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# CONGRUENCE PROPERTIES OF FUNCTIONS RELATED TO THE PARTITION FUNCTION

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**In this paper we describe a straightforward and almost entirely elementary method for establishing congruence properties of certain functions that are related to the partition function.**

For integer $k$ define $p_k(n)$ by

$$\prod_{m=1}^{\infty}(1-x^m)^k = \sum_{n=0}^{\infty} p_k(n)x^n .$$

In particular, $p_{-1}(n)$ is $p(n)$, the partition function and $p_{24}(n-1)$ is Ramanujan's $\tau$-function.

We are interested in congruences of the form

$$(1) \qquad p_k(np+b) \equiv 0 \pmod{p} \text{ for all } n \geq 1$$

for prime $p$, as typified by the partition congruences

$$(2) \qquad p(5n+4) \equiv 0 \pmod{5},$$

$$(3) \qquad p(7n+5) \equiv 0 \pmod{7}$$

and

$$(4) \qquad p(11n+6) \equiv 0 \pmod{11}$$

discovered by Ramanujan and proved in **[13]** and **[14]**. Ramanujan also conjectured that if $24b \equiv 1 \pmod{q}$ and $q = 5^{\alpha}7^{\beta}11^{\gamma}$ then $p(qn+b) \equiv 0 \pmod{q}$. He was able to supply proofs for $q = 25, 49$ in **[13]** and $q = 121$ in an unpublished manuscript **[15]**. Ramanujan's conjecture was incorrect as stated for powers of 7 and Watson **[16]** proved a modified version; if $24b \equiv 1 \pmod{5^{\alpha}7^{2\beta}}$ then $p(5^{\alpha}7^{2\beta}n+b) \equiv 0 \pmod{5^{\alpha}7^{\beta+1}}$. Watson's proofs have been simplified by Hirschhorn and Hunt **[6]** and Garvan **[4]**. Lehner **[9]** dealt with $q = 1331$ and the proof of the conjecture was completed by Atkin **[1]**.

Congruences modulo powers of 13 have been considered by Atkin and O'Brien **[2]**. A general treatment of $p_k(n)$ modulo powers of 2, 3, 5, 7 and 13 is given in Atkin **[3]**, modulo powers of 11 in

Gordon [5] and modulo powers of 17 in a forthcoming paper by Hughes [7].

In everything that follows, $p$ is a prime number $\geq 5$. The variable $x$ always satisfies $|x| < 1$ to ensure absolute convergence and we write $f(x) \equiv g(x) \pmod{p}$ to mean that $f(x) - g(x)$ is a power series in $x$ with integer coefficients that are all divisible by $p$.

Euler's pentagonal number theorem,

$$\prod_{m=1}^{\infty}(1 - x^m) = \sum_{n=-\infty}^{\infty}(-1)^n x^{(3n^2+n)/2},$$

and Jacobi's identity,

$$\prod_{m=1}^{\infty}(1 - x^m)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{(n^2+n)/2} \tag{5}$$

completely determine $p_1(n)$ and $p_3(n)$. Also it suffices to consider $k$ modulo $p$ because, as is easily shown, if $p_k(n)$ satisfies a congruence of the form (1) for some prime $p$ then the same is true for $p_{k\pm p}(n)$.

With certain values of $k$, other than 0, 1 and 3, it is possible to establish congruences by well-known methods which are entirely elementary. For instance, Ramanujan's original proofs of (2) and (3) in [13] are easily extended to show that (1) holds when

$$k = 4, \quad p \equiv 5 \pmod{6}, \quad 6b + 1 \equiv 0 \pmod{p} \quad \text{and when}$$
$$k = 6, \quad p \equiv 3 \pmod{4}, \quad 4b + 1 \equiv 0 \pmod{p}.$$

For an alternative proof of (2), the congruence

$$p_9(5m + 4) \equiv 0 \pmod{5},$$

follows from

$$p_9(n) = \sum_{r=0}^{n}\sum_{s=0}^{n-r} p_3(r)p_3(s)p_3(n - r - s).$$

By (5), if $n \equiv 4 \pmod{5}$ and the $r, s$ term of the double sum is non-zero then

$$p_3(r)^2 + p_3(s)^2 + p_3(n - r - s)^2 = 8n + 3 \equiv 0 \pmod{5},$$

which cannot be true unless at least one of the terms on the left-hand side is divisible by 5. But then $p_9(n)$ will also be a multiple of 5.

In Table 1 we give an exhaustive list of congruences of the form (1) for $p \leq 199$ and $2 \leq k \leq p - 1$, $k \neq 3, 4, 6$.

A theorem of Newman [10] established using modular function theory states that if $k = 4, 6, 8, 10, 14, 26$, $p$ is a prime $> 3$ such that $k(p+1) \equiv 0 \pmod{24}$ and $b = k(p^2-1)/24$ then $p_k(n) \equiv 0 \pmod{p}$ for $n \equiv b \pmod{p}$. This theorem disposes of all the $k = 8$ cases in Table 1 as well as $k = 10$, 14 and 26 when $p \equiv 11 \pmod{12}$. Another of Newman's results [11] is that for even $k$, $4 \leq k \leq 24$ and prime $p > 3$ such that $b = k(p-1)/24$ is an integer,

$$p_k(np+b) \equiv p_k(n)p_k(b) \qquad \pmod{p}.$$

Thus $k = 19$, $p = 12$ and $k = 22$, $p = 61$ in Table 1 reduce to single congruences. Newman's method is described in Chapter 7 of Knopp [8].

In [14], Ramanujan gives proofs of (4) by two different methods one of which we extend in order to deal with any congruence of the form (1) for which $24b + k \equiv 0 \pmod{p}$. In particular we can prove all the entries in Table 1 (see next page).

We illustrate the method with $k = 10$, $p = 19$, $b = 17$ and for convenience we use the same notation as Ramanujan. Let

$$\phi_{r,s}(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s x^{mn},$$
$$P = 1 - 24\phi_{0,1}(x),$$
$$Q = 1 + 240\phi_{0,3}(x)$$

and

$$R = 1 - 504\phi_{0,5}(x).$$

It is well known from the theory of the Dedekind eta-function that

$$12^3 x \prod_{m=1}^{\infty}(1-x^m)^{24} = Q^3 - R^2. \tag{6}$$

In fact, $P$, $Q$ and $R$ are the normalised Eisenstein series $E_2$, $E_4$ and $E_6$. They are related to the discriminant $\Delta$ and the invariants $g_2(\tau)$ and $g_3(\tau)$ by

$$P = \frac{1}{2\pi i}\frac{\Delta'(\tau)}{\Delta(\tau)}, \quad Q = \frac{3}{4}\frac{g_2(\tau)}{\pi^4} \quad \text{and} \quad R = \frac{27}{8}\frac{g_3(\tau)}{\pi^6},$$

where $x = e^{2\pi i\tau}$ for $\tau$ in the upper half plane.

TABLE 1

$p_k(np+b) \equiv 0 \pmod p$

| | k | | | | | | |
|---|---|---|---|---|---|---|---|
| p | 8 | 10 | 12 | 14 | 18 | 22 | 26 |
| 11 | 7 | 6 | – | – | – | – | – |
| 17 | 11 | | | 15 | – | – | – |
| 19 | | 17 | 9 | | | – | – |
| 23 | 15 | 13 | | 9 | 5 | | – |
| 29 | 19 | | | 26 | | | |
| 31 | | 28 | | | | | |
| 41 | 27 | | | 37 | | | |
| 43 | | 39 | | | | | |
| 47 | 31 | 27 | | 19 | | | 42 |
| 53 | 35 | | | 48 | | | |
| 59 | 39 | 34 | | 24 | | | 53 |
| 61 | | | | | | 55 | |
| 67 | | 61 | | | | | |
| 71 | 47 | 41 | | 29 | | | 64 |
| 79 | | 72 | | | | | |
| 83 | 55 | 48 | | 34 | | | 75 |
| 89 | 59 | | | 81 | | | |
| 101 | 67 | | | 92 | | | |
| 103 | | 94 | | | | | |
| 107 | 71 | 62 | | 44 | | | 97 |
| 113 | 75 | | | 103 | | | |
| 127 | | 116 | | | | | |
| 131 | 87 | 76 | | 54 | | | 119 |
| 137 | 91 | | | 125 | | | |
| 139 | | 127 | | | | | |
| 149 | 99 | | | 136 | | | |
| 151 | | 138 | | | | | |
| 163 | | 149 | | | | | |
| 167 | 111 | 97 | | 69 | | | 152 |
| 173 | 115 | | | 158 | | | |
| 179 | 119 | 104 | | 74 | | | 163 |
| 191 | 127 | 111 | | 79 | | | 174 |
| 197 | 131 | | | 180 | | | |
| 199 | | 182 | | | | | |

In **[12]**, Ramanujan establishes in a direct and elementary manner a number of identities involving $P$, $Q$ and $R$, including

$$QR = 1 - 264\phi_{0,9}(x), \tag{7}$$

$$441Q^3 + 250R^2 = 691 + 65520\phi_{0,11}(x), \tag{8}$$

$$P^2 - Q = 12\theta P, \tag{9}$$

$$PQ - R = 3\theta Q \tag{10}$$

and

$$PR - Q^2 = 2\theta R, \tag{11}$$

where $\theta$ is the differential operator $x\,d/dx$.

Now to prove that $P_{10}(19n+17) \equiv 0 \pmod{19}$ for all $n \geq 0$, it suffices to show that the same is true for $p_{48}(19n+17)$. By (6) this is equivalent to showing that in

$$(Q^3 - R^2)^2 = \sum_{n=1}^{\infty} c(n) x^n,$$

the coefficients $c(19), c(38), \ldots$ are multiples of 19 and one way of doing this is to find a power series $f(x)$ with integer coefficients satisfying

$$(Q^3 - R^2)^2 \equiv 12\theta f(x) \pmod{19}.$$

We succeed because of the identity

$$\begin{aligned}(12)\quad 12\theta(9P^3Q^4 &+ 16P^3QR^2 + 13P^2Q^3R + 7P^2R^3 \\ &+ 5PQ^5 + 13PQ^2R^2 + 18Q^4R + 14QR^3) \\ = (Q^3 - R^2)^2 &+ 19(9P^4Q^4 + 16P^4QR^2 - 4P^3Q^3R + 4P^3R^3 \\ &- 3P^2Q^2R^2 + 6PQ^4R + 10PQR^3 \\ &- 6Q^6 - 29Q^3R^2 - 3R^4)\end{aligned}$$

which is easily verified using (9), (10) and (11).

To obtain an identity like (12) we consider the matrix $A^{\alpha,\beta,\gamma}_{\lambda,\mu,\nu}$ defined by equating coefficients of $P^\lambda Q^\mu R^\nu$ in

$$\sum_{\substack{\lambda,\mu,\nu\geq 0 \\ \lambda+2\mu+3\nu=6s}} P^\lambda Q^\mu R^\nu A^{\alpha,\beta,\gamma}_{\lambda,\mu,\nu} = 12\theta P^\lambda Q^\mu R^\nu$$

as $\alpha$, $\beta$ and $\gamma$ run through the non-negative integers satisfying $\alpha + 2\beta + 3\gamma = 6s - 1$. Here $s$ satisfies

$$24s \equiv k \pmod{p}.$$

Next we solve the linear congruences

$$\sum_{\substack{\alpha,\beta,\gamma\geq 0 \\ \alpha+2\beta+3\gamma=6s-1}} A^{\alpha,\beta,\gamma}_{\lambda,\mu,\nu} a_{\alpha,\beta,\gamma} \equiv t_{\lambda,\mu,\nu} \pmod{p}$$

for $a_{\alpha,\beta,\gamma}$, where

$$t_{0,\mu,\nu} = (-1)^{\nu/2}\binom{s}{\nu/2},$$
$$t_{\lambda,\mu,\nu} = 0 \quad \text{for } \lambda \geq 1$$

TABLE 2

$12\theta P^{\alpha}Q^{\beta}R^{\gamma}$ for $\alpha + 2\beta + 3\gamma = 11$

| $\lambda\ \mu\ \nu$ | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 0 4 | −4 | | | | | | | | | | | | | | | |
| 0 3 2 | −18 | −16 | −1 | | | | | | | | | | | | | |
| 0 6 0 | 0 | −6 | 0 | −1 | | | | | | | | | | | | |
| 1 1 3 | 22 | 0 | −8 | 0 | −2 | | | | | | | | | | | |
| 1 4 1 | | 22 | −12 | −20 | 0 | −2 | | | | | | | | | | |
| 2 2 2 | | | 21 | 0 | −18 | −12 | −3 | | | | | | | | | |
| 2 5 0 | | | | 21 | 0 | −6 | 0 | −3 | | | | | | | | |
| 3 0 3 | | | | | 20 | 0 | −4 | 0 | | | | | | | | |
| 3 3 1 | | | | | | 20 | −12 | −16 | −4 | | | | | | | |
| 4 1 2 | | | | | | | 19 | 0 | −8 | −5 | | | | | | |
| 4 4 0 | | | | | | | | 19 | −6 | 0 | −5 | | | | | |
| 5 2 1 | | | | | | | | | 18 | −12 | 12 | −6 | | | | |
| 6 0 2 | | | | | | | | | | 17 | 0 | −4 | | | | |
| 6 3 0 | | | | | | | | | | | 17 | −6 | −7 | | | |
| 7 1 1 | | | | | | | | | | | | 16 | −8 | −8 | | |
| 8 2 0 | | | | | | | | | | | | | 15 | −6 | −9 | |
| 9 0 1 | | | | | | | | | | | | | | 14 | −4 | |
| 10 1 0 | | | | | | | | | | | | | | | 13 | −11 |
| 12 0 0 | | | | | | | | | | | | | | | | 11 |
| $\alpha$ | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 8 | 9 | 11 |
| $\beta$ | 1 | 4 | 2 | 5 | 0 | 3 | 1 | 4 | 2 | 0 | 3 | 1 | 2 | 0 | 1 | 0 |
| $\gamma$ | 3 | 1 | 2 | 0 | 3 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 0 |

and, as before, $\lambda + 2\mu + 3\nu = 6s$. Then $a_{\alpha,\beta,\gamma}$ are the required coefficients, for

$$
\begin{aligned}
12\theta \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=6s-1}} a_{\alpha,\beta,\gamma}P^{\alpha}Q^{\beta}R^{\gamma}
&\equiv \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=6s-1}} \sum_{\substack{\lambda,\mu,\nu\geq 0\\ \lambda+2\mu+3\nu=6s}} P^{\lambda}Q^{\mu}R^{\nu}A^{\alpha,\beta,\gamma}_{\lambda,\mu,\nu}a_{\alpha,\beta,\gamma}\\
&\equiv \sum_{\substack{\lambda,\mu,\nu\geq 0\\ \lambda+2\mu+3\nu=6s}} t_{\lambda,\mu,\nu}P^{\lambda}Q^{\mu}R^{\nu} \equiv (Q^3 - R^2)^s \pmod{p}.
\end{aligned}
$$

The case $s = 2$ is illustrated in Table 2.

What is interesting is perhaps not the actual method, for it merely involves routine computations, but rather the existence of the identity itself. It seems that there is no simpler expression of the form $12\theta f(x)$ that will serve our purpose.

In the other case for $p = 19$, namely $k = 12$, the corresponding expression is somewhat longer. The exponent of $Q^3 - R^2$ is $10$ and we are dealing with $P^{\alpha}Q^{\beta}R^{\gamma}$ where $\alpha + 2\beta + 3\gamma = 59$. The result of solving the congruences is

$$\begin{aligned}12\theta(&4P^4Q^{26}R + 16P^4Q^{23}R^3 + 12P^4Q^{20}R^5 + 17P^4Q^{17}R^7\\&+ 10P^4Q^{14}R^9 + 8P^4Q^{11}R^{11} + 16P^4Q^8R^{13} + 6P^4Q^5R^{15}\\&+ 13P^4Q^2R^{17} + 5P^3Q^{28} + P^3Q^{25}R^2 + 8P^3Q^{22}R^4\\&+ 2P^3Q^{19}R^6 + 5P^3Q^{16}R^8 + 5P^3Q^{13}R^{10} + 4P^3Q^{10}R^{12}\\&+ 7P^3Q^7R^{14} + 2P^3Q^4R^{16} + 9P^3QR^{18} + 9P^2Q^{27}R\\&+ 7P^2Q^{24}R^3 + 13P^2Q^{21}R^5 + 2P^2Q^{18}R^7 + 7P^2Q^{15}R^9\\&+ 5P^2Q^{12}R^{11} + 7P^2Q^9R^{13} + 16P^2Q^6R^{15} + 15P^2Q^3R^{17}\\&+ 18P^2R^{19} + 4PQ^{29} + 6PQ^{26}R^2 + PQ^{23}R^4\\&+ 14PQ^{20}R^6 + 8PQ^{17}R^8 + 8PQ^{14}R^{10} + PQ^8R^{14}\\&+ 13PQ^5R^{16} + 12PQ^2R^{18} + 15Q^{28}R + 4Q^{25}R^3\\&+ 13Q^{22}R^5 + 16Q^{19}R^7 + 3Q^{16}R^9 + 10Q^{13}R^{11}\\&\qquad + 15Q^{10}R^{13} + 5Q^7R^{15} + 7Q^4R^{17} + 14QR^{19})\\&\equiv (Q^3 - R^2)^{10} \pmod{19}.\end{aligned}$$

In one of his proofs of (4), Ramanujan uses (7) and (8) as well as

$$\begin{aligned}Q(PQ - R) &= 720\phi_{1,8}(x),\\2PQ^2 - P^2R - QR &= 1728\phi_{2,7}(x),\\P^3Q - 3P^2R + 3PQ^2 - QR &= 3456\phi_{3,6}(x)\end{aligned}$$

and

$$15PQ^2 - 20P^2R + 10P^3Q - 4QR - P^5 = 20736\phi_{4,5}(x)$$

in order to establish

$$(Q^3 - R^2)^5 \equiv -5\phi_{1,8}(x) + 3\phi_{2,7}(x) + 3\phi_{3,6}(x) - \phi_{4,5}(x) \pmod{11}$$

in which it is clear that the coefficients of $x^{11n}$ on the right-hand side are divisible by 11.

Alternatively, using our method we obtain

$$\begin{aligned}12\theta(&10P^3Q^{13} + P^3Q^{10}R^2 + 7P^3Q^7R^4 + 7P^3Q^4R^6 + 5P^3QR^8\\&+ 4P^2Q^{12}R + 10P^2Q^9R^3 + 8P^2Q^6R^5 + 9P^2R^9 + 5PQ^{14}\\&+ 6PQ^{11}R^2 + 8PQ^8R^4 + 2PQ^5R^6 + 3PQ^2R^8\\&\qquad + 10Q^{13}R + Q^{10}R^3 + Q^7R^5 + 10Q^4R^7 + 3QR^9)\\&\equiv (Q^3 - R^2)^5 \pmod{11}.\end{aligned}$$

For the other $p = 11$ case, namely $k = 8$, $b = 7$ we use

$$\begin{aligned}12\theta(3P^2Q^9R + 9P^2Q^6R^3 + 8P^2Q^3R^5 + 5P^2R^7 + 7PQ^{11}\\ + 6PQ^8R^2 + 5PQ^5R^4 + 9PQ^2R^6 + 6Q^{10}R + 8QR^7)\\ \equiv (Q^3 - R^2)^4 \pmod{11}.\end{aligned}$$

In a similar manner we can complete the proof of all the congruences in Table 1 except for $k = 26$, $p \neq 179$ where, as can be verified by computation, it turns out that there is no formula of the form

$$12\theta \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} P^\alpha Q^\beta R^\gamma \equiv (Q^3 - R^2)^{p-b} \pmod{p}. \tag{13}$$

In fact we obtain

$$\begin{aligned}12\theta \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} P^\alpha Q^\beta R^\gamma\\ \equiv (Q^3 - R^2)^{(p+13)/12} + u(p) P^{11} Q^{(p-27)/4} R(Q^3 - R^2)\\ \pmod{p}\end{aligned} \tag{14}$$

for some $a_{\alpha,\beta,\gamma}$ and $u(p) \pmod{p}$. As noted above, $u(179) = 0$.

Nevertheless, using the same method we can show that, for $p \equiv 11 \pmod{12}$, $47 \leq p \leq 197$, there are congruences of the form

$$12\theta \sum_{\alpha,\beta,\gamma} a_{\alpha,\beta,\gamma} P^\alpha Q^\beta R^\gamma \equiv Q^p (Q^3 - R^2)^{p-b} \pmod{p} \tag{15}$$

which have the desired property. Indeed, $Q^{-p}$ is congruent modulo $p$ to a power series in $x^p$. So multiplying the right-hand side of (15) by $Q^{-p}$ preserves the divisibility by $p$ of the coefficients of $x^{np}$. For example with $p = 47$, $k = 26$, $b = 42$ we have

$$\begin{aligned}12\theta \sum_{\alpha=0}^{11} \sum_{\substack{\beta=\alpha\\ \beta\equiv\alpha \pmod 3}}^{(123-\alpha)/2} a_{\alpha,\beta} P^\alpha Q^\beta R^{(123-\alpha-2\beta)/3}\\ \equiv Q^{47}(Q^3 - R^2)^5 \pmod{47}\end{aligned}$$

where the coefficients $a_{\alpha,\beta}$ are given by Table 3.

Of course the congruences in Table 1 are really statements about Cauchy powers of Ramanujan's $\tau$-function and can be established using modular function theory as already indicated. The author conjectures that, corresponding to every congruence of the form (1) there is a congruence (13), except possibly when $p \equiv 11 \pmod{12}$ and $k = 26$ in which case both (14) and (15) apply.

TABLE 3

| | $\alpha$ | | | | | $\alpha$ | | | | | $\alpha$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\beta$ | 0 | 3 | 6 | 9 | $\beta$ | 1 | 4 | 7 | 10 | $\beta$ | 2 | 5 | 8 | 11 |
| 0 | 15 | | | | 1 | 6 | | | | 2 | 31 | | | |
| 3 | 21 | 35 | | | 4 | 41 | 41 | | | 5 | 45 | 12 | | |
| 6 | 7 | 32 | 3 | | 7 | 40 | 21 | 19 | | 8 | 41 | 42 | 20 | |
| 9 | 21 | 0 | 8 | 33 | 10 | 38 | 2 | 27 | 4 | 11 | 33 | 16 | 20 | 30 |
| 12 | 17 | 7 | 15 | 29 | 13 | 6 | 10 | 46 | 13 | 14 | 46 | 22 | 22 | 2 |
| 15 | 19 | 23 | 13 | 31 | 16 | 11 | 0 | 40 | 2 | 17 | 22 | 36 | 9 | 2 |
| 18 | 18 | 4 | 16 | 42 | 19 | 44 | 3 | 23 | 25 | 20 | 28 | 11 | 44 | 44 |
| 21 | 45 | 15 | 29 | 9 | 22 | 23 | 31 | 45 | 29 | 23 | 37 | 14 | 31 | 28 |
| 24 | 33 | 39 | 36 | 29 | 25 | 22 | 33 | 21 | 12 | 26 | 24 | 2 | 1 | 17 |
| 27 | 41 | 1 | 35 | 17 | 28 | 37 | 8 | 25 | 25 | 29 | 5 | 28 | 41 | 43 |
| 30 | 25 | 37 | 38 | 45 | 31 | 45 | 28 | 27 | 16 | 32 | 26 | 44 | 10 | 27 |
| 33 | 26 | 15 | 3 | 27 | 34 | 18 | 38 | 4 | 46 | 35 | 9 | 39 | 25 | 8 |
| 36 | 40 | 16 | 45 | 26 | 37 | 31 | 41 | 36 | 1 | 38 | 13 | 3 | 18 | 33 |
| 39 | 33 | 1 | 34 | 14 | 40 | 12 | 33 | 3 | 3 | 41 | 38 | 27 | 28 | 46 |
| 42 | 41 | 36 | 1 | 43 | 43 | 19 | 25 | 36 | 21 | 44 | 13 | 3 | 45 | 5 |
| 45 | 32 | 34 | 14 | 44 | 46 | 2 | 31 | 25 | 5 | 47 | 33 | 39 | 35 | 43 |
| 48 | 24 | 15 | 2 | 38 | 49 | 45 | 34 | 16 | 24 | 50 | 38 | 43 | 11 | 14 |
| 51 | 4 | 9 | 45 | 22 | 52 | 22 | 46 | 18 | 29 | 53 | 14 | 16 | 28 | 44 |
| 54 | 7 | 30 | 38 | 4 | 55 | 37 | 3 | 6 | 7 | 56 | 17 | 11 | 17 | 26 |
| 57 | 30 | 29 | 7 | 23 | 58 | 25 | 6 | 42 | | 59 | 45 | 9 | | |
| 60 | 29 | 16 | | | 61 | 13 | | | | | | | | |

Further congruences can be established by the same method. For example each of the following functions is congruent modulo $p$ to a power series of the form $12\theta f(x)$.

$$
\begin{aligned}
p = 11: \quad & (Q^3 - R^2)^8 + 4P^6Q^{18}(Q^3 - R^2), \\
& (Q^3 - R^2)^{10} + 6P^8Q^{23}(Q^3 - R^2), \\
p = 13: \quad & (Q^3 - R^2)^9 + 5P^2Q^5(Q^{21} - R^{14}), \\
& Q^{13}(Q^3 - R^2)^7 + 2P^4Q^{11}(Q^{21} - R^{14}), \\
p = 17: \quad & (Q^3 - R^2)^4 + 3P^{12}R^2(Q^3 - R^2), \\
& (Q^3 - R^2)^5 + 6P^7Q^7R(Q^3 - R^2), \\
& (Q^3 - R^2)^8 + 10P^9Q^{15}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{11} + 9P^{11}Q^{23}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{12} + 10P^6Q^{30}(Q^3 - R^2), \\
& Q^{17}(Q^3 - R^2)^{12} + 12P^6Q^{23}(Q^{27} - R^{18}), \\
p = 19: \quad & (Q^3 - R^2)^{13} + P^2Q^5(Q^{33} - R^{22}), \\
p = 23: \quad & (Q^3 - R^2)^9 + 4P^{14}Q^{17}(Q^3 - R^2), \\
& (Q^3 - R^2)^{16} + 21P^6Q^{42}(Q^3 - R^2), \\
& (Q^3 - R^2)^{20} + 4P^8Q^{53}(Q^3 - R^2),
\end{aligned}
$$

$$\begin{aligned}
p = 29: \quad & (Q^3 - R^2)^8 + 19P^7Q^{16}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{13} + 2P^9Q^{30}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{18} + 20P^{11}Q^{44}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{20} + P^6Q^{54}(Q^3 - R^2), \\
& (Q^3 - R^2)^{25} + 9P^8Q^{68}(Q^3 - R^2), \\
& Q^{29}(Q^3 - R^2)^6 + 5P^{12}Q^{38}(Q^3 - R^2), \\
p = 31: \quad & (Q^3 - R^2)^6 + 29P^8Q^{11}(Q^3 - R^2), \\
& (Q^3 - R^2)^{21} + 2P^2Q^{11}(Q^{51} - R^{34}), \\
& (Q^3 - R^2)^{30} + 16P^{17}Q^{77}R(Q^3 - R^2), \\
p = 37: \quad & (Q^3 - R^2)^4 + 16P^9Q^3R(Q^3 - R^2), \\
& (Q^3 - R^2)^7 + 32P^8Q^{14}(Q^3 - R^2), \\
& (Q^3 - R^2)^{10} + 36P^7Q^{22}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{25} + P^2Q^{17}(Q^{57} - R^{38}), \\
p = 41: \quad & (Q^3 - R^2)^{11} + 40P^7Q^{25}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{18} + 30P^9Q^{45}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{25} + 22P^{11}Q^{65}R(Q^3 - R^2), \\
& (Q^3 - R^2)^{28} + 4P^6Q^{78}(Q^3 - R^2), \\
& (Q^3 - R^2)^{35} + 34P^8Q^{98}(Q^3 - R^2), \\
p = 43: \quad & (Q^3 - R^2)^8 + 8P^8Q^{17}(Q^3 - R^2), \\
& (Q^3 - R^2)^{29} + 4P^2Q^{17}(Q^{69} - R^{46}), \\
p = 47: \quad & (Q^3 - R^2)^{32} + 34P^6Q^{90}(Q^3 - R^2), \\
& (Q^3 - R^2)^{40} + 25P^8Q^{113}(Q^3 - R^2)
\end{aligned}$$

and

$$p = 541: \quad (Q^3 - R^2)^{136}.$$

Finally we have a general result:

THEOREM. *Suppose $p = 6t + 1$ is prime. Then there exist integer coefficients $a_\beta$ such that*

$$12x\frac{d}{dx} \sum_{\substack{\beta,\gamma \\ 2\beta+3\gamma=5p}} a_\beta Q^\beta R^\gamma \equiv (Q^3 - R^2)^{(5p+1)/6} - \binom{4t}{t} Q^{p-1}(Q^{3(p+1)/2} - R^{p+1}) \pmod{p}.$$

*Proof*. If $\beta$ and $\gamma$ are related by $2\beta + 3\gamma = 5p$ then

$$12x\frac{d}{dx}Q^{\beta}R^{\gamma} \equiv 4\beta Q^{\beta-1}R^{\gamma-1}(Q^3 - R^2) \qquad (\text{mod } p).$$

Writing $w$ for the integer $\frac{5p+1}{6}$, we have to solve the following set of congruences modulo $p$.

$$\begin{aligned}
4a_1 &\equiv (-1)^w, \\
16a_4 - 4a_1 &\equiv (-1)^{w-1}\binom{w}{w-1}, \\
&\ldots, \\
4(p-3)a_{p-3} - 4(p-6)a_{p-6} &\equiv (-1)^{3t+2}\binom{w}{3t+2}, \\
-4(p-3)a_{p-3} &\equiv (-1)^{3t+1}\binom{w}{3t+1} + \binom{4t}{t}, \\
4(p+3)a_{p+3} &\equiv (-1)^{3t}\binom{w}{3t}, \\
4(p+6)a_{p+6} - 4(p+3)a_{p+3} &\equiv (-1)^{3t-1}\binom{w}{3t-1}, \\
&\ldots, \\
4\frac{5p-3}{2}a_{(5p-3)/2} - 4\frac{5p-9}{2}a_{(5p-9)/2} &\equiv -\binom{w}{1}, \\
-4\frac{5p-3}{2}a_{(5p-3)/2} &\equiv 1 - \binom{4t}{t}.
\end{aligned}$$

A solution is possible since

$$\begin{aligned}
1 &- \binom{w}{1} + \cdots + (-1)^{3t}\binom{w}{3t} - \binom{4t}{t} \\
&\equiv 1 + \binom{t}{1} + \cdots + \binom{4t-1}{3t} - \binom{4t}{t} \qquad (\text{mod } p) \\
&= 0.
\end{aligned}$$

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