ASYMPTOTIC EXPANSION AT A CORNER FOR THE
CAPILLARY PROBLEM: THE SINGULAR CASE

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Consider the solution of the capillary surface equation near a corner of the base domain. It is shown that there exists an asymptotic expansion of the height rise of the surface in a wedge when $\alpha + \gamma < \pi/2$, where 2α is the corner angle and $0 \le \gamma < \pi/2$ the contact angle between the surface and the container wall. The asymptotic does not depend on the particular solution considered. In particular, the leading singularity which was discovered by Concus and Finn is equal to the solution up to $O(r^3)$.

1. Introduction. We consider the non-parametric capillary problem in the presence of gravity. One seeks a surface $S \colon u = u(x)$, defined over a base domain $\Omega \subset \mathbb{R}^2$, such that S meets vertical cylinder walls over the boundary $\partial\Omega$ in a prescribed constant angle γ. This problem leads to the equations, see Finn [6],

$$(1.1) \qquad\qquad \operatorname{div} Tu = \kappa u \quad \text{in } \Omega,$$

$$(1.2) \qquad \nu \cdot Tu = \cos\gamma \quad \text{on the smooth parts of } \partial\Omega,$$

where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}},$$

$\kappa = \text{const} > 0$ and ν is the exterior unit normal on the smooth parts of $\partial\Omega$.

Let $x = 0$ be a corner of Ω with the interior angle 2α satisfying

$$(1.3) \qquad\qquad 0 < 2\alpha < \pi.$$

We assume that the corner is bounded by lines near $x = 0$ and that $\Omega_{R_0} = \Omega \cap B_{R_0}$ coincides with the circular sector

$$\{x \in \mathbb{R}^2 \mid x_1 \tan\alpha \ge |x_2|\} \cap B_{R_0}$$

for a sufficiently small $R_0 > 0$, where

$$B_R = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < R\}.$$

Furthermore, we assume that the contact angle satisfies

$$(1.4) \qquad\qquad 0 \le \gamma < \pi/2.$$

Concus and Finn [2] have shown that u is bounded near the corner if and only if $\alpha + \gamma \geq \pi/2$ is satisfied.

The existence of a solution at an isolated corner when $\alpha + \gamma > \pi/2$ was proved by Emmer [5] and in the case $\alpha + \gamma \leq \pi/2$ by Finn and Gerhardt [7].

When $\alpha + \gamma > \pi/2$ there exists an asymptotic expansion of u near the corner, see [9]. In the borderline case $\alpha + \gamma = \pi/2$ Tam [13] obtained that the normal vector to the surface S is continuous up to the corner.

In this note we will give an asymptotic expansion in the case

$$(1.5) \qquad\qquad \alpha + \gamma < \pi/2,$$

where the solution u is singular at the corner.

The proof is based on the following comparison principle of Concus and Finn, see [3] or [6, Chapter 5]. This principle remains valid for unbounded domains too, see Finn and Hwang [8].

For $0 < R \leq R_0$ we set $\Omega_R = \Omega \cap B_R$, $\Sigma_R = (\partial\Omega \cap B_R) \setminus \{0\}$ and $\Gamma_R = \Omega \cap \partial B_R$.

THEOREM 1.1 (*Concus and Finn* [3]). *Let* v, $w \in C^2(\Omega_R \cup \Gamma_R)$ *with the properties*

  (i) $\operatorname{div} Tv - \kappa v \leq \operatorname{div} Tw - \kappa w$ *in* Ω_R,
  (ii) $v \geq w$ *on* Γ_R,
  (iii) $\nu \cdot Tv \geq \nu \cdot Tw$ *on* Σ_R *as a limit from points of* Ω_R.
Then $v \leq w$ *in* Ω_R.

This theorem is a consequence of a more general comparison principle, see Finn [6, Chapter 5].

Let r, θ be polar coordinates centered at $x = 0$, set $k = \sin\alpha/\cos\gamma$ and

$$(1.6) \qquad\qquad h(\theta) = \frac{\cos\theta - \sqrt{k^2 - sin^2\theta}}{\kappa k}.$$

Then, it was shown by Concus and Finn [3] that under the assumptions (1.3), (1.4) and (1.5) there exist positive constants r_0 and A only depending on α, γ, κ and not on the particular solution u considered such that

$$(1.7) \qquad\qquad \left| u - \frac{h(\theta)}{r} \right| \leq A \quad \text{in } \Omega_{r_0}.$$

The author [10] improved this estimate by showing that in fact there exist positive constants r_0, A and ε not dependent on the particular

solution u considered such that

(1.8) $\left| u - \dfrac{h(\theta)}{r} \right| \leq A r^{\varepsilon}$ in Ω_{r_0}.

The proofs of (1.7) and (1.8) are based on the above comparison principle.

The estimate (1.8) shows that for fixed θ the height $u(x)$ is asymptotically a hyperbola. For $\theta = \pm \alpha$ one obtains the curves of contact on the container wall. With respect to an experiment performed by Taylor [14] in 1712 compare Finn [6, Note 4, p. 131], see also Boys [1, p. 26] and Minkowski [12, p. 577] for the assertion that the trace of the surface on the container wall is a hyperbola.

2. **Asymptotic expansion.** The proof of the existence of the following expansion is completely based on the comparison principle of Concus and Finn. In contrast to the bounded case $\alpha + \gamma > \pi/2$, the terms in the expansion do not depend on the particular solution u considered.

In the following we set

$$h_{-1}(\theta) = h(\theta),$$

where h is defined by (1.6).

Let u be a solution of (1.1), (1.2) and assume that (1.3), (1.4) and (1.5) hold. Under these assumptions we have

THEOREM 2.1. *For a given nonnegative integer* m *there exist positive constants* r_0, A *and* $m + 1$ *functions* $h_{4l-1}(\theta)$, $l = 0, \ldots, m$, *analytical on* $(-\alpha, \alpha)$ *and bounded on* $[-\alpha, \alpha]$, *such that*

$$\left| u(x) - \sum_{l=0}^{m} h_{4l-1}(\theta) r^{4l-1} \right| \leq A r^{4m+3}$$

in Ω_{r_0}. *Moreover, the constants* r_0, A *and the functions* h_{4l-1} *do not depend on the particular solution* u *considered.*

REMARK 2.1. The functions $h_{4l-1}(\theta)$ are solutions of regular boundary value problems of second order (3.16), (3.17) with analytic coefficients and an analytic right-hand side, provided $0 < \gamma$ is satisfied, see Lemma 3.2. Therefore, the functions h_{4l-1} are analytical on the closed interval $[-\alpha, \alpha]$. In the case $\gamma = 0$ the boundary value problems are singular, see §§4 and 5.

In particular, we obtain from $m = 0$ an improvement of the estimate (1.8).

COROLLARY 2.1. *Under the assumptions* (1.3), (1.4) *and* (1.5) *there exist positive constants* r_0 *and* A *not dependent on the particular solution considered such that*

$$\left| u(x) - \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{\kappa k r} \right| \leq A r^3$$

in Ω_{r_0}.

REMARK 2.2. It is interesting that the above asymptotic expansion has the same qualitative behavior as does the asymptotic expansion for the isolated singular solution of $\operatorname{div} Tu = -\kappa u$ studied by Concus and Finn [4]. In particular, both expansions contain only powers in r^{4l-1} with nonnegative integers l.

3. Proof of Theorem 2.1 if $0 < \gamma < \pi/2$. The result follows by induction from the next three lemmas.

For a nonnegative integer n we define

$$(3.1) \qquad v_n(x) = \sum_{l=0}^{n} h_{4l-1}(\theta) r^{4l-1}$$

with analytic functions $h_{4l-1}(\theta)$ on $[-\alpha, \alpha]$.

We assume that v_n satisfies

$$(3.2) \qquad \operatorname{div} Tv_n - \kappa v_n = O(r^{4n+3}) \quad \text{in } \Omega_{R_0},$$

$$(3.3) \qquad \nu \cdot Tv_n - \cos\gamma = O(r^{4n+4}) \quad \text{on } \Sigma_{R_0}.$$

By writing $g = O(r^\beta)$, we mean $|g| \leq cr^\beta$, where c is independent of $x \in \Omega_{R_0}$.

The calculations in the proof of the next lemma are natural extensions of computational results of Concus and Finn [2, 3] and of the author [10].

LEMMA 3.1. *Let* v_n *be a function of the above type* (3.1) *satisfying* (3.2) *and* (3.3). *Then there exist positive constants* A *and* r_0, $0 < r_0 \leq R_0$, *not dependent on the solution* u *considered such that*

$$|u(x) - v_n(x)| \leq A r^{4n+3}$$

in Ω_{r_0}.

Proof. Let

$$(3.4) \qquad w = v + Aq(\theta)r^\lambda, \qquad v = v_n,$$

where $A = $ const, $\lambda = $ const > 0 and $q \in C^2[-\alpha, \alpha]$. The positive constant λ and the function $q(\theta)$ will be determined later independent of A.

In polar coordinates we have

$$\operatorname{div} Tw = r^{-1}\left\{\left(\frac{rw_r}{\sqrt{1+|Dw|^2}}\right)_r + \left(\frac{r^{-1}w_\theta}{\sqrt{1+|Dw|^2}}\right)_\theta\right\},$$

where $|Dw|^2 = w_r^2 + r^{-2}w_\theta^2$.

The definition of w yields

$$\operatorname{div} Tw = r^{-1}\left\{\left[\frac{rv_r + A\lambda qr^\lambda}{\sqrt{1+|Dv|^2}}\left(1+\frac{Q}{1+|Dv|^2}\right)^{-1/2}\right]_r\right.$$
$$\left. + \left[\frac{r^{-1}v_\theta + Aq'r^{\lambda-1}}{\sqrt{1+|Dv|^2}}\left(1+\frac{Q}{1+|Dv|^2}\right)^{-1/2}\right]_\theta\right\}$$

where

$$Q = 2A\lambda v_r qr^{\lambda-1} + 2Aq'v_\theta r^{\lambda-2} + A^2\lambda^2 q^2 r^{2\lambda-2} + A^2 q'^2 r^{2\lambda-2}.$$

We assume that for a given sufficiently small $\eta > 0$ the inequality

(3.5) $$|A|r_0^{\lambda+1} \leq \eta$$

holds. Then, after some calculations, we obtain

(3.6) $$\operatorname{div} Tw = \operatorname{div} Tv + AL_\lambda[q]r^\lambda + \eta_1 + \eta_2$$

in Ω_{r_0}, with

(3.7) $$L_\lambda[q] = (a_1(\theta)q' + a_0(\theta)q)' + b_1(\theta)q' + b_0(\theta)q,$$

where $a_1 = h^2(h^2 + h'^2)^{-3/2}$, $a_0 = \lambda hh'(h^2 + h'^2)^{-3/2}$, $b_1 = (\lambda + 2)hh'(h^2 + h'^2)^{-3/2}$ and $b_0 = \lambda(\lambda + 2)h'^2(h^2 + h'^2)^{-3/2}$ and the quantities η_1 and η_2 satisfy

$$|\eta_1| \leq c_1|A|r^{\lambda+4}, \qquad |\eta_2| \leq c_2 A^2 r^{2\lambda+1}$$

where the constants c_1 and c_2 do not depend on A or r.

From (3.4) and (3.6) we obtain

(3.8) $$\operatorname{div} Tw - \kappa w = A(L_\lambda[q] - \kappa q)r^\lambda + \eta_1 + \eta_2 + \eta_3$$

in Ω_{r_0}, where

$$|\eta_3| \leq c_3 r^{4n+3},$$

c_3 independent of A or r.

For $\nu \cdot Tw$ on Σ_{R_0} we have in polar coordinates

$$\nu \cdot Tw = \text{sign}\,\theta \cdot \frac{r^{-1}w_\theta}{\sqrt{1+|Dw|^2}}, \qquad \theta = \pm\alpha.$$

Under the assumption (3.5) it follows after calculation that

$$(3.9) \qquad \nu \cdot Tw - \cos\gamma = AG_\lambda[q]r^{\lambda+1} + \mu_1 + \mu_2 + \mu_3$$

on Σ_{r_0}, where

$$(3.10) \qquad\qquad G_\lambda[q] = a_1(\theta)q' + a_0(\theta)q$$

and

$$|\mu_1| \leq k_1|A|r^{\lambda+5}, \qquad |\mu_2| \leq k_2A^2r^{2\lambda+2}, \qquad |\mu_3| \leq k_3r^{4n+4}$$

with constants k_j not dependent on A or r.

We insert $q = h(\theta)^{-\lambda} + \varepsilon_0$ with $\lambda = 4n + 3$ into $L_\lambda[q]$ and $G_\lambda[q]$, where ε_0 is a sufficiently small positive constant such that $L_\lambda[q] - \kappa q < 0$ remains valid on $[-\alpha, \alpha]$.

If $A > 0$, then this choice and (3.8), (3.9) imply that there are positive constants c_0 and k_0 not dependent on r or A such that

$$\text{div}\,Tw - \kappa w \leq -c_0Ar^\lambda + c_1Ar^{\lambda+4} + c_2A^2r^{2\lambda+1} + c_3r^\lambda$$

in Ω_{r_0} and

$$Tw - \cos\gamma \geq k_0Ar^{\lambda+1} - k_1Ar^{\lambda+5} - k_2A^2r^{2\lambda+2} - k_3r^{\lambda+1}$$

on Σ_{r_0}. From (3.5) it follows that

$$(3.11) \qquad \text{div}\,Tw - \kappa w \leq -r^\lambda[A(c_0 - c_1r_0^4 - c_2\eta) - c_3]$$

in Ω_{r_0}, and

$$(3.12) \qquad \nu \cdot Tw - \cos\gamma \geq r^{\lambda+1}[A(k_0 - k_1r_0^4 - k_2\eta) - k_3]$$

on Σ_{r_0}. The estimate (1.7) or (1.8) implies that for

$$(3.13) \qquad\qquad A = A_0r_0^{-\lambda},$$

where A_0 is a fixed sufficiently large positive constant which does not depend on r_0 or on the particular solution u considered, we have

$$(3.14) \qquad\qquad u \leq w \quad \text{on } \Gamma_{r_0}.$$

From (3.11), (3.12) and (3.13) we conclude that there exist sufficiently small $\eta > 0$ and $r_0 > 0$ such that the estimate (3.5) and the inequalities

$$\operatorname{div} Tw - \kappa w \leq 0 \qquad \text{in } \Omega_{r_0},$$
$$\nu \cdot Tw \geq \cos \gamma \qquad \text{on } \Sigma_{r_0},$$
$$w \geq u \qquad \text{on } \Gamma_{r_0}$$

hold. The comparison principle of Concus and Finn (see Theorem 1.1) implies $u \leq w$ in Ω_{r_0}, or

$$u(x) \leq v_n(x) + C r^{4n+3},$$

where $C = A h(\theta)^{-(4n+3)}$.

If $A < 0$, then we obtain by the same reasoning a lower bound for u in Ω_{r_0}. Thus, the lemma is proved.

The next lemma concerns the existence of approximate solutions in the sense of (3.1), (3.2) and (3.3).

LEMMA 3.2. *Assume that* $v_m(x)$, m *a nonnegative integer, satisfies* (3.1), (3.2) *and* (3.3) *with* $n = m$. *Then there exists an analytic function* $h_{4m+3}(\theta)$ *on* $[-\alpha, \alpha]$ *such that*

$$v_{m+1}(x) = v_m(x) + h_{4m+3}(\theta) r^{4m+3}$$

satisfies (3.2) *and* (3.3) *with* $n = m + 1$.

Proof. Set $v = v_m$ and $\lambda = 4m + 3$. We seek a function $q(\theta)$ such that $w = v(x) + q(\theta) r^{\lambda}$ is an approximate solution in the sense of (3.2) and (3.3) with $n = m + 1$.

Following the proof of Lemma 3.1, we obtain

$$\operatorname{div} Tw - \kappa w = \operatorname{div} Tv - \kappa v + (L_{\lambda}[q] - \kappa q) r^{\lambda} + O(r^{\lambda+4})$$

in Ω_{r_0}, and

$$\nu \cdot Tw = \nu \cdot Tv + G_{\lambda}[q] r^{\lambda+1} + O(r^{\lambda+5})$$

on Σ_{r_0}, provided $r_0 > 0$ is sufficiently small.

Since $v = v_m$ is given by (3.1) and satisfies (3.2) and (3.3), we have for an analytic function $f_{\lambda}(\theta)$ on $[-\alpha, \alpha]$

(3.15) $$\operatorname{div} Tv - \kappa v = f_{\lambda}(\theta) r^{\lambda} + O(r^{\lambda+4})$$

in Ω_{r_0}, and for constants $g_\lambda^{(1)}$, $g_\lambda^{(2)}$

$$\nu \cdot Tv = \cos\gamma + g_\lambda^{(i)} r^{\lambda+1} + O(r^{\lambda+5})$$

on $\Sigma_{r_0}^{(i)}$, where $\Sigma^{(1)}$ denotes the upper part and $\Sigma^{(2)}$ the lower part of Σ.

Thus, we seek a solution $q(\theta)$ of the following linear and regular boundary value problem.

(3.16) $\qquad L_\lambda[q] - \kappa q = f_\lambda(\theta) \quad \text{in } (-\alpha, \alpha),$

(3.17) $\qquad G_\lambda[q] = \begin{cases} g_\lambda^{(1)}, & \theta = \alpha, \\ g_\lambda^{(2)}, & \theta = -\alpha, \end{cases}$

where the differential operators L_λ and G_λ are defined by (3.7) and (3.10).

The lemma is proved if there exists a solution of (3.16) and (3.17). We will show that the homogeneous problem

(3.18) $\qquad L_\lambda[q] - \kappa q = 0 \quad \text{in } (-\alpha, \alpha),$

(3.19) $\qquad G_\lambda[q] = 0 \quad \text{if } \theta = \pm\alpha,$

has only the null solution. Consequently, there is a unique solution of (3.16), (3.17).

Let $q_0(\theta)$ be a solution of the homogeneous problem. We replace $v = v_m$ by $v_m + q_0(\theta)r^\lambda \ln r$ in w of the proof of Lemma 3.1; see (3.4). Following the calculations in that proof, we obtain

$$\text{div } Tw = \text{div } Tv + AL_\lambda[q]r^\lambda + \eta_1 + \eta_2$$
$$= \text{div } Tv_m + L_\lambda[q_0]r^\lambda \ln r + AL_\lambda[q]r^\lambda + \eta_1 + \eta_2 + \eta_3$$

where the η_j satisfy the same estimates as in the proof of Lemma 3.1.

From the definition of w we obtain

$$\text{div } Tw - \kappa w = (L_\lambda[q_0] - \kappa q_0)r^\lambda \ln r$$
$$+ A(L_\lambda[q] - \kappa q)r^\lambda + \eta_1 + \eta_2 + \eta_3.$$

For the boundary operator we find

(3.21) $\qquad \nu \cdot Tw - \cos\gamma = G_\lambda[q_0]r^{\lambda+1} \ln r + AG_\lambda[q]r^{\lambda+1}$
$$+ \mu_1 + \mu_2 + \mu_3,$$

also the μ_j satisfy the same estimates as in the proof of Lemma 2.1.

Because q_0 solves the homogeneous problem (3.18), (3.19), the estimates (3.20) and (3.21) imply by the same reasoning as in the second part of the proof of Lemma 3.1 the asymptotic formula

$$u(x) = v_m(x) + q_0(\theta)r^\lambda \ln r + O(r^\lambda),$$

where $\lambda = 4m + 3$.

According to Lemma 2.1, we have on the other hand

$$u(x) = v_m(x) + O(r^\lambda).$$

These expansions force that $q_0(\theta) = 0$ for all $\theta \in [-\alpha, \alpha]$. Thus, the lemma is proved.

REMARK 3.1. The functions $h_{4l-1}(\theta)$ are even with respect to $\theta = 0$ since $h_{-1}(\theta) = h(\theta)$ is even. We omit the details.

REMARK 3.2. The solvability of a boundary value problem of the type (3.16), (3.17) was discussed also in [11] by using asymptotic expansions.

The next lemma, see Finn [6, Chapter 5], which yields the basis of the induction, follows after some calculation.

LEMMA 3.3. *The singular function $v_0 = h(\theta)r^{-1}$ satisfies* (3.2) *and* (3.3), *also in the case $\gamma = 0$.*

4. Proof of Theorem 2.1 if $\gamma = 0$. In this case the function $h'(\theta)$, h is given by (1.6), becomes singular at the ends of the interval $(-\alpha, \alpha)$ whereas h remains bounded from below by a positive constant.

More precisely, there are positive constants c_0, c_1 and ε_0, $0 < \varepsilon_0 < \alpha$, such that

$$(4.1) \qquad c_0(\alpha^2 - \theta^2)^{-1/2} \le h'(\theta)\,\mathrm{sign}\,\theta \le c_1(\alpha^2 - \theta^2)^{-1/2},$$
$$(4.2) \qquad c_0(\alpha^2 - \theta^2)^{-3/2} \le h''(\theta) \le c_1(\alpha^2 - \theta^2)^{-3/2}$$

on $(-\alpha, -\alpha + \varepsilon_0) \cup (\alpha - \varepsilon_0, \alpha)$.

Let $v_n(x)$ be given by (3.1) with analytic functions $h_{4l-1}(\theta) \equiv q$ in $(-\alpha, \alpha)$ satisfying

$$(4.3a) \qquad \sup |q(\theta)| < \infty,$$
$$(4.3b) \qquad \sup[(\alpha^2 - \theta^2)^{1/2}|q'(\theta)|] < \infty,$$
$$(4.3c) \qquad \sup[(\alpha^2 - \theta^2)^{3/2}|q''(\theta)|] < \infty,$$

where the supremum is taken over $(-\alpha, \alpha)$.

LEMMA 4.1. *Let v_n be a function of type* (3.1), *where h_{4l-1} satisfies* (4.3a)–(4.3c) *and* (3.2). *Then the conclusion of Lemma 3.1 is valid.*

Proof. Let w be given by (3.4), where the functions h_{4l-1} and q satisfy (4.3a)–(4.3c). We have

$$w_\theta = r^{-1}h' + \sum_{l=1}^{n} h'_{4l-1} r^{4l-1} + Aq'r^\lambda$$

and

$$1 + |Dw|^2 = r^{-4}\left(r^4 + \left[-h + \sum_{l=1}^{n}(4l-1)h_{4l-1}r^{4l} + A\lambda q r^{\lambda+1} \right]^2 \right.$$
$$\left. + \left[h' + \sum_{l=1}^{n} h'_{4l-1} r^{4l} + Aq'r^{\lambda+1} \right]^2 \right).$$

Inserting these expressions into

$$\nu \cdot Tw = \operatorname{sign}\theta \cdot \frac{r^{-1}w_\theta}{\sqrt{1+|Dw|^2}},$$

we obtain from the properties (4.1), (4.3a) and (4.3b) and under the assumption (3.5) that

(4.4) $$\nu \cdot Tw = 1 \quad \text{on } \Sigma_{r_0}$$

as a limit from points of Ω_{r_0}.

After some calculation we arrive at (3.8) also in the case $\gamma = 0$.

Combining (3.8) and (4.4), Lemma 3.1 follows also in the case $\gamma = 0$ as in the second part of the proof of that lemma.

One obtains the estimate (3.8) by the same calculations as in the proof of Lemma 3.1, provided that $\theta \in (-\alpha + \varepsilon_0, \alpha - \varepsilon_0)$ is satisfied.

It is not clear from that computation whether the estimates (3.8) remain valid up to the ends of the intervals $(-\alpha, \alpha)$. If $\theta \in (-\alpha, -\alpha + \varepsilon_0] \cup [\alpha - \varepsilon_0, \alpha)$, then we calculate div Tw as follows.

With the abbreviations

$$H_0 = -h(\theta) + \sum_{l=1}^{n}(4l-1)h_{4l-1}(\theta)r^{4l} + A\lambda q(\theta)r^{\lambda+1},$$

$$H_1 = h'(\theta) + \sum_{l=1}^{n} h'_{4l-1}(\theta)r^{4l} + Aq'(\theta)r^{\lambda+1}$$

we obtain

$$\operatorname{div} Tw = r^{-1}([rH_0(r^4 + H_0^2 + H_1^2)^{-1/2}]_r + [H_1(r^4 + H_0^2 + H_1^2)^{-1/2}]_\theta).$$

Since

$$[H_1(r^4 + H_0^2 + H_1^2)^{-1/2}]_\theta$$
$$= H_{1,\theta}(r^4 + H_0^2 + H_1^2)^{-3/2}(r^4 + H_0^2),$$

it follows from (4.1), (4.2) and (4.3a)–(4.3c) that the constants c_j in the estimates (3.8) remain bounded near the ends of the interval.

This is valid also for the functions $f_{4m+3}(\theta)$ in the expansion (3.15).

LEMMA 4.2. *Assume that* $v_m(x)$, *m a nonnegative integer, satisfies* (3.1), (4.3a)–(4.3c) *and* (3.2) *with* $n = m$. *Then there exists a function* $h_{4m+3}(\theta)$ *satisfying* (4.3a)–(4.3c) *such that the function* v_{m+1} *from Lemma 3.2 solves the approximate differential equation* (3.2) *with* $n = m + 1$.

Proof. As in the proof of Lemma 3.2 we arrive at the differential equation (3.16) for $q = h_{4m+3}$ with the right-hand side f_{4m+3}.

The boundary conditions are given by (4.3a)–(4.3c).

By the same reasoning as in the second part of the proof of Lemma 3.2 one concludes that the associated homogeneous problem has only the solution $q = 0$ on $(-\alpha, \alpha)$.

From the lemma of the next section it follows that there is a unique solution of (3.16) under the boundary conditions (4.3a)–(4.3c). Thus, Lemma 4.2 is proved.

The result of Theorem 2.1 if $\gamma = 0$ follows from the above lemmas and from Lemma 3.3 as in the previous section.

5. The singular boundary value problem. From the behavior (4.1), (4.2) of h one sees that coefficients of the differential equation (3.16) degenerate at the ends of the interval if $\gamma = 0$.

LEMMA 5.1. *Under the boundary conditions* (4.3a)–(4.3c) *there exists a unique solution* $q \in C^2(-\alpha, \alpha)$ *of the singular differential equation* (3.16) *with the right-hand side* $f_\lambda(\theta) = f_{4m+3}(\theta)$.

The proof of this lemma is based on standard methods for such singular boundary value problems. For the convenience of the reader we will sketch the proof.

Let $\phi \in C_0^1(-\alpha, \alpha)$. We define an associated bilinear form to $L_\lambda[q]$ by

$$a[q, \phi] = \int_{-\alpha}^{\alpha} (a_1 q' \phi' + a_0 q \phi' - b_1 q' \phi - b_0 q \phi + a_0 q \phi) \, d\theta.$$

Instead of (3.16) we consider

$$(5.1) \qquad a[q, \phi] + \kappa(q, \phi) = -(f, \phi)$$

for all $\phi \in C_0^1(-\alpha, \alpha)$, where $(f, \phi) = \int_{-\alpha}^{\alpha} f(\theta)\phi(\theta) \, d\theta$.

Define the Hilbert space $\widehat{H}_1$ as the closure of $C_0^1(-\alpha, \alpha)$ with respect to the norm

$$\|\phi\|_1 = \left(\int_{-\alpha}^{\alpha} a_1(\theta)\phi'^2 \, d\theta + \|\phi\|_0^2 \right)^{1/2},$$

where $\|\phi\|_0^2 = \int_{-\alpha}^{\alpha} \phi^2 \, d\theta$.

From the special structure of the coefficients of $L_\lambda[q]$ and the estimate (4.1) for $h'(\theta)$ one concludes that there are positive constants c_0 and c_1 such that

$$a[\phi, \phi] \geq c_1 \|\phi\|_1^2 - c_0 \|\phi\|_0^2$$

for all $\phi \in \widehat{H}_1$.

Consequently, according to the lemma of Lax and Milgram, the problem

$$q \in \widehat{H}_1 : a[q, \phi] + c(q, \phi) = -(f, \phi)$$

for all $\phi \in \widehat{H}_1$, has a unique solution for each $f \in L^2$, provided c is a sufficiently large positive constant.

Since $c(q, \phi)$ is a compact perturbation of $a[q, \phi] + \kappa(q, \phi)$ with respect to the above $\widehat{H}_1$ norm, it follows that the associated operator T to $a[q, \phi] + \kappa(q, \phi)$ considered as a mapping from $\widehat{H}_1$ in L^2 has index zero, that means

$$\text{index } T = \text{codim}(\text{range } T) - \dim(\text{kernel } T) = 0.$$

Thus, there is a unique solution of (5.1) for each $f \in L^2$, provided the associated homogeneous problem has only the null solution.

The lemma is proved if each solution $q \in \widehat{H}_1$ of (5.1) belongs to $C^2(-\alpha, \alpha)$ and satisfies (4.3a)–(4.3c). The definition of $\widehat{H}_1$ implies

$$(5.2) \qquad \sup[(\alpha^2 - \theta^2)|q(\theta)|] < \infty$$

for $q \in \widehat{H}_1$, which is an easy consequence of a Sobolev embedding lemma in one dimension. The supremum is taken over $(-\alpha, \alpha)$. In fact, one has the stronger inequality

$$(5.3) \qquad \sup[(\alpha^2 - \theta^2)^{3/4}|q(\theta)|] < \infty,$$

which is a consequence of the Hardy inequality.

From the special structure of the coefficients of the differential equation (3.16) and since the right-hand side f is bounded on $(-\alpha, \alpha)$, we obtain the desired estimates (4.3a)–(4.3c) after three iterations by using the equation (3.16) and the estimate (5.2). With (5.3) instead of (5.2) we need only two iterations.

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