w\colon K(Z\,,\,n) \to K(Z_2\,;\,n+2\,,\,n+7\,,\,n+8\,,\,n+12) = K_0$$

be defined by

$$w^*(\iota_{n+2}) = \mathrm{Sq}^2\iota_n\,;\; w^*(\iota_{n+7}) = \mathrm{Sq}^7\iota_n\,;$$
$$w^*(\iota_{n+8}) = \mathrm{Sq}^8\iota_n\,;\; w^*(\iota_{n+12}) = \mathrm{Sq}^{8,4}\iota_n.$$

If E is the fiber of w, we have the following diagram

$$(1.3)\qquad \begin{array}{ccc} \Omega K_0 & & \\ \downarrow{\scriptstyle j} & & \\ E & & \\ \downarrow & & \\ K(Z\,,\,n) & \xrightarrow{\;w\;} & K_0 \end{array}$$

and there exist elements $v_j \in PH^{n+j}(E\,;\,Z_2)$ defined by the relations (1.2).

A calculation shows that the element

$$z = \mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2}+\mathrm{Sq}^{6,3})v_7 + (\mathrm{Sq}^8+\mathrm{Sq}^{6,2})v_8$$
$$+\,\mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4v_{12} + Sq^3v_{13} + \mathrm{Sq}^2v_{14} + \mathrm{Sq}^1v_{15}$$

lies in $PH^{16+n}(E) \cap \ker(j^*) = p^*PH^{16+n}(K(Z_2\,,\,n))$. It follows that $z = cp^*(\mathrm{Sq}^{16}\iota_n)$, where $c \in Z_2$. For $n = 16$ there is a commutative diagram

$$
\begin{array}{ccccc}
 & & \Omega K_0 & & \\
 & & \downarrow{\scriptstyle j} & & \\
 & \overset{\tilde{f}}{\nearrow} & E & & \\
 & & \downarrow & & \\
K(Z, 2) & \xrightarrow{f} & K(Z, n) & \xrightarrow{w} & K_0
\end{array}
$$

where $f^*(\iota_{16}) = \iota_2^8$. Now consider $\overline{w}: K(Z_2, 16) \to K_0 \times K(Z_2, 17)$ given by the same formulas as w on the fundamental classes in K_0 and such that $\overline{w}^*(\iota_{17}) = \mathrm{Sq}^1 \iota_{16}$. Let $\overline{E}$ be the fiber of $\overline{w}$.

There exists a commutative diagram

(1.4)

$$
\begin{array}{ccccc}
 & & \Omega K_0 & \longrightarrow & \Omega K_0 \times K(Z_2, 16) \\
 & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \bar{j}} \\
 & \overset{\tilde{f}}{\nearrow} & E & \xrightarrow{\overline{h}} & \overline{E} \\
 & & \downarrow & & \downarrow \\
K(Z, 2) & \xrightarrow{f} & K(Z, 16) & \xrightarrow{h} & K(Z_2, 16) \\
 & & {\scriptstyle w}\downarrow & & {\scriptstyle \overline{w}}\downarrow \\
 & & K_0 & \longrightarrow & K_0 \times K(Z_2, 17)
\end{array}
$$

Further, there is another lifting $\tilde{h}: K(Z, 2) \to \overline{E}$ of hf that has its H-deviation

$$D\tilde{h}: K(Z, 2) \times K(Z, 2) \to K(Z_2, 16)$$

given by $[D\tilde{h}] = \iota_2^4 \otimes \iota_2^4$. This holds because

$$B(hf)^* B\overline{w}^*(\iota_{18}) = \mathrm{Sq}^{9,4,2}\iota_3 = (\mathrm{Sq}^{4,2}\iota_3)^2$$

and because $B(hf)^* B\overline{w}^*$ is zero on the fundamental classes in K_0.

In $PH^*(\overline{E})$ there exist elements $\overline{v}_j$ such that $\overline{h}^*(\overline{v}_j) = v_j$. The components of $\overline{v}_j$ in $H^*(K(Z_2, 16))$ are:

(1.5) $\overline{v}_3$: $\mathrm{Sq}^3\iota_{16}$; $\overline{v}_7$: 0; $\overline{v}_8$: $\mathrm{Sq}^8\iota_{16}$; $\overline{v}_{10}$: 0;

$\overline{v}_{12}$: $\mathrm{Sq}^{8,4}\iota_{16}$; $\overline{v}_{13}$: $\mathrm{Sq}^{4,9}\iota_{16}$; $\overline{v}_{14}$: 0; $\overline{v}_{15}$: $\mathrm{Sq}^{15}\iota_{16}$.

It follows that $\tilde{h}^*(\overline{v}_j)$ is primitive except for $\tilde{h}^*(\overline{v}_8)$ which has

$$\overline{\Delta}\tilde{h}^*(\overline{v}_8) = \iota_2^8 \otimes \iota_2^4 + \iota_2^4 \otimes \iota_2^8 = \overline{\Delta}(\iota_2^{12}).$$

Because $H^*(K(Z, 2))$ is trivial in odd degrees and is Z_2 in even degrees, we conclude

$$\tilde{h}^*(\overline{v}_j) = 0 \quad \text{if } j \neq 8, \quad \text{and}$$
$$\tilde{h}^*(\overline{v}_8) = \iota_2^{12}.$$

Therefore if

$$\overline{z} = \mathrm{Sq}^{11,2}\overline{v}_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})\overline{v}_7 + (\mathrm{Sq}^8 + \mathrm{Sq}^{6,2})\overline{v}_8 + \mathrm{Sq}^{4,2}\overline{v}_{10} + \mathrm{Sq}^4\overline{v}_{12} + \mathrm{Sq}^3\overline{v}_{13} + \mathrm{Sq}^2\overline{v}_{14} + \mathrm{Sq}^1\overline{v}_{15},$$

it follows that

$$(1.6) \qquad \tilde{h}^*(\overline{z}) = \mathrm{Sq}^8\tilde{h}^*(\overline{v}_8) = \mathrm{Sq}^8(\iota_2^{12}) = \iota_2^{16} = \mathrm{Sq}^{16}(\iota_2^8).$$

Now $\tilde{h}$ and $\overline{h}\tilde{f}$ both lift hf, so

$$\overline{h}\tilde{f} = \tilde{h} + \overline{j}F,$$

for some

$$F = (F_1, F_2): K(Z, 2) \to \Omega K_0 \times K(Z_2, 16).$$

But ΩK_0 is odd-dimensional with the exception of $K(Z_2, 22)$. If $\tilde{f}$ is altered by F_1, then

$$\overline{h}\tilde{f} = \tilde{h} + \overline{j}F_2,$$

and $[F_2] = d\iota_2^8$, $d \in Z_2$.

Using (1.5) we calculate that for all j

$$(1.7) \qquad F_2^*\overline{j}^*(\overline{v}_j) = 0, \quad \text{and hence} \quad \tilde{h}^*(\overline{v}_j) = \tilde{f}^*(v_j).$$

Therefore

$$\tilde{f}^*(z) = \tilde{h}^*(\overline{z}) = \mathrm{Sq}^{16}(\iota_2^8).$$

It follows that $c = 1$. □

Theorem 1.2. $QH^{23}(X) = 0$.

Proof. By the restrictions on the degrees of generators of $H^*(X)$, $H^i(X \wedge X) = 0$ for $i = 7, 15$, and 31. So by the Steenrod connections, all generators in degrees less than 63 may be chosen to be primitive. Further, in degrees < 40, $H^*(X)$ is an exterior algebra in which

$$(1.8) \qquad QH^k(X) = 0, \quad k \neq 7, 15, 23, 27, 29, 31, 39.$$

The lowest-dimensional possible non-trivial Steenrod operation is Sq^8 acting on $H^{15}(X)$. So let $x_{23} = \mathrm{Sq}^8 x_{15} \neq 0$. By (1.8), Sq^2, Sq^8, and $\mathrm{Sq}^{8,4}$ are all zero on x_{23}, and $\mathrm{Sq}^7 x_{23} = \mathrm{Sq}^{15} x_{15}$, which is zero by hypothesis. Thus the factorization (1.1) applies to $\mathrm{Sq}^{16} x_{23}$. We now construct the universal example.

Let $p_0\colon E_0 \to K(Z, 23)$ be the fiber of the map

$$g\colon K(Z, 23) \to K(Z_2; 25, 30, 31, 35)$$

given by

$$\begin{aligned} g^*(\iota_{25}) &= \mathrm{Sq}^2(\iota_{23}), \\ g^*(\iota_{30}) &= \mathrm{Sq}^7(\iota_{23}), \\ g^*(\iota_{31}) &= \mathrm{Sq}^8(\iota_{23}), \quad \text{and} \\ g^*(\iota_{35}) &= \mathrm{Sq}^{8,4}(\iota_{23}). \end{aligned}$$

Next, define $p_1\colon E_1 \to E_0$ to be the fiber of the map

$$g_0\colon E_0 \to K_0 = K(Z_2; 26, 30, 33, 35, 36, 37, 38; \overline{32}, \overline{33}, \overline{35})$$

given by

$$g_0^*(\iota_{23+m}) = v_m \qquad (m \neq 8),$$

and

$$g_0^*(\bar{\iota}_{31+k}) = \mathrm{Sq}^k v_8 \qquad (k = 1, 2, 4).$$

Consider the element in $H^{47}(K_0)$:

$$\begin{aligned} \chi = {} & \mathrm{Sq}^8[\mathrm{Sq}^{11,2}\iota_{26} + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})\iota_{30} + \mathrm{Sq}^{4,2}\iota_{33} + \mathrm{Sq}^4\iota_{35} \\ & \qquad + \mathrm{Sq}^3\iota_{36} + \mathrm{Sq}^2\iota_{37} + \mathrm{Sq}^1\iota_{38}] \\ & + \mathrm{Sq}^{15}\bar{\iota}_{32} + (\mathrm{Sq}^{14} + \mathrm{Sq}^{10,4})\bar{\iota}_{33} + \mathrm{Sq}^{12}\bar{\iota}_{37}. \end{aligned}$$

Applying g_0^* to χ, we get

$$\begin{aligned} g_0^*(\chi) &= \mathrm{Sq}^8[\mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})v_7 + \mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4 v_{12} \\ & \qquad + \mathrm{Sq}^3 v_{13} + \mathrm{Sq}^2 v_{14} + \mathrm{Sq}^1 v_{15}] \\ & \quad + (\mathrm{Sq}^{15,1} + \mathrm{Sq}^{14,2} + \mathrm{Sq}^{12,4} + \mathrm{Sq}^{10,4,2})v_8 \\ &= \mathrm{Sq}^8[\mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})v_7 + \mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4 v_{12} \\ & \qquad + \mathrm{Sq}^3 v_{13} + \mathrm{Sq}^2 v_{14} + \mathrm{Sq}^1 v_{15} + (\mathrm{Sq}^8 + \mathrm{Sq}^{6,2})v_8] \\ &= \mathrm{Sq}^8\mathrm{Sq}^{16} p_0^*(\iota_{23}) \\ &= (\mathrm{Sq}^{24} + \mathrm{Sq}^{23,1} + \mathrm{Sq}^{22,2} + \mathrm{Sq}^{20,4}) p_0^*(\iota_{23}). \end{aligned}$$

The values of the last three operations on ι_{23} are in the kernel of p_0^*. So

$$g_0^*(\chi) = \mathrm{Sq}^{24} p_0^*(\iota_{23}).$$

Hence there exists an element $v \in H^{46}(E_1)$ such that $\overline{\Delta}(v) = p_1^* p_0^*(\imath_{23}) \otimes p_1^* p_0^*(\imath_{23})$ and $j_1^*(v) = \sigma^*(\chi)$, where j_1 is the fiber of p_1.

We now need to map X into E_1. Let $f: X \to K(Z, 23)$ be such that $f^*(\imath_{23}) = x_{23}$. We remark that f can be chosen to be an H-map, since $H^{23}(X \wedge X; Z) = 0$. Since the composition $g \circ f$ is nullhomotopic, there exists a lifting $f_0: X \to E_0$ of f. The H-deviation of f_0 factors through j_0, the fiber of p_0, say $Df_0 = j_0 \circ \widetilde{D}_0$. The map $\widetilde{D}_0$ corresponds to a set of classes in $H^k(X \wedge X)$, $k = 24, 29, 30$, and 34.

We shall work in P_2X, the projection plane of X. Recall that there is an exact triangle [3]

$$\text{(1.9)} \qquad \begin{array}{ccc} H^*(P_2X) & \longrightarrow & IH^*(X) \\ & \nwarrow \quad \swarrow & \\ & IH^*(X) \otimes IH^*(X) & \end{array}$$

that relates P_2X to X. This implies that

$$\text{(1.10)} \qquad H^k(P_2X) = 0 \qquad (17 \le k \le 22).$$

Let $u_{16} \in H^{16}(P_2X)$ correspond to x_{15} and set $u_{24} = \mathrm{Sq}^8 u_{16}$. By (1.10) and the Adem relations, Sq^2, Sq^8, and $\mathrm{Sq}^{8,4}$ are all zero on u_{24}. So by [3], the components of $\widetilde{D}_0$ in degrees 24, 30, and 34 are all zero. Thus $\widetilde{D}_0 \in H^{29}(X \wedge X)$, so it is a sum of terms of the form $x_7 \otimes x_7' x_{15}$, $x_7 x_7' \otimes x_{15}$, and twists of these terms. Consider the elements $f_0^* \circ g_0^*(\imath)$, where $\imath$ is one of the fundamental classes of K_0. We have

$$\overline{\Delta}(f_0^* \circ g_0^*(\imath)) = (Df_0)^* g_0^*(\imath) = \widetilde{D}_0^* \circ j_0^* \circ g_0^*(\imath).$$

Referring to the matrix relation (1.2), we see that the only possible non-zero values can be when $\imath = \imath_{37}$, when

$$\widetilde{D}_0^* \circ j_0^* \circ g_0^*(\imath) = \mathrm{Sq}^8 \widetilde{D}_0^*(\imath_{29}).$$

Hence the images under $f_0^* \circ g_0^*$ of all the fundamental classes of K_0, with the possible exception of $\imath_{37}$, are primitive, so for degree reasons they must be zero. We might possibly have

$$f_0^* \circ g_0^*(\imath_{37}) = \sum x_{i,7} x'_{i,7} x_{i,23}.$$

But since $\mathrm{Sq}^8: H^{15}(X) \to H^{23}(X)$ is onto, we may alter the lift f_0 by the action of the fiber on the map $\tilde{f}: X \to K(Z_2, 29)$ given by

$$\tilde{f}^*(\imath_{29}) = \sum x_{i,7} x'_{i,7} x_{i,15}$$

so as to make, for the altered f_0, $f_0^* \circ g_0^*(\imath_{37}) = 0$. Thus there exists a lifting $f_1\colon X \to E_1$.

We now consider the element $f_1^*(v) \in H^{46}(X)$. We have

$$\overline{\Delta}(f_1^*(v)) = (f_1^* \otimes f_1^*)(\overline{\Delta} v) + (Df_1)^*(v) = x_{23} \otimes x_{23} + (Df_1)^*(v).$$

There is no term in $H^*(X)$ whose coproduct has $x_{23} \otimes x_{23}$ as a summand. Now

$$Df_1 = \theta + j_1 \circ \widetilde{D}_1,$$

where $\theta\colon X \wedge X \to E_1$ is a map given by applying the Cartan formula, Theorem 3.1 of [7], to Df_0. The map θ factors through cohomology classes in $H^*(X \wedge X)$ of which one factor is a primary or secondary operation applied to a decomposable element, and, by the Cartan formulae for primary and secondary operations, such operations cannot hit x_{23}. Also, $(j_1 \circ \widetilde{D}_1)^*(v)$ lies in the image of Steenrod operations applied to elements of degrees $\neq 30$ or 38, so $x_{23} \otimes x_{23}$ cannot be in this image. Thus Sq^8 is identically zero on $H^{15}(X)$ and hence $QH^{23}(X) = 0$. □

COROLLARY 1.3. *$H^*(X)$ is an exterior algebra on generators concentrated in degrees of the form $2^d - 1$ for $d \geq 3$. Further, the action of the Steenrod algebra on $H^*(X)$ is trivial.*

Proof. By the Steenrod connections, any element of $QH^*(X)$ not in a degree of the form $2^d - 1$ lies in the image of Steenrod operations applied to generators in degrees of the form $2^d - 1$. By Theorem 1.2 and the Steenrod Connections, it follows that

$$\mathrm{Sq}^{2^i} QH^{2^d-1}(X) = 0 \quad \text{for } i = 0, 1, 2, 3.$$

By [**1**], Sq^{2^i} factors through secondary operations for $i \geq 4$ if x_{2^d-1} lies in the kernel of Sq^{2^j} for $0 \leq j \leq i - 1$.

So consider the first nontrivial Steenrod operation, say $\mathrm{Sq}^{2^i} x_{2^d-1}$. By the Cartan formula, $\mathrm{Sq}^{2^i} x_{2^d-1}$ is primitive, so it must be a generator. By the Steenrod connections we must have $i = d-1$. By Theorem 1.2 we must have $d \geq 5$, so $i \geq 4$. But this implies $\mathrm{Sq}^{2^{d-1}} x_{2^d-1}$ is in the image of Steenrod operations of lower degree, which cannot happen. Thus the action of the Steenrod algebra on $H^*(X)$ is trivial. Hence $H^*(X)$ is an exterior algebra on generators in degrees of the form $2^d - 1$, $d \geq 3$. □

2. $H^{15}(X)$.

THEOREM 2.1. $H^{15}(X) = 0$.

Proof. Let x_{15} be a nonzero element of $H^{15}(X)$. We define a cohomology operation as follows. Consider the diagram:

$$
\begin{array}{ccccc}
 & & E_2 & & \\
 & \nearrow^{f_2} & \downarrow p_2 & & \\
(2.1) & & E_1 & \xrightarrow{g_2} & K(Z_2;\, 18,\, 19,\, 22,\, 23,\, 24,\, 30) \\
 & \nearrow^{f_1} & \downarrow p_1 & & \\
X & \xrightarrow{f} & K(Z,\, 15) & \xrightarrow{g_1} & K(Z_2;\, 17,\, 19,\, 23)
\end{array}
$$

which is associated with a factorization of Sq^{16} as

$$\mathrm{Sq}^{16} = \sum \alpha_{ij}\varphi_{ij}$$

in which the α_{ij} are Steenrod operations and the φ_{ij} are the secondary operations of Adams, **[1]**, and is constructed as follows.

The map g_1 is given by the formulas

$$g_1^*(\iota_{15+2^k}) = \mathrm{Sq}^{2^k}(\iota_{15}), \qquad k = 1\,,\, 2\,,\, 3.$$

The map g_2 is given by the formulas

$$g_2^*(\iota_{14+2^i+2^j}) = v_{ij}\,,$$

where v_{ij} is an element in $H^*(E_1)$ that represents the secondary operation φ_{ij}.

The map f represents the element x_{15}. The lift f_1 exists since all Steenrod operations are zero on x_{15}, by Corollary 1.3. Now the H-deviation of f_1 factors through the fiber of p_1, namely $K(Z_2;\, 16,\, 18,\, 22)$. Hence the reduced coproducts of the $f_1^*(v_{ij})$ are in the image of Steenrod operations, which are all zero. Hence the $f_1^*(v_{ij})$ are primitive, so they are all zero. Therefore the lift f_2 exists.

In $H^{30}(E_2)$ there is an element v whose reduced coproduct is $p_2^*p_1^*(\iota_{15}) \otimes p_2^*p_1^*(\iota_{15})$. We shall show that the reduced coproduct of $f_2^*(v)$ contains a term $x_{15} \otimes x_{15}$, which will be a contradiction. Let us write the factorization of the H-deviation Df_1 of f_1 as

$$Df_1 = \widetilde{D} \circ j_1.$$

The map $\widetilde{D}$ determines elements in degrees 16, 18, and 22 of $X \wedge X$. Checking possibilities, we see that the components in degrees 16 and

18 are zero, while we may express the component in degree 22 as

$$\widetilde{D}^*(\iota_{22}) = \sum x_{7,i} \otimes x_{15,i}$$

for elements $x_{7,i}$ and $x_{15,i}$ in degrees 7 and 15 respectively. Use of the Cartan formula, [7], now enables us to express the H-deviation of f_2 as the sum of terms in the image of Steenrod operations (which are all zero) together with terms of the form

$$\psi_i(x_{7,i}) \otimes x_{15,i},$$

where the ψ_i are secondary operations. We need to check that it cannot happen for $x_{15,i}$ and $\psi_i(x_{7,i})$ both to be x_{15}. To determine the secondary operations involved here, we may consider the diagram

$$\begin{array}{ccccc} & & G & & \\ & \overset{h_1}{\nearrow} & \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{h} & K(Z, 7) & \xrightarrow{\mathrm{Sq}^2, \mathrm{Sq}^4} & K(Z_2; 9, 11). \end{array}$$

Using either the Serre or the Eilenberg-Moore spectral sequence we see that a basis for $H^{15}(G)$ is given by elements in the image of Steenrod operations together with an element $\tilde{w}_{0,3}$ that restricts to the fiber of π to be $(\mathrm{Sq}^5 + \mathrm{Sq}^{4,1})\iota_{10}$. So we need to determine whether $h_1^*(\tilde{w}_{0,3})$ can be x_{15}.

For dimensional reasons, h_1 is an H-map. Hence it determines a map $\hat{h}_1\colon P_2X \to BG$, where BG denotes the classifying space of G. If $h_1^*(\tilde{w}_{0,3}) = x_{15}$, then $y_{16} = \hat{h}_1^*(B\tilde{w}_{0,3})$ is a representative in $H^*(P_2X)$ of the primitive class x_{15}. In **[10]**, Corollary 1.3, we derived the formula (in the cohomology of BG)

$$(B\pi^*(\iota_8))^3 \equiv \mathrm{Sq}^8(B\tilde{w}_{0,3}), \quad \text{modulo}\, \mathrm{Im}(\mathrm{Sq}^{12}, \mathrm{Sq}^{6,3}, \mathrm{Sq}^{4,2,1}).$$

In general, three-fold cup products in $H^*(P_2X)$ are all zero. By the hypotheses on X and (1.9), $H^*(P_2X) = 0$ in degrees 12, 15, and 17. So $\mathrm{Sq}^8\hat{h}_1^*(B\tilde{w}_{0,3}) = 0$. By **[13]**, $\mathrm{Sq}^8(y_{16}) = \sum y_{8,i}y_{16,i}$, where the $y_{8,i}$ and $y_{16,i}$ correspond to $x_{7,i}$ and $x_{15,i}$, respectively. So we obtain that $\psi_i(x_{7,i})$ cannot contain x_{15} as a summand; hence the reduced coproduct

$$\overline{\Delta} h_2^*(v) = x_{15} \otimes x_{15},$$

which, as stated above, is a contradiction. □

3. $QH^{2^k-1}(X)$. By Corollary 1.3 and Theorem 2.1, $H^*(X)$ is an exterior algebra on generators in degrees 7 and $2^d - 1$, for $d \geq 5$, and has trivial action of the Steenrod algebra.

THEOREM 3.1. *$QH^*(X)$ is concentrated in degree 7.*

Proof. Let $x = x_{2^k-1}$, $k \geq 5$, be a generator of lowest degree greater than seven. Let $\xi H^*(X)$ denote the image of the cup-squaring map $\xi(x) = x^2$. Since $H_*(X)$ is associative, we may assume by [**8**] that $\bar{\Delta}x \in \xi H^*(X) \otimes H^*(X)$, which is trivial since $\xi H^*(X) = 0$. Hence x may be chosen to be primitive. We shall construct an operation similar to that in the proof of Theorem 1.4. Consider the following diagram:

(3.1)
$$\begin{array}{ccccc} & & E_2 & & \\ & \nearrow^{f_2} & \downarrow{p_2} & & \\ & & E_1 & \xrightarrow{g_2} & K_2 \\ & \nearrow^{f_1} & \downarrow{p_1} & & \\ X & \xrightarrow{f} & K(Z, 2^k_1) & \xrightarrow{g_1} & K_1, \end{array}$$

in which $K_1 = \prod_i K(Z_2; 2^k - 1 + 2^n)$, $1 \leq n \leq k-1$, and

$$g_1^*(\iota_{2^k-1+2^n}) = \mathrm{Sq}^{2^n}\iota_{2^k-1},$$

and in which $K_2 = \prod_{i,j} K(Z_2; 2^k - 2 + 2^i + 2^j)$, and g_2 represents the secondary operations φ_{ij} associated with a factorization of Sq^{2^k}.

By Corollary 1.3, all Steenrod equations vanish on x, so $g_1 f \simeq *$ and the lift f_1 exists.

We note that in degrees below $2^k - 1$, $H^*(X)$ is concentrated in degrees divisible by seven. Since x is primitive, f is an H-map. Therefore $D_{g_2 f_1}$ factors through the fiber of p_1. Hence the formula for the H-deviation of a composition yields that $D_{g_2 f_1}$ is in the image of primary operations in $H^*(X \wedge X)$, so it is zero by Corollary 1.3. Hence $g_2 f_1$ is represented by primitive elements of $H^*(X)$ in degrees not of the form $2^d - 1$. Since all primitives are concentrated in degrees of the form $2^d - 1$, $g_2 f_1$ is nullhomotopic, and the lift f_2 exists.

To simplify the situation, we loop the entire diagram to obtain

$$\begin{array}{ccccc} & & \Omega E_2 & & \\ & \overset{\Omega f_1}{\nearrow} & \downarrow \Omega p_2 & & \\ & & \Omega E_1 & \xrightarrow{\Omega g_2} & \Omega K_2 \\ & \overset{\Omega f_2}{\nearrow} & \downarrow \Omega p_1 & & \\ \Omega X & \xrightarrow{\Omega f} & K(Z\,,\, 2^k - 2) & \xrightarrow{\Omega g_1} & \Omega K_1 . \end{array} \tag{3.2}$$

Note. The c-invariant was introduced in **[14]** as the obstruction to an H-map between two homotopy-commutative H-spaces preserving the homotopy-commutative structure. There are various choices for this invariant, which depend on the choice of homotopy realizing the H-map. It was observed that if Y and Z are H-spaces and h: $Y \to Z$ a map, then the composition

$$\sum \Omega Y \wedge \sum \Omega Y \xrightarrow{\varepsilon\wedge\varepsilon} Y \wedge Y \xrightarrow{Dh} Z \tag{3.3}$$

has as its double adjoint $\Omega Y \wedge \Omega Y \to \Omega Z$ a particular choice for the c-invariant $c(\Omega h)$. In the sequel we shall always make this choice for our c-invariants.

We have a suspension element v in $H^{2^{k+1}-1}(\Omega E_2)$ such that

$$c(v) = (\Omega(p_1 p_2))^* \iota_{2^k-2} \otimes (\Omega(p_1 p_2))^* \iota_{2^k-2}.$$

We shall consider the c-invariant of the element

$$(\Omega f_2)^*[v] \in H^{2^{k+1}-1}(\Omega X) = 0.$$

Let $u_{2^k-2} = \sigma^*(x_{2^k-1})$. Then, applying (3.3) to the formula for the H-deviation for a composition of maps, we obtain

$$0 = c((\Omega f_2)^*[v]) = u_{2^k-2} \otimes u_{2^k-2} + c(\Omega f_2)^*[v].$$

Since x_{2^k-1} is primitive, u_{2^k-2} is a c-class. Hence $c(\Omega f_1)$ factors as

$$\Omega X \wedge \Omega X \xrightarrow{\tilde{c}} \Omega^3 K_1 \to \Omega^2 E_1 .$$

We have a commutative diagram

$$\begin{array}{ccc} & & \Omega^2 E_2 \\ & \overset{c(\Omega f_2)}{\nearrow} & \downarrow \Omega^2 p_2 \\ \Omega X \wedge \Omega X & \xrightarrow[c(\Omega f_1)]{} & \Omega^2 E_1 \end{array} \tag{3.4}$$

Now $c(\Omega f_1)$ is adjoint to

$$\begin{array}{ccccc} \sum\Omega X \wedge \sum\Omega x & \longrightarrow & \Omega K_1 & \longrightarrow & E_1, \\ & \searrow & \swarrow \scriptstyle{\widetilde{D}_1} & & \\ & & X \wedge X & & \end{array}$$

hence $[c(\Omega f_1)] \in (PH^*(\Omega X) \otimes PH^*(\Omega X))^{2^k+2^n-4}$.

According to [6], there is an isomorphism of coalgebras

$$\mathrm{Tor}_{H^*(X)}(Z_2, Z_2) \cong H^*(\Omega X).$$

It follows that $H^*(\Omega X)$ in degrees less than $2^k - 2$ is a divided polynomial coalgebra on primitive elements of degree 6. Therefore

$$(3.5)\quad [c\Omega f_1)] \in PH^6(\Omega X)\otimes PH^{2^k-2}(\Omega X)+PH^{2^k-2}(\Omega X)\otimes PH^6(\Omega X).$$

Further, the indecomposables of $H^*(\Omega X)$ in degrees less than $2^k - 2$ are concentrated in degrees of the form $3 \cdot 2^r$. But if $k > 4$, no Steenrod operation on an element in one of these degrees can hit an indecomposable in degree $2^k - 2$, so u_{2^k-2} is not in the image of the Steenrod algebra.

An analysis of the Cartan formula [7] for secondary operations applied to diagrams 3.4 and 3.5 yields that $u_{2^k-2} = \psi(u_6)$, where ψ is a secondary operation defined on 6-dimensional primitives in the kernel of all Steenrod operations. We proceed to study all such operations. Note that ψ has degree $2^k - 8$. The possibilities come from the suspension elements in $H^{2^k-2}(G)$, where G is the space defined as follows. Let G' be defined to be the fiber of the horizontal map g' in the diagram

$$\begin{array}{ccc} G' & & \\ \downarrow & & \\ K(Z, 2^k-1) & \xrightarrow[g']{Sq^2, Sq^4, \ldots, Sq^{2^{k-1}}} & \Pi K(Z_2; 2^k - 1 + 2^n) \end{array}$$

Now set

$$G = \Omega^{2^k-7}G' \quad \text{and} \quad g = \Omega^{2^k-7}g'.$$

So G is fibered as $\pi\colon G \to K(Z, 6)$. We shall see that in $H^{2^k-2}(G)$, $\mathrm{im}(\sigma^*) \subset \overline{A(2)} \cdot H^*(G)$. For, if an element ψ of $H^{2^k-2}(G)$ is a stable operation, then by [1] ψ can be expressed as a sum

$$\psi = \sum \alpha_{ij} v_{ij},$$

in which the v_{ij} represent the operations ψ_{ij} applied to $\pi^*(\iota_6)$. We note that none of the v_{ij} occurs in degree $2^k - 2$.

If $v \in H^{2^k-2}(G)$ represents an unstable operation, then it must be in the image of $(\sigma^*)^N$ but not in the image of $(\sigma^*)^{N+1}$, for some N. Write $v = (\sigma^*)^N[\hat{v}]$, $\hat{v} \in H^{2^k-2+N}(B^N G)$. Since $\hat{v}$ is not a suspension, its a_m-obstruction **[12]** must be non-zero for some m. Such an obstruction must arise from having

$$\iota_{N+7}^m \in \mathrm{Im}(B^{N+1}g)^*$$

for some m of the form $m = 2^r$.

If $r > 1$, then $\iota_{N+7}^m = \mathrm{Sq}^{2^{r-1}(N+7)}\gamma\iota_{N+7}$, where

$$\gamma = \mathrm{Sq}^{2^{r-2}(N+7)} \cdots \mathrm{Sq}^{N+7}.$$

If $r = 1$, then $N = 2^k - 14$, so that $\iota_{N+7}^2 = \mathrm{Sq}^1\gamma\iota_{N+7}$, where $\gamma = \mathrm{Sq}^{2^k-8}$. In either case there is a relation

$$\gamma = \sum \alpha_n \mathrm{Sq}^{2^n}, \quad \alpha_n \in A(2),$$

so there exists an element $w \in H^{2^k-3}(G)$ that restricts to the fiber to be $\sum \alpha_n \iota_{2^n+5}$. Hence a representative of v is given by $\mathrm{Sq}^1 w$ if $r = 1$ and by $\mathrm{Sq}^{2^{r-1}(N+7)}w$ if $r > 1$.

Thus $\psi(u_6)$ must be in the image of the Steenrod operations. This implies that u_{2^k-2} lies in the image of Steenrod operations which is a contradiction. Since

$$\sigma^*\colon QH^{2^k-1}(X) \to PH^{2^k-2}(\Omega X)$$

is monic, we conclude that $QH^{2^k-1}(X) = 0$. □

Proof of the Main Theorem. We now know that $H^*(X)$ is an exterior algebra on seven-dimensional generators. If $H^*(X; Z)$ has odd torsion, then for some odd prime p, there is an even generator of the form $\beta_1 P^n x_{2n+1}$ by **[9]**. Applying the Bockstein spectral sequence, this yields an odd generator in the rational cohomology of degree $(2np+2)p^d - 1$ for $d \geq 1$. But

$$(2np+2)p^d - 1 > 7$$

so $H^*(X; Z)$ has no odd torsion. Hence it is torsion-free. Therefore

$$H^*(X; Z) \cong \Lambda(x_1, \ldots, x_r)$$

where $\deg(x_i) = 7$.

We now use the Hurewicz isomorphism to obtain our desired homotopy equivalence

$$S^7 \times \cdots \times S^7 \xrightarrow{f} X. \qquad \square$$

REFERENCES

[1] J. F. Adams, *On the nonexistence of elements of Hopf invariant one*, Ann. of Math., **72** (1960), 20–103.

[2] W. Browder, *On differential Hopf algebras*, Trans. Amer. Math. Soc., **107** (1963), 153–178.

[3] W. Browder and E. Thomas, *On the projective plane of an H-space*, Illinois J. Math., **7** (1963), 492–502.

[4] D. Goncalves, Mod 2 *homotopy-associative H-spaces*, Geometric Applications of Homotopy Theory I, Proceedings Evanston, Springer Lec. Notes **657** (1977), 198–216.

[5] J. Hubbuck, *Generalized cohomology operations and H-spaces of low rank*, Trans. Amer. Math. Soc., **141** (1969), 335–360.

[6] R. Kane, *On loop spaces without p-torsion*, Pacific J. Math., **60** (1975), 189–201.

[7] J. Lin, *Higher order operations in the* mod 2 *cohomology of a finite H-space*, Amer. J. Math., **105** (1983), 855–937.

[8] ——, *Steenrod connections and connectivity in H-spaces*, Mem. Amer. Math. Soc., **369** (1987), 1–87.

[9] ——, *Torsion in H-spaces* II, Ann. of Math., **107** (1978), 41–88.

[10] J. Lin and F. Williams, *Two torsion and homotopy-associative H-spaces*, J. Math. Kyoto Univ., **30** (1990), 523–541.

[11] F. Sigrist and U. Suter, *Eine Anwendung der K-theorie in der Theorie der H-Raume*, Comment. Math. Helv., **47** (1972), 36–52.

[12] J. D. Stasheff, *Homotopy associativity of H-spaces*, I, II, Trans. Amer. Math. Soc., **108** (1963), 275–312.

[13] F. Williams, *The H-deviation of a lift*, Proc. Amer. Math. Soc., **104** (1988), 1291–1295.

[14] A. Zabrodsky, *Cohomology operations and homotopy commutative H-spaces*, in *The Steenrod Algebra and its Applications*, Springer Lecture Notes in Math., **168** (1970), 308–317.

Received January 18, 1990 and in revised form October 30, 1991. The first author was supported in part by a grant from the National Science Foundation.

UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA 92043

AND

NEW MEXICO STATE UNIVERSITY
LAS CRUCES, NM 88003

PACIFIC JOURNAL OF MATHEMATICS

Volume 156 No. 1 November 1992