

Pacific Journal of Mathematics

GENERALIZED HORSESHOE MAPS AND INVERSE LIMITS

SARAH ELIZABETH HOLTE

GENERALIZED HORSESHOE MAPS AND INVERSE LIMITS

SARAH E. HOLTE

The now-classical example due to Smale, the horseshoe map, displays interesting dynamics as well as a topologically complicated attractor. In 1986 Marcy Barge showed that the full attracting sets of horseshoe maps are homeomorphic to inverse limits of the unit interval with a single bonding map. Here we extend Barge's results to a more general class of maps.

1. Introduction. In [Ba], Barge describes the attracting sets of horseshoe maps as inverse limits of the unit interval with a single bonding map. Topologically these spaces are chainable continua known as Knaster continua.

In this paper we consider a more general class of maps which we will refer to as generalized horseshoe maps. We will show that the attractors of these maps are homeomorphic to inverse limits of the unit interval with a single bonding map. Both the generalized horseshoe map and the bonding map which defines the inverse limit space described above "follow a pattern" in a sense we will define in the next section. In §3 we will prove two theorems about inverse limit spaces which will be needed in the proof of the main result given in §4. In the final section of the paper we will give some examples, and show that the horseshoe maps which Barge studied in [Ba] are special cases of the generalized horseshoes we consider here. For basic information on attractors and inverse limits see [S].

2. Preliminaries. Let I denote the unit interval and $\{f_n\}_{n=1}^{\infty}$ be a sequence of maps of I into I . Let

$$(I, f_n) = \{(x_0, x_1, \dots) : x_n \in I \text{ and } f_n(x_{n+1}) = x_n, n = 1, 2, \dots\}$$

be the inverse limit space with bonding maps f_n and topology induced by the metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

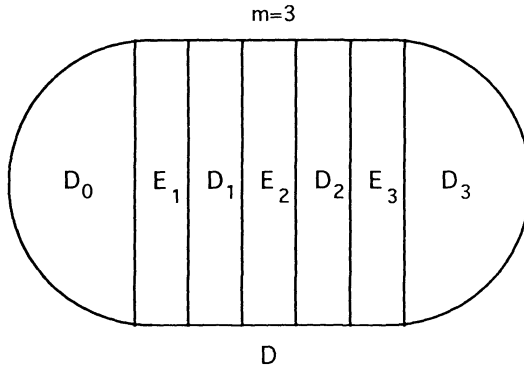


FIGURE 1

For $n = 0, 1, \dots$, let $\pi_n: (I, f_n) \rightarrow I$ be defined by $\pi_n((x_0, x_1, \dots)) = x_n$. It is often the case that we wish to consider inverse limit spaces with a single bonding map f , i.e., $f_n = f$ for $n = 1, 2, \dots$. Let (I, f) denote such an inverse limit space.

Next, let $Q: \{0, 1, 2, \dots, m\} \rightarrow \{0, 1, 2, \dots, m\}$ be a function such that $Q(j) \neq Q(j+1)$, $0 \leq j \leq m-1$, and $\{0, m\} \subset \text{range } Q$. We will use the notation $Q = (Q(0), Q(1), \dots, Q(m))$ to denote the map $Q: \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, m\}$. For example, $Q = (0, 2, 1)$ denotes the map $Q: \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ where $Q(0) = 0$, $Q(1) = 2$, and $Q(2) = 1$. Let D denote $(I \times I) \cup D_0 \cup D_m$ where D_0 and D_m are half disks attached to the opposites sides $\{0\} \times I$ and $\{1\} \times I$ respectively. Subdivide $I \times I$ as follows: For $1 \leq j \leq m-1$, let $D_j = [\frac{4j-1}{4m}, \frac{4j+1}{4m}] \times I$. Let $E_1 = [0, \frac{3}{4m}] \times I$, $E_m = [1 - \frac{3}{4m}, 1] \times I$, and for $2 \leq j \leq m-1$, let $E_j = [\frac{4j-3}{4m}, \frac{4j-1}{4m}] \times I$. The space D is pictured in Figure 1.

Let $\pi: D \rightarrow I$ be defined by $\pi(D_0) = 0$, $\pi(D_m) = 1$, and $\pi|_{I \times I}$ be projection onto the first coordinate. Define $p: I \rightarrow I$ as follows: $p(0) = 0$, $p(1) = 1$, $p(\pi(D_j)) = \frac{j}{m}$, $1 \leq j \leq m-1$, and p is linear on $\bigcup_{j=1}^m \pi(E_j)$. Let $P = p \circ \pi$. Note that $P: D \rightarrow I$ and $P(D_j) = \frac{j}{m}$ for $0 \leq j \leq m$.

We say that a map $F_Q: D \rightarrow D$ follows Q if F_Q is a homeomorphism of D into D which satisfies the following conditions (see Figure 2):

- (i) $F_Q(P^{-1}(P(z))) \subset P^{-1}(P(F_Q(z)))$ for each $z \in D$,
- (ii) $F_Q(D_j) \subset \text{interior } D_{Q(j)}$, $0 \leq j \leq m$,
- (iii) $\text{diam } F_Q^k(P^{-1}(P(z))) \rightarrow 0$ uniformly in z as $k \rightarrow \infty$.

If $F_Q: D \rightarrow D$ follows Q we say that F_Q is a Q -horseshoe map. Let Λ_Q denote the set $\bigcap_{k=0}^{\infty} F_Q^k(D)$ where F_Q is a Q -horseshoe map.

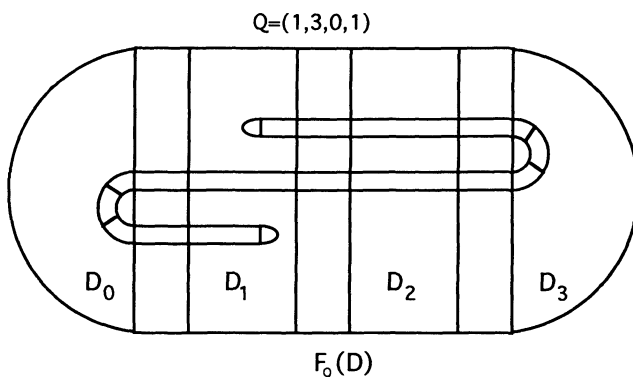


FIGURE 2

We say that $g_Q: I \rightarrow I$ follows Q if $g_Q(\frac{j}{m}) = \frac{Q(j)}{m}$, $0 \leq j \leq m$, and g_Q is linear on $[\frac{j}{m}, \frac{j+1}{m}]$, $0 \leq j \leq m-1$. Let (I, g_Q) denote the inverse limit space of I with the single bonding map g_Q . The following theorem is our main result, and relates Λ_Q and (I, g_Q) .

THEOREM 2.1. *Suppose $Q = (Q(0), Q(1), \dots, Q(m))$ is a function such that $Q(j) \neq Q(j+1)$, $0 \leq j \leq m-1$, and $\{0, m\} \subset \text{range } Q$. If F_Q is a Q -horseshoe map, and $g_Q: I \rightarrow I$ follows Q , then Λ_Q is homeomorphic to (I, g_Q) .*

We will prove this theorem in §4. To do so, two results about inverse limits of the interval are needed. These results constitute §3.

3. Inverse limits.

THEOREM 3.1. *Let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be sequences of surjective self-maps of $I = [0, 1]$. Suppose that $A = \{0 = a_0 < a_1 < \dots < a_m = 1\}$ is a finite subset of I such that for each $n \in \mathbb{N}$, f_n and g_n are both strictly increasing or strictly decreasing on $[a_j, a_{j+1}]$, $f_n(a_j) = g_n(a_j)$, $0 \leq j \leq m$, and f_n and g_n are both invariant on A . Then (I, f_n) is homeomorphic to (I, g_n) .*

Proof. Let I_j denote the interval $[a_j, a_{j+1}]$. We will show that if $(x_0, x_1, \dots) \in (I, f_n)$ then there exists a unique point $(y_0, y_1, \dots) \in (I, g_n)$ such that $y_0 = x_0$ and $x_n \in I_j$ if and only if $y_n \in I_j$. Then we can define $\phi: (I, f_n) \rightarrow (I, g_n)$ by setting $\phi((x_0, x_1, \dots))$ equal to the unique point of (I, g_n) described above. To complete the proof of Theorem 3.1 we will show that ϕ is one-to-one, onto, and continuous.

To define ϕ , let (x_0, x_1, \dots) be an element of (I, f_n) . We inductively define a nested sequence $\{Q_n\}_{n=0}^\infty$ of closed, nonempty subsets

of (I, g_n) with the following properties: if $(y_0, y_1, \dots) \in Q_n$, then $y_0 = x_0$, and $y_i \in I_j$ if and only if $x_i \in I_j$ for $0 \leq i \leq n$. Let $Q_0 = \pi_0^{-1}(x_0) \subset (I, g_n)$. Then Q_0 is closed and nonempty. Now suppose $Q_n \subset Q_{n-1} \subset \dots \subset Q_0$ satisfy the above properties. Define Q_{n+1} as follows: let (y_0, y_1, \dots) be an element of Q_n , $I_{j(n)}$ an interval which contains y_n and x_n , and $I_{j(n+1)}$ an interval which contains x_{n+1} . Then $f_n(x_{n+1}) = x_n \in I_{j(n)}$ so that $f_n(I_{j(n+1)}) \cap I_{j(n)} \neq \emptyset$. Also, f_n is invariant on A , so $f_n(I_{j(n+1)}) = [a_{k_1}, a_{k_2}]$, where a_{k_1} and a_{k_2} are elements of A . Thus $I_{j(n)} \subset f_n(I_{j(n+1)})$ or $I_{j(n)} \cap f_n(I_{j(n+1)}) = \{x_n\}$. If $I_{j(n)} \subset f_n(I_{j(n+1)}) = g_n(I_{j(n+1)})$ then $y_n \in I_{j(n)} \subset g_n(I_{j(n+1)})$, so there exists $y_{n+1} \in I_{j(n+1)}$ such that $g_n(y_{n+1}) = y_n$. In this case, set $Q_{n+1} = \pi_{n+1}^{-1}(y_{n+1}) \subset (I, g_n)$. If $I_{j(n)} \cap f_n(I_{j(n+1)}) = \{x_n\}$, then $x_n = a_{j(n)}$ or $x_n = a_{j(n+1)}$; so $y_n = x_n$. Let $y_{n+1} = x_{n+1}$ and $Q_{n+1} = \pi_{n+1}^{-1}(y_{n+1}) \subset (I, g_n)$. Obviously Q_{n+1} is closed and nonempty and it is easy to check that $Q_{n+1} \subset Q_n$.

Since each Q_n is closed and nonempty, and the sets Q_0, Q_1, \dots are nested, it follows that there exists $(y_0, y_1, \dots) \in \bigcap Q_n$. Suppose that (y'_0, y'_1, \dots) is another point of $\bigcap Q_n$. Let i be the first coordinate so that $y'_i \neq y_i$. Then $i > 0$ since $y'_0 = y_0 = x_0$. Also, y_i and y'_i are both elements of some interval I_j and $g_{i-1}(y_i) = g_{i-1}(y'_i) = y_{i-1}$. But this contradicts the fact that g_{i-1} is one-to-one on I_j . Therefore there is only one point in $\bigcap Q_n$. Thus we define $\phi: (I, f_n) \rightarrow (I, g_n)$ by setting $\phi((x_0, x_1, \dots))$ equal to the unique point (y_0, y_1, \dots) in (I, g_n) such that $x_0 = y_0$ and $x_n \in I_j$ if and only if $y_n \in I_j$.

The same construction shows that given a point (y_0, y_1, \dots) in (I, g_n) we can find a unique point $(x_0, x_1, \dots) \in (I, f_n)$ such that $x_0 = y_0$, and $x_n \in I_j$ if and only if $y_n \in I_j$. It follows that ϕ is one-to-one and onto. We now show that ϕ is continuous.

Let $(x_0, x_1, \dots) \in (I, f_n)$, $(y_0, y_1, \dots) = \phi((x_0, x_1, \dots))$, and L be the minimum of the lengths of the intervals $[a_j, a_{j+1}]$. Given $\varepsilon > 0$, choose N so that $\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$. For each $n \in \mathbb{N}$ and j between 0 and m , $g_n^{-1}: g_n(I_j) \rightarrow I_j$ is a homeomorphism since $g_n|_{I_j}$ is one-to-one. From now on, let g_n^j denote $g_n^{-1}: g_n(I_j) \rightarrow I_j$. Note that if $y_{n+1} \in I_j$, then $y_{n+1} = g_n^j(y_n)$. Next, for each i , $0 \leq i \leq N+1$, define L_i as follows: if $x_i \in A$, let $L_i = L$. If $x_i \notin A$, then $x_i \in (a_j, a_{j+1})$. In this case, let $L_i = \min\{a_{j+1} - x_i, x_i - a_j\}$. Note that if $|x_i - x'_i| \leq L_i$ then x_i and x'_i both lie in some I_j .

As we noted above, $g_{N-1}^j: g_{N-1}(I_j) \rightarrow I_j$ is a homeomorphism. Thus, for each j , $1 \leq j \leq m$, we may choose δ_{N-1}^j so that if y and

y' are elements of $g_{N-1}(I_j)$ with $|y - y'| < \delta_{N-1}^j$, then $|g_{N-1}^j(y) - g_{N-1}^j(y')| < \frac{\varepsilon}{2N}$. Let $\delta_{N-1} = \min\{\delta_{N-1}^1, \dots, \delta_{N-1}^m\}$. Similarly, for each j , $1 \leq j \leq m-1$, choose δ_{N-2}^j so that if y and y' are elements of $g_{N-2}(I_j)$ with $|y - y'| < \delta_{N-2}^j$, then $|g_{N-2}^j(y) - g_{N-2}^j(y')| < \min\{\frac{\varepsilon}{2N}, \delta_{N-1}\}$. Let $\delta_{N-2} = \min\{\delta_{N-2}^1, \dots, \delta_{N-2}^m\}$.

Continue in this way to choose $\delta_{N-(i+1)}^j$ so that if y and y' are elements of $g_{N-(i+1)}(I_j)$ with $|y - y'| < \delta_{N-(i+1)}^j$, then $|g_{N-(i+1)}^j(y) - g_{N-(i+1)}^j(y')| < \min\{\frac{\varepsilon}{2N}, \delta_{N-i}\}$. Let

$$\delta_{N-(i+1)} = \min\{\delta_{N-(i+1)}^1, \dots, \delta_{N-(i+1)}^m\}.$$

Thus we obtain $\delta_0, \delta_1, \dots, \delta_{N-1}$ such that if y and y' are elements of $g_i(I_j)$ with $|y - y'| < \delta_i$, then $|g_i^j(y) - g_i^j(y')| < \min\{\delta_{i+1}, \frac{\varepsilon}{2N}\}$.

Finally let $\delta = \min\{\delta_0, L_0, L_1/2, \dots, L_{N+1}/2^{N+1}, \varepsilon/2N\}$. Now suppose that $(x'_0, x'_1, \dots) \in (I, f_n)$ such that

$$d((x_0, x_1, \dots), (x'_0, x'_1, \dots)) < \delta.$$

Let (y'_0, y'_1, \dots) denote $\phi((x'_0, x'_1, \dots))$. Since

$$d((x_0, x_1, \dots), (x'_0, x'_1, \dots)) < \delta \leq L_i/2^i$$

for $0 \leq i \leq N+1$, it follows that $|x_i - x'_i| < L_i$. Therefore, there exists $I_{j(i)}$ such that x_i and x'_i are both elements of $I_{j(i)}$. This implies that y_i and y'_i are both elements of $I_{j(i)}$.

We now show inductively that $|y_i - y'_i| < \min\{\delta_i, \frac{\varepsilon}{2N}\}$ for $0 \leq i \leq N$. First, $|y_0 - y'_0| = |x_0 - x'_0| < \delta \leq \min\{\delta_0, \frac{\varepsilon}{2N}\}$. Now suppose that $|y_i - y'_i| < \min\{\delta_i, \frac{\varepsilon}{2N}\}$. Let $I_{j(i+1)}$ be an interval which contains y_{i+1} and y'_{i+1} . Then y_i and y'_i are elements of $g_i(I_{j(i+1)})$. Furthermore, $|y_i - y'_i| < \delta_i$ by the induction hypothesis, so $|g_i^j(y_i) - g_i^j(y'_i)| < \min\{\delta_{i+1}, \frac{\varepsilon}{2N}\}$. But $g_i^j(y_i) = y_{i+1}$ and $g_i^j(y'_i) = y'_{i+1}$. Therefore $|y_{i+1} - y'_{i+1}| < \min\{\delta_{i+1}, \frac{\varepsilon}{2N}\}$. It follows that

$$\begin{aligned} d((y_0, y_1, \dots), (y'_0, y'_1, \dots)) &= \sum_{i=0}^{\infty} \frac{|y_i - y'_i|}{2^i} \\ &= \sum_{i=0}^{N-1} \frac{|y_i - y'_i|}{2^i} + \sum_{i=N}^{\infty} \frac{|y_i - y'_i|}{2^i} \leq \sum_{i=0}^{N-1} \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus ϕ is continuous, and this completes the proof of Theorem 3.1.

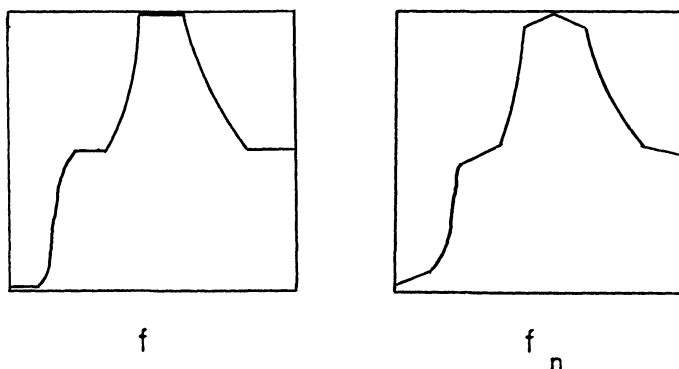


FIGURE 3

THEOREM 3.2. Let $f: I \rightarrow I$, $A = \{0 = a_0 < a_1 < \cdots < a_m = 1\}$, and $B \subset I$ which has a finite number of nondegenerate components. Suppose that f , A , and B satisfy the following conditions:

- (i) f is constant on each component of B ,
- (ii) f is strictly monotone on each component of $I - B$,
- (iii) $f|_{[a_j, a_{j+1}]}$ is monotone, $0 \leq j \leq m - 1$,
- (iv) f is invariant on A ,
- (v) $A \subset B$ and A intersects each component of B in at most one point.

If $g: I \rightarrow I$ satisfies $g(a_j) = f(a_j)$ for $a_j \in A$, and g is linear on $[a_j, a_{j+1}]$, $1 \leq j \leq m - 1$, then (I, f) is homeomorphic to (I, g) .

Proof. We will use the following notation: if B_{i_1} and B_{i_2} are components of B such that $x < y$ for each $x \in B_{i_1}$ and $y \in B_{i_2}$ then write $B_{i_1} < B_{i_2}$. Let $B_1 < B_2 < \cdots < B_r$ be the components of B . Then each B_i is an interval, $[b_i, c_i]$. Note that $0 = b_1$ and $1 = c_r$ since $\{0, 1\} \subset B$. Let $L = \min\{b_{i+1} - c_i : 1 \leq i \leq r - 1\}$ and choose N so that if $n \geq N$, then $\frac{1}{n} < \frac{L}{2}$. For each $n \geq N$, let $b_1^n = b_1 = 0$, $b_i^n = b_i - \frac{1}{n}$, $2 \leq i \leq r$, $c_r^n = c_r = 1$, and $c_i^n = c_i + \frac{1}{n}$, $1 \leq i \leq r - 1$.

Define f_n as follows: $f_n(x) = f(x)$ if $x \in A \cup \bigcup_{i=1}^{r-1} [c_i^n, b_{i+1}^n]$. If $B_i \cap A = \{a_j\}$, define f_n to be linear on $[b_i^n, a_j]$ and $[a_j, c_i^n]$. If $B_i \cap A = \emptyset$, define f_n to be linear on $[b_i^n, c_i^n]$. The graphs of f and f_n are pictured in Figure 3.

It is a straightforward check that $f_n \rightarrow f$ uniformly as $n \rightarrow \infty$. Thus it follows from Theorem 3 of [Br] that (I, f) is homeomorphic to (I, f_{n_k}) where $\{f_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{f_n\}_{n=1}^\infty$. In order to show that (I, f) is homeomorphic to (I, g) we show that (I, g) is

homeomorphic to (I, f_{n_k}) . For each $k \in \mathbb{N}$ let $g_k = g$. We will show that $\{f_{n_k}\}_{k=1}^\infty$, $\{g_k\}_{k=1}^\infty$, and A satisfy the conditions of Theorem 1. First note that for each $k \in \mathbb{N}$ and $a_j \in A$, $g_k(a_j) = g(a_j) = f(a_j) = f_{n_k}(a_j)$. Therefore f_{n_k} and g_k agree on A . Furthermore, f is invariant on A and so f_{n_k} and g_k are also invariant on A .

Next we check that g is strictly monotone on $[a_j, a_{j+1}]$ for each j between 0 and $m-1$. Since g is linear on $[a_j, a_{j+1}]$, it suffices to show that $g(a_j) \neq g(a_{j+1})$. Suppose that $g(a_j) = g(a_{j+1})$. Then $f(a_j) = f(a_{j+1})$. Condition (iii) in the hypothesis of the theorem says that $f|_{[a_j, a_{j+1}]}$ is monotone, so $f(a_j) = f(a_{j+1})$ implies that f is constant on $[a_j, a_{j+1}]$. Also, $a_j \in B_{i_1} = [b_{i_1}, c_{i_1}]$ and $a_{j+1} \in B_{i_2} = [b_{i_2}, c_{i_2}]$, where $i_1 < i_2$. Thus, $[c_{i_1}, b_{i_2}] \subset [a_j, a_{j+1}]$, which implies that f is constant on $[c_{i_1}, b_{i_2}]$. But $[c_{i_1}, b_{i_2}]$ must contain at least one component of $I - B$, and f is strictly monotone on each component of $I - B$. We have reached a contradiction and so it must be the case that $g(a_j) \neq g(a_{j+1})$.

Finally, we check that each f_{n_k} is strictly increasing (decreasing) on $[a_j, a_{j+1}]$ if g is strictly increasing (decreasing) on $[a_j, a_{j+1}]$. Suppose that g is strictly increasing on $[a_j, a_{j+1}]$. Then f is increasing on $[a_j, a_{j+1}]$ and

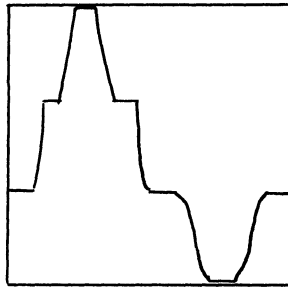
$$[a_j, a_{j+1}] = [a_j, c_{i_1}^{n_k}] \cup [c_{i_1}^{n_k}, b_{i_1+1}^{n_k}] \cup \cdots \cup [c_{i+s-1}^{n_k}, b_{i+s}^{n_k}] \cup [b_{i+s}^{n_k}, a_{j+1}].$$

It is a straightforward check that f_{n_k} is strictly increasing on each of these subintervals of $[a_j, a_{j+1}]$. The case where g is strictly decreasing is proved similarly.

Since Theorem 3.1 applies, it follows that (I, f_{n_k}) and (I, g_k) are homeomorphic. Furthermore, $(I, g_k) = (I, g)$ and (I, f_{n_k}) is homeomorphic to (I, f) . Thus (I, g) is homeomorphic to (I, f) .

4. Proof of Theorem 2.1. We are now ready for the proof of our main result, Theorem 2.1. Suppose $Q = (Q(0), Q(1), \dots, Q(m))$ is a function such that $Q(j) \neq Q(j+1)$, $0 \leq j \leq m-1$, and $\{0, m\} \subset \text{range } Q$ and let F_Q be a Q -horseshoe. Define $f_Q: I \rightarrow I$ by $f_Q(x) = P(F_Q(P^{-1}(x)))$. The graph of f_Q for $Q = (1, 3, 0, 1)$ is pictured in Figure 4 (see next page) (F_Q is pictured in Figure 2).

It is easy to check that f_Q is well defined, continuous, and that $P \circ F_Q = f_Q \circ P$. Thus we may define $\hat{P}: \Lambda_Q \rightarrow (I, f_Q)$ by $\hat{P}(z) = (P(z), P(F_Q^{-1}(z)), P(F_Q^{-2}(z)), \dots)$. It follows from the proof of Theorem 1 in [Ba] that \hat{P} is a homeomorphism. Thus Λ_Q is homeomorphic to (I, f_Q) .



f_Q

FIGURE 4

Next, we use Theorem 3.2 to show that (I, f_Q) is homeomorphic to (I, g_Q) . To this end, let $A = \{\frac{j}{m} : 0 \leq j \leq m\}$ and let $B = \bigcup_{j=0}^m P(F_Q^{-1}(D_j \cap F_Q(D)))$. Then f_Q , A , and B satisfy the conditions of Theorem 3.2. Therefore (I, f_Q) is homeomorphic to (I, g) where $g(a_j) = f_Q(a_j)$ for $a_j \in A$ and g is linear on $[a_j, a_{j+1}]$. But

$$f_Q(a_j) = P(F_Q(P^{-1}(a_j))) = P(F_Q(D_j)) \subset P(D_{Q(j)}) = \frac{Q(j)}{m}.$$

Therefore $g(a_j) = f_Q(a_j) = \frac{Q(j)}{m}$ and g is linear on $[a_j, a_{j+1}]$. Thus g follows Q , and the theorem is proved.

5. Examples. For our first example, we show that the horseshoe maps studied in [Ba] are special cases of the generalized horseshoes considered here. Consider

$$Q = \begin{cases} (0, m, 0, m, \dots, m, 0) : m \text{ even,} \\ (0, m, 0, m, \dots, 0, m) : m \text{ odd.} \end{cases}$$

Then any Q -horseshoe map, F_Q , is an m -fold horseshoe map described in [Ba]. Its attracting set is a Knaster continuum. Next consider $Q = (0, 2, 1)$. Then F_Q and g_Q are pictured in Figure 5. It is well known that (I, g_Q) is homeomorphic to the $\sin(\frac{1}{x})$ continuum, and thus the attracting set of F_Q is homeomorphic to this continuum.

Finally consider $Q = (1, 2, 0)$. Then F_Q and g_Q are pictured in Figure 6. It is well known that (I, g_Q) is homeomorphic to the three point indecomposable continuum described in [HY], pages 141–142. Thus the attracting set of F_Q is homeomorphic to this continuum.

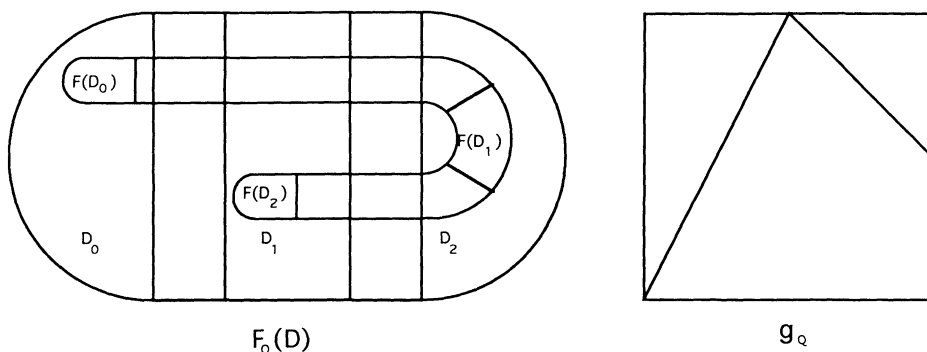


FIGURE 5

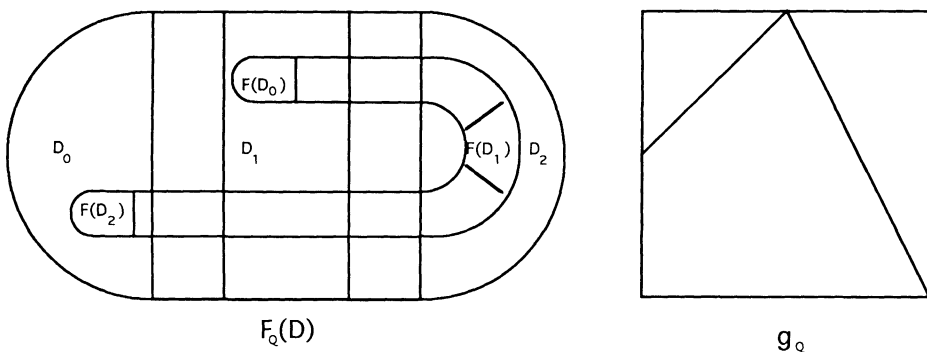


FIGURE 6

REFERENCES

- [Ba] Marcy Barge, *Horseshoe maps and inverse limits*, Pacific J. Math., **121** (1986), 29–39.
- [Br] Morton Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc., **11** (1960), 478–483.
- [HY] John G. Hocking and Gail S. Young, *Topology*, Dover Publications, Inc., New York, 1961.
- [S] Richard Schori, *Chaos: An introduction to some topological aspects*, in *Continuum Theory and Dynamical Systems*, Morton Brown, Editor, American Mathematical Society, Providence, Rhode Island, 1991, pp. 149–161.

Received February 1, 1991 and in revised form May 28, 1991.

UNIVERSITY OF MISSOURI
 ROLLA, MO 65401

PACIFIC JOURNAL OF MATHEMATICS

Founded by

E. F. BECKENBACH (1906–1982) F. WOLF (1904–1989)

EDITORS

V. S. VARADARAJAN
(Managing Editor)
University of California
Los Angeles, CA 90024-1555
vsv@math.ucla.edu

HERBERT CLEMENS
University of Utah
Salt Lake City, UT 84112
clemens@math.utah.edu

F. MICHAEL CHRIST
University of California
Los Angeles, CA 90024-1555
christ@math.ucla.edu

THOMAS ENRIGHT
University of California, San Diego
La Jolla, CA 92093
tenright@ucsd.edu

NICHOLAS ERCOLANI
University of Arizona
Tucson, AZ 85721
ercolani@math.arizona.edu

R. FINN
Stanford University
Stanford, CA 94305
finn@gauss.stanford.edu

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720
vfr@math.berkeley.edu

STEVEN KERCKHOFF
Stanford University
Stanford, CA 94305
spk@gauss.stanford.edu

C. C. MOORE
University of California
Berkeley, CA 94720

MARTIN SCHARLEMANN
University of California
Santa Barbara, CA 93106
mgscharl@henri.ucsb.edu

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the 1991 *Mathematics Subject Classification* scheme which can be found in the December index volumes of *Mathematical Reviews*. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024-1555.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$190.00 a year (10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1992 by Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS

Volume 156 No. 2 December 1992

Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x = Ax + B$	201
LUIS ALÍAS, ANGEL FERRANDEZ and PASCUAL LUCAS	
Lie algebras of type D_4 over number fields	209
BRUCE ALLISON	
Subsemigroups of completely simple semigroups	251
ANNE ANTONIPPILLAI and FRANCIS PASTIJN	
Studying links via closed braids. VI. A nonfiniteness theorem	265
JOAN BIRMAN and WILLIAM W. MENASCO	
Minimal orbits at infinity in homogeneous spaces of nonpositive curvature	287
MARÍA J. DRUETTA	
Generalized horseshoe maps and inverse limits	297
SARAH ELIZABETH HOLTE	
Determinantal criteria for transversality of morphisms	307
DAN LAKSOV and ROBERT SPEISER	
Four dodecahedral spaces	329
PETER LORIMER	
Semifree actions on spheres	337
MONICA NICOLAU	
Conformal deformations preserving the Gauss map	359
ENALDO VERGASTA	
Hecke eigenforms and representation numbers of arbitrary rank lattices	371
LYNNE WALLING	