

Pacific Journal of Mathematics

CONFORMAL DEFORMATIONS PRESERVING THE GAUSS
MAP

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In this work, given a conformal immersion $f: M^n \rightarrow \mathbb{R}^N$ of a Riemannian manifold M^n into a euclidean space \mathbb{R}^N , we establish conditions for the existence of another conformal immersion $\bar{f}: M^n \rightarrow \mathbb{R}^N$ with the same Gauss map as f . In particular, for $n = 2$ and $N = 3$, these conditions are described by means of a partial differential equation on the principal curvatures of f .

0. Introduction. Let M^n be a connected n -dimensional Riemannian manifold and let $f: M^n \rightarrow \mathbb{R}^N$ be a conformal immersion. We denote by $F: M^n \rightarrow G_{n,N}$ the Gauss map of f , which assigns to each point $p \in M^n$ the n -dimensional tangent space $f_*(T_p M)$ in the Grassmannian $G_{n,N}$. We consider the following problem: Under what conditions does there exist another conformal immersion $\bar{f}: M^n \rightarrow \mathbb{R}^N$ such that the Gauss map of \bar{f} coincides with the Gauss map of f , up to a congruence in $G_{n,N}$ induced by a congruence in \mathbb{R}^N ? When this occurs we say that \bar{f} is a *G-deformation* of f . This situation is equivalent to considering conformal immersions f and \bar{f} with parallel tangent spaces $f_*(T_p M)$ and $\bar{f}_*(T_p M)$ in \mathbb{R}^N , which we will always assume. The analogous problem for isometric immersions f and \bar{f} was considered by Dajczer and Gromoll [D&G].

In §1 we characterize our situation by means of a tensor field and a differentiable function satisfying certain conditions (see Proposition 1.5). This result will be used in §2, where we treat the above problem for $n = 2$.

For surfaces, we also consider the *oriented Gauss map* $F^*: M^2 \rightarrow G_{2,N}^*$, where now $f_*(T_p M)$ is seen as an oriented 2-plane in the oriented Grassmannian $G_{2,N}^*$. In regard to the above problem we have two different situations. The first one is when f and \bar{f} have the same oriented Gauss map. In this case, it was shown by Hoffman and Osserman [H&O-2] that either f and \bar{f} are minimal surfaces or \bar{f} coincides with f up to homothety and translation in \mathbb{R}^N . The other situation is when, for any local orientation in M^2 , the oriented Gauss maps of f and \bar{f} differ by the orientation-reversing congruence in $G_{2,N}^*$. In this case we call \bar{f} a *G*-deformation* and say that

f is G^* -deformable. If f is not totally umbilic, we show that a G^* -deformation is unique up to homothety and translation (Theorem 2.1). When $N = 4$, we also prove that G^* -deformable immersions must have flat normal bundle. For $N = 3$, Theorem 2.4 characterizes G^* -deformable immersions by means of a condition on their principal curvatures. We apply Theorem 2.4 to obtain \bar{f} when f is a rotation surface, a cyclid of Dupin or a surface with constant mean curvature. A similar result is obtained for constant mean curvature surfaces in the euclidean sphere S^3 .

For hypersurfaces in \mathbb{R}^{n+1} , $n \geq 3$, the problem considered here will be treated in a forthcoming paper. Most of the results contained in these two works were announced in [Ve] and were obtained in my doctoral thesis. I wish to express my deep gratitude to Professor M. Dajczer for valuable advice and constant encouragement. I also thank the referee for many helpful suggestions.

1. Conformal deformation in \mathbb{R}^N preserving the Gauss map. Let us denote by $\langle \cdot \rangle_\circ$ the Riemannian metric on M^n and by A_ξ the second fundamental form of the conformal immersion $f: M^n \rightarrow \mathbb{R}^N$ in the normal direction ξ , defined by

$$(1.1) \quad f_* A_\xi X = -(\tilde{\nabla}_X \xi)^t,$$

where $(\cdot)^t$ denotes the tangent projection along f and $\tilde{\nabla}$ is the Levi-Civita connection of the canonical metric $\langle \cdot, \cdot \rangle$ on the euclidean space \mathbb{R}^N . We denote also by $\langle \cdot, \cdot \rangle$ the metric on M^n induced by f , defined by $\langle \cdot, \cdot \rangle = e^{2\varphi_1} \langle \cdot, \cdot \rangle_\circ$, where $e^{2\varphi_1}: M^n \rightarrow \mathbb{R}$ is the conformal factor of f .

Let $\bar{f}: M^n \rightarrow \mathbb{R}^N$ be a G -deformation of f with conformal factor $e^{2\varphi_2}$. We define an orthogonal tensor field $T: TM \rightarrow TM$ by

$$(1.2) \quad T = e^{-\varphi} f_*^{-1} \circ P \circ \bar{f}_*,$$

where $\varphi = \varphi_2 - \varphi_1$ and, for each $q \in M^n$,

$$(1.3) \quad P_q: T_{\bar{f}(q)} \mathbb{R}^N \rightarrow T_{f(q)} \mathbb{R}^N$$

denotes the parallel transport in \mathbb{R}^N . For any vector field V along \bar{f} we have

$$(1.4) \quad \tilde{\nabla}_X P V = P \tilde{\nabla}_X V,$$

where X is any tangent field on M^n . We denote by ∇ the Levi-Civita connection on M^n relative to the metric $\langle \cdot, \cdot \rangle$ and by $\nabla \varphi$ the gradient of φ with respect to this metric. The following result gives

necessary and sufficient conditions on T and φ for the existence (at least locally) of a G -deformation of f .

PROPOSITION 1.5. *Let $f: M^n \rightarrow \mathbb{R}^N$ be a conformal immersion.*

(i) *If \bar{f} is a G -deformation of f , then*

$$(1.6) \quad \nabla_X T = T \circ (X \wedge \nabla \varphi)$$

and

$$(1.7) \quad A_\xi \circ T = T^{-1} \circ A_\xi$$

for any tangent field X and normal field ξ . Moreover the second fundamental form \bar{A} of \bar{f} is given by

$$(1.8) \quad \bar{A}_\xi = e^{-\varphi} T^{-1} \circ A_{P\xi}.$$

(ii) *If M^n is simply connected and there exist an orthogonal tensor field T and a differentiable function φ satisfying (1.6) and (1.7), then for any $q_0 \in M^n$,*

$$\bar{f}(q) = \int_{q_0}^q e^\varphi f_* T$$

defines a G -deformation of f .

Proof. We will make use of the Gauss formula

$$(1.9) \quad \tilde{\nabla}_X f_* Y = f_* \nabla_X Y + \alpha(X, Y),$$

where $X, Y \in TM$ and $\alpha(X, Y)$ denotes the normal component of $\tilde{\nabla}_X f_* Y$. Recall the relationship between α and A_ξ , given by

$$\langle \alpha(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Levi-Civita connection $\bar{\nabla}$ of the metric on M^n induced by \bar{f} (see [Ku], p. 316) is given by

$$\bar{\nabla}_X Y = \nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y \rangle \nabla \varphi.$$

Thus we can write

$$(1.10) \quad (\tilde{\nabla}_X \bar{f}_* Y)^t = \bar{f}_* (\nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y \rangle \nabla \varphi).$$

From (1.4), (1.9) and (1.10) we obtain

$$\begin{aligned} f_* \nabla_X T Y &= f_* (-X(\varphi) e^{-\varphi} f_*^{-1} P \bar{f}_* Y + e^{-\varphi} \nabla_X f_*^{-1} P \bar{f}_* Y) \\ &= -X(\varphi) e^{-\varphi} P \bar{f}_* Y + e^{-\varphi} (\tilde{\nabla}_X P \bar{f}_* Y)^t \\ &= -X(\varphi) e^{-\varphi} P \bar{f}_* Y \\ &\quad + e^{-\varphi} P \bar{f}_* (\nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y \rangle \nabla \varphi) \\ &= f_* (T \nabla_X Y + Y(\varphi) T X - \langle X, Y \rangle T \nabla \varphi), \end{aligned}$$

and this proves (1.6).

If ξ is a vector field normal to f , by (1.1) and (1.4) we have

$$\bar{f}_* \bar{A}_\xi X = -(\tilde{\nabla}_X \xi)^t = -(P^{-1} \tilde{\nabla}_X P \xi)^t = P^{-1} f_* A_{P\xi} X.$$

Thus $\bar{A}_\xi = e^{-\varphi} T^{-1} A_{P\xi}$. Now (1.7) follows from the fact that \bar{A}_ξ and $A_{P\xi}$ are self-adjoint.

In order to prove (ii), we compute the exterior differential of the 1-form $e^\varphi f_* T$ defined on M^n with values in \mathbb{R}^N :

$$\begin{aligned} d(e^\varphi f_* T)(X, Y) &= \tilde{\nabla}_X e^\varphi f_* T Y - \tilde{\nabla}_Y e^\varphi f_* T X - e^\varphi f_* T([X, Y]) \\ &= X(\varphi) e^\varphi f_* T Y + e^\varphi \tilde{\nabla}_X f_* T Y - Y(\varphi) e^\varphi f_* T X \\ &\quad - e^\varphi \tilde{\nabla}_Y f_* T X - e^\varphi f_* T(\nabla_X Y - \nabla_Y X). \end{aligned}$$

Now we use (1.9) to get

$$\begin{aligned} d(e^\varphi f_* T)(X, Y) &= e^\varphi (\alpha(X, TY) - \alpha(Y, TX)) \\ &\quad + e^\varphi f_* (\nabla_X T Y - T \nabla_X Y - \nabla_Y T X + T \nabla_Y X \\ &\quad + X(\varphi) T Y - Y(\varphi) T X). \end{aligned}$$

By (1.6) the above equality becomes

$$d(e^\varphi f_* T)(X, Y) = e^\varphi (\alpha(X, TY) - \alpha(Y, TX)).$$

But, for each vector ξ normal to f we have

$$\begin{aligned} \langle \alpha(X, TY) - \alpha(Y, TX), \xi \rangle \\ = \langle A_\xi X, TY \rangle - \langle A_\xi T X, Y \rangle = \langle (T^{-1} A_\xi - A_\xi T) X, Y \rangle \end{aligned}$$

and this vanishes by (1.7). Thus $e^\varphi f_* T$ is a closed 1-form on M^n . Since M^n is simply connected, we can define $\bar{f}: M^n \rightarrow \mathbb{R}^N$ by

$$\bar{f}(q) = \int_{q_0}^q e^\varphi f_* T.$$

Then $\bar{f}_* = e^\varphi f_* T$ and $\langle \bar{f}_* X, \bar{f}_* Y \rangle = e^{2\varphi} \langle X, Y \rangle$. So \bar{f} is a G -deformation of f . \square

REMARK 1.11. As an immediate consequence of (1.6), we see that φ is constant along M^n if and only if T is a parallel tensor field with respect to the metric $\langle \cdot, \cdot \rangle$. When this occurs, f and \bar{f} induce the same metric on M^n , up to a constant factor. Thus, in this case the problem considered here is equivalent to considering isometric immersions f and \bar{f} with the same Gauss map. This was done by Dajczer and Gromoll in [D&G], where the orthogonal tensor field (1.2) becomes $T = f_*^{-1} P \bar{f}_*$, is parallel and satisfies (1.7).

2. Conformal deformations of surfaces preserving the Gauss map. In this section we study conformal surfaces in \mathbb{R}^N that are G^* -deformable. In this case, the above tensor field T must satisfy the additional condition $\det T = -1$ on M^2 . We also obtain a result for surfaces with constant mean curvature in the euclidean sphere S^3 . We begin with a uniqueness result.

THEOREM 2.1. *Let $f: M^2 \rightarrow \mathbb{R}^N$ be a conformal immersion which is not totally umbilic. If there exists a G^* -deformation \bar{f} , then \bar{f} is unique up to homothety and translation in \mathbb{R}^N .*

Proof. Let \bar{M}^2 denote M^2 with the opposite orientation. Denote the Gauss maps of $f: M^2 \rightarrow \mathbb{R}^N$ and $\bar{f}: \bar{M}^2 \rightarrow \mathbb{R}^N$ by F and \bar{F} , respectively. Then $F = \bar{F}$ as maps of M^2 (without orientation) into $G_{2,N}^*$. Now apply Theorem 1.1 of [H&O-1] and the basic uniqueness result in [H&O-2]. \square

Using some results of [We-1] and [We-2] we prove the following two theorems.

THEOREM 2.2. *Let $f: M^2 \rightarrow \mathbb{R}^4$ be a G^* -deformable conformal immersion. Then the normal bundle of f is flat.*

Proof. We may assume the Gauss map $F: M^2 \rightarrow G_{2,4}^*$ is an immersion since the curvature of the normal bundle is zero anywhere F fails to be regular. Then as in the proof of the previous theorem $F: M^2 \rightarrow G_{2,4}^*$ and $\bar{F}: \bar{M}^2 \rightarrow G_{2,4}^*$ are equal. Using Corollary 3 on p. 464 of [We-2], and the notation there, the existence of $f: M^2 \rightarrow \mathbb{R}^4$ implies

$$\varepsilon_1(g) + \rho_1(g) = \varepsilon_2(g) + \rho_2(g);$$

from the existence of $\bar{f}: \bar{M}^2 \rightarrow \mathbb{R}^4$, it follows that

$$\varepsilon_1(g) - \rho_1(g) = \varepsilon_2(g) - \rho_2(g),$$

where g is the metric induced on M^2 by f . Thus $\varepsilon_1(g) = \varepsilon_2(g)$ and by Corollary 2 on p. 464 of [We-2] it follows that the normal bundle is flat. \square

THEOREM 2.3. *Let $f: M^2 \rightarrow \mathbb{R}^N$ with $N \geq 5$ be a conformal immersion. If there exists a point of M^2 which is not an inflection point of f , then f is not G^* -deformable.*

Proof. This follows immediately from Proposition 5 of [We-1] and the observation that if \bar{f} existed then as above $f: M^2 \rightarrow \mathbb{R}^N$ and $\bar{f}: \bar{M}^2 \rightarrow \mathbb{R}^N$ would have the same Gauss map. \square

The main result of this section is the following.

THEOREM 2.4. *Let $f: M^2 \rightarrow \mathbb{R}^3$ be a conformal immersion without umbilic points and let ν and ω be unit principal vector fields of f , with eigenvalues λ and μ respectively.*

(i) *If f is G^* -deformable, then*

$$(2.5) \quad (\lambda - \mu)(\nu(\omega(\lambda)) + \omega(\nu(\mu))) + \nu(\mu)\omega(\mu) - \nu(\lambda)\omega(\lambda) = 0.$$

(ii) *If M^2 is simply connected and (2.5) is satisfied, then f is G^* -deformable.*

Proof. Let p be any point M^2 and ξ be a unit normal field defined on a neighborhood of p . Since f has no umbilic points, there exist differentiable functions λ and μ , and orthonormal tangent fields ν and ω , defined on a neighborhood of p , such that

$$(2.6) \quad A_\xi \nu = \lambda \nu, \quad A_\xi \omega = \mu \omega.$$

From Codazzi equation

$$\nabla_\nu(\mu \omega) - A_\xi \nabla_\nu \omega = \nabla_\omega(\lambda \nu) - A_\xi \nabla_\omega \nu$$

we get

$$(2.7) \quad \nu(\mu) = (\lambda - \mu)\langle \nabla_\omega \nu, \omega \rangle, \quad \omega(\lambda) = (\mu - \lambda)\langle \nabla_\nu \omega, \nu \rangle.$$

Let us suppose that f is G^* -deformable. Then there exist a differentiable function $\varphi: M^2 \rightarrow \mathbb{R}$ and an orthogonal tensor field T with $\det T = -1$, satisfying (1.6) and (1.7). Let ν_1 and ω_1 be orthonormal tangent fields on a neighborhood of p such that $T\nu_1 = \nu$ and $T\omega_1 = -\omega$. Then by (1.7) we have

$$A_\xi \nu_1 = A_\xi T^{-1} \nu = T A_\xi \nu.$$

Thus $A_\xi \nu_1$ is parallel to ν_1 . So ν_1 and ω_1 are principal directions, and this determines T up to sign. From now on we will suppose that $T\nu = \nu$ and $T\omega = -\omega$. Now (1.6) is equivalent to

$$(2.8) \quad \nu(\varphi) = -2\langle \nabla_\omega \nu, \omega \rangle, \quad \omega(\varphi) = -2\langle \nabla_\nu \omega, \nu \rangle.$$

From (2.7) and (2.8) we have

$$(2.9) \quad \nu(\varphi) = -\frac{2\nu(\mu)}{\lambda - \mu}, \quad \omega(\varphi) = \frac{2\omega(\lambda)}{\lambda - \mu}.$$

By (2.8) we get

$$\begin{aligned} [\nu, \omega](\varphi) &= (\nabla_\nu \omega)(\varphi) - (\nabla_\omega \nu)(\varphi) \\ &= \langle \nabla_\nu \omega, \nu \rangle \nu(\varphi) - \langle \nabla_\omega \nu, \omega \rangle \omega(\varphi) \\ &= 0. \end{aligned}$$

Thus we must have

$$\nu(\omega(\varphi)) - \omega(\nu(\varphi)) = 0$$

or, using (2.9),

$$(2.10) \quad \nu \left(\frac{\omega(\lambda)}{\lambda - \mu} \right) + \omega \left(\frac{\nu(\mu)}{\lambda - \mu} \right) = 0,$$

which is equivalent to (2.5). Note that equation (2.5) is invariant by change of sign of the vector fields ξ , ν and ω . Thus it is valid everywhere on M^2 .

Suppose now that (2.5) is satisfied on M^2 . We define the tangent vector field

$$\delta = -2\langle \nabla_\omega \nu, \omega \rangle \nu - 2\langle \nabla_\nu \omega, \nu \rangle \omega$$

and observe that δ does not depend on the unit vector fields ξ , ν and ω satisfying (2.6). Now we define in M^2 the 1-form γ given by

$$(2.11) \quad \gamma(X) = \langle \delta, X \rangle.$$

Using (2.7) we compute

$$\begin{aligned} d\gamma(\nu, \omega) &= \nu(\gamma(\omega)) - \omega(\gamma(\nu)) - \gamma(\nabla_\nu \omega - \nabla_\omega \nu) \\ &= \frac{2}{\lambda - \mu} (\nu(\omega(\lambda)) + \omega(\nu(\mu))) \\ &\quad + \frac{2}{(\lambda - \mu)^2} (\nu(\mu)\omega(\mu) - \nu(\lambda)\omega(\lambda)). \end{aligned}$$

By (2.5) the 1-form γ is closed. Since M^2 is simply connected, there exists $\varphi: M^2 \rightarrow \mathbb{R}$ such that $\nabla \varphi = \delta$, that is,

$$(2.12) \quad \nu(\varphi) = -2\langle \nabla_\omega \nu, \omega \rangle, \quad \omega(\varphi) = -2\langle \nabla_\nu \omega, \nu \rangle.$$

We define the tensor field T by $T\nu = \nu$ and $T\omega = -\omega$. Then T is orthogonal, $\det T = -1$ and (1.7) is satisfied. From (2.12) it is easy to show that (1.6) is satisfied. By Proposition 1.5, f is G^* -deformable. \square

The next result is needed for the proof of some of the corollaries to Theorem 2.4.

PROPOSITION 2.13. *If (x, y) are principal coordinates on an open set $U \subset M^2$, then on U (2.5) is equivalent to*

$$(2.14) \quad \left(\frac{(\lambda + \mu)_x}{\lambda - \mu} \right)_y + \left(\frac{(\lambda + \mu)_y}{\lambda - \mu} \right)_x = 0.$$

Proof. Let E and G be positive functions such that

$$(2.15) \quad \frac{\partial}{\partial x} = E\omega \quad \text{and} \quad \frac{\partial}{\partial y} = G\omega.$$

Since $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ we obtain

$$\begin{aligned} E\omega(G) - EG\langle \nabla_{\omega}\omega, \omega \rangle &= 0, \\ -G\omega(E) + EG\langle \nabla_{\omega}\omega, \omega \rangle &= 0, \end{aligned}$$

and then by (2.7) we have

$$(2.16) \quad G_x = \frac{G\mu_x}{\lambda - \mu}, \quad E_y = -\frac{E\lambda_y}{\lambda - \mu}.$$

From (2.15) and (2.16) it follows that

$$\begin{aligned} \omega(\omega(\mu)) &= \frac{\mu_{xy}}{EG} + \frac{\mu_x\lambda_y}{(\lambda - \mu)EG}, \\ \omega(\omega(\lambda)) &= \frac{\lambda_{xy}}{EG} - \frac{\mu_x\lambda_y}{(\lambda - \mu)EG}, \end{aligned}$$

and then (2.5) and (2.14) are equivalent. \square

In the next two corollaries, we will consider f as the inclusion map of an open subset of M^2 in \mathbb{R}^3 .

COROLLARY 2.17. *Let M^2 be a rotation surface which does not meet its axis of symmetry. Then f is G^* -deformable. If M^2 is not totally umbilic then \bar{f} is unique up to homothety and translation; also, $\bar{f}(M)$ is again a rotation surface. If M^2 is part of a sphere and $\bar{f}(M)$ is a rotation surface, then $\bar{f}(M)$ is part of a catenoid.*

Proof. We take on $f(M^2)$ the parametrization $\phi: (0, 2\pi) \times I \rightarrow \mathbb{R}^3$ given by

$$\phi(x, y) = (\alpha(y) \cos x, \alpha(y) \sin x, \beta(y)),$$

where $(\alpha(y), \beta(y))$ is a plane curve defined on an open interval $I \subset \mathbb{R}$ and satisfying $\alpha(y) \neq 0$ for any $y \in I$.

Now define $\varphi: M^2 \rightarrow \mathbb{R}$ by $e^\varphi = 1/\alpha^2(y)$. One can immediately verify that

$$(2.18) \quad \bar{\phi}(x, y) = \left(\frac{1}{\alpha(y)} \cos x, \frac{1}{\alpha(y)} \sin x, \bar{\beta}(y) \right),$$

where

$$\bar{\beta}(y) = \int_{y_0}^y \frac{\beta'(t)}{\alpha^2(y)} dt,$$

satisfy $\bar{\phi}_x = e^\varphi \phi_x$ and $\bar{\phi}_y = -e^\varphi \phi_y$. Thus \bar{f} defined by $\bar{f}(\phi(x, y)) = \bar{\phi}(x, y)$ is a G^* -deformation. Now, if M^2 is part of a sphere, then up to homothety and translation f is the normal Gauss map of f into the unit sphere S^2 . So $\bar{f}(M^2)$ must be a minimal surface, hence it is part of a catenoid. \square

A surface in \mathbb{R}^3 that is the envelope of a family of spheres tangent to three fixed spheres in \mathbb{R}^3 is called a cyclid of Dupin. These surfaces can be characterized by the fact that they are the surfaces without umbilic points whose principal curvatures are constant along the respective curvature lines (see [C&R], pp. 151–166).

COROLLARY 2.19. *Let M^2 be a cyclid of Dupin and U an open simply connected subset of M^2 . Then f restricted to U is G^* -deformable.*

Proof. If (x, y) are principal coordinates, then the respective principal curvatures λ and μ satisfy $\lambda_x = \mu_y = 0$. Thus (2.14) is verified and we can apply Theorem 2.4. \square

REMARK 2.20. In the preceding corollary, we have by (2.9),

$$\varphi = \log(c(\lambda - \mu)^2)$$

for some positive constant $c \in \mathbb{R}$. By (1.8), the principal curvatures of \bar{f} are

$$\bar{\lambda} = \frac{c\lambda}{(\lambda - \mu)^2}, \quad \bar{\mu} = \frac{-c\mu}{(\lambda - \mu)^2}.$$

Thus, in general, the new surface $\bar{f}(U)$ is not a cyclid of Dupin.

COROLLARY 2.21. *Let $f: M^2 \rightarrow \mathbb{R}^3$ be an oriented minimal surface without umbilic points and let $N: M^2 \rightarrow S^2 \subset \mathbb{R}^3$ be the normal Gauss map. Then f is G^* -deformable and $\bar{f} = N$ up to homothety and translation.*

Proof. Taking principal coordinates (x, y) , we have

$$N_x = -\lambda \frac{\partial}{\partial x}, \quad N_y = \lambda \frac{\partial}{\partial y}.$$

Thus the corollary is a consequence of Theorem 2.1. \square

COROLLARY 2.22. *Let $f: M^2 \rightarrow \mathbb{R}^3$ be an oriented surface free of umbilic points, with constant mean curvature $H \neq 0$, and let $N: M^2 \rightarrow S^2 \subset \mathbb{R}^3$ be the normal Gauss map. Then f is G^* -deformable and \bar{f} is the parallel surface $g = f + \frac{1}{H}N$, up to homothety and translation.*

Proof. Taking principal coordinates (x, y) , we have

$$g_x = f_x + \frac{1}{H}N_x = \left(1 - \frac{\lambda}{H}\right) f_x = \left(\frac{\mu - \lambda}{\lambda + \mu}\right) f_x$$

and

$$g_y = \left(\frac{\lambda - \mu}{\lambda + \mu}\right) f_y.$$

Thus g is a G^* -deformation of f . We observe that, since

$$N_x = -\lambda f_x = \lambda \left(\frac{\lambda + \mu}{\lambda - \mu}\right) \bar{f}_x$$

and

$$N_y = -\mu f_y = -\mu \left(\frac{\lambda + \mu}{\lambda - \mu}\right) \bar{f}_y,$$

one sees that the mean curvature of \bar{f} is also H . \square

We conclude this work with a result analogous to the preceding corollary, for a constant mean curvature surface in S^3 .

PROPOSITION 2.23. *Let $f: M^2 \rightarrow S^3$ be an oriented surface free of umbilic points, with constant mean curvature H , and let $N: M^2 \rightarrow S^3$ be a vector field normal to f . Then f (seen as a surface in \mathbb{R}^4) is G^* -deformable and \bar{f} is the parallel surface*

$$\bar{f} = \frac{1}{(H^2 + 1)^{1/2}} (N + Hf): M^2 \rightarrow S^3,$$

up to homothety and translation.

Proof. It is analogous to the preceding proof. \square

REMARK 2.24. One can easily check that the above immersion \bar{f} has constant mean curvature in S^3 . When $H = 0$, $-\bar{f}$ is the polar map of the minimal immersion f , as defined by Lawson [La].

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Received May 2, 1988 and in revised form August 25, 1991.

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Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Carmel Valley, California 93924, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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PACIFIC JOURNAL OF MATHEMATICS

Volume 156 No. 2 December 1992

Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x = Ax + B$	201
LUIS ALÍAS, ANGEL FERRANDEZ and PASCUAL LUCAS	
Lie algebras of type D_4 over number fields	209
BRUCE ALLISON	
Subsemigroups of completely simple semigroups	251
ANNE ANTONIPPILLAI and FRANCIS PASTIJN	
Studying links via closed braids. VI. A nonfiniteness theorem	265
JOAN BIRMAN and WILLIAM W. MENASCO	
Minimal orbits at infinity in homogeneous spaces of nonpositive curvature	287
MARÍA J. DRUETTA	
Generalized horseshoe maps and inverse limits	297
SARAH ELIZABETH HOLTE	
Determinantal criteria for transversality of morphisms	307
DAN LAKSOV and ROBERT SPEISER	
Four dodecahedral spaces	329
PETER LORIMER	
Semifree actions on spheres	337
MONICA NICOLAU	
Conformal deformations preserving the Gauss map	359
ENALDO VERGASTA	
Hecke eigenforms and representation numbers of arbitrary rank lattices	371
LYNNE WALLING	