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**THE MODULI OF RATIONAL WEIERSTRASS FIBRATIONS
OVER P^1 : SINGULARITIES**

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The Weierstrass equation $y^2 = x^3 + ax + b$, where a and b are rational functions of one variable, defines a fibration over \mathbf{P}^1 , which we call a Weierstrass fibration. We consider the moduli space W of rational Weierstrass fibrations over \mathbf{P}^1 . In this paper we determine the singular locus of W and we compute the general singularities. We work over \mathbf{C} , but it seems possible to generalize our methods to characteristic $p \neq 2, 3$.

Introduction. In [Mi] Miranda has constructed moduli spaces W_N , $N \geq 0$, for Weierstrass fibrations over \mathbf{P}^1 whose zero section has self intersection number $-N$ in the associated elliptic surface. Seiler has generalized and extended this work in [Sei2] and [Sei3]. For $N = 1$, we have the moduli space of rational fibrations $W = W_1$. The points of W parametrize isomorphism classes of rational Weierstrass fibrations over \mathbf{P}^1 with at most rational double point singularities whose associated elliptic surface (= minimal resolution of singularities) has only reduced fibers. By passing to the associated elliptic surface, W can be viewed as parametrizing isomorphism classes of relatively minimal elliptic surfaces over \mathbf{P}^1 admitting a section which have only reduced fibers. The basic definitions and constructions are reviewed in §1.

To determine the singular locus of W , we first find the locus S of Weierstrass fibrations that have non-negligible (= nontrivial) automorphisms. By means of the Weierstrass equation, this boils down to finding stable pairs of Weierstrass coefficients whose isotropy group with respect to the action of $G = \mathrm{GL}_2/\pm I$ is nontrivial. This work is the content of §2 and culminates in Theorem 1 where the 7 irreducible components of S are listed.

The general singularities turn out to be cyclic quotient singularities. We compute and classify them with the help of the slice theorem and work of Prill [Pr] in Theorem 2, §3.

This work is part of my Ph.D. thesis. I want to thank my advisor M. Artin and Rick Miranda for their help.

1. Generalities. All varieties we consider are defined over the field of complex numbers \mathbf{C} . Unless otherwise stated all topological notions refer to the Zariski topology. We refer the reader to [Mi], [Ka] and [M-S] for proofs of the following facts in this section.

Let S be a variety. Let $p: Y \rightarrow S$ be a flat proper morphism of irreducible varieties whose fibers are of one of the following types:

- (a) an elliptic curve,
- (b) a rational curve with a node,
- (c) a rational curve with a cusp.

Let σ be a section of p not touching the nodes and cusps of the fibers. The quadruple (Y, S, p, σ) is called a *Weierstrass fibration over S* . We usually denote Weierstrass fibrations by Y/S when there is no risk of confusion.

A morphism of a Weierstrass fibration (Y, S, p, σ) into a Weierstrass fibration (Y', S', p', σ') is given by a pair of morphisms $f: Y \rightarrow Y'$ and $\varphi: S \rightarrow S'$ such that $p' \circ f = \varphi \circ p$ and $f \circ \sigma = \sigma' \circ \varphi$.

When $S = C$ is a complete nonsingular connected curve, a Weierstrass fibration with nonsingular general fiber and only rational double point singularities is called a *Weierstrass model*. As is well known, a Weierstrass model Y/C can be described by a *Weierstrass equation* over C , i.e. there exists an invertible sheaf \mathcal{L} over C and sections a of $\mathcal{L}^{\otimes 4}$ and b of $\mathcal{L}^{\otimes 6}$ such that Y is isomorphic to the hypersurface in $\mathbf{P}(O_C \otimes \mathcal{L}^{\otimes(-2)} \oplus \mathcal{L}^{\otimes(-3)})$ given by $y^2 = x^3 + ax + b$. The morphism $J = J(a, b) = 4a^3/(4a^3 + 27b^2)$ of C into \mathbf{P}^1 is called the *J-invariant*.

Let $S = \mathbf{P}^1$. Choose coordinates t, s such that $t = 1, s = 0$ is the point at infinity. Call V_n the set of homogeneous functions of degree n on \mathbf{P}^1 viewed as homogeneous forms of degree n in t, s . Call G the quotient group $\mathbf{GL}_2/(\pm I)$. We use the same notation for a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in \mathbf{GL}_2 and for its image in G . We also use the notation $\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}$ for diagonal matrices, $\alpha = \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}$ for scalar matrices and $\begin{pmatrix} & \beta \\ \gamma & \end{pmatrix}$ for matrices with zeros in the main diagonal. Let $f(t, s) \in V_n$ and g be an element of G with matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We define

$$(f \cdot g)(t, s) = f(\alpha t + \beta s, \gamma t + \delta s).$$

This defines a right action of G on V_n . The pair of coefficients (a, b) of a rational Weierstrass model over \mathbf{P}^1 can be interpreted as an element of $V_4 \times V_6$.

In this way we get an injection of the set of isomorphism classes of rational Weierstrass models over \mathbf{P}^1 into $(V_4 \times V_6)/G$, where G acts

by means of its actions on V_4 and V_6 . Denote by X the open set of \mathbf{SL}_2 -stable (= finite stabilizer and closed orbit) elements of $V_4 \times V_6$ (to be called just *stable* from now on). The quotient algebraic variety $W = X/G$ is called the *moduli of rational Weierstrass fibrations over \mathbf{P}^1* . We denote by $\pi: X \rightarrow W$ the canonical map. Under the above injection points of W correspond to classes of Weierstrass models whose associated elliptic surface has reduced fibers. For $f \in V_n$ and $\tau \in \mathbf{P}^1$ denote by $v_\tau(f)$ the order of vanishing of f at τ . An element $(a, b) \in V_4 \times V_6$ is stable if and only if the following *numerical criterion* holds:

$$\min(3v_\tau(a), 2v_\tau(b)) < 6$$

for all $\tau \in \mathbf{P}^1$.

Let $x = (a, b) \in X$. Denote by Y_x the Weierstrass fibration with equation $\eta^2 = \xi^3 + a\xi + b$. Denote by $\text{Stab } x$ the *isotropy group* (= stabilizer) of x with respect to the action of G . Denote by $\text{Aut}_{WF}(Y_x/\mathbf{P}^1)$ the automorphism group of the Weierstrass fibration Y_x/\mathbf{P}^1 and by N the normal subgroup of *negligible* automorphisms, i.e., those of the form

$$\eta = \pm \eta', \quad \xi = \xi', \quad t = t', \quad s = s'.$$

Define $\text{Aut}_{RWF}(Y_x/\mathbf{P}^1) = (\text{Aut}_{WF}(Y_x/\mathbf{P}^1))/N$, the *reduced automorphism group of Y_x/\mathbf{P}^1* . Given $g \in \text{Stab } x$ with matrix $\lambda \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\lambda \neq 0$, $\alpha\delta - \beta\gamma = 1$, the formulas

$$\eta = \lambda^{-3}\eta', \quad \xi = \lambda^{-2}\xi', \quad t = \alpha t' + \beta s', \quad s = \gamma t' + \delta s'$$

define an element of $\text{Aut}_{RWF}(Y_x/\mathbf{P}^1)$ denoted by $\text{Aut } g$. The following proposition follows from well known facts.

PROPOSITION 1. *The canonical group homomorphism $\text{Stab } x \rightarrow \text{Aut}_{RWF}(Y_x/\mathbf{P}^1)$, $g \mapsto \text{Aut } g$ is bijective.*

We view the J -invariant $J(x) = J(a, b) = 4a^3/(4a^3 + 27b^2)$ as a morphism of \mathbf{P}^1 into \mathbf{P}^1 . We denote by $\text{Aut } J(x)$ the group of deck transformations of $J(x): \mathbf{P}^1 \rightarrow \mathbf{P}^1$. For g an element of G with matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ we denote by Pg the linear fractional transformation $z \mapsto (\alpha z + \beta)/(\gamma z + \delta)$, viewed as an element of $\mathbf{PGL}_2 = \text{Aut } \mathbf{P}^1$. The proof of the following easy corollary is left to the reader.

COROLLARY. *Suppose that $x \in X$ has nonconstant J -invariant. The canonical group homomorphism $\text{Stab } x \rightarrow \text{Aut } J(x)$, $g \mapsto Pg$ is injective.*

REMARK. In fact the homomorphism of the above corollary is *bijective*, but the proof is more involved.

2. Components of S . Recall that $\pi: X \rightarrow W$ is the canonical morphism. Define

$$S = \pi\{x \in X \mid \text{Stab } x \neq 1\}.$$

By the corollary to Proposition 1, this set is the locus in moduli of Weierstrass fibrations with nontrivial automorphisms. In this section we determine the irreducible components of the closed set S .

Let the group Γ operate on the set E . Let H be a subgroup of Γ . We denote

$$E^H = \{x \in E \mid xg = x \text{ for all } g \in H\}.$$

For $g \in \Gamma$, define $E^g = E^{(g)}$, where (g) is the group generated by g ; we remark that E^g is the set of x in E such that $xg = x$. When $E = X$, $\Gamma = G$, H subgroup of G , $g \in G$, we use the notations

$$\text{Inv } H = X^H, \quad \text{Inv } g = X^g.$$

It is clear that

$$S = \bigcup \{\pi(\text{Inv } g) \mid g \in G, g \neq 1, g \text{ of finite order}\}.$$

LEMMA 1. *Let $g \in G$ be of finite order. The sets $\text{Inv } g$ and $\pi(\text{Inv } g)$ are irreducible closed in X and W respectively.*

REMARK. It follows from Lemma 1 that the maximal elements among the $\pi(\text{Inv } g)$ are the irreducible components of S . Since a Noetherian topological space has a finite number of irreducible components, the set S is closed.

Proof of Lemma 1. We have

$$\text{Inv } g = (V_4 \times V_6)^g \cap X$$

where $(V_4 \times V_6)^g$ is a sub-vector space of $V_4 \times V_6$ and X is open in $V_4 \times V_6$. It follows that $\text{Inv } g$ is irreducible and closed. Consequently $\pi(\text{Inv } g)$ is irreducible. We have not used the fact that g is of finite order up to here.

Now let C be the conjugacy class of g . Since g is of finite order it follows from [Bo, pp. 227–228] that C is closed. Moreover G acts properly on X by [GIT, p. 41, Converse 1.13] and the fact that $\pi: X \rightarrow W = X/G$ is affine. Hence the morphism

$$\begin{aligned} X \times G &\xrightarrow{\psi} X \times X, \\ (x, h) &\mapsto (xh, x) \end{aligned}$$

is proper.

Denote by Δ_X the diagonal morphism of X into $X \times X$. It follows that the set $\Delta_X^{-1}(\psi(X \times C))$ is closed. Since

$$\Delta_X^{-1}(\psi(X \times C)) = \{x \in X \mid xg = x \text{ for some } g \in C\}$$

is G -saturated, it is clear that

$$\pi(\text{Inv } g) = \pi(\Delta_X^{-1}(\psi(X \times C)))$$

is closed. □

For any prime number p , let R_p be a system of representatives of the equivalence classes of elements of $\mathbf{F}_p^* - \{1\} = (\mathbf{Z}/p\mathbf{Z}) - \{0, 1\}$ with respect to the equivalence relation between elements u, v of $\mathbf{F}_p^* - \{1\}$ defined by the condition “ $u = v$ or $u = v^{-1}$ ”. Moreover we define $\zeta_n = e^{2\pi i/n}$.

LEMMA 2. *We have*

$$S = \bigcup \pi(\text{Inv } g)$$

where g runs over the following list:

$$\begin{pmatrix} i & \\ & i \end{pmatrix}, \quad \begin{pmatrix} \zeta_3 & \\ & \zeta_3 \end{pmatrix}, \\ \begin{pmatrix} \zeta_p & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}, \quad l \in R_p, \quad p = 3, 5, 7, 11. \\ \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} -i & \\ & i \end{pmatrix},$$

The inclusion

$$\bigcup \pi(\text{Inv } g) \subset S$$

is obvious. Now let $u \in S$. There are two cases:

- (i) $J(x) = 0$ (resp. $J(x) = 1$) for all $x \in \pi^{-1}(u)$.
- (ii) $J(x)$ is nonconstant for all $x \in \pi^{-1}(u)$.

Case (i). The conditions $J(x) = 0$ and $J(x) = 1$ are equivalent to $x \in \text{Inv } \zeta_3$ and $x \in \text{Inv } i$ respectively. We conclude in this case that

$$u \in \bigcup \pi(\text{Inv } g)$$

where $g = i, \zeta_3$.

Case (ii). Since $J(x)$ is nonconstant, it follows from the rationality of the Weierstrass model determined by x , that $\deg J(x) \leq 12$,

where $\deg J(x)$ denotes the degree of the cover $J(x): \mathbf{P}^1 \rightarrow \mathbf{P}^1$. The following argument shows that every element φ of $\text{Aut } J(x)$ has order $\leq d = \deg J(x) \leq 12$. Take a classical nonempty open set U in \mathbf{P}^1 such that $(J(x))^{-1}(U)$ is a disjoint union of d copies of U . Suppose that V is one of such copies. Then the sequence $V = \varphi^0(V), \varphi^1(V), \dots, \varphi^d(V)$ has a repetition, say

$$\varphi^i(V) = \varphi^j(V) \quad \text{for } 0 \leq i < j \leq d.$$

Thus

$$V = \varphi^{j-i}(V)$$

which implies, since φ is an analytic function, that φ has order $\leq j - i \leq d$. We conclude by the corollary to Proposition 1 that every element of $\text{Stab } x$ has order ≤ 12 . Now we notice the following facts.

(α) If $u \in \pi(\text{Inv } g)$, there exists $x \in \pi^{-1}(u)$ such that $x \in \text{Inv } g$. Thus $\text{Stab } x \supset (g)$. It follows that g has order ≤ 12 by the above considerations.

(β) If $u \in \pi(\text{Inv } g)$, there exists $x \in \pi^{-1}(u)$ such that $x \in \text{Inv } g$. Since $J(x)$ is nonconstant, $x = (a, b)$ with $a \neq 0$, $b \neq 0$. Suppose g were scalar with matrix $\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$. It follows that

$$ag = \lambda^4 a = a, \quad bg = \lambda^6 b = b$$

which implies $\lambda^4 = \lambda^6 = 1$. Thus $\lambda^2 = 1$, which contradicts the fact that $g \neq 1$ in G . Consequently g is nonscalar.

(γ) Given g of finite order there exists $g' \in (g)$ of prime order such that

$$\pi(\text{Inv } g) \subset \pi(\text{Inv } g').$$

(δ) Given g of finite order there exists a diagonal element g' conjugate to g such that

$$\pi(\text{Inv } g) = \pi(\text{Inv } g').$$

(ε) If (g) is conjugate to (g') , then

$$\pi(\text{Inv } g) = \pi(\text{Inv } g').$$

We conclude from (α) to (ε) that

$$u \in \bigcup \pi(\text{Inv } g),$$

where g runs through a system of representatives of the equivalence classes of nonscalar diagonal elements of G of prime order ≤ 12 with respect to the equivalence relation between elements g, g' of

G defined as follows. We say that g is *equivalent* to g' if (g) is conjugate to (g') .

Let g be of prime order $p \leq 12$ with matrix $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, $\lambda_1 \neq \lambda_2$. In case $p = 2$, we have either $\lambda_1^2 = \lambda_2^2 = 1$ or $\lambda_1^2 = \lambda_2^2 = -1$. Thus g is equivalent to one of $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$, $\begin{pmatrix} -i & \\ & i \end{pmatrix}$. In case p is odd, suppose first that $\lambda_1^p = \lambda_2^p = 1$. If $\lambda_2 = 1$, then $\lambda_1 \neq 1$. There exists an integer μ such that

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^\mu = \begin{pmatrix} \zeta_p & \\ & 1 \end{pmatrix}.$$

Thus g is equivalent to $\begin{pmatrix} \zeta_p & \\ & 1 \end{pmatrix}$. The case $\lambda_1 = 1$ reduces to the previous one by conjugation with the matrix

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

If $\lambda_1 \neq 1$, $\lambda_2 \neq 1$, there exists an integer μ such that

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^\mu = \begin{pmatrix} \lambda_1^\mu & \\ & \zeta_p \end{pmatrix}.$$

For some integer $l \neq 0, 1$

$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^\mu = \begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}.$$

Thus g is equivalent to

$$\begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}, \quad l \neq 0, 1.$$

When $\lambda_1^p = \lambda_2^p = -1$, set $\lambda_i' = -\lambda_i$, $i = 1, 2$ and reduce to the previous case.

The proof of Lemma 2 is finished by the observation that whenever $m \cdot l = 1 \pmod{p}$,

$$\begin{pmatrix} \zeta_p^m & \\ & \zeta_p \end{pmatrix} \text{ is equivalent to } \begin{pmatrix} \zeta_p^l & \\ & \zeta_p \end{pmatrix}. \quad \square$$

For $g \in G$, we have

$$\text{Inv } g = (V_4 \times V_6)^g \cap X = (V_4^g \times V_6^g) \cap X.$$

Let g be diagonal. The g -invariant monomials of V_n form a vector basis of V_n^g . Thus a general element of V_n^g is given by a linear

combination with general coefficients of elements from such a basis. A general element of $V_4^g \times V_6^g$ is just a pair of general elements of V_4^g and V_6^g . Such a general element is also a general element of $\text{Inv } g$ since X is open. It is stable if some specialization is stable.

Now we choose the following R_p for $p = 3, 5, 7, 11$:

$$\begin{aligned} R_3 &= \{2\}, \\ R_5 &= \{2, 4\}, \\ R_7 &= \{2, 3, 6\}, \\ R_{11} &= \{2, 3, 5, 7, 10\}. \end{aligned}$$

In Table 1 we give bases of g -invariant monomials of V_4^g and V_6^g for the different values of g that appear in Lemma 2 subject to the above choice of R_p 's, except for the cases $g = i, \zeta_3$ which are trivial. We also indicate for which values of g the set $\text{Inv } g$ is nonempty.

TABLE 1. *Invariant Monomials.* We list all g -invariant monomials of degrees 4 and 6

p	g	degree 4	degree 6	g -invariant pairs
2	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	t^4, t^2s^2, s^4	t^6, t^4s^2, t^2s^4, s^6	some stable
	$\begin{pmatrix} -i & \\ & i \end{pmatrix}$	t^4, t^2s^2, s^4	t^5s, t^3s^3, ts^5	some stable
3	$\begin{pmatrix} \zeta_3 & \\ & 1 \end{pmatrix}$	t^3s, s^4	t^6, t^3s^3, s^6	some stable
	$\begin{pmatrix} \zeta_3^2 & \\ & \zeta_3 \end{pmatrix}$	t^2s^2	t^6, t^3s^3, s^6	some stable
5	$\begin{pmatrix} \zeta_5 & \\ & 1 \end{pmatrix}$	s^4	t^5s, s^6	some stable
	$\begin{pmatrix} \zeta_5^2 & \\ & \zeta_5 \end{pmatrix}$	ts^3	t^4s^2	some stable
	$\begin{pmatrix} \zeta_5^4 & \\ & \zeta_5 \end{pmatrix}$	s^4	s^6	all unstable
7	$\begin{pmatrix} \zeta_7 & \\ & 1 \end{pmatrix}$	s^4	s^6	all unstable
	$\begin{pmatrix} \zeta_7^2 & \\ & \zeta_7 \end{pmatrix}$	t^3s	ts^5	some stable
	$\begin{pmatrix} \zeta_7^3 & \\ & \zeta_7 \end{pmatrix}$	No solutions	No solutions	
	$\begin{pmatrix} \zeta_7^6 & \\ & \zeta_7 \end{pmatrix}$	t^2s^2	t^3s^3	all semistable
11	$\begin{pmatrix} \zeta_{11}^{10} & \\ & \zeta_{11} \end{pmatrix}$	t^2s^2	t^3s^3	all semistable
No other solutions for $p = 11$.				

TABLE 2. *Components of S . Here Γ is the component $\pi(\text{Inv } g)$, $x = (a, b)$ is an element of the general orbit over Γ*

Γ	g	$x = (a, b)$	$\text{Stab } x$	$\deg J(x)$	$\dim \Gamma$
A	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{cases} a = (t^2 - s^2)(t^2 - k^2 s^2) \\ b = c(t^2 - m^2 s^2)(t^2 - n^2 s^2)(t^2 - p^2 s^2) \end{cases}$	C_2	12	5
B	$\begin{pmatrix} -i & \\ & i \end{pmatrix}$	$\begin{cases} a = (t^2 - s^2)(t^2 - k^2 s^2) \\ b = cts(t^2 - m^2 s^2)(t^2 - n^2 s^2) \end{cases}$	C_2	12	4
C	$\begin{pmatrix} \zeta_3 & \\ & 1 \end{pmatrix}$	$\begin{cases} a = (t^3 - s^3)s \\ b = c(t^3 - m^3 s^3)(t^3 - n^3 s^3) \end{cases}$	C_3	12	3
D	$\begin{pmatrix} \zeta_5 & \\ & 1 \end{pmatrix}$	$\begin{cases} a = s^4 \\ b = c(t^5 - s^5)s \end{cases}$	C_5	10	1
E	$\begin{pmatrix} \zeta_5^2 & \\ & \zeta_5 \end{pmatrix}$	$\begin{cases} a = ts^3 \\ b = t^4 s^2 \end{cases}$	C_5	5	0
F	$\begin{pmatrix} \zeta_7^2 & \\ & \zeta_7 \end{pmatrix}$	$\begin{cases} a = t^3 s \\ b = ts^5 \end{cases}$	C_7	7	0
G	$\begin{pmatrix} \zeta_3 & \\ & \zeta_3 \end{pmatrix}$	$\begin{cases} a = 0 \\ b = t(t-s)s(t-ms)(t-ns)(t-ps) \end{cases}$	C_3	$J \equiv 0$	3

THEOREM 1. *The irreducible components of S are listed in Table 2. Suppose g and x are entries in a row of Table 2 with x an element of the general orbit over $\Gamma = \pi(\text{Inv } g)$. Then $\text{Stab } x = (g)$.*

In Table 2 the element $x = (a, b)$ is obtained by taking a general element of $\text{Inv } g$ constructed from Table 1 and eliminating parameters redundant with respect to the action of G . The resulting parameters are chosen in such a way as to make explicit the zeros of a and b .

Keeping in mind the remark after Lemma 1, we first prove that the sets $\pi(\text{Inv } g)$ for g in Table 2 are an irredundant decomposition of S . Among the $\pi(\text{Inv } g)$ in Lemma 2 the following inclusions hold:

$$\begin{aligned}
 (\alpha) \quad & \pi \text{Inv} \begin{pmatrix} i & \\ & i \end{pmatrix} \subset \pi \text{Inv} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \\
 (\beta) \quad & \pi \text{Inv} \begin{pmatrix} \zeta_3^2 & \\ & \zeta_3 \end{pmatrix} \subset \pi \text{Inv} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}.
 \end{aligned}$$

(α) By putting $c = 0$ in the element of the general orbit over $\pi \text{Inv} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ we get

$$\begin{cases} a = (t^2 - s^2)(t^2 - k^2 s^2), \\ b = 0. \end{cases}$$

By dimension considerations we get that the locus $J \equiv 1$ which equals $\pi \text{Inv} \begin{pmatrix} i & \\ & i \end{pmatrix}$ is contained in $\pi \text{Inv} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$.

(β) Since $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ is conjugate to $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ we have

$$\pi \operatorname{Inv} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \pi \operatorname{Inv} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}.$$

But the general element of $\operatorname{Inv} \begin{pmatrix} \zeta_3^2 & \\ & \zeta_3 \end{pmatrix}$ is invariant under $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.

To check that there are no other inclusions we use systems of eigenvalues. More precisely, let R be the equivalence relation on \mathbf{C}^2 generated by the relations “ $(x', y') = (y, x)$ ” and “ $(x', y') = (-x, -y)$ ” both between the elements (x, y) and (x', y') of \mathbf{C}^2 . Thus the class of (x, y) consists of the elements (x, y) , (y, x) , $(-x, -y)$ and $(-y, -x)$. The system of eigenvalues of an element of finite order $g \in G$ will be considered as an element of \mathbf{C}^2/R .

For a subgroup of finite order H of G we denote by $\operatorname{Eigenval}(H)$ the set of systems of eigenvalues of elements of H . Clearly $\operatorname{Eigenval}(H)$ depends only on the conjugacy class of H .

Let g_1, g_2 appear in Table 2 and suppose that $\pi(\operatorname{Inv} g_1) \subset \pi(\operatorname{Inv} g_2)$. Suppose x_1 is the element of the general orbit over $\pi(\operatorname{Inv} g_1)$, given in Table 2. Then $\pi(x_1) \in \pi(\operatorname{Inv} g_2)$, which implies that $x_1 \in \operatorname{Inv} h^{-1} g_2 h$ for some h . By (ii) it follows that

$$(g_1) = \operatorname{Stab} x_1 \supset (h^{-1} g_2 h).$$

Thus $\operatorname{Eigenval}(g_1) \supset \operatorname{Eigenval}(g_2)$. The reader can easily check case by case that this can only happen when $g_1 = g_2$.

Now it remains to prove that $\operatorname{Stab} x \subset (g)$, for g, x satisfying the conditions of the theorem. The inclusion $(g) \subset \operatorname{Stab} x$ is obvious. In cases D, E, F, suppose $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \operatorname{Stab} x$, $x = (a, b)$. By comparing coefficients in the equations $ah = a$ and $bh = b$, one concludes that $h \in (g)$. The remaining cases depend on a series of lemmas.

As usual, we identify \mathbf{P}^1 with $\mathbf{C} \cup \{\infty\}$ and automorphisms of \mathbf{P}^1 with linear fractional transformations. Moreover, given a set E of $n \geq 3$ distinct points of \mathbf{P}^1 , every automorphism of \mathbf{P}^1 stabilizing the set E is determined by the induced permutation of E . We indicate such automorphisms by giving only the induced permutation. We omit the proof of the following well-known lemmas.

LEMMA 3. *Every automorphism of \mathbf{P}^1 that permutes the points $0, 1, \infty, m, n, p$, where m, n, p are in general position, is the identity.*

LEMMA 4. *The group of automorphisms of \mathbf{P}^1 that permute the points $0, 1, \infty, k$, for general k is Klein's four-group consisting of $(0\ 1)(\infty\ k)$, $(0\ \infty)(1\ k)$, $(1\ \infty)(0\ k)$ and the identity e .*

LEMMA 5. *The group of automorphisms of \mathbf{P}^1 that permute the points $1, \zeta_3, \zeta_3^2, \infty$ is the tetrahedral group.*

LEMMA 6. *The group of automorphisms of \mathbf{P}^1 that permute the points $1, -1, k, -k$, for general k is Klein's four-group consisting of $(1\ -1)(k\ -k)$, $(1\ k)(-1\ -k)$, $(1\ -k)(-1\ k)$ and the identity e .*

Now we return to the proof of Theorem 1. We omit Case B because it is similar to case A. We treat case G first.

Case G. Suppose $bh = b$ for $b = t(t-s)s(t-ms)(t-ns)(t-ps)$, m, n, p in general position and $h \in G$. By Lemma 3, we infer that h has the matrix $\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$ since the automorphism of \mathbf{P}^1 induced by h permutes the zeros of b . Thus $\lambda^6 = 1$.

Case A. Suppose $ah = a$ for $a = (t^2 - s^2)(t^2 - k^2s^2)$, k general, $h \in G$. By Lemma 5 the linear fractional transformation Ph is one of $(1\ -1)(k\ -k)$, $(1\ k)(-1\ -k)$, $(1\ -k)(-1\ k)$ or the identity. But $(1\ k)(-1\ -k)$ and $(1\ -k)(-1\ k)$ cannot stabilize the set $\{m, -m, n, -n, p, -p\}$ for m, n, p in general position.

Case C. Suppose $ah = a$ for $h \in G$. By Lemma 6, Ph belongs to the tetrahedral group permuting the points $1, \zeta_3, \zeta_3^2, \infty$, which is isomorphic to the alternating group of the set $\{1, \zeta_3, \zeta_3^2, \infty\}$. Taking into account the form of the elements of this group ([Se], p. 41), for m, n in general position the subgroup stabilizing the set of zeros of b is $((1\ \zeta_3\ \zeta_3^2))$. \square

3. Singularities. In this section we prove that S is the singular locus of W and we determine the general singularities.

All the representations we consider in the following are finite dimensional linear representations over \mathbf{C} of finite groups.

We need the notion of isomorphism of two representations $\rho: H \rightarrow \mathbf{GL}(V)$ and $\rho': H' \rightarrow \mathbf{GL}(V')$ of not necessarily identical groups H, H' . The definition is obvious. The representation $\rho: H \rightarrow \mathbf{GL}(V)$ is called *small* if no element in the image of ρ has 1 as eigenvalue of multiplicity $\dim V - 1$. We gather in the following proposition the results we need from [Pr].

PROPOSITION 2. *Two small faithful representations $\rho: H \rightarrow \mathbf{GL}(V)$ and $\rho': H' \rightarrow \mathbf{GL}(V')$ are isomorphic if and only if the germs of analytic space $(V/H, 0)$ and $(V'/H', 0)$ are isomorphic.*

A small faithful representation $\rho: H \rightarrow \mathbf{GL}(V)$ is identically equal to the identity if and only if $(V/H, 0)$ is nonsingular.

Now let u be a point of W and x an element of X such that $u = \pi(x)$. Put $H = \text{Stab } x$ and let N be an H -invariant complement to $T_x(xG)$ in $T_x(X)$. By the slice theorem ([Sch], p. 56) and the fact that π is affine, there exists an isomorphism of germs of analytic space $(W, u) \xrightarrow{\sim} (N/H, 0)$. Call $\rho = \rho_{x,N}$ the representation of H defined by its action on N and $\rho^f = \rho_{x,N}^f$ the faithful representation of $H/\text{Ker } \rho$ induced by ρ . The isomorphism class of the germ (W, u) depends only on the isomorphism class of the representation ρ . We say that the representation $\rho = \rho_{x,N}$ is *associated* to the point $u = \pi(x)$.

THEOREM 2. *The set S is the singular locus of W . Representations associated to the general singularities, which are given in Table 3, are faithful and small.*

The following corollary is immediate by Proposition 2.

COROLLARY. *The isomorphism classes of the associated representations classify the general singularities up to isomorphism.*

TABLE 3. Associated Representations. Here Γ is the component of $\pi(\text{Inv } g)$, x is the element of the general orbit over Γ such that $\text{Stab } x = (g)$, $\rho = \rho_{x,N}$ is a representation of $\text{Stab } x$ associated to $u = \pi(x)$

Γ	$\text{Stab } x$	Eigenvalues of $\rho(g)$							
A	C_2	1,	1,	1,	1,	1,	-1,	-1,	-1
B	C_2	1,	1,	1,	1,	-1,	-1,	-1,	-1
C	C_3	1,	1,	1,	ζ_3 ,	ζ_3 ,	ζ_3 ,	ζ_3^2 ,	ζ_3^2
D	C_5	ζ_5 ,	ζ_5 ,	ζ_5^2 ,	ζ_5^2 ,	ζ_5^2 ,	ζ_5^3 ,	ζ_5^3 ,	ζ_5^4
E	C_5	1,	ζ_5 ,	ζ_5 ,	ζ_5^2 ,	ζ_5^2 ,	ζ_5^3 ,	ζ_5^3 ,	ζ_5^4
F	C_7	ζ_7 ,	ζ_7^2 ,	ζ_7^2 ,	ζ_7^3 ,	ζ_7^4 ,	ζ_7^5 ,	ζ_7^6 ,	ζ_7^6
G	C_3	1,	1,	1,	ζ_3 ,	ζ_3 ,	ζ_3 ,	ζ_3 ,	ζ_3

Proof of Theorem 2. It is clear that the representations in Table 3 are faithful and small. We have to prove that they are associated to the general points of the components of S .

First of all we recall some generalities on infinitesimals of first order. Let X be an analytic space, $x \in X$. Let $\mathbf{C}[\varepsilon]$ be the algebra of dual numbers. Let $\text{Specan } \mathbf{C}[\varepsilon]$ be the analytic space with only one point o and with local ring $\mathbf{C}[\varepsilon]$ at that point. We use the notation

$$\begin{aligned} X(\mathbf{C}[\varepsilon])_x &= \text{Hom}((\text{Specan } \mathbf{C}[\varepsilon], o), (X, x)) \\ &= \text{Hom}_{\mathbf{C}\text{-alg loc}}(\mathcal{O}_{X, x}, \mathbf{C}[\varepsilon]) \end{aligned}$$

for the set of $\mathbf{C}[\varepsilon]$ -valued points of X at x .

The map

$$\begin{aligned} \text{Der}_{\mathbf{C}}(\mathcal{O}_{X, x}, \mathbf{C}) &\rightarrow \text{Hom}_{\mathbf{C}\text{-alg loc}}(\mathcal{O}_{X, x}, \mathbf{C}[\varepsilon]), \\ t &\rightarrow u = x + \varepsilon t \end{aligned}$$

establishes a bijection of $T_x(X)$ onto $X(\mathbf{C}[\varepsilon])_x$.

Now denote by X the set of stable elements of $V_4 \times V_6$ and by G the group $\mathbf{GL}_2/(\pm I)$ as before. Let $x \in X$. The orbital map $\rho: G \rightarrow X$, $g \mapsto xg$ is étale because $\text{Stab } x$ is a finite set. We get the following commutative diagram:

$$\begin{array}{ccc} G(\mathbf{C}[\varepsilon])_1 & \xrightarrow{\rho} & X(\mathbf{C}[\varepsilon])_x \\ \varphi \uparrow & & \uparrow \psi \\ \mathbf{M}_2(\mathbf{C}) & \xrightarrow{d\rho_1} & V_4 \times V_6 \end{array}$$

where we identify $T_1(G) = T_I(\mathbf{GL}_2) = \mathbf{M}_2(\mathbf{C}) = 2 \times 2$ matrices, $T_x(X) = T_x(V_4 \times V_6) = V_4 \times V_6$, $T_x(G) = \text{Im } d\rho_1$ and φ and ψ are the bijections described above. Note that ρ and $d\rho_1$ are injective. We have a canonical basis t_1, \dots, t_4 of $\text{Im } d\rho_1$ namely the image of the canonical basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathbf{M}_2(\mathbf{C})$. It can be computed explicitly from the equation

$$x \cdot (I + \varepsilon E_i) = x + \varepsilon t_i$$

which follows from

$$(\rho \circ \varphi)(e_i) = x \cdot (I + \varepsilon E_i)$$

and

$$(\psi \circ d\rho_1)(e_i) = x + \varepsilon t_i.$$

Now let us explain how to choose N . Let $\theta: V_4 \times V_6 \xrightarrow{\sim} \mathbf{C}^{12}$ be the isomorphism defined by the canonical basis, $(t^4, 0), (t^3s, 0), \dots, (s^4, 0), (0, t^6), \dots$, of $V_4 \times V_6$. Let $t_i = (\sum \alpha_{kl}^i t^k s^l), (\sum \beta_{mn}^i t^m s^n)$. Let A be the matrix

$$\left(\begin{array}{c|c} \alpha_{4,0}^1 \cdots \alpha_{0,4}^1 & \beta_{6,0}^1 \cdots \beta_{0,6}^1 \\ \alpha_{4,0}^4 \cdots \alpha_{0,4}^4 & \beta_{6,0}^4 \cdots \beta_{0,6}^4 \end{array} \right).$$

The row space of A is $\text{Im } d\rho_1$. We choose a square submatrix B of A such that $\det B \neq 0$. The submatrix B is gotten from A by deleting a row $(\alpha_{k,l}^i)$ (resp. a row $(\beta_{m,n}^i)$) if and only if $(k, l) \in D$ (resp. $(m, n) \in E$) for well determined sets D, E . The subspace $N = N_B$ generated by $(t^k s^l, 0), (k, l) \in D$ and $(0, t^m s^n), (m, n) \in E$ is a complement of $\text{Im } d\rho_1$. This is obvious by considering their images under θ .

To calculate the matrix A we use the following formulas, where $f = \sum a_{ij} t^i s^j$ is an element of V_n .

$$\begin{aligned} f \cdot (I + \varepsilon E_1) &= f + \varepsilon \sum f_{ij} i t^i s^j, \\ f \cdot (I + \varepsilon E_2) &= f + \varepsilon \sum f_{ij} i t^{i-1} s^{j+1}, \\ f \cdot (I + \varepsilon E_3) &= f + \varepsilon \sum f_{ij} j t^{i+1} s^{j-1}, \\ f \cdot (I + \varepsilon E_4) &= f + \varepsilon \sum f_{ij} j t^i s^j. \end{aligned}$$

We indicate the explicit choice of the square submatrix B in each case by underlining the corresponding columns of A . The reader should keep in mind Table 2.

Case A. By setting

$$\begin{aligned} \kappa &= -(1 + k^2), \\ \lambda &= k^2, \\ \mu &= -(m^2 + n^2 + p^2), \\ \nu &= m^2 n^2 + m^2 p^2 + n^2 p^2, \\ \pi &= -m^2 n^2 p^2, \end{aligned}$$

the general element x can be written

$$\begin{cases} a = t^4 + \kappa t^2 s^2 + \lambda s^4, \\ b = c(t^6 + \mu t^4 s^2 + \nu t^2 s^4 + \pi s^6). \end{cases}$$

The matrix A has the form

$$\left[\begin{array}{ccccc|ccccc} 4 & 0 & 2\kappa & 0 & 0 & 6c & 0 & 4\mu c & 0 & 2\nu c & 0 & 0 \\ 0 & 4 & 0 & 2\kappa & 0 & 0 & 6c & 0 & 4\mu c & 0 & 4\nu c & 0 \\ 0 & 2\kappa & 0 & 4\lambda & 0 & 0 & 2\mu c & 0 & 4\nu c & 0 & 6\pi c & 0 \\ \underline{0} & \underline{0} & \underline{2\kappa} & \underline{0} & 4\lambda & 0 & 0 & 2\mu c & 0 & 4\nu c & 0 & 6\pi c \end{array} \right]$$

and the submatrix B consists of the underlined columns. It is clear that $\det B \neq 0$ for k, m, n, p general enough.

The action of $g = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ on N_B is given by

$$s^4; \quad t^6, \quad t^5s, \quad t^4s^2, \quad t^3s^3, \quad t^2s^4, \quad ts^5, \quad s^6 \\ +1, +1, -1, +1, -1, +1, -1, +1$$

where the first line is the canonical basis of N_B which consists of eigenvectors of g and the second line are the corresponding eigenvalues.

In the following cases we just indicate the matrices A, B .

Case B. Here

$$\kappa = -(1 + k^2), \quad \lambda = k^2, \quad \mu = -(m^2 + n^2), \quad \nu = m^2n^2.$$

$$\left[\begin{array}{ccccc|ccccc} 4 & 0 & 2\kappa & 0 & 0 & 0 & 6c & 0 & 3\mu c & 0 & \nu c & 0 \\ 0 & 4 & 0 & 2\kappa & 0 & 0 & 0 & 6c & 0 & 3\mu c & 0 & \nu c \\ 0 & 2\kappa & 0 & 4 & 0 & c & 0 & 3\mu c & 0 & 5\nu c & 0 & 0 \\ \underline{0} & \underline{0} & \underline{2\kappa} & \underline{0} & \underline{4} & \underline{0} & \underline{c} & \underline{0} & \underline{3\mu c} & \underline{0} & \underline{5\nu c} & \underline{0} \end{array} \right].$$

Case C. Here

$$\mu = (m^3 + n^3), \quad \nu = m^3n^3.$$

$$\left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & -4 & 0 & 0 & 0 & 3\mu c & 0 & 0 & 6\nu c \\ 0 & 3 & 0 & 0 & 0 & 6c & 0 & 0 & 3\mu c & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 6c & 0 & 0 & 3\mu c & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & \underline{0} & \underline{0} & \underline{3\mu c} & \underline{0} & \underline{0} & \underline{6\nu c} & \underline{0} \end{array} \right].$$

Case D.

$$\left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 5c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & -6c & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & \underline{c} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{-6c} \end{array} \right].$$

Case E.

$$\left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{0} & \underline{3} & \underline{0} & 0 & 0 & \underline{2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{array} \right].$$

Case F.

$$\left[\begin{array}{cccc|cccc} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ \underline{0} & \underline{1} & \underline{0} & 0 & 0 & 0 & 0 & 0 & 0 & \underline{5} & 0 \end{array} \right] .$$

Case G. Here

$$\begin{aligned} \mu &= -1 - m - n - p, \\ \nu &= m + n + p + mn + mp + np, \\ \pi &= pmn - mp - np - mnp, \\ \rho &= mnp, \end{aligned}$$

$$\left[\begin{array}{c|ccccccc} & 0 & 5 & 4\mu & 3\nu & 2\pi & \rho & 0 \\ 0 & 0 & 0 & 5 & 4\mu & 3\nu & 2\pi & \rho \\ & 0 & \mu & 2\nu & 3\pi & 4\rho & 0 & 0 \\ & 0 & \underline{0} & \mu & 2\nu & \underline{3\mu} & \underline{4\rho} & \underline{0} \end{array} \right] .$$

The reader can check by specialization that the underlined matrix B has $\det B \neq 0$ for m, n, p general enough.

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