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**ON THE EXTENSION OF LIPSCHITZ FUNCTIONS FROM  
BOUNDARIES OF SUBVARIETIES TO STRONGLY  
PSEUDOCONVEX DOMAINS**

K. ADACHI AND HIROSHI KAJIMOTO

# ON THE EXTENSION OF LIPSCHITZ FUNCTIONS FROM BOUNDARIES OF SUBVARIETIES TO STRONGLY PSEUDOCONVEX DOMAINS

KENZŌ ADACHI AND HIROSHI KAJIMOTO

**In this paper, we study the principal value integral on boundaries of subvarieties in strongly pseudoconvex domains and using it, we give a condition for the extendability of Lipschitz functions.**

**Introduction.** Let  $D$  be a strongly pseudoconvex domain in  $C^n$  with  $C^\infty$  boundary. Henkin [6] and Ramírez [12] obtained independently the support function  $g(\zeta, z)$  for  $D$  which depends holomorphically on  $z$ , and then, using this support function, they obtained the integral formula for holomorphic functions in  $\overline{D}$ . On the other hand, Stout [14], when  $p = 1$ , and then, Hatziafratis [5], when  $p$  is arbitrary, obtained the integral formula for a certain subvariety  $V$  of codimension  $p$  in  $D$ . By using the support function  $g(\zeta, z)$  and the integral formula for  $V$ , we can obtain the kernel  $\Omega(\zeta, z)$  for  $(\zeta, z) \in \partial V \times \overline{D}$ . In this paper, we shall define the principal value integral P.V.  $\int_{\partial V} f(\zeta) \Omega(\zeta, z)$  for a Lipschitz function  $f$  on  $\partial V$  and  $z \in \partial V$ . The definition of the principal value integral is the same as that of Alt [2] when  $V = D$  (cf. Dolbeault [4]). By using the principal value integral we can give the condition for a Lipschitz function on  $\partial V$  to be the boundary value of a function that is holomorphic in  $D$  and continuous on  $\overline{D}$ . Finally we end the introduction by giving an example which shows that the Lipschitz continuity is necessary in order to define the principal value integral.

**EXAMPLE.** Define  $\varphi \in C^\infty(0, \infty)$  such that

$$\varphi(\theta) = \begin{cases} 1 & \text{if } 0 < \theta \leq \frac{\pi}{4}, \\ 0 & \text{if } \theta \geq \frac{\pi}{2}. \end{cases}$$

Extend  $\varphi$  to an odd function on  $R \setminus \{0\}$ . Let  $D$  be the unit disc in  $\mathbb{C}$  and  $f$  be a function on  $\partial D$  such that

$$f(e^{i\theta}) = \begin{cases} \frac{\varphi(\theta)}{\log|\theta|} & \text{if } 0 < |\theta| \leq \pi, \\ 0 & \text{if } \theta = 0. \end{cases}$$

Then  $f$  is continuous but not Lipschitz continuous on  $\partial D$ . Compute the principal value integral at  $1 \in \partial D$ . We find

$$\begin{aligned}
 \text{P.V.} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta) d\zeta}{\zeta - 1} \\
 &= \lim_{\varepsilon \rightarrow 0+} \frac{1}{2\pi} \int_{\varepsilon \leq |\theta| \leq \pi} \frac{f(e^{i\theta}) e^{i\theta}}{e^{i\theta} - 1} d\theta \\
 &= \lim_{\varepsilon \rightarrow 0+} \frac{1}{2\pi} \int_{\varepsilon}^{\pi} f(e^{i\theta}) \frac{e^{i\theta} + 1}{e^{i\theta} - 1} d\theta \\
 &= \lim_{\varepsilon \rightarrow 0+} \frac{1}{2\pi} \int_{\varepsilon}^{\pi/4} \frac{1}{\log \theta} \frac{e^{i\theta} + 1}{e^{i\theta} - 1} d\theta + (\text{finite value}).
 \end{aligned}$$

But we have

$$\frac{1}{\log \theta} \frac{e^{i\theta} + 1}{e^{i\theta} - 1} \sim \frac{-2i}{\theta \log \theta}$$

when  $\theta \rightarrow 0+$ . This shows that the principal value integral of  $f$  at 1 does not converge.

**1. The integral formula on subvarieties.** Let  $D$  be a bounded strongly pseudoconvex domain in  $C^n$  with  $C^\infty$  boundary. Let  $\rho$  be a defining function of  $D$ , i.e.,  $D = \{z: \rho(z) < 0\}$ . We set

$$\begin{aligned}
 F(\zeta, z) &= \sum_{i=1}^n 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) (z_i - \zeta_i) \\
 &\quad + \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) (z_i - \zeta_i)(z_j - \zeta_j).
 \end{aligned}$$

According to the construction of Henkin [6] (cf. Henkin-Leiterer [8], p. 108), there exists a pseudoconvex neighborhood  $\tilde{D}$  of  $\bar{D}$ , a neighborhood  $W$  of  $\partial D$ , and a  $C^\infty$  function  $g: W \times \tilde{D} \rightarrow C$  such that for each  $\zeta \in W$ ,  $g(\zeta, z)$  is holomorphic in  $\tilde{D}$ . For  $r > 0$ , define  $\Delta_r = \{(\zeta, z) \in W \times \tilde{D}: |\zeta - z| < r\}$ . Then there exist a constant  $\sigma_1 > 0$  and a non-vanishing  $C^\infty$  function  $Q(\zeta, z)$  on  $\Delta_{\sigma_1}$  such that  $g(\zeta, z) = F(\zeta, z)Q(\zeta, z)$  for  $(\zeta, z) \in \Delta_{\sigma_1}$  and  $g(\zeta, z) \neq 0$  for  $(\zeta, z) \in W \times \tilde{D} \setminus \Delta_{\sigma_1}$ . Moreover  $g(\zeta, z)$  admits a division

$$g(\zeta, z) = \sum_{j=1}^n g_j(\zeta, z)(z_j - \zeta_j)$$

with  $g_j: W \times \tilde{D} \rightarrow C$  of class  $C^\infty$  and holomorphic in the second variable. Let  $h_1, \dots, h_p$  ( $p < n$ ) be holomorphic functions in  $\tilde{D}$ .

Define

$$\tilde{V} = \{z \in \tilde{D}: h_1(z) = \cdots = h_p(z) = 0\}, \quad V = \tilde{V} \cap D.$$

Assume that

$$(1) \quad \partial h_1 \wedge \cdots \wedge \partial h_p \wedge \partial \rho \neq 0 \quad \text{on } \partial V.$$

By Hefer's theorem, there exist holomorphic functions  $h_{ij}(\zeta, z)$  for  $(\zeta, z) \in \tilde{D} \times \tilde{D}$  such that

$$h_i(\zeta) - h_i(z) = \sum_{j=1}^n h_{ij}(\zeta, z)(\zeta_j - z_j), \quad i = 1, \dots, p.$$

Define

$$\alpha^h(\zeta, z) = \begin{vmatrix} g_1 & h_{11} & \cdots & h_{p1} & \bar{\partial}_\zeta g_1 & \cdots & \bar{\partial}_\zeta g_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ g_n & h_{1n} & \cdots & h_{pn} & \bar{\partial}_\zeta g_n & \cdots & \bar{\partial}_\zeta g_n \end{vmatrix},$$

$$|\nabla h(\zeta)|^2 = \sum_{1 \leq j_1 < \cdots < j_p \leq n} \left| \frac{\partial(h_1, \dots, h_p)}{\partial(\zeta_{j_1}, \dots, \zeta_{j_p})}(\zeta) \right|^2,$$

$$\beta^h(\zeta) = (-1)^{p(p+1)/2} |\nabla h(\zeta)|^{-2} \begin{vmatrix} \frac{\partial h_1}{\partial \zeta_1} & \cdots & \frac{\partial h_p}{\partial \zeta_1} & d\zeta_1 & \cdots & d\zeta_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial h_1}{\partial \zeta_n} & \cdots & \frac{\partial h_p}{\partial \zeta_n} & d\zeta_n & \cdots & d\zeta_n \end{vmatrix}$$

and

$$K_V(\zeta, z) = c(n, p) \alpha^h(\zeta, z) \wedge \beta^h(\zeta),$$

where

$$c(n, p) = (-1)^{p(n+1)} (-1)^{n(n-1)/2} \frac{1}{(n-p)!(2\pi i)^{n-p}}.$$

Let  $n - p = k$ . We define the kernel  $\Omega(\zeta, z)$  by

$$\Omega(\zeta, z) = \frac{K_V(\zeta, z)}{g(\zeta, z)^k}.$$

Let  $A(D)$  (resp.  $A(V)$ ) be the space of functions that are holomorphic in  $D$  (resp.  $V$ ) and continuous on  $\bar{D}$  (resp.  $\bar{V}$ ). Then Hatziafratis [5] proved the following.

**THEOREM 1.** *For  $f \in A(V)$  and  $z \in V$ , the integral formula*

$$(2) \quad f(z) = \int_{\partial V} f(\zeta) \Omega(\zeta, z)$$

*holds.*

Now we begin by proving the following lemma.

LEMMA 1. If  $h_1(z) = z_{k+1}, \dots, h_p(z) = z_n$ , then we have

$$K_V(\zeta, z) = (-1)^{k(k-1)/2} \frac{(k-1)!}{(2\pi i)^k} \cdot \sum_{j=1}^k (-1)^{j-1} g_j \bigwedge_{\substack{i=1 \\ i \neq j}}^k \bar{\partial}_\zeta g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k.$$

*Proof.* By the definition of  $\alpha^h$ ,  $\beta^h$ , we have

$$\begin{aligned} \alpha^h(\zeta, z) &= \begin{vmatrix} g_1 & 0 \cdots 0 & \bar{\partial}_\zeta g_1 \cdots \bar{\partial}_\zeta g_1 \\ \vdots & \vdots & \vdots \\ g_p & 0 \cdots 0 & \vdots \\ & 1 \cdots 0 & \vdots \\ \vdots & \ddots & \\ g_n & 0 \cdots 1 & \bar{\partial}_\zeta g_n \cdots \bar{\partial}_\zeta g_n \end{vmatrix} = (-1)^p \begin{vmatrix} g_1 & \bar{\partial}_\zeta g_1 \cdots \bar{\partial}_\zeta g_1 \\ \vdots & \vdots \\ g_k & \bar{\partial}_\zeta g_k \cdots \bar{\partial}_\zeta g_k \end{vmatrix} \\ &= (-1)^{n-k} (k-1)! \sum_{j=1}^k (-1)^{j-1} g_j \bigwedge_{i \neq j} \bar{\partial}_\zeta g_i, \\ \beta^h(\zeta) &= (-1)^{\frac{p(p+1)}{2}} \begin{vmatrix} 0 \cdots 0 & d\zeta_1 \cdots d\zeta_1 \\ \vdots & \vdots \\ 0 \cdots 0 & \vdots \\ & 1 \cdots 0 \\ & \ddots \\ 0 \cdots 1 & d\zeta_n \cdots d\zeta_n \end{vmatrix} = (-1)^{\frac{p(p+1)}{2}} \begin{vmatrix} d\zeta_1 \cdots d\zeta_1 \\ \vdots \\ d\zeta_k \cdots d\zeta_k \end{vmatrix} \\ &= (-1)^{\frac{(n-k)(n-k+1)}{2}} k! d\zeta_1 \wedge \cdots \wedge d\zeta_k. \end{aligned}$$

Therefore we have

$$\begin{aligned} K_V(\zeta, z) &= (-1)^{(n-k)(n+1)} (-1)^{n(n-1)/2} \frac{1}{k!(2\pi i)^k} \alpha^h(\zeta, z) \wedge \beta^h(\zeta, z) \\ &= (-1)^{k(k-1)/2} \frac{(k-1)!}{(2\pi i)^k} \sum_{j=1}^k (-1)^{j-1} g_j \bigwedge_{i \neq j} \bar{\partial}_\zeta g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k. \end{aligned}$$

This completes the proof of Lemma 1.

For  $\sigma > 0$ , we set  $S_{\zeta, \sigma} = \{z : |z - \zeta| < \sigma\}$ . Then we have the following.

LEMMA 2. Let  $\sigma$ ,  $0 < \sigma < \sigma_1$ , be sufficiently small. Then for  $\zeta \in \partial V$ ,  $z \in \tilde{D} \cap S_{\zeta, \sigma}$ , we can choose a local coordinate system in such a way that

$$\Omega(\zeta, z) = \frac{2^k}{(2\pi i)^k} \frac{\partial \rho(\zeta) \wedge (\bar{\partial} \partial \rho)^{k-1}(\zeta) + e(\zeta, z)}{F(\zeta, z)^k},$$

where  $e(\zeta, z)$  is a  $(k, k-1)$  form satisfying  $e(\zeta, z) = O(|\zeta - z|)$ .

*Proof.* By a local holomorphic change of coordinates, we may assume that  $h_1 = z_{k+1}, \dots, h_p = z_n$ . We have (cf. Anderson and Berndtsson [3], Lemma 3),

$$\begin{aligned} & \sum_{j=1}^k (-1)^{j-1} \frac{g_j \wedge_{i \neq j} \bar{\partial}_\zeta g_i \wedge d\zeta_1 \wedge \dots \wedge d\zeta_k}{g(\zeta, z)^k} \\ &= c(k) \frac{(\sum_{j=1}^k g_j d\zeta_j) \wedge (\sum_{j=1}^k \bar{\partial}_\zeta g_j \wedge d\zeta_j)^{k-1}}{g(\zeta, z)^{6k}}, \end{aligned}$$

where  $c(k) = (-1)^{k(k-1)/2} \frac{1}{(k-1)!}$ .

If  $|\zeta - z| < \sigma$ , then we obtain

$$\begin{aligned} & \sum_{i=1}^n g_i(\zeta, z)(z_i - \zeta_i) \\ &= \sum_{i=1}^n \left\{ \left( 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) + \sum_{j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_j - \zeta_j) \right) Q(\zeta, z) \right\} (z_i - \zeta_i). \end{aligned}$$

Therefore we have

$$g_i(\zeta, \zeta) = 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) Q(\zeta, \zeta), \quad 1 \leq i \leq n.$$

By Lemma 1, we obtain

$$\begin{aligned} \Omega(\zeta, z) &= \frac{1}{(2\pi i)^k} \frac{(\sum_{j=1}^k 2 \frac{\partial \rho}{\partial \zeta_j}(\zeta) Q(\zeta, \zeta) d\zeta_j) \wedge (2 \sum_{j=1}^k \bar{\partial}_\zeta (\frac{\partial \rho}{\partial \zeta_j}(\zeta) Q(\zeta, \zeta)) \wedge d\zeta_j)^{k-1}}{F(\zeta, z)^k Q(\zeta, z)^k} \\ &\quad + \frac{e(\zeta, z)}{F(\zeta, z)^k Q(\zeta, z)^k} \\ &= \frac{2^k}{(2\pi i)^k} \frac{\partial \rho(\zeta) \wedge (\bar{\partial} \partial \rho)^{k-1}(\zeta) + e(\zeta, z)}{F(\zeta, z)^k}. \end{aligned}$$

This completes the proof of Lemma 2.

Now we prove the following lemma which will be used in the proof of Theorem 2 in order to calculate the principal value of the kernel function.

LEMMA 3. For  $\zeta, z \in \partial V$ , it holds that

$$F(\zeta, z) - \overline{F(z, \zeta)} = O(|\zeta - z|^3).$$

*Proof.* We may assume that  $h_1 = z_{k+1}, \dots, h_p = z_n$ . By the Taylor expansion, we have

$$\begin{aligned} \frac{1}{2}F(\zeta, z) &= \sum_{i=1}^k \frac{\partial \rho}{\partial \zeta_i}(\zeta)(z_i - \zeta_i) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j) \\ &= \sum_{i=1}^k \left( \frac{\partial \rho}{\partial \zeta_i}(z) + \sum_{j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(z)(\zeta_j - z_j) \right. \\ &\quad \left. + \sum_{j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \bar{\zeta}_j}(z)(\bar{\zeta}_j - \bar{z}_j) \right) (z_i - \zeta_i) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(\zeta_i - z_i)(\zeta_j - z_j) + O(|\zeta - z|^3). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \frac{1}{2}\overline{F(\zeta, z)} &= \sum_{i=1}^k \frac{\partial \rho}{\partial \bar{\zeta}_i}(\zeta)(\bar{\zeta}_i - \bar{z}_i) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j}(\zeta)(\bar{\zeta}_i - \bar{z}_i)(\bar{\zeta}_j - \bar{z}_j). \end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \frac{1}{2}(F(\zeta, z) - \overline{F(z, \zeta)}) \\
&= - \sum_{i=1}^k \frac{\partial \rho}{\partial \zeta_i}(z)(\zeta_i - z_i) - \sum_{i=1}^k \frac{\partial \rho}{\partial \bar{\zeta}_i}(z)(\bar{\zeta}_i - \bar{z}_i) \\
&\quad - \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(z)(\zeta_i - z_i)(\zeta_j - z_j) \\
&\quad - \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial \bar{\zeta}_i \partial \bar{\zeta}_j}(z)(\bar{\zeta}_i - \bar{z}_i)(\bar{\zeta}_j - \bar{z}_j) \\
&\quad - \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \bar{\zeta}_j}(z)(\zeta_i - z_i)(\bar{\zeta}_j - \bar{z}_j) + O(|\zeta - z|^3) \\
&= -\rho(\zeta) + \rho(z) + O(|\zeta - z|^3) = O(|\zeta - z|^3).
\end{aligned}$$

This completes the proof of Lemma 3.

**2. The principal value of the kernel function.** Let  $z \in \partial V$  and  $f$  be a continuous function on  $\partial V$ . If

$$\lim_{\varepsilon \rightarrow 0+} \int_{\partial V \cap \{\zeta : |g(\zeta, z)| > \varepsilon\}} f(\zeta) \Omega(\zeta, z)$$

exists, then we stand for the above limit by

$$\text{P.V.} \int_{\partial V} f(\zeta) \Omega(\zeta, z).$$

Now we are going to prove the following theorem which was obtained by Kerzman and Stein [9] when  $k = n$ .

**THEOREM 2.** For  $z \in \partial V$ , it holds that

$$\text{P.V.} \int_{\zeta \in \partial V} \Omega(\zeta, z) = \frac{1}{2}.$$

*Proof.* Let  $q \in \partial V$  be fixed. We may assume that for  $\delta > 0$  sufficiently small,

$$\tilde{V} \cap S_{q,\delta} = \{z \in S_{q,\delta} : z_{k+1} = \cdots = z_n = 0\}.$$



For  $z \in \tilde{V} \cap S_{q,\delta}$ , we have

$$\begin{aligned} \rho(z) = 2 \operatorname{Re} & \left( \sum_{i=1}^k \frac{\partial \rho}{\partial z_i}(q)(z_i - q_i) \right. \\ & \left. + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial z_i \partial z_j}(q)(z_i - q_i)(z_j - q_j) \right) \\ & + \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(q)(z_i - q_i)(\bar{z}_j - \bar{q}_j) + O(|z - q|^3). \end{aligned}$$

For simplicity, we may assume that  $q = 0$ . By (1), we can find a new local coordinate system  $w_1, \dots, w_n$  by letting

$$w_1 = 2 \sum_{i=1}^k \frac{\partial \rho}{\partial z_i}(0) z_i + \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0) z_i z_j,$$

and choosing  $w_1, \dots, w_n$  suitably such that  $w(0) = 0$ . Then we have

$$\rho(w) = \operatorname{Re} w_1 + \sum_{i,j=1}^k \frac{\partial^2 \rho}{\partial w_i \partial \bar{w}_j}(0) w_i \bar{w}_j + O(|w|^3).$$

By a unitary transformation  $w'_i = (w'_1, \dots, w'_n)$ , we obtain

$$\rho(w') = \operatorname{Re} \left( \sum_{j=1}^k a_j w'_j \right) + \sum_{i=1}^k \frac{\partial^2 \rho}{\partial w'_i \partial \bar{w}'_i}(0) |w'_i|^2 + O(|w'|^3),$$

where  $(a_1, \dots, a_k)$  is a non zero vector. Again we can find a new local coordinate system  $\zeta = (\zeta_1, \dots, \zeta_n)$  such that

$$(3) \quad \rho(\zeta) = \operatorname{Re} \zeta_1 + \sum_{i=1}^k |\zeta_i|^2 + O(|\zeta|^3).$$

We set  $\zeta_j = x_j + iy_j$ . Then we have

$$\begin{aligned} & \partial \rho(0) \wedge (\bar{\partial} \partial \rho)^{k-1}(0) \\ &= \left( \sum_{j=1}^k \frac{\partial \rho}{\partial \zeta_j}(0) d\zeta_j \right) \wedge \left( \sum_{i=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \bar{\zeta}_i}(0) d\bar{\zeta}_i \wedge d\zeta_i \right)^{k-1}. \end{aligned}$$

Since  $d\rho = 0$  on  $\partial V$ , we have

$$\sum_{j=1}^k \frac{\partial \rho}{\partial x_j}(0) dx_j + \sum_{j=1}^k \frac{\partial \rho}{\partial y_j}(0) dy_j = 0.$$

Therefore we have

$$\begin{aligned}
 & \partial \rho(0) \wedge (\bar{\partial} \partial \rho)^{k-1}(0) \\
 &= \frac{i}{2} \sum_{j=1}^k \left( \frac{\partial \rho}{\partial x_j}(0) dy_j - \frac{\partial \rho}{\partial y_j}(0) dx_j \right) \\
 & \quad \wedge (2i)^{k-1} \left( \sum_{j=2}^k \frac{\partial^2 \rho}{\partial w_j \partial \bar{w}_j}(0) dx_j \wedge dy_j \right)^{k-1} \\
 &= 2^{k-2} i^k (k-1)! dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_k \wedge dy_k.
 \end{aligned}$$

Thus, by Lemma 2, we obtain

(4)

$$\begin{aligned}
 & \Omega(\zeta, z) \\
 &= \frac{2^{k-2} (k-1)! dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_k \wedge dy_k + O(|\zeta| + |\zeta - z|)}{\pi^k F(\zeta, z)^k}.
 \end{aligned}$$

Let  $\nu$  be the unit inner normal of  $\partial V$  at 0. Then we have

$$\nu = - \left( \frac{\partial \rho}{\partial x_1}(0), \frac{\partial \rho}{\partial y_1}(0), \dots, \frac{\partial \rho}{\partial y_n}(0) \right) = (-1, 0, \dots, 0).$$

We set, for  $\delta > 0$  sufficiently small,  $z = \nu \delta$ . Then we have

$$\begin{aligned}
 & F(\zeta, z) - F(\zeta, 0) \\
 &= \sum_{i=1}^k (-\zeta_i + \nu_i \delta) \left( 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) + \sum_{j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) (-\zeta_j + \nu_j \delta) \right) \\
 & \quad - \sum_{i=1}^k (-\zeta_i) \left( 2 \frac{\partial \rho}{\partial \zeta_i}(\zeta) + \sum_{j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) (-\zeta_j) \right) \\
 &= \sum_{i=1}^k 2\nu_i \delta \frac{\partial \rho}{\partial \zeta_i}(\zeta) \\
 & \quad + \sum_{i,j=1}^k (\nu_i \nu_j \delta^2 - \nu_i \zeta_j \delta - \nu_j \zeta_i \delta) \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta) \\
 &= -\delta + O(\delta|\zeta| + \delta^2).
 \end{aligned}$$

On the other hand we have

$$F(0, \zeta) = \sum_{i=1}^k \zeta_i \left( 2 \frac{\partial \rho}{\partial \zeta_i}(0) + \sum_{j=1}^k \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(0) \zeta_j \right) = \zeta_1 = x_1 + iy_1,$$

and

$$0 = \rho(\zeta) = x_1 + \sum_{i=2}^k |\zeta_i|^2 + O(|\zeta|^3).$$

Thus we obtain

$$F(0, \zeta) = - \sum_{i=1}^k |\zeta_i|^2 + iy_1 + O(|\zeta|^3).$$

Taking account of Lemma 3, we have

$$\begin{aligned} F(\zeta, z) &= F(\zeta, 0) - \delta + O(|\zeta|\delta + \delta^2) \\ &= \overline{F(0, \zeta)} - \delta + O(|\zeta|^3 + |\zeta|\delta + \delta^2) \\ &= -iy_1 - \sum_{i=2}^k |\zeta_i|^2 - \delta + O(|\zeta|^3 + |\zeta|\delta + \delta^2). \end{aligned}$$

We set

$$\beta(0, \varepsilon) = \lim_{\delta \rightarrow 0} \int_{\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}} \frac{K_V(\zeta, \nu\delta)}{F(\zeta, \nu\delta)^k}.$$

Then, by (4), we have

$$\beta(0, \varepsilon) = \lim_{\delta \rightarrow 0} \int_{\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}} \frac{2^{k-2}(k-1)!\sigma(d\zeta) + O(|\zeta| + \delta)}{\pi^k F(\zeta, \nu\delta)^k}$$

where  $\sigma(d\zeta)$  is the surface element of  $\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}$ . We set  $\zeta' = (\zeta_2, \dots, \zeta_k)$ . It holds from (3) that  $(\partial\rho/\partial x_1)(0) \neq 0$ . Therefore by the implicit function theorem,  $\rho = 0$  can be represented by  $x_1 = \varphi(y_1, \zeta')$ , where  $\varphi$  is a smooth function satisfying  $\varphi(0, 0) = 0$ . Thus we have  $x_1 = O(|y_1| + |\zeta'|)$ . Hence there exists a constant  $c_1 > 0$  such that

$$||\zeta'|^2 + iy_1| - |F(\zeta, 0)|| \leq c_1(|\zeta'| + |y_1|)^3.$$

Therefore, for  $\varepsilon > 0$  sufficiently small, we have

$$\begin{aligned} &\left\{ \zeta \in \partial V : ||\zeta'|^2 + iy_1| < \frac{\delta}{2} \right\} \\ &\subset \{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\} \subset \left\{ \zeta \in \partial V : ||\zeta'|^2 + iy_1| < \frac{3}{2}\varepsilon \right\}. \end{aligned}$$

Thus we have

$$\beta(0, \varepsilon) = \frac{2^{k-2}(k-1)!}{\pi^k} \lim_{\delta \rightarrow 0} \int_{\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}} \frac{\sigma(d\zeta)}{F(\zeta, \nu\delta)^k},$$

provided the limit of the right-hand side exists. We set  $A = \delta + |\zeta'|^2 + iy_1$ . Then we have  $F(\zeta, \nu\delta) = A + \mathcal{E}$  where  $\mathcal{E} = O(|\zeta|^3 + \delta|\zeta| + \delta^2)$ . Then for some constant  $c_2 > 0$ , it holds that

$$\left| \frac{\mathcal{E}}{A} \right| \leq c_2(|\zeta'| + |y_1| + \delta).$$

Thus if we choose  $\delta$  and  $\varepsilon$  sufficiently small, then we have

$$\left| \frac{\mathcal{E}}{A} \right| \leq \frac{1}{2}.$$

On the other hand we have

$$\frac{1}{F(\zeta, \nu\delta)^k} = \frac{1}{(A + \mathcal{E})^k} = \frac{1}{A^k} + \mathcal{E}',$$

where  $\mathcal{E}'$  satisfies, for some constant  $c_3 > 0$ ,

$$|\mathcal{E}'| \leq \frac{c_3(|\zeta'| + |y_1| + \delta)}{|A|^k}.$$

It holds that

$$\int_{\{|iy_1 + |\zeta'|^2| < \varepsilon\}} \frac{|\zeta'| + |y_1| + \delta}{|A|^k} \sigma(d\zeta) \rightarrow 0 \quad (\text{as } \delta \rightarrow 0, \varepsilon \rightarrow 0).$$

Thus we obtain

$$\beta(0, \varepsilon) = \frac{2^{k-2}(k-1)!}{\pi^k} \lim_{\delta \rightarrow 0} \int_{\{|iy_1 + |\zeta'|^2| < \varepsilon\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + \delta)^k}.$$

The dilation  $(y-1, \zeta') \rightarrow (\delta y_1, \sqrt{\delta}\zeta')$  gives

$$\lim_{\varepsilon \rightarrow 0} \beta(0, \varepsilon) = \frac{2^{k-2}(k-1)!}{\pi^k} \lim_{R \rightarrow \infty} \int_{\{|iy_1 + |\zeta'|^2| < R\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + 1)^k}.$$

The calculation of this limit is contained in Korányi and Vagi [10] as follows. Consider the integral

$$I(\varepsilon, R) = \int_{\{\varepsilon < |iy_1 + |\zeta'|^2| < R\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + 1)^k}, \quad 0 < \varepsilon < R.$$

Introduce polar coordinates in  $C^{k-1}$ ,

$$\rho = |\zeta'|, \quad \omega = \frac{\zeta'}{|\zeta'|}, \quad \sigma(d\zeta) = \rho^{2k-3} dy_1 d\rho d\omega$$

where  $d\omega$  is the surface element in the unit sphere  $S^{2k-3}$  in  $C^{k-1}$ . We denote the volume of the unit sphere  $S^{2k-3}$  by  $|S^{2k-3}|$ . Next we make the variable change

$$u = \rho^2, \quad du = 2\rho d\rho$$

and then introduce polar coordinates in the  $u, y_1$  halfplane,

$$u + iy_1 = se^{i\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \varepsilon < s < R, \\ du dy_1 = s ds d\theta.$$

We find

$$\begin{aligned} I(\varepsilon, R) &= |S^{2k-3}| \int_{\{\varepsilon^2 < y_1^2 + \rho^4 < R^2\}} \frac{\rho^{2k-3} d\rho dy_1}{(1 + \rho^2 + iy_1)^k} \\ &= \frac{|S^{2k-3}|}{2} \int_{\{\varepsilon^2 < y_1^2 + u^2 < R, \ 0 \leq u\}} \frac{u^{k-1} du dy_1}{(1 + u + iy_1)^k} \\ &= \frac{|S^{2k-3}|}{2} \int_{-\pi/2}^{\pi/2} \cos^{k-2} \theta d\theta \int_{\varepsilon}^R \frac{s^{k-1} ds}{(1 + se^{i\theta})^k}. \end{aligned}$$

The variable change  $\theta \rightarrow \theta - \frac{\pi}{2}$ ,  $s \rightarrow \frac{1}{\rho}$  finally gives

$$I(\varepsilon, R) = \frac{|S^{2k-3}|}{2} \int_0^\pi \sin^{k-2} \theta d\theta \int_{1/R}^{1/\varepsilon} \frac{d\rho}{\rho(\rho - ie^{i\theta})}.$$

Then Korányi and Vagi ([10], Lemma 6.2, p. 613) gives

$$\lim_{\substack{\varepsilon \rightarrow 0+ \\ R \rightarrow \infty}} I(\varepsilon, R) = \frac{\pi^{k-1}}{(k-2)!} \cdot \frac{\pi}{2^{k-1}(k-1)!} = \frac{\pi^k}{2^{k-1}(k-1)!}.$$

Hence we obtain

$$\lim_{\varepsilon \rightarrow 0+} \beta(0, \varepsilon) = \frac{1}{2},$$

which completes the proof of Theorem 2.

**3. The continuous extension to the boundary.** The following proposition is proved essentially by Adachi [1] (cf. Henkin [7]). But for the comparison with the principal value integral, we give the sketch of the proof.

**PROPOSITION 1.** *Define, for  $z \in \overline{D} \setminus \partial V$ ,*

$$H(z) = \int_{\zeta \in \partial V} \Omega(\zeta, z).$$

*Then the function*

$$\tilde{H}(z) = \begin{cases} H(z) & (z \in \overline{D} \setminus \partial V), \\ 1 & (z \in \partial V) \end{cases}$$

*belongs to  $A(D)$ .*

*Proof.* Since  $H(z)$  is holomorphic in  $\overline{D}|\partial V$ , it is sufficient to show that, for  $z^0 \in \partial V$ ,

$$\lim_{z \rightarrow z^0, z \in \overline{D}|\partial V} H(z) = 1.$$

We may assume that

$$\tilde{V} \cap S_{z^0, \sigma_1} = \{z \in S_{z^0, \sigma_1} : z_{k+1} = \dots = z_n = 0\}.$$

By (1), we may assume, without loss of generality, that  $(\partial \rho / \partial \zeta_1)(z^0) \neq 0$ . For  $z \in S_{z^0, \sigma_1}$ , we consider the system of equations for  $\zeta^0 = (\zeta_1^0, \dots, \zeta_n^0)$  of the following form:

$$(5) \quad \begin{cases} \sum_{i=1}^n \frac{\partial \rho}{\partial \zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0, \\ \zeta_i^0 = z_i \ (i = 2, \dots, k), \quad \zeta_{k+1}^0 = \dots = \zeta_n^0 = 0. \end{cases}$$

We set  $\varepsilon = (|z_{k+1}|^2 + \dots + |z_n|^2)^{1/2}$ . Then by Adachi [1], there exist positive constants  $\sigma_2$  ( $< \sigma_1$ ),  $\gamma_1$  and  $\gamma_2$  such that for any  $\sigma \leq \sigma_2$  and any  $z \in S_{z^0, \sigma/2} \cap (\overline{D}|\partial V)$ , there exists a unique solution  $\zeta^0 = \zeta^0(z)$  of the system (5) which belongs to the set  $S_{z^0, \sigma} \cap V$  and satisfies the following.

$$(6) \quad \varepsilon \leq |z - \zeta^0| \leq \gamma_1 \varepsilon,$$

$$(7) \quad \left| \sum_{i=1}^n \frac{\partial g}{\partial z_i}(\zeta, z)(\zeta_i^0 - z_i) \right| \leq \gamma_2 \varepsilon (|\zeta - z| + \varepsilon).$$

From the integral formula (2) we have

$$H(\zeta^0) = 1.$$

Hence it is sufficient to show that

$$\lim_{z \rightarrow z^0, z \in \overline{D}|\partial V} |H(z) - H(\zeta^0)| = 0.$$

Let  $z \in S_{z^0, \sigma/2} \cap (\overline{D}|\partial V)$ . Let  $V'$  be an open subset in  $\tilde{V}$  with smooth boundary such that  $\overline{V} \subset V' \subset \overline{V}' \subset \tilde{V}$ . By using Stokes' formula, we have

$$H(z) = \int_{\partial V'} \Omega(\zeta, z) - \int_{V' - V} \overline{\partial}_\zeta \Omega(\zeta, z).$$

Define

$$\Psi(z) = \int_{(V' - V) \cap S_{z^0, \sigma}} \overline{\partial}_\zeta \Omega(\zeta, z).$$

It is sufficient to show that

$$\lim_{z \rightarrow z^0, z \in \overline{D} \setminus \partial V} |\Psi(z) - \Psi(\zeta^0)| = 0.$$

We can write  $\Psi(z)$  in the following form

$$\begin{aligned} \Psi(z) &= \int_{(V'-V) \cap S_{z^0, \sigma}} \frac{A(\zeta, z)}{g(\zeta, z)^k} \\ &\quad + \int_{(V'-V) \cap S_{z^0, \sigma}} \sum_{j=1}^n \frac{(\zeta_j - z_j) B_j(\zeta, z)}{g(\zeta, z)^{k+1}}, \end{aligned}$$

where  $A(\zeta, z)$ ,  $B_j(\zeta, z)$  are  $(k, k)$  forms that are smooth in  $(\zeta, z)$  and holomorphic in  $z$ . By using (6), (7), there are positive constants  $\gamma_3$  and  $\gamma_4$  such that

$$\begin{aligned} &\left| \frac{d\Psi(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \\ &\leq \gamma_3 \int_{(V'-V) \cap S_{z^0, \sigma}} \frac{\varepsilon dV}{|g(\zeta, z)|^{k+1}} \\ &\quad + \gamma_4 \int_{(V'-V) \cap S_{z^0, \sigma}} \frac{|\zeta - z| \varepsilon (|\zeta - z| + \varepsilon) dV}{|g(\zeta, z)|^{k+2}}. \end{aligned}$$

By the estimates obtained by Henkin [7], we have for some constant  $\gamma_5 > 0$

$$(8) \quad \left| \frac{d\Psi(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \leq \gamma_5 (\varepsilon |\log \varepsilon| + \varepsilon).$$

Let

$$z(\theta) = \zeta^0 + \theta(z - \zeta^0) \quad \text{for } \theta \in [0, 1].$$

Then the uniqueness of the solution of the system (5) implies  $\zeta^0(z(\theta)) = \zeta^0(z)$ . Therefore from (8), we have, for some constant  $\gamma_6 > 0$ ,

$$\begin{aligned} &\left| \frac{d\Psi(\zeta^0 + \lambda\theta(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \\ &= \left| \frac{d\Psi(\zeta^0(z(\theta)) + \lambda(z(\theta) - \zeta^0(z(\theta))))}{d\lambda} \right|_{\lambda=1} \\ &\leq \gamma_6 (\theta \varepsilon |\log(\theta \varepsilon)| + \theta \varepsilon). \end{aligned}$$

Thus we obtain for some constant  $\gamma_7 > 0$ ,

$$\begin{aligned} |\Psi(z) - \Psi(\zeta^0)| &= \left| \int_0^1 \frac{d}{d\theta} \Psi(\zeta^0 + \theta(z - \zeta^0)) d\theta \right| \\ &= \left| \int_0^1 \frac{1}{\theta} \left( \frac{d\Psi(\zeta^0 + \lambda\theta(z - \zeta^0))}{d\lambda} \right)_{\lambda=1} d\theta \right| \\ &\leq \gamma_7 \int_0^1 (\varepsilon |\log \varepsilon| + \varepsilon |\log \theta| + \varepsilon) d\theta \rightarrow 0 \quad (\varepsilon \rightarrow 0), \end{aligned}$$

which completes the proof of Proposition 1.

**4. The extension of Lipschitz functions from the boundary.** In order to prove Lemma 5, we need the following lemma which is the modified version of Lemma 3.1 of Henkin [6].

**LEMMA 4.** *Let  $t = (t_1, \dots, t_{2k}) \in R^{2k}$ ,  $\varepsilon > 0$ ,  $0 < \delta < 1$ ,  $t' = (t_2, \dots, t_{2k})$ . Then we have*

$$I_1 = \int_{\{\delta^2 \leq |t'|^2 + \varepsilon^2 \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{[(|t'|^2 + \varepsilon^2)^2 + t_2^2]^{k/2}} \leq \gamma \log \frac{\gamma}{\delta},$$

where  $\gamma$  is the constant which is independent of  $\varepsilon$  and  $\delta$ .

*Proof.* (a) In case  $\varepsilon^2 \leq \frac{1}{2}\delta^2$ . Since  $\delta^2 \leq |t'|^2 + \varepsilon^2$ , we have  $|t'|^2 \geq \frac{1}{2}\delta^2$ . Therefore we have

$$I_1 \leq \int_{\{\delta^2/2 \leq |t'|^2 \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{(|t'|^2 + t_2)^{k/2}}.$$

If  $k = 1$ , then we have

$$I_1 \leq \int_{\delta/2}^1 \frac{dt_2}{t_2} \leq \gamma \log \frac{\gamma}{\delta}.$$

If  $k \geq 2$ , by using polar coordinates, we have for some  $\gamma_1 > 0$ ,



$$\begin{aligned}
I_1 &\leq \gamma_1 \int_{\delta/2}^1 dr \int_0^\pi \frac{r^{2k-2} \sin^{2k-3} \varphi d\theta}{(r^4 + r^2 \cos^2 \varphi)^{k/2}} \\
&= \gamma_1 \int_{\delta/2}^1 dr \int_0^\pi \frac{r^{k-2} \sin^{2k-3} \varphi d\varphi}{(r^2 + \cos^2 \varphi)^{k/2}} \\
&\leq \gamma_1 \int_{\delta/2}^1 dr \int_0^\pi \frac{\sin^{2k-3} \varphi}{r^2 + \cos^2 \varphi} d\varphi \\
&\leq \gamma_1 \int_{\delta/2}^1 dr \int_{-1}^1 \frac{ds}{r^2 + s^2} = \gamma_1 \int_{\gamma/2}^1 \frac{2}{r} \tan^{-1} \left( \frac{1}{r} \right) dr \\
&\leq \pi \gamma_1 \int_{\delta/2}^1 \frac{dr}{r} \leq \gamma \log \frac{\gamma}{\delta}.
\end{aligned}$$

(b) In case  $\varepsilon^2 \geq \frac{1}{2}\delta^2$ . Then we have

$$I_1 \leq \int_{\{|t'| \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{[(|t'|^2 + \varepsilon^2)^2 + t_2^2]^{k/2}}.$$

If  $k = 1$ , then we have for some  $\gamma_2 > 0$ ,

$$I_1 \leq \gamma_2 \int_0^1 \frac{dt_2}{t_2 + \delta^2} \leq \gamma \log \frac{\gamma}{\delta}.$$

If  $k \geq 2$ , then we obtain for some  $\gamma_3 > 0$ ,

$$\begin{aligned}
I_1 &\leq \gamma_3 \int_0^1 dr \int_0^\pi \frac{r^{2k-2} \sin^{2k-3} \varphi d\theta}{[(r^2 + \delta^2)^2 + r^2 \cos^2 \varphi]^{k/2}} \\
&\leq \gamma_3 \int_0^1 dr \int_0^\pi \frac{\sin^{2k-3} \varphi d\varphi}{(r + \delta^2/r)^2 + \cos^2 \varphi} \\
&\leq \gamma_3 \int_0^1 dr \int_{-1}^1 \frac{ds}{(r + \delta^2/r)^2 + s^2} \\
&\leq \pi \gamma_3 \int_0^1 \frac{dr}{\delta^2/r + r} \leq \frac{\pi \gamma_3}{2} \int_{\delta^2}^{1+\delta^2} \frac{d\lambda}{\lambda} \leq \gamma \log \frac{\gamma}{\delta}.
\end{aligned}$$

This completes the proof of Lemma 4.

Define, for  $\delta > 0$ ,

$$(\partial V)_\delta = \{w : |w - \zeta| < \delta \text{ for some } \zeta \in \partial V\}.$$

Then we have

**LEMMA 5.** *There exists  $\delta_0 > 0$  such that for any  $z \in \overline{D}$  and any  $\delta$  ( $0 < \delta < \delta_0$ ), it holds that*

$$I_2 = \int_{(S_{z, \delta_0} | S_{z, \delta}) \cap \partial V} \frac{\sigma(d\zeta)}{|F(\zeta, z)|^k} \leq \gamma \log \frac{\gamma}{\delta}.$$

where  $\gamma$  is independent of  $z$  and  $\delta$ .

*Proof.* There exist positive constants  $\delta_0$  and  $\gamma_1$  such that

$$-\operatorname{Re} F(\zeta, z) \geq \rho(\zeta) - \rho(z) + \gamma_1 |\zeta - z|^2 \quad \text{for } z \in (\partial V)_{\delta_0}, \zeta \in S_{z, \delta_0}.$$

We may assume that  $z \in (\partial V)_{\delta_0} \cap \overline{D}$ ,  $d\rho \neq 0$  on  $(\partial V)_{\delta_0}$ , and that

$$\tilde{V} \cap S_{z, \delta_0} = \{w \in S_{z, \delta_0} : w_{k+1} = \dots = w_n = 0\}.$$

We can find a new local coordinate system  $t = (t_1, \dots, t_{2n})$  by letting

$$\begin{aligned} t_1 + it_2 &= \rho(\zeta) - \rho(z) + i \operatorname{Im} F(\zeta, z), \\ t_{2j-1} + it_{2j} &= \zeta_j - z_j \quad \text{for } j = k+1, \dots, n, \end{aligned}$$

and choosing  $t_3, \dots, t_{2k}$  suitably such that  $t(z) = 0$ . Then there exist positive constants  $\gamma_2, \gamma_3$  and  $\gamma_4$  such that

$$\begin{aligned} \gamma_2 |\zeta - z|^2 &\leq |t|^2 \leq \gamma_3 |\zeta - z|^2, \\ |F(\zeta, z)| &\geq \gamma_4 [(t_1 + |t|^2)^2 + t_2^2]^{1/2}. \end{aligned}$$

Define  $\varepsilon^2 = |\rho(z)| + |z_{k+1}|^2 + \dots + |z_n|^2$ . Taking account of the relation

$$\begin{aligned} (S_{z, \delta_0} | S_{z, \delta}) \cap \partial V &\subset \{t : t_1 = -\rho(z), \gamma_2 \delta^2 \leq |t|^2 \leq \gamma_3^2 \delta_0^2, \\ &\quad t_{2j-1} + it_{2j} = -z_j \quad (j = k+1, \dots, n)\}, \end{aligned}$$

we have, together with Lemma 4, for some  $\gamma_5 > 0$ ,

$$I_2 \leq \gamma_5 \int_{\{\gamma_2 \delta^2 \leq |t|^2 \leq \gamma_3^2 \delta_0^2\}} \frac{dt_2 \cdots dt_{2k}}{[(t_2^2 + \dots + t_{2k}^2 + \varepsilon^2)^2 + t_2^2]^{k/2}} \leq \gamma \log \frac{\gamma}{\delta},$$

which completes the proof of Lemma 5.

We set

$$K_V(\zeta, z) = N_V(\zeta, z) \sigma(d\zeta).$$

Then we have the following.

**LEMMA 6.** *Let  $0 < \alpha \leq 1$ . Then there exists a positive constant  $\delta_0$  such that for any  $z \in \overline{D}$  and any  $\delta$  ( $0 < \delta < \delta_0$ ),*

$$I_3 = \int_{S_{z,\delta} \cap \partial V} \frac{|\zeta - z|^\alpha |N_V(\zeta, z)|}{|g(\zeta, z)|^k} \sigma(d\zeta) \leq \gamma \delta^\alpha \log \frac{\gamma}{\delta},$$

where  $\gamma$  is independent of  $\delta$  and  $z$ .

*Proof.* There exists a positive constant  $\gamma_1$  such that

$$I_3 \leq \gamma_1 \int_{S_{z,\delta} \cap \partial V} \frac{|\zeta - z|^\alpha \sigma(d\zeta)}{|F(\zeta, z)|^k}.$$

Thus, by Lemma 5, we have

$$\begin{aligned} I_3 &\leq \gamma_1 \sum_{i=0}^{\infty} \int_{(S_{z,\delta 2^{-i}} | S_{z,\delta 2^{-(i+1)}}) \cap \partial V} \frac{|\zeta - z|^\alpha \sigma(d\zeta)}{|F(\zeta, z)|^k} \\ &= \sum_{i=0}^{\infty} \left( \frac{i+1}{2^{i\alpha}} \log 2 \right) \gamma_1 \delta^\alpha + \gamma_1 \delta^\alpha \log \left( \frac{\gamma_2}{\delta} \right) \left( \sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \right) \\ &\leq \gamma \delta^\alpha \log \frac{\gamma}{\delta}, \end{aligned}$$

which completes the proof of Lemma 6.

Let  $F$  be a closed subset of  $C^n$ . According to the definition of Stein [13], we define the Lipschitz space for  $0 < \alpha \leq 1$  such that

$$\text{Lip}(\alpha, F) = \{f: |f(x)| \leq M, |f(x) - f(y)| \leq M|x - y|^\alpha, x, y \in F\}.$$

From the extension theorem (Stein [13], Theorem 3, p. 174),  $f \in \text{Lip}(\alpha, F)$  can be regarded as an element of  $\text{Lip}(\alpha, C^n)$ . We shall prove the following theorem which was proved by Martinelli in the case when the kernel is Bochner-Martinelli kernel (cf. Martinelli [11], Dolbeault [4]).

**THEOREM 3.** *Let  $f \in \text{Lip}(\alpha, \partial V)$ . Then it holds that for any  $z \in \partial V$ ,*

$$\begin{aligned} &\lim_{t \rightarrow z, t \in \overline{D} \cap \partial V} \int_{\zeta \in \partial V} (f(\zeta) - f(z)) \Omega(\zeta, t) \\ &= \int_{\zeta \in \partial V} (f(\zeta) - f(z)) \Omega(\zeta, z). \end{aligned}$$

*Proof.* Since  $f \in \text{Lip}(\alpha, \partial V)$ , the integral of the right-hand side converges. In view of Lemma 6, for  $\varepsilon > 0$ , there exists  $\delta > 0$  such

that for any  $w \in \overline{D}$ , we have

$$\int_{S_{w, 3\delta} \cap \partial V} \frac{|f(\zeta) - f(w)| |N_V(\zeta, w)|}{|g(\zeta, z)|^k} \sigma(d\zeta) < \varepsilon.$$

We set

$$T(\zeta, w) = \frac{(f(\zeta) - f(w))N_V(\zeta, w)}{g(\zeta, w)^k},$$

$$A_\delta = \{(\zeta, w) \in \partial V \times \overline{D} : |w - \zeta| \geq \delta\}.$$

Since  $T(\zeta, w)$  is continuous on  $A_\delta$ , there exists  $\rho$  ( $0 < \rho < \delta$ ) such that

$$|T(\zeta, z) - T(\zeta, w)| < \varepsilon, \quad |f(z) - f(w)| < \varepsilon$$

$$\text{for } |w - z| < \rho, \quad |\zeta - z| \geq 2\rho, \quad \zeta \in \partial V.$$

By Proposition 1, there exists a constant  $\gamma_1$  such that

$$|H(t)| \leq \gamma_1 \quad \text{for } t \in \overline{D} \cap \partial V.$$

Thus we have for  $|t - z| < \rho$ ,  $t \in \overline{D} \cap \partial V$ ,

$$\begin{aligned} & \left| \int_{\zeta \in \partial V} (f(\zeta) - f(z)) \Omega(\zeta, t) - \int_{\zeta \in \partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) \right| \\ &= \left| \int_{\partial V} T(\zeta, t) \sigma(d\zeta) + (f(t) - f(z)) H(t) - \int_{\partial V} T(\zeta, z) \sigma(d\zeta) \right| \\ &\leq \gamma_1 |f(t) - f(z)| + \int_{\partial V \setminus S_{z, 2\delta}} |T(\zeta, t) - T(\zeta, z)| \sigma(d\zeta) \\ &\quad + \int_{S_{z, 2\delta} \cap \partial V} |T(\zeta, z)| \sigma(d\zeta) + \int_{S_{t, 3\delta} \cap \partial V} |T(\zeta, t)| \sigma(d\zeta) \\ &\leq \gamma_1 \varepsilon + \varepsilon \int_{\partial V \setminus S_{z, 2\delta}} \sigma(d\zeta) + \varepsilon + \varepsilon \leq \gamma_2 \varepsilon. \end{aligned}$$

This completes the proof of Theorem 3.

Now we shall prove the following theorem which shows that any Lipschitz function on  $\partial V$  is the continuous boundary value of a function of  $A(D)$  by adding  $\int_{\partial V} (f(\zeta) - f(\cdot)) K_V(\zeta, \cdot)$ .

**THEOREM 4.** *Let  $f \in \text{Lip}(\alpha, \partial V)$ ,  $0 < \alpha \leq 1$ . Define*

$$\tilde{f}(z) = f(z) + \int_{\zeta \in \partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) \quad \text{for } z \in \partial V,$$

and

$$F(z) = \int_{\zeta \in \partial V} f(\zeta) \Omega(\zeta, z) \quad \text{for } z \in \overline{D} \setminus \partial V.$$

Then the holomorphic function  $F(z)$  can be extended continuously to  $\overline{D}$  and it has the boundary value  $F|_{\partial V} = \tilde{f}$ .

*Proof.* For  $z \in \partial V$ , we have from Theorem 3,

$$\begin{aligned} \lim_{t \rightarrow z, t \in \overline{D} \setminus \partial V} (F(t) - \tilde{f}(z)) \\ = \lim_{t \rightarrow z, t \in \overline{D} \setminus \partial V} \left( \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, t) \right. \\ \left. - \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) \right) = 0. \end{aligned}$$

Since  $F(z)$  is holomorphic in  $\overline{D} \setminus \partial V$ ,  $F \in A(D)$ . This completes the proof of Theorem 4.

The boundary value  $\tilde{f}$  is also represented by

$$\tilde{f}(z) = \frac{1}{2}f(z) + \text{P.V.} \int_{\partial V} f(\zeta) \Omega(\zeta, z)$$

in view of the following.

**LEMMA 7.** *Let  $f \in \text{Lip}(\alpha, \partial V)$  and  $z \in \partial V$ . Then we have*

$$\text{P.V.} \int_{\partial V} f(\zeta) \Omega(\zeta, z) = \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) + \frac{1}{2}f(z).$$

*Proof.* We set, for  $\varepsilon > 0$ ,  $M(\varepsilon) = \partial V \cap \{\zeta : |g(\zeta, z)| > \delta\}$ . In view of Lemma 6 and Theorem 2, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{M(\varepsilon)} f(\zeta) \Omega(\zeta, z) &= \lim_{\varepsilon \rightarrow 0} \int_{M(\varepsilon)} (f(\zeta) - f(z)) \Omega(\zeta, z) \\ &\quad + f(z) \lim_{\varepsilon \rightarrow 0} \int_{M(\varepsilon)} \Omega(\zeta, z) \\ &= \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) + \frac{1}{2}f(z). \end{aligned}$$

This completes the proof of Lemma 7.

Now we are going to prove the following which is a main theorem in this paper.

**THEOREM 5.** *Let  $f \in \text{Lip}(\alpha, \partial V)$ ,  $0 < \alpha \leq 1$ . If  $f$  satisfies for any  $z \in \partial V$ ,*

$$\text{P.V.} \int_{\partial V} f(\zeta) \Omega(\zeta, z) = \frac{1}{2} f(z),$$

*then there exists a function  $F \in A(D)$  such that  $F|_{\partial V} = f$ .*

*Proof.* We set

$$F(z) = \int_{\partial V} f(\zeta) \Omega(\zeta, z) \quad \text{for } z \in \overline{D} \setminus \partial V.$$

Then  $F$  can be extended continuously to  $\overline{D}$  and satisfies, for  $z \in \partial V$ ,

$$\begin{aligned} F(z) &= f(z) + \int_{\partial V} (f(\zeta) - f(z)) \Omega(\zeta, z) \\ &= \frac{1}{2} f(z) + \text{P.V.} \int_{\partial V} f(\zeta) \Omega(\zeta, z) = f(z), \end{aligned}$$

which completes the proof of Theorem 5.

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(Managing Editor)  
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vsv@math.ucla.edu

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University of California  
Los Angeles, CA 90024-1555  
christ@math.ucla.edu

HERBERT CLEMENS  
University of Utah  
Salt Lake City, UT 84112  
clemens@math.utah.edu

THOMAS ENRIGHT  
University of California, San Diego  
La Jolla, CA 92093  
tenright@ucsd.edu

NICHOLAS ERCOLANI  
University of Arizona  
Tucson, AZ 85721  
ercolani@math.arizona.edu

R. FINN  
Stanford University  
Stanford, CA 94305  
finn@gauss.stanford.edu

VAUGHAN F. R. JONES  
University of California  
Berkeley, CA 94720  
vfr@math.berkeley.edu

STEVEN KERCKHOFF  
Stanford University  
Stanford, CA 94305  
spk@gauss.stanford.edu

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