Pacific Journal of Mathematics

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PAWEL GAJER

Volume 157 No. 2 February 1993

PACIFIC JOURNAL OF MATHEMATICS
Vol. 157, No. 2, 1993

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PAWEŁ GAJER

Spaces of metrics of positive scalar curvature are studied modulo a concordance relation. It is shown that the set of concordance classes of metrics with positive scalar curvature on a closed manifold of dimension ≥ 6 depends only on the dimension, the first Stiefel-Whitney class of the manifold, and the cokernel of a homomorphism $\pi_2(M^n) \to \widetilde{KO}(S^2)$. In addition, for every nonnegative integer i the ith concordance group of metrics of positive scalar curvature is defined and it is shown that for a spin manifold the group is nontrivial when $n + i = 4k + 3, 8k, 8k + 1, k \geq 1$.

Two metrics g_0 and g_1 with positive scalar curvature on M^n are *concordant*, written $g_0 \cong g_1$, if there is a metric g of positive scalar curvature on $M^n \times [0, 1]$ such that $g|_{M^n \times \{i\}} = g_i$ for $i = 0, 1$, and g is a product near $M^n \times \partial[0, 1]$. It is easy to see that concordance is an equivalence relation. The set of its equivalence classes on the space $\mathbb{PSC}(M^n)$ of metrics of positive scalar curvature on M^n will be denoted by $\pi_0^c(\mathbb{PSC}(M^n))$. For $n \geq 3$ the connected sum operation induces a group structure on $\pi_0^c(\mathbb{PSC}(S^n))$ **[G]**. It will be proved that for every simply-connected spin manifold M^n of dimension $n \geq 6$ the group acts freely and transitively on $\pi_0^c(\mathbb{PSC}(M^n))$. This is a special case of the following result.

THEOREM 2.1. *Let M^n be a closed manifold of dimension $n \geq 6$. There exists a group depending only on the dimension, the first Stiefel-Whitney class of M^n, and the cokernel of the homomorphism*

$$\pi_2(M^n) \to \widetilde{KO}(S^2) \quad \textit{given by} \quad [\varphi] \mapsto [\varphi^* TM^n]$$

that acts freely and transitively on the set $\pi_0^c(\mathbb{PSC}(M^n))$.

The group occurring in the statement of Theorem 2.1 is essentially Hajduk's obstruction group for the existence of metrics of positive scalar curvature (for more details see §2 and **[H2]**).

Every smooth map $g\colon [0, 1] \to \mathbb{PSC}(M^n)$ induces a concordance between $g(0)$ and $g(1)$ **[GL2**, Proposition 4.43**]**. In other words, there is a surjective map $\pi_0(\mathbb{PSC}(M^n)) \to \pi_0^c(\mathbb{PSC}(M^n))$. For every $i \geq 1$, there exists an ith-dimensional counterpart $\pi_i(\mathbb{PSC}(M^n), g) \to \pi_i^c(\mathbb{PSC}(M^n), g)$ of the map, which is a group homomorphism. The groups $\pi_i^c(\mathbb{PSC}(M^n), g)$ are called concordance groups of positive scalar curvature metrics on M^n, with the base metric g. The following theorem is one of the main results of this paper.

THEOREM 3.1. *If M^n is a closed spin manifold, then $\pi_i^c(\mathbb{PSC}(M^n), g) \neq 0$ for every $n + i = 4k + 3$, $8k$, $8k + 1$, $k \geq 1$.*

This is a refinement of the following theorem due to Hitchin: *if M^n is a closed spin manifold, then $\pi_i(\mathbb{PSC}(M^n)) \neq 0$ for $i = 0, 1$ and $n + i = 8k$, $8k + 1$, $k \geq 1$.*

The paper is organized as follows. In §1 the notion of a handle metric is introduced and it is proved that an arbitrary metric of positive scalar curvature is isotopic to one that decomposes into a sum of a handle metric and a concordance (see Theorem 1.1). Section 2 contains the proof of Theorem 2.1. Concordance groups of positive scalar curvature metrics are studied in §3.

I would like to thank Marek Lewkowicz for many valuable conversations on deformations of metrics of positive scalar curvature. A special thanks is due to Bogusław Hajduk for sharing with me his ideas on obstruction groups for the existence of metrics with positive scalar curvature and for encouraging me during the preparation of this paper.

1. Deformations of $\mathbb{PSC}$ metrics. In the section the notions of handle and GL metrics are introduced and the following results on deformations of positive scalar curvature metrics are proved.

THEOREM 1.1. *Let W^{n+1} be a bordism between closed manifolds M^n and N^n. If W^{n+1} has a handle decomposition without handles of codimension less than three, then every metric g on W^{n+1} of positive scalar curvature and a product near the boundary is isotopic to one which decomposes into the sum $g_H \cup c$ of a handle metric g_H and a concordance c between $g_H|_{N^n}$ and $g|_{N^n}$.*

THEOREM 1.2. *Let V^i be a submanifold of $M^n \backslash \partial M^n$ of codimension greater than two with a trivial normal bundle. Every metric of positive scalar curvature on M^n is isotopic to one which is GL along V^i.*

In this paragraph GL metrics along submanifolds are defined. Handle metrics will appear later as a by-product of the following construction of GL metrics. Let V^i and M^n be as in Theorem 1.2. Let us first consider the situation where V^i is a point x. If $n \geq 3$, then it is possible to deform an arbitrary metric of positive scalar curvature on M^n in a neighborhood D of x by the following Gromov-Lawson construction (cf. [**GL1**, the proof of Theorem A]). First we take a hypersurface of revolution L in $D \times \mathbb{R}_+$ with a metric of positive scalar curvature on it which at the right end of L is of the form $g_{\text{can}}^{n-1} + dt^2$, where g_{can}^k is the standard metric on the unit sphere S^k. Then we glue $\mathbb{D}^n$ to the end of L and extend the metric from L to $\mathbb{D}^n$ by the "torpedo" metric g_{tp}^n. The metric obtained in this way on M^n will be called *GL around* x. Assume now that V^i is a closed submanifold of M^n. We say that a metric of a positive scalar curvature on M^n is *GL along* V^i if there is a neighborhood of V^i in M^n such that the metric induced on the neighborhood is of the form $g_V + g_{\text{GL}}^{n-i}$, where g_{GL}^{n-i} is a GL metric on the disk $\mathbb{D}^{n-i}$ and g_V is a metric on V^i (not necessarily of positive scalar curvature). If V^i has a nonempty boundary ∂V^i, then the GL metric along V^i is defined in the following way. Take $\varepsilon > 0$ such that the exponential map on ε-disk bundles $\nu_\varepsilon(\partial V^i)$ and $\nu_\varepsilon(V^i)$ of the normal bundles of ∂V^i and V^i in M^n is an embedding. In the sequel these disk bundles will be identified with their images by the exponential map. The normal bundle to ∂V^i splits into the sum $\nu(V^i)|_{\partial V^i} \oplus \text{span}(X)$, where X is a field of outward vectors on ∂V^i orthogonal to $\nu(V^i)$ and $\text{span}(X)$ is the line bundle on ∂V^i induced by X. Consider a neighborhood $\nu_{+,\varepsilon}(\partial V^i) \cup \nu_\varepsilon(V^i)$ of V^i in M^n, with $\nu_{+,\varepsilon}(\partial V^i) = \{(v, tX) \in (\nu(V^i) \oplus \text{span}(X)) \cap \nu_\varepsilon(\partial V^i) : t > 0\}$. A metric g of positive scalar curvature on M^n is *GL along* V^i if there exists $\varepsilon_0 \in (0, \varepsilon)$ such that the restriction of g to the neighborhood $\nu_{+,\varepsilon_0}(\partial V^i) \cup \nu_{\varepsilon_0}(V^i)$ of V^i is of the form

$$(g|_{\partial V^i} + g_{\text{GL}}^{n-i+1}|_{D_+^{n-i+1}}) \cup (g_V + g_{\text{GL}}^{n-i}).$$

The proof of Theorem 1.2 in the case when V^i is a closed submanifold of M^n follows exactly the proof of Theorem $2'$ in [**G**]. When V^i has a nonempty boundary one has to deform a given metric on $\nu_{+,\varepsilon}(\partial V^i)$ and $\nu_\varepsilon(V^i)$ at the same time using the technique of the proof of Theorem $2'$.

Proof of Theorem 1.1. Let W^{n+1} be as in Theorem 1.1. The modification of a given metric of positive scalar curvature on W^{n+1} to a handle one (which will be defined along the proof) will have an inductive character; with induction on the number of handles of W^{n+1}. The inductive step follows.

Let $W^{n+1} = (M^n \times I) \cup_j H^{i+1}$, where $j\colon \partial(\mathbb{D}^{i+1}) \times \mathbb{D}^{n-i} \to M^n \times \{1\}$ is the gluing map for H^{i+1}, and let g be a metric of positive scalar curvature on W^{n+1} product in a neighborhood of the boundary. Choose $\gamma > 0$ such that g is product in the collar $M^n \times [0, \gamma]$ of $M^n \times \{0\}$ in $M^n \times I$. By an abuse of notation $\mathbb{D}^{i+1}$ will denote the disk $(\mathbb{D}^{i+1} \times \{0\}) \cup_j (j(\partial(\mathbb{D}^{i+1} \times \{0\}) \times (\gamma/2, 1]))$, where $\mathbb{D}^{i+1} \times \{0\}$ is the left disk of H^{i+1}. We can apply Theorem 1.2 to the disk $\mathbb{D}^{i+1}$ and deform g in a neighborhood $\nu_{+,\varepsilon}(\partial\mathbb{D}^{i+1}) \cup (\mathbb{D}^{i+1})$ of $\mathbb{D}^{i+1}$ to a metric of the form

$$(g|_{\partial D^{i+1}} + g_{\mathrm{GL}}^{n-i+1}|_{D_+^{n-i+1}}) \cup (g|_{D^{i+1}} + g_{\mathrm{GL}}^{n-i}).$$

There exist: $\mu > 0$ and $\varepsilon_0 \in (0, \varepsilon)$ such that g restricted to $\nu_{+,\varepsilon_0}(\partial\mathbb{D}^{i+1}) \cup \nu_{\varepsilon_0}(\mathbb{D}^{i+1})$ is of the form

$$(g|_{\partial D^{i+1}} + \mu g_{\mathrm{can}}^{n-i+1}|_{D_+^{n-i+1}}) \cup (g|_{D^{i+1}} + \mu g_{tp}^{n-i}).$$

If μ is small enough, then the metric can be isotoped to

$$g' = (g_{\mathrm{can}}^{i} + \mu g_{\mathrm{can}}^{n-i+1}|_{D_+^{n-i+1}}) \cup (g_{tp}^{i+1} + \mu g_{tp}^{n-i}).$$

This metric can be described in another way as follows. Embed $\mathbb{D}^{i+1}$ in $\mathbb{R}^{i+2}$ such that the induced on $\mathbb{D}^{i+1}$ metric will be "torpedo". Consider the embedding induced by the sequence of inclusions

$$\begin{aligned}\mathbb{D}^{i+1} \times (-\infty, 0] &\subset \mathbb{R}^{i+2} \times (-\infty, 0] \\ &\subset \mathbb{R}^{i+2} \times \mathbb{R} \subset \mathbb{R}^{i+2} \times \mathbb{R} \times \mathbb{R}^{n-i+1} = \mathbb{R}^{n+4}\end{aligned}$$

and put $P = \{x \in \mathbb{R}^{n+4} | \operatorname{dist}(x, \mathbb{D}^{i+1} \times (-\infty, 0]) = \mu\}$. There exists $\delta > 0$ such that the metric induced by the Euclidean metric of $\mathbb{R}^{n+4}$ on $\widehat{P} = P \cap (\mathbb{R}^{i+2} \times (-\delta, 0] \times \mathbb{R}^{n-i+1})$ will coincide with g' if we smooth the corner along $P \cap (\mathbb{R}^{i+2} \times \{0\} \times \mathbb{R}^{n-i+1})$. Let us now embed $\mathbb{R}^{i+2} \times \mathbb{R} \times \mathbb{R}^{n-i+1}$ into $\mathbb{R}^{i+2} \times \mathbb{R} \times \mathbb{R}^{i+1} \times \mathbb{R}$ and rotate $\mathbb{D}^{i+1}$ in this space around $\mathbb{R}^{i+2} \times \{0\} \times \mathbb{R}^{n-i+1} \times \{0\}$ by $\pi/2$ radians. The trace S of this rotation has as a boundary $D_L \cup D_R$, where D_L and D_R are the images of $\mathbb{D}^{i+1}$ in $\mathbb{R}^{n+4}$ before and after rotation. Let T be a manifold obtained from $\mathbb{D}^{i+1} \times (-\infty, 0]$ by gluing to $\mathbb{D}^{i+1} \times \{0\}$ the trace of the rotation S along D_L, and then gluing to the resulting manifold the product $((\partial\mathbb{D}^{i+1} \times (-\infty, 0]) \cup \mathbb{D}^{i+1}) \times [0, \delta)$ along $(\partial\mathbb{D}^{i+1} \times (-\infty, 0]) \cup D_R$, and smoothing the resulting corners. Consider $P' = \{x \in \mathbb{R}^{n+4} | \operatorname{dist}(x, T) = \mu\}$. The rotation of $\mathbb{D}^{i+1}$ induces an isotopy between g' and the metric induced by the Euclidean metric of $\mathbb{R}^{n+4}$ on P'. Let us extend the isotopy to an isotopy between the GL metric along $\mathbb{D}^{i+1}$ and a metric $\hat{g}$. There exists $\varepsilon_1 \in (0, \delta)$ such that the metric g_H induced on $\nu_{+,\varepsilon_0}(\partial\mathbb{D}^{i+1}) \cup \nu_{\varepsilon_0}(\partial\mathbb{D}^{i+1} \times [0, \varepsilon_1))$ by $\hat{g}$ has positive scalar curvature and is product near the boundary. The metric induced by $\hat{g}$ on the complement of $\nu_{+,\varepsilon_0}(\partial\mathbb{D}^{i+1}) \cup \nu_{\varepsilon_0}(\partial\mathbb{D}^{i+1} \times [0, \varepsilon_1))$ is a concordance c between $g_H|_{N^n}$ and $g|_{N^n}$, where $N^n = \partial(W^{n+1})\backslash M^n$. Hence g is isotopic to a metric $g_H \cup c$. The metric g_H is determined by $g_M = g|_{M^n\times\{0\}}$ and H^{i+1} in the unique way only up to isotopy. Hence every metric isotopic to g_H will be called in the sequel the *handle metric induced by* g_M *and the handle* H^{i+1}. More generally, if $(M^n \times I) \cup H^{\lambda_0} \cup \cdots \cup H^{\lambda_k}$ is a handle decomposition on M^n of a bordism W^{n+1} without handles of codimension less than three, then the inductive usage of the handle metric construction produces a metric of positive scalar curvature on W^{n+1} which we shall call the *handle metric induced by* g *and the handle decomposition* $(M^n \times I) \cup H^{\lambda_0} \cup \cdots \cup H^{\lambda_k}$.

The next theorem is a straightforward consequence of the handle metric construction.

SURGERY THEOREM (**[GL1, SY, G]**). *Let* (M^n, g) *be a closed Riemannian manifold with positive scalar curvature and let* W^{n+1} *be a bordism between* M^n *and* N^n *such that* W^{n+1} *admits a handle decomposition on* M^n *without handles of codimension less than three, then there exists a metric with positive scalar curvature on* W^{n+1}, *which extends* g *and is product on a collar of* $M^n \cup N^n$.

The following result is a version of Theorem 1.1 for manifolds with boundary.

THEOREM 1.3. *Let* M^n *be a closed manifold with a handle decomposition with all handles of codimension greater than two. Then every concordance* g *between metrics* g_0, g_1 *of positive scalar curvature on* M^n *is isotopic to a metric of the form* $(g_H + dt^2) \cup c \cup g|_{M_{\leq 2}}$, *where* $M_{\leq 2}$ *is the sum of handles of codimension* ≤ 2 *and* c *is a concordance on* $\partial M_{\leq 2}$.

The proof of Theorem 1.3 goes along the lines of the proof of Theorem 1.1. Let M^n and g be as in Theorem 1.3. It is possible to extend g from $M^n \times I$ to a metric $\hat{g}$ of positive scalar curvature on

$M^n \times \mathbb{R}$ putting on $M^n \times (-\infty, 0]$ and $M^n \times [1, +\infty)$ the appropriate product metrics. Then, using Theorem 1.2, one can inductively deform $\hat{g}$ along the products $\mathbb{D}^{\lambda_i} \times \mathbb{R}$, where $\mathbb{D}^{\lambda_i}$ is an extended left disk of h^{λ_i}, getting the required isotopy.

2. Concordances of $\mathbb{PSC}$ metrics. One of the basic corollaries of the Surgery Theorem is that existence of a positive scalar curvature metric on a manifold M^n depends on the 2-connected bordism class of M^n. In this section it is shown that the set of concordance classes of metrics with positive scalar curvature on M^n depends only on the dimension, the first Stiefel-Whitney class of M^n, and the cokernel of a homomorphism $\pi_2(M^n) \to \widetilde{KO}(S^2)$.

THEOREM 2.1. *Let M^n be a closed manifold of dimension greater than five. There exists a group depending only on the dimension, the first Stiefel-Whitney class of M^n, and the cokernel of the homomorphism*

$$\pi_2(M^n) \to \widetilde{KO}(S^2) \quad \text{given by} \quad [\phi] \mapsto [\phi^* TM^n]$$

that acts on the set $\pi_0^c(\mathbb{PSC}(M^n))$ in a transitive and free way.

When M^n is simply-connected and spin, the group is $\pi_0^c(\mathbb{PSC}(S^n))$. In general the group is essentially Hajduk's obstruction group for the existence of metrics of positive scalar curvature, which construction will be described later.

A bordism W^{n+1} between closed manifolds M^n and N^n is called *2-connected* if the groups $\pi_i(W^{n+1}, M^n)$ and $\pi_i(W^{n+1}, N^n)$ are trivial for $i = 1, 2$.

PROPOSITION 2.2. *If M^n and N^n are 2-connected bordant closed manifolds of dimension greater than four, then the sets of concordance classes of metrics of positive scalar curvature on M^n and N^n are in a bijective correspondence.*

The proof of Proposition 2.2 is based on the following lemma.

LEMMA 2.3. *Let W^{n+1} be a bordism between M^n and N^n with a handle decomposition on M^n without handles of codimension less than three, and let g_0 and g_1 be two metrics of positive scalar curvature on W^{n+1} product in a collar of the boundary. If $g_0|_{M^n}$ and $g_1|_{M^n}$ are concordant, then the metrics $g_0|_{N^n}$ and $g_1|_{N^n}$ are concordant as well.*

Proof. Let c be a concordance between the metrics $g_0|_{M^n}$ and $g_1|_{M^n}$. The metric $g_0 \cup c \cup g_1$ on $W^{n+1} \cup_{M^n} (M^n \times I) \cup_{M^n} (-W^{n+1})$ has positive scalar curvature and is a product near the boundary. Since W^{n+1} is the 2-connected bordism, the manifold

$$W^{n+1} \cup_{M^n} (M^n \times I) \cup_{M^n} (-W^{n+1})$$

is relatively 2-connected bordant to $N^n \times I$. By the Surgery Theorem, the above bordism and the metric $g_0 \cup c \cup g_1$ induce a concordance between $g_0|_{N^n}$ and $g_1|_{N^n}$. This completes the proof of Lemma 2.3.

Proof of Proposition 2.2. Let W^{n+1} be a 2-connected bordism between M^n and N^n. By Morse-Smale theory, there are handle decompositions of W^{n+1} on M^n and on N^n without handles of codimension less than three. Thus, every metric g of positive scalar curvature on M^n induces a metric $S(g)$ on N^n, and conversely, every metric g' on N^n induces a metric $S^{-1}(g')$ on M^n. By Lemma 2.3, S and S^{-1} depend only on concordance classes of metrics of positive scalar curvature and are inverse to one another.

For every smooth manifold M^n and a positive integer $i < \dim M^n$ the kernel of the homomorphism

$$\pi_i(M^n) \to \widetilde{KO}(S^i), \quad [\varphi] \mapsto [\varphi^* TM^n]$$

describes the part of $\pi_i(M^n)$ which can be killed by surgery. For $i = 1$ it is the first Stiefel-Whitney class of M^n. For $i = 2$ it is the second Stiefel-Whitney class of M^n, if M^n is a simply-connected manifold.

Let π be a finitely presentable group and let $\omega = (\omega_1, \omega_2)$ where $\omega_1 \in \operatorname{Hom}(\pi, \mathbb{Z}/2\mathbb{Z})$ and $\omega_2 \in \mathbb{Z}/2\mathbb{Z}$. We say that M^n is a (π, ω)-manifold if $\pi_1(M^n) \cong \pi$, $\omega_1(M^n) = \omega_1$, and the homomorphism $\pi_2(M^n) \to \widetilde{KO}(S^2)$ is an isomorphism when $\pi_2 = 1$, and $\pi_2(M^n)$ is trivial when $\omega_2 = 0$. If M^n is a closed manifold of dimension ≥ 5, then surgery killing the kernel of $\pi_2(M^n) \to \widetilde{KO}(S^2)$ induces a 2-connected bordism between M^n and a (π, ω)-manifold with $\pi = \pi_1(M^n)$, $\omega_1 = \omega_1(M^n)$, and ω_2 depending on the cokernel of the homomorphism $\pi_2(M^n) \to \widetilde{KO}(S^2)$.

The remaining part of the section is devoted to a proof of the following result.

THEOREM 2.4. *For every (π, ω)-manifold M^n of dimension $n \geq 6$ there is a transitive and free action of a group $\pi_0^c(n, \pi, \omega)$ on $\pi_0^c(\mathbb{PSC}(M^n))$.*

It is clear that Proposition 2.2 together with Theorem 2.4 imply Theorem 2.1.

$\pi_0^c(n, \pi, \omega)$ are Hajduk's obstruction groups for the existence of metrics of positive scalar curvature on (π, ω)-manifolds. Originally Hajduk defined them for spin manifolds. The following few paragraphs describe adaptation of his construction to (π, ω)-manifolds.

For every presentation α of π, a homomorphism $\pi \to \mathbb{Z}/2\mathbb{Z}$, an element of $\mathbb{Z}/2\mathbb{Z}$, and a number $n \geq 5$ there will be defined a (π, ω)-manifold with boundary $\mathbb{D}^n(\pi, \alpha, \omega)$ such that $\mathbb{D}^n(\pi, \alpha, \omega) = \mathbb{D}^n$ when π and ω are trivial and the following result holds.

LEMMA 2.5 ([H2]). *Let M^n be a (π, ω)-manifold of dimension ≥ 5. Then for an arbitrary presentation α of π there is an embedding $\varphi: \mathbb{D}^n(\pi, \alpha, \omega) \to M^n$ and a handle decomposition of M^n such that φ maps the canonical handle decomposition of $\mathbb{D}^n(\pi, \alpha, \omega)$ onto the union of all handles of codimension less than three.*

Later on we will see that if $\mathbb{T}^n(\pi, \alpha, \omega)$ is the doubling of $\mathbb{D}^n(\pi, \alpha, \omega)$, then the set of concordance classes of metrics of positive scalar curvature on $\mathbb{T}^n(\pi, \alpha, \omega)$ is a group, and it does not depend on the presentation α of π.

This paragraph contains the definition of $\mathbb{D}^n(\pi, \alpha, \omega)$. Assume that $n \geq 5$ and let π be a group with a presentation $\alpha = (a_1, a_2, \dots, a_t | r_1, r_2, \dots, r_s)$ and $\omega_1 \in \operatorname{Hom}(\pi, \mathbb{Z}/2\mathbb{Z})$. We can rearrange the order of the generators such that

$$\omega_1(a_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq l, \\ 1 & \text{for } i+1 \leq i \leq t. \end{cases}$$

It is well known that for every $n \geq 1$ there are only two, up to isomorphism, n-dimensional vector bundles ε^n, γ^n over S^1. Let $\mathbb{D}(\varepsilon^n)$ and $\mathbb{D}(\gamma^n)$ be the unit disk bundles associated with ε^n and γ^n respectively. The boundary connected sum of l copies of $\mathbb{D}(\varepsilon^{n-1})$ and $t-l$ copies of $\mathbb{D}(\gamma^{n-1})$ is a manifold V_1 with the fundamental group free of l generators. For every relator r_i of π there is a smooth embedding $\tilde{r}_i: S^1 \to \partial V_1$, whose image in $\pi_1(V_1)$ coincides with r_i. Since ω_1 or r_i is trivial, the normal bundle to $r_i(S^i)$ in ∂V_1 is trivial. Let R_i be a trivialization of the normal bundle. The manifold V_2 is obtained from V_1 by attaching the trivial disk bundle over $\mathbb{D}^2$ to ∂V_1 along R_i maps. The diffeomorphism type of V_2 does not depend on the choice of the trivializations R_i. For $\omega_2 = 0$ let $\mathbb{D}^n(\pi, \alpha, \omega) = V_2$ and for $\omega_2 = 1$ let $\mathbb{D}^n(\pi, \alpha, \omega)$ be the boundary connected sum of V_2 with a nontrivial $(n-2)$-dimensional disk bundle over S^2 (for

$n \geq 3$ there are only two isomorphism classes of vector bundles over S^2).

Let $\mathbb{T}^n(\pi, \alpha, \omega) = \mathbb{D}^n(\pi, \alpha, \omega) \cup_\partial (-\mathbb{D}^n(\pi, \alpha, \omega))$ be the doubling of $\mathbb{D}^n(\pi, \alpha, \omega)$.

LEMMA 2.6 ([**H2**]). *If $n \geq 6$ and α_0, α_1 are presentations of a group π, then the sets of concordance classes of metrics of positive scalar curvature on $\mathbb{T}^n(\pi, \alpha_0, \omega)$ and $\mathbb{T}^n(\pi, \alpha_1, \omega)$ are in a bijective correspondence.*

Lemma 2.6 follows from Proposition 2.2 and the fact that for $n \geq 6$ and two arbitrary presentations α_1, α_2 of π the spaces $\mathbb{T}^n(\pi, \alpha_1, \omega)$ and $\mathbb{T}^n(\pi, \alpha_2, \omega)$ are 2-connected bordant one to another (for details see [**H2**]). In the sequel we use the notation $\pi_0^c(n, \pi, \omega)$ for the set of metrics of positive scalar curvature on $\mathbb{T}^n(\pi, \alpha, \omega)$.

The group structure on $\pi_0^c(n, \pi, \omega)$ is induced by the following operation. Let s_0, s_1 be metrics of positive scalar curvature on $\mathbb{T}^n(\pi, \alpha, \omega)$. The handle metric g_{can} induced by the canonical handle decomposition of $\mathbb{D}^n(\pi, \alpha, \omega)$ will be called canonical. By Theorem 1.1, s_0 and s_1 are isotopic to metrics of the form $g_{\text{can}} \cup c_0' \cup g_{\text{can}}$ and $g_{\text{can}} \cup c_1 \cup g_{\text{can}}$ respectively. Define $[s_0] \circ [s_1] = [g_{\text{can}} \cup c_0 \cup c_1 \cup g_{\text{can}}]$. The operation is well defined by Theorem 1.3. The neutral element is given by the trivial concordance $g_{\text{can}}|_{\partial D_n(\pi, \alpha, \omega)} + dt^2$ on $\partial D_n(\pi, \alpha, \omega)$ itself. The inverse to $[g]$ is $[-g]$ by the following result.

CANCELLATION LEMMA. *Let c be a concordance between metrics g_0 and g_1 of positive scalar curvature on a closed manifold M^n of dimension greater than four and let $-c = \varphi^*(c)$ where*

$$\varphi: (M^n \times [-1, 1]) \to (M^n \times [-1, 1]): (x, t) \mapsto (x, -t).$$

Then the concordance $c \cup_{g_1} (-c)$ is relatively concordant to the trivial one $g_0 + dt^2$.

Proof. Consider the product metric $g + dt^2$ on $(M^n \times I) \times I$ where $I = [-1, 1]$. Smoothing the metric (cf. [**G**]) around the corners $M^n \times \{-1\} \times \{-1\}$ and $M^n \times \{-1\} \times \{1\}$ of $(M^n \times I) \times I$ produces the metric $c \cup_{g_1} (-c)$ on $M^n \times I = \partial((M^n \times I) \times I) \backslash (M^n \times \{1\} \times (-1, 1))$ and the product metric $g_0 + dt^2$ on $M^n \times \{1\} \times I$. Take $(M^n \times [1, 2]) \times [-0.5, 0.5]$ with the product metric $g_0 + dl^2 + dt^2$ and glue it to $(M^n \times I) \times I$ identifying the common parts and smoothing the corners. Obtain in this way the metric, which gives the required relative concordance.

By Theorem 1.1, an arbitrary metric g of positive scalar curvature on a (π, ω)-manifold M^n is isotopic to a metric of the form $g_H \cup c_g \cup g_{\text{can}}$ where g_H is the handle metric induced by all handles of codimension greater than two, g_{can} is the canonical handle metric on $\mathbb{D}^n(\pi, \alpha, \omega)$, and c_g is a concordance. At the same time, an arbitrary metric s of positive scalar curvature on $\mathbb{T}^n(\pi, \alpha, \omega)$ is isotopic to a metric of the form $g_{\text{can}} \cup g_{\text{can}}$. Consider a pairing $\pi_0^c(n, \pi, \omega) \times \pi_0^c(\mathbb{PSC}(M^n)) \to \pi_0^c(\mathbb{PSC}(M^n))$ which assigns to $[s] \in \pi_0^c(n, \pi, \omega)$ and $[g] \in \pi_0^c(\mathbb{PSC}(M^n))$ the class $[g_H \cup c_g \cup c_s \cup g_{\text{can}}]$. It is well defined and constitutes an action by Theorem 1.3 and the Cancellation Lemma.

Let us see that the action is free, i.e., if $[s] \cdot [g] = [g]$, then s is concordant to the standard metric $g_{\text{can}} \cup g_{\text{can}}$ on $\mathbb{T}^n(\pi, \alpha, \omega)$. By Theorem 1.1, $[s] \cdot [g] = [g]$ if and only if $g_H \cup c_g \cup c_s \cup g_{\text{can}}$ is concordant to $g_H \cup c_g \cup g_{\text{can}}$. Theorem 1.3 implies that the concordance is isotopic to a metric of the form

$$(g_H + dt^2) \cup \hat{c} \cup (g_{\text{can}} + dt^2)$$

where $\hat{c}$ is a relative concordance between c_g and $c_g \cup c_s$. The metric s is concordant to the standard metric on $\mathbb{T}^n(\pi, \alpha, \omega)$ if and only if c_s is relatively concordant to the trivial concordance $g_{\text{can}}|_{\partial D_n(\pi, \alpha, \omega)}$. By the Cancellation Lemma,

$$g_{\text{can}}|_{\partial D_n(\pi, \alpha, \omega)} + dt^2 \cong -c_g \cup c_g \cong -c_g \cup c_g \cup c_s \cong c_s .$$

Let us now see that the action is transitive. It will be shown that for two arbitrary metrics g_0 and g_1 of positive scalar curvature on a (π, ω)-manifold M^n there is a metric s of positive scalar curvature on $\mathbb{T}^n(\pi, \alpha, \omega)$ such that the metric obtained by the action of s on g_0 is concordant to g_1. Take representatives $g_H \cup c_{g_0} \cup g_{\text{can}}$, $g_H \cup c_{g_1} \cup g_{\text{can}}$, and $g_H \cup c_s \cup g_{\text{can}}$ of concordance classes of g_0, g_1, and s respectively. By the Cancellation Lemma, $c_{g_0} \cup (-c_{g_0})$ is relatively concordant to the trivial concordance. Hence, the metric given by the action of $g_{\text{can}} \cup (-c_{g_0} \cup c_{g_1}) \cup g_{\text{can}}$ on g_0 is concordant to g_1. This completes the proof of Theorem 2.1.

3. Concordance groups of $\mathbb{PSC}$ metrics. This section contains a definition of the concordance groups of metrics of positive scalar curvature, some of their elementary properties, and a proof of the following result.

THEOREM 3.1. *If M^n is a closed spin manifold of dimension ≥ 6, then the i^{th} concordance group $\pi_i^c(\mathbb{PSC}(M^n), g)$ of metrics of positive* *scalar curvature on* M^n *is nontrivial for* $n+i=4k+3$, $8k$, $8k+1$ *where* $k \geq 1$ *and* $i \neq 1, 2$.

Let $\{p\}$ stand for the south pole on S^i, let dx^2 denote the Euclidean flat metric of $\mathbb{R}^i$, and let g be a fixed metric of positive scalar curvature on M^n.

Construction of the concordance groups of metrics of positive scalar curvature. Let $\pi_i^c(\mathbb{PSC}(M^n), g)$ be the set of concordance classes of metrics of positive scalar curvature on $S^i \times M^n$ which are of the form $dx^2 + g$ in a neighborhood of $\{p\} \times M^n$ in $S^i \times M^n$. The group operation on $\pi_i^c(\mathbb{PSC}(M^n), g)$ is defined as follows. Let g_0 and g_1 be representatives of classes $\alpha_0, \alpha_1 \in \pi_i^c(\mathbb{PSC}(M^n), g)$. There is a neighborhood $D \times M^n$ of $\{p\} \times M^n$ in $S^i \times M^n$ such that g_0, g_1 restricted to it are of the form $dx^2 + g$. Let $\tilde{g}$ be a metric of positive scalar curvature on $(S^i \# S^i) \times M^n$ given by identification of the metrics $g_0|_{(S^i \times M^n)\setminus\{p\}\times M^n}$ and $g_1|_{(S^i \times M^n)\setminus\{p\}\times M^n}$ along the open set $(D\setminus\{p\}) \times M^n$. Define $\alpha_0 \cdot \alpha_1$ as the concordance class of the metric $(\varphi \times \mathrm{id}_M)^*(\tilde{g})$ where φ is a diffeomorphism of S^i onto $S^i \# S^i$ sending the south pole p of S^i into a point q of S^i, $\# S^i$ where the metric $\tilde{g}$ is flat. For $i = 1$, the set $D\setminus\{p\}$ has two connected components, and it is important from which one the point q is chosen, therefore, for $i = 1$, let the point q come from the third quadrant of the first sphere. It is easy to see that the operation is well defined and determines a group structure with the neutral element induced by the class of a metric $g_{\mathrm{fl}} + g$ where g_{fl} is a metric obtained from the standard sphere metric by making it flat in a neighborhood of the south pole of S^i.

The groups $\pi_i^c(\mathbb{PSC}(M^n), g)$ depend only on concordance class of the diffeomorphism φ, and this is why they are Abelian for $i > 1$. It is easy to see that for concordant metrics g_0, g_1 of positive scalar curvature on M^n, the groups $\pi_i^c(\mathbb{PSC}(M^n), g_0)$ and $\pi_i^c(\mathbb{PSC}(M^n), g_1)$ are isomorphic. One can also check that the map $\pi_0(\mathbb{PSC}(M^n)) \to \pi_0^c(\mathbb{PSC}(M^n))$ has its analogs $\pi_i(\mathbb{PSC}(M^n), g) \to \pi_i^c(\mathbb{PSC}(M^n), g)$ for $i \geq 1$, which are group homomorphisms.

Proof of Theorem 3.1. Let M^n be a simply-connected closed spin manifold and let $i = 0$. By Theorem 2.1 there is a bijective correspondence between elements of $\pi_0^c(\mathbb{PSC}(M^n))$ and $\pi_0^c(\mathbb{PSC}(S^n))$. Let $\tilde{\pi}_0^c(\mathbb{PSC}(S^n))$ be the subgroup of $\pi_0^c(\mathbb{PSC}(S^n))$ consisting of concordance classes of those metrics that are boundary restrictions of metrics with positive scalar curvature on compact spin manifolds. Hajduk defined a homomorphism $a\colon \tilde{\pi}_0^c(\mathbb{PSC}(S^n)) \to KO_{n+1}(pt)$ and noticed that a is an isomorphism if the Gromov-Lawson conjecture is true **[H1]**. Since the conjecture was proved to be true [S] the group $\tilde{\pi}_0^c(\mathbb{PSC}(S^n))$ are nontrivial for $n = 4k+3$, $8k$, $8k+1$, $k \geq 1$. This proves Theorem 3.1 for simply-connected spin manifolds and $i = 0$.

If M^n be a spin manifold, then the product $S^i \times M^n$ is spin as well. Therefore it is possible to kill the first two homotopy groups of $S^i \times M^n$ obtaining a simply-connected spin manifold W^{n+1}. When $n + i \geq 6$ the surgery is performed in codimension ≥ 3 and thus it induces a map $S\colon \mathbb{PSC}(S^i \times M^n) \to \mathbb{PSC}(W^{n+1})$. By Lemma 2.3, if $g_0, g_1 \in \mathbb{PSC}(S^i \times M^n)$ are concordant, then $S(g_0)$, $S(g_1)$ are concordant as well, in particular, S induces a map $\pi_i^c(\mathbb{PSC}(M^n)) \to \pi_0^c(\mathbb{PSC}(W^{n+1}))$. By Theorem 2.1 this map induces another one $\pi_0^c(n, \pi, 0) \to \pi_0^c(\mathbb{PSC}(S^n))$, which will be still denoted by S. Fix a metric g of positive scalar curvature on $\mathbb{T}^n(\pi, \alpha, 0)$. If the connected sum $-S(g)\# g$ is taken outside the support of the surgeries defining S, then $S(-S(g)\# g) = -S(g)\# S(g)$. By the Cancellation Lemma, $-S(g)\# S(g)$ is concordant to the standard metric on S^n. Therefore, for every metric s of positive scalar curvature on S^n the image of $s\# S(g)\# g$ under S is concordant to s. This completes the proof of Theorem 3.1.

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Received April 4, 1991 and in revised form February 13, 1992.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 157 No. 2 February 1993