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A SPECTRAL THEORY FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

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A SPECTRAL THEORY FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

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The main objective of this paper is to develop a notion of joint spectrum for complex solvable Lie algebras of operators acting on a Banach space, which generalizes Taylor joint spectrum (T.J.S.) for several commuting operators.

I. Introduction. We briefly recall the definition of Taylor spectrum. Let $\bigwedge(\mathbb{C}^n)$ be the complex exterior algebra on *n* generators e_1, \ldots, e_n , with multiplication denoted by \bigwedge . Let \dot{E} be a Banach space and $a = (a_1, \ldots, a_n)$ be a mutually commuting *n*-tuple of bounded linear operators on E(m.c.o.). Define $\bigwedge_k^n(E) = \bigwedge_k(\mathbb{C}^n) \otimes_{\mathbb{C}} E$, and for $k \ge 1$, D_{k-1} by:

$$D_{k-1}: \bigwedge_{k}^{n}(E) \to \bigwedge_{h-1}^{n}(E)$$

$$D_{k-1}(x \otimes e_{i_1} \wedge \dots \wedge e_{i_k})$$

= $\sum_{j=1}^k (-1)^{j+1} x \cdot a_{i_j} \otimes \dots \otimes e_{i_1} \wedge \dots \wedge \tilde{e}_{i_j} \wedge \dots \wedge e_{i_k}$

where \sim means deletion. Also define $D_k = 0$ for $k \leq 0$.

It is easily seen that $D_k D_{k+1} = 0$ for all k, that is, $\{\bigwedge_k^n(E), D_k\}_{k \in \mathbb{Z}}$ is a chain complex, called the Koszul complex associated with a and E and denoted by R(E, a). The *n*-tuple a is said to be invertible or nonsingular on E, if R(E, a) is exact, i.e., Ker $D_k = \operatorname{ran} E_{k+1}$ for all k. The Taylor spectrum of a on E is $\operatorname{Sp}(a, E) = \{\lambda \in \mathbb{C}^n : a - \lambda \text{ is not invertible}\}.$

Unfortunately, this definition depends very strongly on a_1, \ldots, a_n and not on the vector subspace of L(E) generated by then $(=\langle a \rangle)$.

As we consider Lie algebras, and then naturally involve geometry, we are interested in a geometrical approach to spectrum which depends on L rather than on a particular set of operators.

This is done in II. Given a solvable Lie subalgebra of L(E), L, we associate to it a set in L^* , Sp(L, E).

This object has the classical properties. Sp(L, E) is compact. If L' is an ideal of L, then Sp(L', E) is the projection of Sp(L, E) in L'^* . Sp(L, E) is non-empty.

Besides, it satisfies other interesting properties.

If $x \in L^2$, then Sp(x) = 0. If L is nilpotent, one has the inclusion

$$\operatorname{Sp}(L, E) \subset \{ f \in [L, L]^{\perp} | \forall x \in L, |f(x)| \le ||x|| \}.$$

However the spectral mapping property is ill behaved.

II. The joint spectrum for solvable Lie algebras of operators. First of all, we establish a proposition which will be used in the definition of Sp(L, E).

From now on, L denotes a complex finite dimensional solvable Lie algebra, and U(L) its enveloping algebra.

Let f belong to L^* such that f([L, L]) = 0, i.e., f is a character of L. Then f defines a one dimensional representation of L denoted by $\mathbb{C}(f)$. Let $\varepsilon(f)$ be the augmentation of U(L) defined by f:

$$\varepsilon(f): U(L) \to \mathbb{C}(f),$$

 $\varepsilon(f)(x) = f(x) \qquad (x \in L).$

Let us consider the pair of spaces and maps $V(L) = (U(L) \otimes \bigwedge L, \overline{d}_{p-1})$, where \overline{d}_{p-1} is the map defined by:

$$\overline{d}_{p-1}\colon U(L)\otimes \bigwedge^p L\to U(L)\otimes \bigwedge^{p-1}L.$$

If $p \ge 1$

$$\overline{d}_{p-1}\langle x_{i_1}\cdots x_{i_p}\rangle = \sum_{k=1}^p (-1)^{k+1} (x_{i_k} - f(x_{i_k}))\langle x_i, \hat{x}_{i_k} x_{i_p}\rangle + \sum_{1 \le k \le l \le p} (-1)^{k+l} \langle [x_{i_k}, x_{i_l}] x_{i_1} \hat{x}_{i_k} \hat{x}_{i_l} x_{i_p}\rangle$$

where \uparrow means deletion. If $p \leq 0$, we also define $\overline{d}_p = 0$. Then

PROPOSITION 1. The pair of spaces and maps V(L) is a chain complex. Furthermore, with the augmentation $\varepsilon(f)$, the complex V(L) is a U(L)-free resolution of $\mathbb{C}(f)$ as a left U(L) module.

We omit the proof of Proposition 1 because it is a straightforward generalization of Theorem 7.1 of [3, XIII, 7].

Let L be as usual, from now on, E denotes a Banach space on which L acts as right continuous operators, i.e., L is a Lie subalgebra

of L(E) with the opposite product. Then, by [3, XIII, 1], E is a right U(L) module.

If f is a character of L, by Proposition 1 and elementary homological algebra, the q-homology space of the complex, $(E \otimes \bigwedge L, d(f))$ is $\operatorname{Tor}_q^{U(L)}(E, \mathbb{C}(f)) \ (= H_q(L, E \otimes \mathbb{C}(f))$.

We now state our definition.

DEFINITION 1. Let L and E be as above the set $\{f \in L^*, f(L^2) = 0 | H_*((L, E \otimes \mathbb{C}(f))) \text{ is non-zero}\}$, is the spectrum of L acting on E, and is denoted by Sp(L, E).

By Proposition 1 and Definition 1, it is clear that, if L is a commutative algebra Sp(L, E) reduces to Taylor joint spectrum.

Let us see an example. Let (E, || ||) be $(\mathbb{C}^2, || ||_2)$ and a, b the operators

$$a = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

It is easily seen that [b, a] = b, and then, the vector space $\mathbb{C}(b) \oplus \mathbb{C}(a) = L$ is a solvable Lie subalgebra of $L(\mathbb{C}^2)$.

Using Definition 1, a standard calculation shows that $\operatorname{Sp}(L, E) = \{f \in (\mathbb{C}^2)^* | f(b) = 0; f(a) = \frac{1}{2}, f(a) = -\frac{3}{2}\}.$

Observe that, $||a|| = \frac{1}{2}$; however, Sp(L, E) is not contained in $\{f \in (\mathbb{C}^2)^* | \forall x \in \mathbb{C}^2 | f(x) | \le ||x|| \}$.

III. Fundamental properties of the spectrum. In this section, we shall see that the most important properties of spectral theory are satisfied by our spectrum.

THEOREM 2. Let L and E be as usual. Then Sp(L, E) is a compact set of L^* .

Proof. Let us consider the family of spaces and maps $(E \otimes \bigwedge^i L, d_{i-1}(f))$ $f \in L^{2^{\perp}}$, where $L^{2^{\perp}} = \{f \in L^* | f(L^2) = 0\}$. This family is a parameterized chain complex on $L^{2^{\perp}}$. By Taylor [6, 2.1] the set $\{f \in L^{2^{\perp}} | (E \otimes \bigwedge^i L, d_{i-1}(f)) \text{ is exact}\} = \operatorname{Sp}(L, E)^c$ is an open set in $L^{2^{\perp}}$. Then, $\operatorname{Sp}(L, E)$ is closed in $L^{2^{\perp}}$ and hence in L^* .

To verify that $\operatorname{Sp}(L, E)$ is a compact set we consider a basis of L^2 and we extend it to a basis of L, $\{X_i\}_{1 \le i \le n}$. If $K = \dim L^2$ and $n = \dim L$ let L_i be the ideal generated by $\{X_j\}_{1 \le j \le n, j \ne i}, i \ge K+1$.

Let f be a character of L and represent it in the dual basis of $\{X_i\}_{1 \le i \le n}$, $\{f_i\}_{1 \le i \le n}$ $f = \sum_{i=K+1}^n \xi_i f_i$. For each i, there is a positive

number r_i such that if $\xi_i \ge r_i$,

$$\operatorname{Tor}_p^{U(L)}(E, C(f)) = H_p\left(E \otimes \bigwedge^i L, d_{i-1}(f)\right) = 0 \quad \forall p.$$

To prove our last statement, we shall construct an homotopy operator for the chain complex $(E \otimes \bigwedge^p L, d_{p-1}(f))$ $(f(L^2) = 0)$.

First of all we observe that

$$E \otimes \bigwedge^{p} L = \left(E \otimes \bigwedge^{p} L_{i} \right) \oplus \left(E \otimes \bigwedge^{p-1} L_{i} \right) \bigwedge \langle X_{i} \rangle.$$

As L_i is an ideal of L, $d_{p-1}(E \otimes \bigwedge^p L_i) \subseteq E \otimes \bigwedge^{p-1} L_i$. On the other hand, there is a bounded operator L_{p-1} such that

$$d_{p-1}(f)(a \wedge \langle X_i \rangle)$$

= $(d_{p-1}(f)a) \wedge \langle X_i \rangle + (-1)^p L_{p-1}a \qquad \left(a \in E \otimes \bigwedge^{p-1} L_i\right).$

It is easy to see that, for each p, there is a basis of $\bigwedge^p L_i$, $\{V_j^p\}$ $1 \le j \le \dim \bigwedge^p L_i$, such that if we decompose

$$E \otimes \bigwedge^{P} L_{i} = \bigoplus_{1 \leq j \leq \dim \bigwedge^{p} L_{i}} E\langle V_{j} \rangle,$$

then L_p has the following form

$$L_{p_{ij}^{n}} = \begin{cases} \alpha_{ij}^{p} & i < j, \\ X_{i} - \xi_{i} + \alpha_{jj}^{p} & i = j, \\ 0 & i > j \text{ where } \alpha_{ij} \in \mathbb{C}. \end{cases}$$

Besides, let K_p be a positive real number such that

$$\bigcup_{1\leq j\leq \dim \bigwedge^p L_i} \operatorname{Sp}(X_i + \alpha_{jj}^p) \subseteq B[0, K_p]$$

and $N_i = \max_{0 \le p \le n-1} \{K_p\}$. Then, as L_p has a triangular form, a standard calculation shows that L_p is a topological isomorphism of Banach spaces if $\xi_i \ge N_i$.

Outside $B[0, N_i]$ we construct our homotopy operator

Sp:
$$E \otimes \bigwedge^{p} L \to E \otimes \bigwedge^{p+1} L$$
,
Sp: $E \otimes \bigwedge^{p-1} L_{i} \wedge \langle X_{i} \rangle = 0$,
Sp: $E \otimes \bigwedge^{p} L_{i} \to E \otimes \bigwedge^{p} L_{i} \wedge \langle X_{i} \rangle$
Sp = $(-1)^{p+1} L_{p}^{-1} \wedge \langle X_{i} \rangle$.

From the definition of L_p , we have the following identity:

$$(-1)^{p+2}S_{p-1}d_{p-1}(f)L_p = d_{p-1}(f) \wedge \langle X_i \rangle.$$

The above identity and a standard calculation shows that Sp in an homotopy operator, i.e., $d_pS_p + S_{p-1}d_{p-1} = I$ and then $S_p(L, E)$ is a compact set.

THEOREM 3 (Projection property). Let L and E be as usual, and I an ideal of L. Let π be the projection map from L^{*} onto I^{*}, then

$$\operatorname{Sp}(I, E) = \pi(\operatorname{Sp}(L, E)).$$

Proof. By [2, 5, 3], there is a Jordan Hölder sequence of L such that I is one of its terms. Then, by means of an induction argument, we can assume $\dim(L/I) = 1$.

Let us consider the connected simply connected complex Lie group G(L) such that its Lie algebra is L [5, LG, V].

Let Ad^{*} be the coadjoint representation of G(L) in L^* : Ad^{*}(g)f = $f \operatorname{Ad}(g^{-1})$, where $g \in G(L)$, $f \in L^*$ and Ad is the adjoint representation of G(L) in L.

Let f belong to Sp(I, E). Then, as I is an ideal of L, by [7, 2.13.4], $\text{Ad}^*(g)f$ belongs to I^* ; besides, it is a character of I. Then, one can restrict the coadjoint action of G(L) to I^* . Moreover, Sp(I, E) is invariant under the coadjoint action of G(L) in I^* , i.e.: if $f \in \text{Sp}(I, E)$, $\text{Ad}^*(g)f \in \text{Sp}(I, E) \quad \forall g \in G(L)$.

In order to prove this fact, it is enough to see:

(I)
$$\operatorname{Tor}_*^{U(I)}(E, C(f)) \cong \operatorname{Tor}_*^{(U(I))}(E, C(h))$$

where $h = \operatorname{Ad}^*(g)f$, $g \in G(L)$.

Let Γ be the ring U(I) and φ the ring morphism

$$\varphi = U(\operatorname{Ad} g) \colon U(I) \to U(I)$$

Let us consider the augmentation modules (C(f), E(f)) and (C(h), E(h)).

Then, a standard calculation shows that the hypothesis of [3, VIII, 3.1] are satisfied, which implies (I).

Thus, if $f \in \text{Sp}(I, E)$, the orbit $G(L) \cdot f \subseteq \text{Sp}(I, E)$. However, Sp(I, E) is a compact set of I^* .

As the only bounded orbits for an action of a complex connected Lie group on a vector space are points; $G(L) \cdot f = f$.

Let \overline{f} be an extension of f to L^* , and consider $\alpha = G(L) \cdot \overline{f}$, the orbit of \overline{f} under the coadjoint action of G(L) in L^* .

As $G(L) \cdot f = f$, as an analytic manifold

(II)
$$\dim \alpha \leq 1$$
.

Now suppose \overline{f} is not a character of L: i.e., $\overline{f}(L^2) \neq 0$.

Let L^{\perp} be the following set: $L^{\perp} = \{x \in L | \overline{f}([X, L]) = 0\}$, and let *n* be the dimension of *L*.

As *I* is an ideal of dimension n - 1, $f(I^2) = 0$ and $f(L^2) \neq 0$, by [2, 5, 3], [1, IV, 4.1] and [4, 1, 1.2.8], we have: $L^{\perp} \subset I$, and dim $L^{\perp} = n - 2$.

Let us consider the analytic subgroup of G(L) such that its Lie algebra is L^{\perp} .

As the Lie algebra of the subgroup $G(L)_{\overline{f}} = \{g \in GL | \mathrm{Ad}^*(g)\overline{f} = \overline{f}\}$ is L^{\perp} , the connected component of the identity of $G(L)_{\overline{f}}$ is $G(L^{\perp})$.

However, by [7, 2.9.1, 2.9.7] $\alpha = G(L) \cdot \overline{f}$ satisfies the following properties: $\alpha \cong G(L)/G(L)_{\overline{f}}$, and $\dim \alpha = \dim G(L) - \dim G(L)_{\overline{f}} = \dim G(L) - \dim (G(L^{\perp})) = \dim L - \dim L^{\perp} = 2$, which contradicts (II).

Then \overline{f} is a character of L.

Thus, any extension \overline{f} of an f in Sp(I, E) is a character of L. However, as in [6], there is a short exact sequence of complexes

$$0 \to \left(\bigwedge^* I \otimes E, d(f)\right)$$
$$\to \left(\bigwedge^* L \otimes E, d(\overline{f})\right) \to \left(\bigwedge^* I \otimes E, d(f)\right) \to 0.$$

As U(I) is a subring with unit of U(L) and the complex involved in Definition 1 differs from the one of [6] by a constant term, Taylor's argument of [6, 13, 3.1] still applies and then $Sp(I, E) = \Pi(Sp(L, E))$.

As a consequence of Theorem 3 we have

THEOREM 4. Let L and E be as usual. Then Sp(L, E) is non-void.

IV. Some consequences. In this section we shall see some consequences of the main theorems.

Let E be a Banach space and L a complex finite dimensional solvable Lie algebra acting on E as bounded operators.

One of the well known properties of Taylor spectrum for an *n*-tuple of m.c.o. acting on E is $Sp(a, E) \subseteq \Pi B[0, ||a_i||]$. In the noncommutative case, as we have seen in §II, this property fails.

However, if the Lie algebra is nilpotent, it is still true.

PROPOSITION 5. Let L be a nilpotent Lie algebra which acts as bounded operators on a Banach space E.

Then, $Sp(L, E) \subset \{f \in L^* \mid |f(x)| \le ||x||, x \in L\}.$

Proof. We proceed by induction on dim L. If dim L = 1, we have nothing to verify.

We suppose true the proposition for every nilpotent Lie algebra L' such that dim L' < n.

If dim L = n, by [2, 4, 1], there is a Jordan Hölder series $S = (L_i)_{0 \le i \le n}$, such that $[L, L_i] \subseteq L_{i-1}$.

Let $\{X_i\}_{1 \le i \le n}$ be a basis of L such that $\{X_j\}_{1 \le j \le i}$ generates L_i . Let L'_{n-1} be the vector subspace generated by $\{X_i\}_{1 \le i \le n}$. As $[L, L'_{n-1}] \subseteq L_{n-2} \subset L'_{n-1}$, L'_{n-1} is an ideal. Besides, $L_{n-1} + L'_{n-1} = L$.

Then, by means of Theorem 4 and the inductive hypothesis, we complete the inductive argument and the proposition.

Now, we deal with some consequences of the projection property.

PROPOSITION 6. Let L and E be as usual.

If I is an ideal contained in L^2 , then Sp(I, E) = 0. In particular $Sp(L^2, E) = 0$.

Proof. By the projection property, $\text{Sp}(I, E) = \Pi(\text{Sp}(L, E))$, where Π is the projection from L^* on I^* . However, as Sp(L, E) is a subset of characters of L, $f|_I = 0$, if $I \subseteq L^2$.

PROPOSITION 7. Let L and E be as in Proposition 5. If Sp(L, E) = 0, then $Sp(x) = 0 \quad \forall x \in L$.

Proof. By means of an induction argument, the ideals L_{n-1} , L'_{n-1} of Proposition 5 and Theorem 3, we conclude the proof.

PROPOSITION 8. Let L and E be as usual. Then, if $x \in L^2$: Sp(x) = 0.

Proof. First of all, recall that if L is a solvable Lie algebra, L^2 is a nilpotent one. Then by Proposition 6 Sp $(L^2, E) = 0$, and by Proposition 7 Sp $(x) = 0 \quad \forall x \in L^2$.

V. Remark about the spectral mapping theorem. Note that the example of §II shows that the projection property fails for subspaces which are not ideals (take $I = \langle x \rangle$). Clearly this implies that the spectral mapping theorem also fails in the noncommutative case.

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