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## CURRENTS, METRICS AND MOISHEZON MANIFOLDS

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**A compact complex manifold  $M$  is Moishezon if and only if there exists an integral closed positive  $(1, 1)$ -current  $\omega$  such that  $\omega \geq \varepsilon\sigma$  and  $\omega$  is smooth outside an analytic subvariety.**

**1. Introduction.** Given a Moishezon manifold  $M$ , it is well known (cf. [Mo], [W]) that there is a bimeromorphic morphism  $\pi: \widetilde{M} \rightarrow M$  such that the manifold  $M$  is projective algebraic. Let  $\tilde{\omega}$  be Kähler form on  $\widetilde{M}$  with  $[\tilde{\omega}] \in H^2(\widetilde{M}, \mathbf{Z})$ . Then the pushforward current  $\omega = \pi_*\tilde{\omega}$  is a  $d$ -closed current on  $M$  such that

- (i)  $[\omega] \in H^2(M, \mathbf{Z})$ ;
- (ii)  $\omega$  is smooth on  $M - S$ , where  $S$  is some proper analytic subset in  $M$ ;
- (iii)  $\omega \geq \varepsilon\sigma$  in the sense of currents, where  $\varepsilon > 0$  is some real number and  $\sigma$  is a fixed positive definite  $(1, 1)$ -form (not necessarily  $d$ -closed) on  $M$ .

Conversely, we shall prove the following

**THEOREM 1.1.** *Let  $M$  be a compact complex manifold of dimension  $n$ . Then  $M$  is Moishezon if and only if there exists a  $d$ -closed  $(1, 1)$ -current  $\omega$  on  $M$  such that the conditions (i), (ii) and (iii) above are satisfied.*

In fact, the above theorem is a weak version of a general conjecture of Shiffman [J] which asked: whether a compact complex manifold  $N$  is Moishezon if and only if there exists a  $d$ -closed  $(1, 1)$ -current satisfying the conditions (i) and (iii) above. The conjecture is to generalize the well-known Kodaira embedding theorem in terms of currents and it is still unknown. Some partial results have been obtained [J]: if  $M$  is complex torus, Shiffman's conjecture is true; if  $S$  is a set of isolated points, Theorem 1.1 follows from an extension theorem of Miyaoka [M]; if  $S$  is special in some sense, Theorem 1.1 is also true. All of these results are proved by smoothing of currents technique, and depends on a fact that the top degree Chern number  $(c_1([\omega])^n, M) > 0$ . However, it is easy to find an example of a current  $\omega$  satisfying (i), (ii)

and (iii) but its top degree Chern number is negative. So the method of smoothing currents cannot prove Theorem 1.1.

Recently, Demailly introduced a very useful notion of singular Hermitian metric on a holomorphic line bundle [D2] and he proved many interesting results. One of them [D2, Proposition 4.2 (b)] is that if  $M$  is a projective algebraic manifold of  $n$ -dimension with a Kähler form  $\sigma$  and if  $L$  is a holomorphic line bundle over  $M$ , then  $L$  admits a singular Hermitian metric with  $c(L) \geq \varepsilon\sigma$  if and only if the Kodaira dimension  $\kappa(L) = n$ . We observed that this result is in fact the special case of Shiffman's conjecture in which  $M$  is projective algebraic (Lemma 2.1). Thus we want to modify Demailly's idea to prove Theorem 1.1. The Demailly's proof is based on the standard  $L^2$ -estimate of  $\bar{\partial}$  over Stein or projective algebraic manifolds. However, in our problem,  $M$  is only a compact complex manifold. By observing that  $M - S$  is complete Kähler (Lemma 4.1), instead of using the standard  $L^2$ -estimate of  $\bar{\partial}$ , we then prove Theorem 1.1 by using a deep generalization of the  $L^2$  estimate theorem by Demailly [D1] on complete Kähler manifolds with non-complete Kähler metric and with singular metric on the line bundle. By a similar method, a special case of Shiffman's conjecture when  $M$  is Kähler is also proved.

**THEOREM 1.2.** *Let  $M$  be a compact Kähler manifold of dimension  $n$ . Then  $M$  is projective algebraic if and only if there exists a  $d$ -closed  $(1, 1)$ -current  $\omega$  on  $M$  such that the conditions (i) and (iii) are satisfied.*

We also study a class  $\mathcal{H}$  of compact complex manifolds as suggested by Harvey and Lawson [HL, §5, problem 2]. We prove the following result.

**THEOREM 1.3.** *Let  $X \in \mathcal{H}$ . Then  $X$  is a Moishezon manifold iff it is projective algebraic. In particular, this holds for any analytic compact smooth family  $X$  of curves with Kähler base space.*

Finally we point out an interesting fact below. Its proof is easy from [K], [NS], [N].

**THEOREM 1.4.** *Let  $M$  be any compact complex manifold. Then the statements are equivalent:*

- (i)  $M$  is a Moishezon manifold;
- (ii) there is a proper analytic subset  $S \subset M$  such that  $M - S$  admits a complete Kähler-Einstein metric with negative Ricci curvature;

(iii) *There is a proper analytic subset  $S \subset M$  such that  $M - S$  admits a complete Kähler-Einstein metric with negative Ricci curvature and with finite volume;*

(iv) *There is a proper analytic subset  $S \subset M$  such that  $tM - S$  admits a complete Kähler metric  $g$  with  $\text{Ricci}(g) \leq -g$ .*

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**2. Singular metric on line bundles.** Let  $L$  be a holomorphic line bundle over a complex manifold  $M$ . A *singular Hermitian metric*  $h$  on  $L$  [D2] is a metric which is given in any local trivialization  $\theta: L|_U \rightarrow U \times \mathbb{C}$  by

$$\|\xi\| = |\theta(\xi)|e^{-h_U(x)}, \quad x \in U, \xi \in L_x,$$

where  $h_U \in L^1_{\text{loc}}(U)$  is an arbitrary function, called the *weight* of the metric with respect to the trivialization  $\theta$ . If  $\theta': L|_{U'} \rightarrow U' \times \mathbb{C}$  is another trivialization with the associated weight  $h'_{U'}$ , and if  $\rho \in \mathcal{O}^*(U \cap U')$  is the transition function, then  $\theta'(\xi) = \rho(x)\theta(\xi)$  for  $\xi \in L_x$ , and  $h'_{U'} = h_U + \log|\rho|$  on  $U \cap U'$ . The curvature form of  $L$  is then given by the  $d$ -closed  $(1, 1)$ -current  $c(L, h) = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} h_U$  on  $U$ , which is independent of the choice of local trivialization. The de Rham cohomology class of  $c(L, h)$  is the image of the first Chern class  $c_1(L) \in H^2(X, \mathbb{Z})$  in  $H^2_{DR}(X, \mathbb{R})$ .

In order to relate any integral  $d$ -closed positive  $(1, 1)$ -current to singular Hermitian metric, we need to have the following lemma in the type of Lefschetz'  $(1, 1)$ -theorem.

**LEMMA 2.1.** *Let  $M$  be a complex manifold of dimension  $n$  and let  $\omega$  be a  $d$ -closed positive  $(1, 1)$ -current on  $M$ . If the de Rham class  $[\omega] \in H^2(M, \mathbb{R})$  is integral, then there exists a holomorphic line bundle  $L$  with a singular Hermitian metric  $h$  such that*

$$\omega = c(L, h).$$

*Proof.* Choose an open cover  $\mathcal{U} = \{U_\alpha\}$  of  $M$  such that  $U_\alpha$  are geometrically convex and then all finite intersections of the sets in  $\mathcal{U}$  are contractible. Also assume that each  $U_\alpha$  is chosen small enough so that there exists a plurisubharmonic function  $h_{U_\alpha}$  on  $U_\alpha$  satisfying

$$\omega = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} h_{U_\alpha} = \frac{1}{2\pi} dd^c h_{U_\alpha} \quad \text{on } U_\alpha$$

where  $d^c = \sqrt{-1}(\bar{\partial} - \partial)$ ; hence  $dd^c = 2\sqrt{-1}\partial\bar{\partial}$ .

It is sufficient to find transition functions  $\{\rho_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$  defining a holomorphic line bundle such that

$$(2.2) \quad h_{U_\beta} = h_{U_\alpha} + \log |\rho_{\alpha\beta}|.$$

In this case,  $h_{U_\alpha}$  is the weight of the singular metric.

Put  $u_j = \frac{1}{2\pi} h_{U_\alpha}$  and  $u_{\alpha\beta} = u_\beta - u_\alpha$ . Because  $u_\beta - u_\alpha$  is pluriharmonic on  $U_\alpha \cap U_\beta$ , it implies that  $u_\beta - u_j$  must be smooth; thus  $u_{\alpha\beta} \in \mathcal{C}_\mathbf{R}^\infty(U_\alpha \cap U_\beta)$ .

By exactly the same argument as in [SS, Lemma 2.36, p. 38], we can construct a family of transition functions  $\{\rho_{\alpha\beta}\}$  satisfying (2.2). Here we only sketch the proof: since  $dd^c u_{\alpha\beta} = 0$ , we can choose  $v_{\alpha\beta} = C_\mathbf{R}^\infty(U_\alpha \cap U_\beta)$  such that  $dv_{\alpha\beta} = d^c u_{\alpha\beta}$ . Then  $c_{\alpha\beta\gamma} = v_{\beta\gamma} - v_{\alpha\gamma} + v_{\alpha\beta}$  defines an element  $\{c_{\alpha\beta\gamma}\} \in Z^2(\mathcal{U}, \mathbf{R})$ . By Leray isomorphism,  $\{c_{\alpha\beta\gamma}\}$  corresponds to the cohomology class  $[\omega]$ . Since  $\{c_{\alpha\beta\gamma}\}$  is integral, there is a 1-cochain  $\{b_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathbf{R})$  such that

$$c_{\alpha\beta\gamma} + b_{\beta\gamma} - b_{\alpha\gamma} + b_{\alpha\beta} = m_{\alpha\beta\gamma} \in \mathbf{Z}.$$

Let  $f_{\alpha\beta} = u_{\alpha\beta} + \sqrt{-1}(u_{\alpha\beta} + b_{\alpha\beta})$ , which is a holomorphic function such that

$$f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} = \sqrt{-1}m_{\alpha\beta\gamma}.$$

Let  $\rho_{\alpha\beta} = \exp(2\pi f_{\alpha\beta})$ . Such  $\{\rho_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$  satisfies (2.2). Thus the lemma is proved.  $\square$

From the proof above, by the standard regularity theorem for elliptic operators, it is easy to obtain the following

**COROLLARY 2.3.** *Let  $M$ ,  $\omega$  be as in Lemma 2.1 and let  $\omega$  be smooth on  $M - S$  for some proper analytic subset  $S \subset M$ . Then there exists a holomorphic line bundle  $L$  over  $M$  with a singular metric  $h$  such that*

$$\omega = c(L, h)$$

*and  $h|_{M-S}$  is smooth, i.e., for any point in  $x \in M$ , there is a neighborhood  $U$  of  $x$  in  $M$  such that the weight  $h_U$  of the singular metric  $h$  is smooth on  $U - S$ .*

Let  $L$  be a holomorphic line bundle over  $M$  admitting a singular metric  $h$  such that the curvature current  $c(L, h) \geq 0$ . For any  $x \in M$ , let  $h_U$  be the weight of the metric on a neighborhood  $U$  of  $x$ ,

we define the *Lelong number* with respect to the singular metric by (cf. [D2])

$$(2.4) \quad v(h_U, x) = \liminf_{z \rightarrow x} \frac{h_U(z)}{\log|z-x|}.$$

Equivalently,

$$v(h_U, x) = \lim_{r \rightarrow 0} v(c(L, h), x, r)$$

where

$$v(c(L, h), x, r) = \frac{1}{(2\pi r^2)^{n-1}} \int_{B(x, r)} c(L, h) \wedge (\sqrt{-1} \partial \bar{\partial} |z|^2)^{n-1}.$$

So we can denote  $v(h_U, x)$  to be  $v(c(L, h), x)$ . We define a set

$$E_c(c(L, h)) = \{x \in M; v(c(L, h), x) \geq c\}$$

which is an analytic subset by a well-known theorem of Siu [Si].

**LEMMA 2.5** (cf. [D2, Lemma 2.8]). *If  $\phi$  is a plurisubharmonic function on  $M$ , then  $e^{-2\phi}$  is integrable in a neighborhood of  $x \in M$  if  $v(\phi, x) < 1$ , and  $e^{-2\phi}$  is non-integrable on any neighborhood of  $x$  if  $v(\phi, x) \geq 1$ .*

**3.  $L^2$  estimate for  $\bar{\partial}$  over complete Kähler manifolds.** In this section we review some results of Demailly [D1] and state a general  $L^2$  estimate for  $\bar{\partial}$  for line bundles with singular metric.

Let  $M$  be a complex manifold of dimension  $n$  with a Kähler metric  $\omega$ . We shall use the same notation  $\omega$  to denote the associated Kähler form. Denote  $dV_\omega = \omega^n/n!$  to be the volume form of  $(X, \omega)$ . The form  $\omega$  defines an operator on  $\bigwedge^{p, q} T^*M$  by

$$\omega(\alpha) = \omega \wedge \alpha \in \bigwedge^{p+1, q+1} T^*M$$

and its adjoint operator  $\Lambda$  is defined by

$$\langle \Lambda \alpha, \beta \rangle = \langle \alpha, \omega(\beta) \rangle$$

for all  $\alpha \in \bigwedge^{p, q} T^*M$ ,  $\beta \in \bigwedge^{p-1, q-1} T^*M$ . Here  $\langle \cdot, \cdot \rangle$  is the inner product given by  $\omega$ .

Let  $L$  be a holomorphic line bundle over  $X$ . Then these operators  $\omega$  and  $\Lambda$  can be extended to the space of  $L$ -valued  $(p, q)$ -forms,  $\bigwedge^{p, q} T^*M \otimes L$ , by the identity map  $\text{id}_L$ . In addition we suppose that  $(L, h)$  is a line bundle over  $M$  with a positive  $C^2$  Hermitian

metric  $h$ , i.e., its first Chern class  $c_1(L, h) > 0$ . For each integer  $q$ ,  $1 \leq q \leq n$ , we define a bilinear form  $c(L, h)_q$

$$c(L, h)_q(\alpha, \beta) = \langle 2\pi c_1(L, h)\Lambda\alpha, \beta \rangle$$

for all  $\alpha, \beta \in \wedge^{n,q} T^*M \otimes L$ . Since  $c_1(L, h) > 0$ , it is known that  $c(L, h)_q$  is positive, for all  $q$  [D1, Lemma 3.1]. For any forms  $\alpha \in \wedge^{n,q} T^*M \otimes L$ , one defines

$$|\alpha|_{c(L, h)_q} = \sup_{\beta} \left\{ \frac{|\langle \alpha, \beta \rangle|}{c(L, h)_q(\beta, \beta)} \right\}$$

where  $0 \neq \beta$  runs through  $\wedge^{n,q} T^*M \otimes L$ . Notice that the number  $|\alpha|_{c(L, h)_q}$  may be equal to infinity. In practice, in order to estimate the term  $|\alpha|_{c(L, h)_q}$ , we have the following result. If  $c(L, h) \geq \lambda\omega \otimes \text{Id}_L$ , where  $\lambda \geq 0$  is a measurable function on  $M$ , then for  $\alpha \in \wedge^{n,q} T^*M \otimes L$ , one has [D1, Lemma 3.2]

$$(3.1) \quad |\alpha|_{c(L, h)_q}^2 \leq \frac{1}{q\lambda} |\alpha|^2.$$

**LEMMA 3.2 [D1, Theorem 4.1].** *Let  $M$  be a complete Kähler manifold of dimension  $n$ . Let  $\omega$  be a Kähler metric which is not necessarily complete. Let  $(L, h)$  be a holomorphic Hermitian line bundle over  $M$  with a  $C^2$  positive Hermitian metric  $h$ . Then for any smooth  $L$ -valued  $(n, q)$ -form  $g$  on  $M$  with*

$$\bar{\partial}g = 0, \quad \int_M |g|^2 dV_\omega < \infty \quad \text{and} \quad \int_X |g|_{c(L, h)_q}^2 dV_\omega < \infty,$$

*there exists a smooth  $L$ -valued  $(n, q - 1)$ -form  $f$  on  $M$  such that*

$$\bar{\partial}f = g, \quad \text{and} \quad \int_M |f|^2 dV_\omega \leq \int_X |g|_{c(L, h)_q}^2 dV_\omega.$$

Notice that  $M$  is complete Kähler, i.e.,  $M$  admits a complete Kähler metric  $g$ , but  $\omega$  may not be equal to  $g$ . The norm  $||$  is defined with respect to  $\omega$  and  $h$ .

Let  $M$  be a complete Kähler manifold of dimension  $n$ . Again let  $\omega$  be a Kähler metric which is not necessarily complete. Let  $L$  be a holomorphic line bundle over  $M$  with a  $C^2$  Hermitian metric  $h$ . Let  $\phi$  be a function on  $M$  such that for any point  $x \in M$ , there is a neighborhood  $U$  of  $x$  in  $M$  such that the restriction of  $\phi$  on  $U$

$$(3.3) \quad \phi|_U = \phi_1 + \phi_2$$

where  $\phi_1$  is a  $C^2$  function on  $U$  and  $\phi_2$  is a plurisubharmonic function on  $U$ . The Lebesgue decomposition of the 0-order current  $\sqrt{-1}\partial\bar{\partial}\phi$  gives

$$\sqrt{-1}\partial\bar{\partial}\phi = \sqrt{-1}(\partial\bar{\partial}\phi)_c + \sqrt{-1}(\partial\bar{\partial}\phi)_s$$

where the singular part  $\sqrt{-1}(\partial\bar{\partial}\phi)_s$  is a positive  $(1, 1)$ -current, and the absolute continuous part  $\sqrt{-1}(\partial\bar{\partial}\phi)_c$  is a semipositive  $(1, 1)$ -form with  $L^1_{\text{loc}}$  coefficients.

We define

$$c(L, e^{-\phi}h) = c(L, h) + \frac{\sqrt{-1}}{\pi}(\partial\bar{\partial}\phi)_c.$$

**LEMMA 3.4 [D1, Theorem 5.1].** *Let  $M$  be a complete Kähler manifold of dimension  $n$ . Let  $\omega$  be a Kähler metric which is not necessarily complete. Let  $L$  be a holomorphic line bundle over  $M$  with a  $C^2$  Hermitian metric  $h$ . Let  $\phi$  be a function which is locally the sum of a  $C^2$  function and a plurisubharmonic function as in (3.3). Suppose  $c(L, e^{-\phi}h) \geq 0$ . Then for any smooth  $L$ -valued  $(n, q)$ -form  $g$  on  $M$  with*

$$\bar{\partial}g = 0 \quad \text{and} \quad \int_M |g|_{c(L, e^{-\phi}h)_q}^2 e^{-2\phi} dV_\omega < \infty,$$

*there exists a smooth  $L$ -valued  $(n, q-1)$ -form  $f$  on  $M$  such that*

$$\bar{\partial}f = g \quad \text{and} \quad \int_M |f|^2 e^{-2\phi} dV_\omega \leq \int_M |g|_{c(L, e^{-\phi}h)_q}^2 e^{-2\phi} dV_\omega.$$

*where  $||$  is defined with respect to  $h$  and  $\omega$ .*

The above lemma leads to a general  $L^2$ -estimate for  $\bar{\partial}$  for any holomorphic line bundle with singular metric as follows.

Let  $M$ ,  $\omega$  be as in Lemma 3.4 above. Let  $L$  be a holomorphic line bundle over  $M$  with a singular Hermitian metric  $h$ . Suppose

$$c(L, h) \geq 0$$

in the sense of currents. Now take and fix any smooth Hermitian metric  $h_0$  and  $L$ ; then on each open subset  $U$  such that  $L|_U$  is trivial, the weight  $h_{0,U}$  of  $h_0$  is a smooth function on  $U$ . Define on each such  $U$  a function

$$(3.5) \quad \varphi_U = h_U - h_{0,U}.$$

It is easy to see that we have in fact defined a function  $\varphi$  on  $M$  globally such that

$$\varphi|_U = \varphi_U.$$



Since  $c(L, h) \geq 0$ , by the proof of Lemma 2.1, we see that any weight  $h_U$  of the metric  $h$  is plurisubharmonic, then the function  $\varphi$  is obviously locally a sum of a  $C^2$ -function and a plurisubharmonic function. Then from Lemma 3.4 we obtain

**COROLLARY 3.6.** *Let  $M, \omega$  be as in Lemma 3.4. Let  $L$  be a holomorphic line bundle over  $M$  with a singular Hermitian metric  $h$  such that  $c(L, h) \geq 0$ . Suppose that  $h_0$  is any smooth Hermitian metric on  $L$  and denote  $\varphi$  to be the function on  $M$  defined by (3.5). Then for any smooth  $L$ -valued  $(n, q)$ -form  $g$  on  $M$  with*

$$\bar{\partial}g = 0 \quad \text{and} \quad \int_M |g|_{c(L, e^{-\varphi}h_0)_q}^2 e^{-2\varphi} dV_\omega < \infty,$$

there exists a smooth  $L$ -valued  $(n, q - 1)$ -form  $f$  on  $M$  such that

$$\bar{\partial}f = g \quad \text{and} \quad \int_M |f|^2 dV_\omega \leq \int_M |g|_{c(L, e^{-\varphi}h_0)_q}^2 e^{-2\varphi} dV_\omega,$$

where  $||$  is defined by  $h$  and  $\omega$ .

*Proof.* Apply Lemma 3.4 to  $(L, h_0)$  and  $\varphi$ , we know that for any  $g$  with  $\bar{\partial}g = 0$  and  $\int_M |g|_{c(L, e^{-\varphi}h_0)_q}^2 e^{-2\varphi} dV_\omega < \infty$ , there exists  $f$  such that

$$\bar{\partial}f = g \quad \text{and} \quad \int_M |f|_{h_0, \omega}^2 e^{-2\varphi} dV_\omega \leq \int_M |g|_{c(L, e^{-\varphi}h_0)_q}^2 e^{-2\varphi} dV_\omega.$$

Notice that  $|f|_{h_0, \omega}^2 e^{-2\varphi} = |f|_{h, \omega}^2$  by (3.5), and the corollary follows. □

**REMARK 3.7.** Suppose that the line bundle  $L$  has a singular metric  $h$  such that  $c(L, h) \geq \varepsilon\omega$ , for some constant  $\varepsilon > 0$ , i.e.,

$$c(L, h)(v, v) \geq \varepsilon\omega(v, v),$$

for any test form  $v$ . Since

$$\begin{aligned} c(L, e^{-\phi}h_0)(v, v) &= \left[ c(L, h_0) + \left( \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \right)_c \right] (v, v) \\ &= \left[ c(L, h_0) + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\phi \right] (v, v) \\ &= c(L, h)(v, v), \end{aligned}$$

we see that

$$c(L, e^{-\phi}h_0)(v, v) \geq \varepsilon\omega(v, v).$$

Therefore by (3.1) we have the estimate

$$|\alpha|_{c(L, e^{-\phi} h_0)_q}^2 \leq \text{const} \cdot |\alpha|^2$$

for any  $\alpha \in \wedge^{n,q} T^*M \otimes L$ .

**4. Complete Kähler metric on  $M - S$ .** In order to prove Theorem 1.1, we wish to apply Lemma 3.4 to  $X - S$ . In general,  $X - S$  may not be complete Kähler, but together with the  $d$ -closed  $(1, 1)$ -current  $\omega$  satisfying (ii) and (iii), we can construct a complete Kähler metric on  $M - S$ . The proof is analogous to [D1, Proposition 1.6].

**LEMMA 4.1.** *Let  $M$  be a compact complex manifold and  $S \subset M$  a proper analytic subset. Let  $\omega$  be a  $d$ -closed  $(1, 1)$ -current satisfying the conditions (ii) and (iii) in §1. Then  $M - S$  admits a complete Kähler metric.*

*Proof.* By [D1, Proposition 1.4], for any complex manifold  $M$  and any analytic subset  $S \subset M$ , there exists a locally integrable function  $\psi$  on  $M$  such that  $\psi$  is smooth on  $M - S$ ;  $\psi(x) < -1$ , for any  $x \in M - S$ ; and  $\psi(x) \rightarrow -\infty$  as  $x$  goes to  $S$ , and there exists a real continuous  $(1, 1)$ -form  $\gamma$  on  $X$  such that

$$(4.2) \quad \sqrt{-1} \partial \bar{\partial} \psi \geq \gamma;$$

(4.3) if  $\alpha > 0$  is a real number,  $e^{-\alpha\psi}$  is non-integrable on a neighborhood of a point  $s \in S$  where the codimension of the germ  $S_s$  satisfies

$$\text{codim } S_s \geq \alpha.$$

Put  $\tilde{\omega} = C\omega - \sqrt{-1} \partial \bar{\partial} \sqrt{-\psi}$ , which is a smooth form on  $M - S$  and is a current on  $M$ . We claim that we can choose the constant  $C > 0$  large enough such that

$$(4.4) \quad (C - 1)\omega + \frac{\sqrt{-1} \partial \bar{\partial} \psi}{2\sqrt{-\psi}} \geq 0 \quad \text{on } M - S.$$

In fact, since  $\gamma$  is continuous on  $M$  and  $\omega \geq \varepsilon\sigma$  on  $M$  (cf. (iii) in §1), and since  $X$  is compact, we can find a constant number  $C > 0$  such that

$$(C - 1)\varepsilon\sigma \geq -\frac{\gamma}{2\sqrt{-\psi}} \quad \text{on } M.$$

Then

$$(C - 1)\omega + \frac{\sqrt{-1} \partial \bar{\partial} \psi}{2\sqrt{-\psi}} \geq (C - 1)\varepsilon\sigma + \frac{\gamma}{2\sqrt{-\psi}} \geq 0$$

on  $M$  in the sense of currents. Thus (4.4) is proved. From (4.4), it yields

$$(4.5) \quad \tilde{\omega} \geq \omega + 4\sqrt{-1}\partial\bar{\partial}(-\psi)^{1/4} \wedge \bar{\partial}(-\psi)^{1/4}$$

because

$$\sqrt{-1}\partial\bar{\partial}(-\sqrt{-\psi}) = \frac{\sqrt{-1}\partial\bar{\partial}\psi}{2\sqrt{-\psi}} + 4\sqrt{-1}\partial(-\psi)^{1/4} \wedge \bar{\partial}(-\psi)^{1/4}.$$

Then  $\tilde{\omega}$  is complete Kähler by the same argument of [D1, proof of Proposition 1.6]. For the reader's convenience, we still give the proof: let  $\delta$  (resp.  $\tilde{\delta}$ ) be the geodesic distance associated to  $\omega$  (resp.  $\tilde{\omega}$ ). For any two  $z_1, z_2 \in M$ ,

$$\delta(z_1, z_2) = \inf \int_0^1 \sqrt{\omega\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right)} dt$$

(similarly one defines  $\tilde{\delta}(z_1, z_2)$ ) where  $u$  runs through the set of all  $C^1$  curves  $u: [0, 1] \rightarrow M - S$  with the ending points  $z_1$  and  $z_2$ . By (4.5),

$$\begin{aligned} \tilde{\omega}\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right) &\geq \omega\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right) + 4\left|\partial\psi\left(\frac{du}{dt}\right)\right|^2 \\ &\geq \omega\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right) + \left|\frac{d(\psi \circ u)}{dt}\right|^2 \end{aligned}$$

because

$$\frac{d(\psi \circ u)}{dt} = d\psi\left(\frac{du}{dt}\right) = 2\operatorname{Re}\partial\psi\left(\frac{du}{dt}\right).$$

Thus

$$\tilde{\delta}(z_1, z_2) \geq \sup(\delta(z_1, z_2), |\psi(z_1) - \psi(z_2)|).$$

Since  $\psi$  is exhaustive, and since a manifold admits a complete metric  $\tilde{\omega}$  if and only if the closed balls defined by geodesic distance  $\tilde{\delta}$  are always compact, we know that  $\tilde{\omega}$  is complete Kähler on  $M - S$ .  $\square$

### 5. Proofs of Theorems 1.1 and 1.2.

**LEMMA 5.1** [D1, Lemma 6.9]. *Let  $\Omega \subset \mathbb{C}^n$  be an open subset, and let  $Y \subset \Omega$  be an analytic subset. If  $w$  is a  $(p, q)$ -form with  $L^1_{\text{loc}}$  coefficients on  $\Omega$ , and  $v$  is a  $(p, q - 1)$ -form with  $L^2_{\text{loc}}$  coefficients on  $\Omega$  such that  $\bar{\partial}v = w$  on  $\Omega - Y$  in the sense of currents, then  $\bar{\partial}v = w$  on  $\Omega$  in the sense of currents.*

*Proof of Theorem 1.1.* By Lemma 2.1 and Corollary 2.2, there exists a holomorphic line bundle  $L$  over  $M$  with a singular metric  $h$  such that  $\omega = c(L, h)$  and  $h$  is smooth on  $U - S$ .

By Lemma 4.1,  $M - S$  is a complete Kähler manifold. Since the restriction of the singular metric  $h$  on  $L$  is smooth over  $M - S$ ,  $L|_{M-S}$  has a smooth Hermitian  $h = h|_{M-S}$ . We consider  $\omega$  as Kähler metric on  $M - S$ . Notice that  $\omega$  is not necessarily complete.

Take and fix a point  $x_0 \in M - S$ . Because  $h$  is smooth on  $M - S$ , we see that the Lelong number is

$$v(c(L, h), x) = 0, \quad x = x_0 \text{ or } x \text{ near } x_0.$$

Let  $\Psi_0$  be a smooth function on  $M - \{x_0\}$  which is equal to  $n \log |z - x_0|$  (in some coordinates) near  $x_0$ .

By the hypothesis (iii),  $c(L, h) \geq \varepsilon \sigma$  on  $M$ , there is some  $m$  such that

$$(5.2) \quad mc(L, h) + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \Psi_0 \geq m\varepsilon \sigma + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \Psi_0 \geq \sigma.$$

Put  $v_m = c(L, h) + (\sqrt{-1}/\pi) \partial \bar{\partial} \Psi_0$ . The Lelong number

$$v(v_m, x_0) = n + 1 \quad \text{and} \quad v(v_m, z) < 1, \quad \text{for } z \neq x_0 \text{ near } x_0.$$

Therefore, by Lemma 2.5,  $e^{-2\Psi_0}$  is non-integrable near  $x_0$ .

Let  $P(z)$  be an arbitrary polynomial of degree 1 in the given coordinates  $V$  of  $x_0$ . Fix a smooth cut-off function  $\chi$  with compact support in  $V$  such that  $\chi = 1$  near  $x_0$ . Fix a non-vanishing local holomorphic section  $g \in H^0(V, K_M \otimes L^m)$ .

Then  $v = P \bar{\partial} \chi \otimes g$  is regarded as a smooth  $\bar{\partial}$ -closed  $L^m$ -valued  $(n, 1)$ -form on  $M$  and hence on  $M - S$  such that

$$\int_{M-S} |v|^2 e^{-2\Psi_0} dV_\omega < \infty$$

where  $\Psi_0$  is constructed as above, and  $||$  is defined by  $\omega$  and by the smooth Hermitian metric  $h$ .

Then by (5.2), we apply Lemma 3.4 and Remark 3.7 to  $M - S$ ,  $(L^m, h^m)$  and  $\Psi_0$ , and then there is a smooth  $L^m$ -valued  $(n, 0)$ -form  $u$  on  $M - S$  such that

$$\bar{\partial} u = v \quad \text{and} \quad \int_{M-S} |u|^2 e^{-2\Psi_0} dV_\omega \leq \int_{M-S} |v|^2 e^{-2\Psi_0} dV_\omega < \infty.$$

Then we claim that  $u$  can be extended as a smooth  $L^m$ -valued  $(n, 0)$ -form on  $M$ . In fact, we apply Lemma 5.1 to prove it. Since  $v$  is a smooth  $L^m$ -valued  $(n, 0)$ -form on  $M$ , it is sufficient to show that  $\forall x \in S$ , there is a neighborhood  $U = U_x$  of  $x$  in  $M$  such that

$$\int_{U-S} |u|_U^2 dV_U < \infty$$

where  $dV_U$  is the Euclidean volume form on  $U$  with respect to a coordinate system and the norm  $|\cdot|_U$  is with respect to  $dV_U$  and  $h$ . Recall  $\omega \geq \varepsilon\sigma$ , and  $\Psi_0$  is smooth near  $S$ ; it yields

$$\int_U |u|_U^2 dV_U = \int_{U-S} |u|_U^2 dV_U \leq \text{constant} \int_{U-S} |u|^2 e^{-2\Psi_0} dV_\omega < \infty.$$

The claim then is proved by the regularity theorem.

Also we claim that  $|u(z)| = o(|z - x_0|)$  near  $x_0$ . In fact, it is true by the fact that  $\int_{M-S} |u|^2 e^{-2\Psi_0} dV < \infty$ , and that  $e^{-2\Psi_0}$  is non-integrable near  $x_0$ , and that  $u$  is holomorphic near  $x_0$ .

Therefore

$$f := \chi P g - u \in H^0(M, K_M \otimes L^m)$$

has the prescribed 1-jet  $Pg$  at  $x_0$ . Thus the Kodaira dimension of  $K_M \otimes L^m = n$ . Hence  $M$  is Moishezon.  $\square$

*Proof of Theorem 1.2.* By the similar procedure as above, replacing  $M - S$  by  $M$ , we can apply Corollary 3.6 and Remark 3.7 to know that  $M$  is Moishezon if and only if there is a  $(1, 1)$ -form  $\omega$  satisfying (i) and (iii). Since  $M$  is Kähler, by Moishezon's theorem [Mo], it is equivalent to  $M$  being projective algebraic. Then Theorem 1.2 follows.  $\square$

**6. Projectivity of a class of Moishezon manifolds.** In 1983, Harvey and Lawson proved a characterization theorem for Kähler manifolds [HL], that is, a compact complex manifold is Kähler if and only if there exists no nontrivial positive current which is a bidimension  $(1, 1)$ -component of a boundary. They also raised several general problems. One of them [HL, §5, problem 2] is as follows: describe the class of compact complex manifolds which satisfy that if there exists a non-trivial positive current which is the bidimension  $(1, 1)$ -component of a boundary, then there exists a non-trivial positive smooth current which is the bidimension  $(1, 1)$ -component of a boundary. We denote this class by  $\mathcal{H}$ . The significance of this problem is that to test whether a given manifold in  $\mathcal{H}$  is Kähler; it suffices to check the pointwise non-negative, smooth  $(n-1, n-1)$ -forms, to see if one is a boundary.

It is worth remarking that investigating the obstruction of a Moishezon manifold to be projective algebraic is an interesting problem in the theory of compact complex manifolds. Classically we know that there is no obstruction for compact complex surfaces (Chow-Kodaira [CK]) and for complex tori (Lefschetz [W]). Moishezon's theorem

[Mo] just means that this obstruction is equivalent to that the manifold is non-Kähler. Recently Peternell showed that this obstruction for 3-dimensional complex manifolds is a positive integral linear combination of irreducible curves which is homologous to zero.

As an important example in  $\mathcal{H}$ , we point out that if  $(X, Y, f)$  is any analytic compact smooth family of curves with Kähler base space  $Y$ , then  $X$  is in  $\mathcal{H}$ . Here we say that  $(X, Y, f)$  is an *analytic compact smooth family of curves* if  $X$  and  $Y$  are compact connected complex manifolds, and  $f: X \rightarrow Y$  is a surjective holomorphic map which is everywhere of maximal rank such that each fiber  $X_y = f^{-1}(y)$  is a connected smooth curve for any  $y \in Y$ . Notice that in this case  $f$  is a submersion. For any analytic compact smooth family of curves  $(X, Y, f)$  with Kähler base space  $Y$ , we know  $X \in \mathcal{H}$  by [HL, Theorem (17)<sup>∞</sup>].

*Proof of the Theorem 1.3.* By applying Moishezon's theorem [Mo] that a Moishezon manifold is projective algebraic if and only if it is Kähler, it suffices to show: if  $X$  is a Moishezon manifold, then  $X$  is a Kähler manifold. Suppose  $X$  is non-Kähler. By the result of Harvey and Lawson [HL, Proposition (12) and Theorem (14)], we know that there exists a non-trivial positive current  $T_0$  on  $X$  which is the bidimension  $(1, 1)$ -component of a boundary. Since  $X \in \mathcal{H}$ , there exists a non-trivial positive smooth current  $T$  which is the bidimension  $(1, 1)$ -component of a boundary. We can write

$$T = \partial S^{2,1} + \bar{\partial} S^{1,2},$$

where  $S^{2,1}$  and  $S^{1,2}$  are some currents of  $X$  of the bidimension  $(2, 1)$  and  $(1, 2)$ , respectively. Notice that these  $S^{2,1}$  and  $S^{1,2}$  may not be smooth.

Since  $X$  is Moishezon, there exists a modification  $\pi: \tilde{X} \rightarrow X$  such that the manifold  $\tilde{X}$  is projective algebraic. Let  $\tilde{\sigma}$  be a Kähler form on  $\tilde{X}$ .

We claim that the push-forward current  $\sigma := \pi_* \tilde{\sigma}$  is a  $d$ -closed  $(1, 1)$ -current on  $X$  satisfying the following property: for any point  $a \in X$ , there exists a neighborhood  $U$  of  $a$  in  $X$  with a local coordinates system  $(U, z^1, \dots, z^n)$  and a positive constant  $C$  such that

$$(6.1) \quad \sigma - C \sum_{j=1}^n \sqrt{-1} dz^j \wedge d\bar{z}^j \geq 0$$

on  $U$  in the sense of currents. In fact, for any point  $a \in M$ , we can find a neighborhood  $U$  of  $a$  in  $M$  with a local coordinate system  $(U, z^1, \dots, z^n)$  and a constant  $C > 0$  such that

$$\tilde{\sigma} - C\pi^* \left( \sum_{j=1}^n \sqrt{-1} dz^j \wedge d\bar{z}^j \right) \geq 0$$

on  $\pi^{-1}(U)$  in the sense of currents. Then on  $U$ , we see  $\pi_*\tilde{\sigma} - C\pi_*\pi^*(\sum_{j=1}^n \sqrt{-1} dz^j \wedge d\bar{z}^j) \geq 0$  in the sense of currents. So we have  $\sigma = \pi_*\tilde{\sigma} \geq C\pi_*\pi^*(\sum_{j=1}^n \sqrt{-1} dz^j \wedge d\bar{z}^j) \geq C \sum_{j=1}^n \sqrt{-1} dz^j \wedge d\bar{z}^j$  on  $U$  in the sense of currents. The claim (6.1) is then proved.

We define the smoothing  $\sigma_\varepsilon$  as follows: Let  $\{U_i\}_{1 \leq i \leq q}$  be any finite open covering of  $X$  and  $\{\varphi_i\}_{1 \leq i \leq q}$  be any partition of unity subordinate to  $\{U_i\}_{1 \leq i \leq q}$ . Suppose that every  $U_i$  is a coordinate chart and that  $U_i$  is identified with a unit ball with center  $0 \in \mathbb{C}^n$  with respect to the coordinate chart. On each  $U_i$ , since it is biholomorphic to the open unit ball, we can write

$$\sigma = \sqrt{-1}\partial\bar{\partial}f_i.$$

Because  $f_i - f_j$  is pluriharmonic on  $U_i \cap U_j$ ,  $\forall i \neq j$ , it implies that  $f_j = f_i$  is smooth. Then we define a global  $d$ -closed smooth real  $(1, 1)$ -form  $P$  on  $X$

$$P := \omega - \sqrt{-1}\partial\bar{\partial} \sum_{i=1}^r \varphi_i f_i$$

because  $P|_{U_j} = \sqrt{-1}\partial\bar{\partial} \sum_{i=1}^r \varphi_i (h_j - h_i)$ ; i.e., we have  $\sigma = P + \sqrt{-1} \sum_i \varphi_i f_i$ . Then smoothing  $\sigma_\varepsilon$  of  $\sigma$ , a  $d$ -closed real  $(1, 1)$ -form on  $X$ , is defined by

$$\sigma_\varepsilon = P + \sqrt{-1}\partial\bar{\partial} \left( \sum_{i=1}^q \chi_{i,\varepsilon} * (\varphi_i f_i) \right),$$

where  $\chi_{i,\varepsilon}$  is the standard approximation of identity defined on  $U_i$ .

Since  $\sigma_\varepsilon - \sigma = \sqrt{-1}\partial\bar{\partial} \{ \sum_{i=1}^q (\chi_{i,\varepsilon} * (\varphi_i f_i) - (\varphi_i f_i)) \}$ , it follows that

$$[\sigma] = [\sigma_\varepsilon] \in H^{1,1}(X, \mathbb{R}) \quad \text{and} \quad \sigma_\varepsilon \rightarrow \sigma, \text{ as } \varepsilon \rightarrow 0$$

in the sense of currents.

By the facts that  $X$  is compact and that  $T$  is smooth, we can make the following computation:

$$\begin{aligned} (\sigma, T) &= \lim_{\varepsilon \rightarrow 0} (\sigma_\varepsilon, T) = \lim_{\varepsilon \rightarrow 0} (\sigma_\varepsilon, \partial S^{2,1} + \bar{\partial} S^{1,2}) \\ &= \lim_{\varepsilon \rightarrow 0} (0 + 0) = 0, \end{aligned}$$

where we used the fact that  $\sigma_\varepsilon$  is  $d$ -closed.

On the other hand, the following claim leads to a contradiction and it completes the proof of the theorem:

$$(6.2) \quad (\sigma, T) > 0.$$

In fact, let the open covering  $\{U_i\}_{1 \leq i \leq q}$  of  $X$  and a partition of unity  $\{\varphi_i\}_{1 \leq i \leq q}$  subordinate to  $\{U_i\}_{1 \leq i \leq q}$  be as before. Since  $T$  is non-trivial and  $\sigma$  satisfies the property (6.1), we assume that the form  $\varphi_1 T \neq 0$  and that  $\sigma \geq C \sum_{j=1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j$  on  $U_1$  for some positive constant number  $C$  in the sense of currents. Then

$$(\sigma, T) = \sum_{i=1}^q (\sigma, \varphi_i T).$$

We claim that for each  $i$ ,  $1 \leq i \leq q$ ,  $(\sigma, \varphi_i T) \geq 0$ . In fact, let  $\chi_{i,\varepsilon}$  be the standard approximation of identity defined on  $U_i$ . Then  $(\sigma, \varphi_i T) = \lim_{\varepsilon \rightarrow 0} (\chi_{i,\varepsilon} * \sigma, \varphi_i T)$ . Notice that  $\chi_{i,\varepsilon} * \sigma$  is a positive  $C^\infty(1, 1)$ -form on  $U_i$  for any  $\varepsilon > 0$ , and that  $\varphi_i T$  is positive  $(n-1, n-1)$ -current on  $U_i$ ; it follows that  $(\chi_{i,\varepsilon} * \sigma, \varphi_i T)$  is non-negative. The claim then is verified by letting  $\varepsilon$  go to zero. Therefore, we have shown

$$(6.3) \quad (\sigma, T) \geq (\sigma, \varphi_1 T).$$

For the positive current  $\sigma - C \sum_{j=1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j$  on  $U_1$ , by applying the same method, we know  $(\sigma - C \sum_{j=1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j, \varphi_1 T) \geq 0$ , i.e.,

$$(6.4) \quad (\sigma, \varphi_1 T) \geq C \left( \sum_{j=1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j, \varphi_1 T \right).$$

Applying Wirtinger's Inequality as well as the argument in [HL, §4], we get

$$(6.5) \quad \left( \sum_{j=1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j, \varphi_1 T \right) = M(\varphi_1 T) > 0,$$

where  $M(\varphi_1 T)$  is the mass of  $T$ . The claim (6.2) follows from (6.3), (6.4), (6.5) above.  $\square$

**7. Moishezon manifolds and Kähler-Einstein metrics.** We prove Theorem 1.4 now. The statements (ii)  $\Rightarrow$  (ii)  $\Rightarrow$  (iv) are trivial. It suffices to show (i)  $\Rightarrow$  (iii) and (iv)  $\Rightarrow$  (v).



*Proof for (i)  $\Rightarrow$  (iii).* Suppose that  $M$  is a Moishezon manifold. Then there is a projective algebraic manifold  $\widetilde{M}$ , a proper surjective holomorphic mapping  $\pi: \widetilde{M} \rightarrow M$ , and an analytic set  $V$  such that the restriction mapping  $\pi|_{\widetilde{M}-\widetilde{V}}: \widetilde{M}-\widetilde{V} \rightarrow M-V$  is bihomomorphic, where  $\widetilde{V} := \pi^{-1}(V)$ . Choose a purely 1-codimensional analytic subset  $\widetilde{S}$  on  $\widetilde{M}$  such that  $\widetilde{V} \subset \widetilde{S}$  and that  $K_{\widetilde{M}} \otimes [\widetilde{S}]$  is ample. By applying Hironaka's resolution of singularities if necessary, we assume without loss of generality that  $\widetilde{S}$  is with simple normal crossings. Then by a result of Kobayashi [K, Theorem 1], we know that  $\widetilde{M}-\widetilde{S}$  admits a complete Kähler-Einstein metric  $\tilde{g}$  which is with negative Ricci curvature and with finite volume. Then we set  $g := ((\pi|_{M-S})^{-1})^* \tilde{g}$ .

*Proof for (iv)  $\Rightarrow$  (i).* Suppose that there is an analytic subset  $S \subset M$  such that  $M-S$  admits a complete Kähler metric  $g$  with  $\text{Ricci}(g) \leq -g$ . By Hironaka's resolution of singularities again if necessary, we can assume  $S$  is a hypersurface. Then by the  $L^2$  Riemann-Roch inequality proved by Nadel and Tsuji [NS] and by [N, Proposition 1.11], it implies

$$\begin{aligned} & \liminf_{k \rightarrow +\infty} \frac{1}{k^n} \dim H^0(M, K_M^{\otimes k} \times [S]^{\otimes(k-1)}) \\ & \geq \liminf_{k \rightarrow +\infty} \frac{1}{k^n} \dim H^0_{(2)}(M, K_{M-S}^{\otimes k}) \\ & \geq \frac{1}{n!} \int_{M-S} c_1(K_{M-S})^n \geq \frac{1}{n!} \int_{M-S} g^n > 0. \end{aligned}$$

Since  $H^0(M, K_M^{\otimes k} \otimes [S]^{\otimes(k-1)}) \subset H^0(M, (K_M \otimes [S])^k)$ , we then see

$$\begin{aligned} & \dim H^0(M, (K_M \otimes [S])^{\otimes k}) \\ & \geq \frac{k^n}{n!} \int_{M-S} g^n + O(k^n) = \left( \frac{1}{n!} \int_{M-S} g^n + \frac{O(k^m)}{k^n} \right) k^n. \end{aligned}$$

This yields (i). □

*Note added in proof.* Recently, Shiffman's conjecture has been proved completely. See: S. Ji and B. Shiffman, *Properties of compact complex manifolds carrying closed positive currents*, to appear in J. Geom. Anal.

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# PACIFIC JOURNAL OF MATHEMATICS

Volume 158    No. 2    April 1993

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- On the extension of Lipschitz functions from boundaries of subvarieties to strongly pseudoconvex domains 201  
K. ADACHI and HIROSHI KAJIMOTO
- On a nonlinear equation related to the geometry of the diffeomorphism group 223  
DAVID DAI-WAI BAO, JACQUES LAFONTAINE and TUDOR S. RATIU
- Fixed points of boundary-preserving maps of surfaces 243  
ROBERT F. BROWN and BRIAN SANDERSON
- On orthomorphisms between von Neumann preduals and a problem of Araki 265  
L. J. BUNCE and JOHN DAVID MAITLAND WRIGHT
- Primitive subalgebras of complex Lie algebras. I. Primitive subalgebras of the classical complex Lie algebras 273  
I. V. CHEKALOV
- $L^n$  solutions of the stationary and nonstationary Navier-Stokes equations in  $R^n$  293  
ZHI MIN CHEN
- Some applications of Bell's theorem to weakly pseudoconvex domains 305  
XIAO JUN HUANG
- On isotropic submanifolds and evolution of quasicauistics 317  
STANISŁAW JANECZKO
- Currents, metrics and Moishezon manifolds 335  
SHANYU JI
- Stationary surfaces in Minkowski spaces. I. A representation formula 353  
JIANGFAN LI
- The dual pair  $(U(1), U(1))$  over a  $p$ -adic field 365  
COURTNEY HUGHES MOEN
- Any knot complement covers at most one knot complement 387  
SHICHENG WANG and YING QING WU



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