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**TANGENTIAL AND NORMAL EULER NUMBERS, COMPLEX  
POINTS, AND SINGULARITIES OF PROJECTIONS FOR  
ORIENTED SURFACES IN FOUR-SPACE**

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# TANGENTIAL AND NORMAL EULER NUMBERS, COMPLEX POINTS, AND SINGULARITIES OF PROJECTIONS FOR ORIENTED SURFACES IN FOUR-SPACE

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**For a compact oriented smooth surface immersed in Euclidean four-space (thought of as complex two-space), the sum of the tangential and normal Euler numbers is equal to the algebraic number of points where the tangent plane is a complex line. This follows from the construction of an explicit homology between the zero-chains of complex points and the zero-chains of singular points of projections to lines and hyperplanes representing the tangential and normal Euler classes.**

**1. Introduction.** For a compact two-dimensional surface smoothly immersed in Euclidean four-dimensional space  $\mathbf{R}^4$ , also thought of as complex two-dimensional space  $\mathbf{C}^2$ , the sum of the Euler number of  $M$  and the Euler number of its normal bundle is the negative of the algebraic number of (isolated) *complex points* where its tangent plane is the graph of a complex linear function. Our aim in this article is to present a geometric proof of this theorem. This theorem, already implicit in the fundamental paper of Chern and Spanier [C-S], has been developed and generalized in papers of Webster [W1], [W2], and employed as a lemma in a number of other investigations, for example the work of Hoffman and Osserman [H-O] and Fiedler [F]. These papers rely on a variety of abstract techniques from algebraic topology and differential geometry, and it is frequently not easy to see the underlying geometric content of the theorem. In this paper, we present a concrete proof of the theorem for oriented surfaces, by interpreting the Euler numbers as singularities of projections in order to obtain 0-chains representing the Poincaré duals of the tangential and normal Euler classes, and by using the Gauss map of the surface to exhibit an explicit homology between these chains and the 0-chain of (indexed) complex points.

Consider briefly the Euler numbers involved in the theorem. The (*tangential*) Euler number of  $M$  is the most well-known topological invariant of an oriented surface, characterized by a theorem of Heinz.

Hopf as the algebraic sum of the zeros of a field of vectors tangent to  $M$ . In terms of singularities of projections, the Euler number is the number of maxima plus the number of minima minus the number of ordinary saddle points for a generic real-valued function on the surface, i.e. a projection of the surface into a line.

Similarly the *normal Euler number* of an immersion of a surface into  $\mathbf{R}^4$  can be defined as the algebraic sum of the zeros of a field of vectors normal to  $M$ . If at each point of  $M$ , we construct a normal vector field by taking the normal component of the fixed unit vector  $\mathbf{v}$ , then this normal vector field is zero precisely at the points of  $M$  where  $\mathbf{v}$  is contained in the tangent plane, so the projection into the three-dimensional hyperplane perpendicular to  $\mathbf{v}$  has a singularity at  $p$ . Thus in terms of singularities of projections, we may express the normal Euler number as the algebraic number of singularities of a projection of  $M$  into a hyperplane.

What is the relationship between these two Euler numbers? It is surprising that the sum of the tangential and the normal Euler numbers has anything to do with the algebraic number of points at which the tangent plane to  $M$  is a complex line, but that is precisely the content of the theorem.

To help to understand this relationship, we turn to the Gauss map of the immersion. If  $X: M^2 \rightarrow \mathbf{R}^4$  is a smooth immersion of an oriented two-dimensional manifold into Euclidean four-space, then translating the tangent plane of  $M^2$  to the origin induces a map,  $GX$ , from  $M^2$  to  $G(2, 4)$ , the *Grassmann manifold of oriented two-planes in four-space*, also called the *Grassmannian*. We will show that the three terms in the theorem count the algebraic number of intersections of the image of the Gauss map of  $M$  with certain surfaces in the Grassmannian. We will show that these surfaces form the boundary of a three-dimensional chain in  $G(2, 4)$ , and the intersection of this chain with the Gauss map of the surface provides an explicit homology among the 0-chains that represent the three terms in the equation.

We begin with an expanded discussion of the Euler numbers in question, interpreting them for curves and surfaces in Euclidean two- and three-dimensional space to indicate the geometric nature of the approach we take to prove the theorem. We then present a detailed tour of the Grassmann manifold  $G(2, 4)$ . With this background, we present several examples illustrating the various concepts, leading up to our proof of the theorem.

**2. Euler numbers and singularities of projections.** In general, the concept of a characteristic class of a vector bundle is quite abstract;

the most common approaches are axiomatic, covering the most general situation [M-S]. But in the case of a submanifold immersed in a Euclidean space, these abstract concepts can be viewed concretely in terms of the singularities of projections into lower-dimensional subspaces. (See [C] and [B-McC] for examples of this approach.)

2A. *A simple closed curve.* Our most elementary example is a smooth embedding of the circle as a simple closed curve in the Euclidean plane with a finite number of inflection points. Associated with any unit vector  $\mathbf{v}$  is the height function obtained by projecting the curve into the line along  $\mathbf{v}$ . The critical points of this height function are the points where the vector tangent to the curve is perpendicular to  $\mathbf{v}$ . For almost any  $\mathbf{v}$ , this height function has finitely many critical points on the curve, an even number of them, alternately maxima and minima as we proceed around the circle. (If this is not true for a particular  $\mathbf{v}$ , we may rotate the curve slightly and obtain a curve that does have the property with respect to  $\mathbf{v}$ .)

We can interpret these properties by using the Gauss map  $G_X$  of the embedding  $X$ , which sends each point of the (counterclockwise oriented) circle to the unit tangent vector to the curve at the point, translated to the origin. Thus the Gauss map  $G_X$  goes from the circle to the circle. If we project, for instance, to the line along the second standard basis vector  $\mathbf{e}_2$ , then the critical points are the points with unit tangent vector  $\pm\mathbf{e}_1$ . We can index a critical point by  $-1$  if the Gauss map crosses from below the  $\mathbf{e}_1$ -axis to above and  $+1$  if it crosses from above to below. The points with their Gauss image in the upper semi-circle form a collection of curves on the original circle, each one beginning with a  $-1$  critical point and ending with a critical point indexed  $+1$ . This collection of arcs on the circle provides an explicit homology between the positive and negative points.

Next suppose we consider another direction,  $\mathbf{v}$ , on the unit circle. As in the case of  $\mathbf{e}_2$ , we may index the points where the tangent vector is perpendicular to  $\mathbf{v}$  to obtain another 0-chain. We claim that this 0-chain is homologous to the one obtained for  $\mathbf{e}_2$ . To obtain an explicit homology, we consider an arc of vectors  $\mathbf{v}(t)$  on the unit circle going from  $\mathbf{v}(0) = \mathbf{e}_2$  to  $\mathbf{v}(1) = \mathbf{v}$ . As  $t$  changes from 0 to 1, the critical points for the projection to  $\mathbf{v}(t)$  trace out a collection of curves on the original circle. As we pass a vector parallel to the tangent line at an inflection point of the curve, two critical points will coalesce into one, or a pair of critical points will arise spontaneously from a single point. In any case, the critical points at the beginning together with

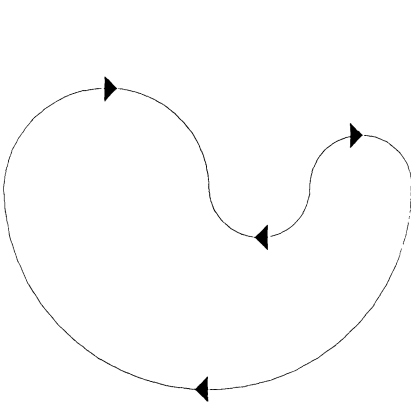


FIGURE 1a

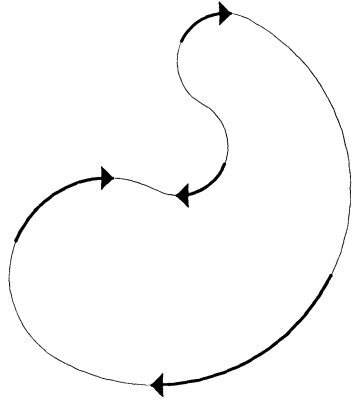


FIGURE 1b

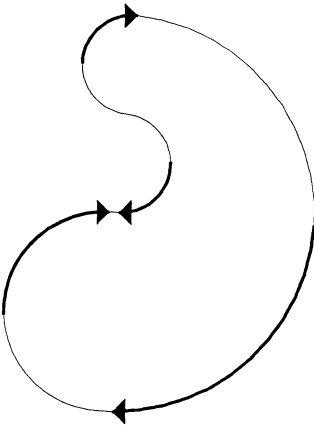


FIGURE 1c

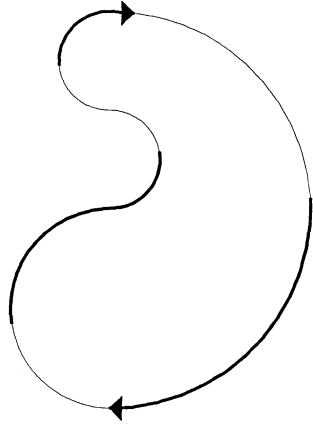


FIGURE 1d

those at the end form the boundary of a collection of oriented curves; these arcs traced on the circle exhibit an explicit homology between the two sets of critical points. (See Figures 1a, 1b, 1c, and 1d.)

In this rather elementary example, we see that the signs of the critical points sum to zero for almost any projection to a line along a unit vector, and that the 0-chains corresponding to projections to two different directions are homologous by an explicit homology obtained from the Gauss mapping. The same results are true for a curve immersed in the plane, or for that matter in  $n$ -dimensional space  $\mathbf{R}^n$ .

2B. *The Euler number of a compact surface in  $\mathbf{R}^3$ .* For a more complicated example, consider a generic immersion of an oriented smooth surface without boundary into Euclidean three-space

$X: M^2 \rightarrow \mathbf{R}^3$ . As above, we wish to show how to determine its (tangential) Euler number by using the Gauss map  $GX$  which sends each point  $p$  of  $M$  to the oriented plane through the origin parallel to the tangent plane to the surface of the point. Following the usual practice, we identify an oriented plane with the unit normal perpendicular to the oriented tangent plane to  $X(M)$  at  $X(p)$ , so the image of  $GX$  lies on the unit sphere in  $\mathbf{R}^3$ . The direction of the unit normal is chosen such that the orientation of the tangent plane followed by the unit normal gives the orientation of all of  $\mathbf{R}^3$ .

To calculate the tangential Euler number, we consider the zeros of a tangential vector field, for example the field that assigns to  $p$  the projection to the third coordinate vector  $\mathbf{e}_3$  into the tangent plane at  $X(p)$ . This vector field is zero exactly when the projection to the line of  $\mathbf{e}_3$  is singular, that is, when the tangent plane at  $X(p)$  is perpendicular to  $\mathbf{e}_3$ . This happens when the image of the Gauss map at  $p$  is either  $\mathbf{e}_3$  or  $-\mathbf{e}_3$ , the north or south pole of the unit sphere  $S^2$ . We assume that for the surface  $X(M)$ , each point  $p$  in the preimage of  $\pm\mathbf{e}_3$  under  $GX$  will have a disc neighborhood on which the Gauss map is one-to-one. (If this is not true, then we may rotate the surface slightly in almost any direction and this property will hold.)

Following Gauss's original definition, we assign the index 1 to  $p$  if the Gauss map in a neighborhood of  $p$  is orientation-preserving and  $-1$  if it is orientation-reversing. The plus sign identifies the points which are maxima or minima of the projection of the surface to the third coordinate axis, and the minus sign identifies the ordinary saddle points. The nature of the critical points can be determined by expressing the surface in the neighborhood of a critical point as the graph of a function  $(x, y, f(x, y))$ . Then the index of the critical point is 1 or  $-1$  depending on the algebraic sign of the Hessian  $f_{xx}f_{yy} - f_{xy}^2$ . The sum of the indices of these points is called the (*tangential*) Euler number of  $M$ . More formally, this indexed set of critical points is defined to be the Poincaré dual of the tangential Euler class of  $M$ .

To see that the Euler number and the Euler class are well-defined, we choose another unit vector  $\mathbf{v}$  and consider the 0-chain of indexed critical points for the projection of the surface to the line along  $\mathbf{v}$ . For most directions  $\mathbf{v}$ , the minor arc  $\mathbf{v}(t)$  of unit vectors on the sphere from  $\mathbf{v}(0) = \mathbf{e}_3$  to  $\mathbf{v}(1) = \mathbf{v}$  will meet the image of the Gauss map of the surface "transversely" so that as we move from  $\mathbf{v}(0)$  to  $\mathbf{v}(1)$ , the indexed critical points of the projection to  $\mathbf{v}(t)$  trace curves on  $M$ , with a finite number of exceptional positions where a maximum

or minimum coalesces with a saddle, or where a saddle and a local extremum arise instantaneously. In any case, the 0-chain at the end minus the 0-chain at the beginning is the boundary of an explicitly constructed oriented 1-chain, so these 0-chains represent the same 0-dimensional homology class, the tangential Euler class, and the sum of their indices is a well-defined integer, the (tangential) Euler number, denoted by  $\chi(M)$ . A similar argument shows that these same results are true for a surface immersed in  $n$ -dimensional space, although the nature of the Gauss map is more complicated in higher dimensions (see below).

The process used in this example illustrates the kind of argument we will use in the proof of the theorem.

2C. *The normal Euler number of an immersed surface in  $\mathbf{R}^4$ .* For an immersion  $X: M \rightarrow \mathbf{R}^4$ , a theorem of Whitney [Wh] states that the sum of the indices of the zeros of any generic normal vector field will be independent of the field used to define it, and this common value is defined to be the normal Euler number of the immersion. As in the case of the tangential Euler number, we may approach the normal Euler number in terms of singularities of projections. One way to obtain a normal vector field on an immersed surface is to start with a fixed unit vector  $\mathbf{v}$  and consider the projection  $N_p\mathbf{v}$  of  $\mathbf{v}$  into the normal space to  $M$  at  $p$ . The zeros of this vector field will occur precisely at the points  $p$  where the vector  $\mathbf{v}$  lies in the tangent space at  $p$ , and for almost all  $\mathbf{v}$ , there will be a finite number of such singular points  $p$ . If we project down  $\mathbf{v}$  into the three-dimensional subspace perpendicular to  $\mathbf{v}$ , then the zeros of  $N_p\mathbf{v}$  are characterized by the fact that the image of the tangent space at  $X(p)$  has dimension less than two. In the generic situation, this will happen only at a finite number of points, and the rank at each of these singular points is one. In three-dimensional space, we can recognize such singularities by the fact that the image of the surface has a “pinch point” or “Whitney umbrella point”, where the image of the tangent plane to the surface in  $\mathbf{R}^4$  is just a line in  $\mathbf{R}^3$ . It is possible to assign to each of these pinch points an index 1 or  $-1$  to obtain a 0-chain  $Z(\mathbf{v})$ . For almost any other unit vector  $\mathbf{w}$ , all but a finite number of points on the arc  $\mathbf{v}(t)$  from  $\mathbf{v}(0) = \mathbf{v}$  to  $\mathbf{v}(1) = \mathbf{w}$  will have generic pinch points as their singularities, and the pinch points of  $\mathbf{v}(t)$  will trace out a collection of arcs on  $M$ . At the singular positions, two pinch points coalesce and disappear or two new pinch points arise instantaneously. This collection of arcs gives an explicit homology between the 0-chains for  $\mathbf{v}(0)$

and  $\mathbf{v}(1)$ , so the homology class of the 0-chain  $Z(\mathbf{v})$  is independent of the direction  $\mathbf{v}$  used to define it. This integral homology class is called the *normal Euler class*, and the sum of the indices of the pinch points will be a well-defined number,  $n(X)$ , the *normal Euler number* of the immersion.

Although the tangential Euler number can be determined using any immersion whatsoever, the normal Euler number depends on the immersion into  $\mathbf{R}^4$ . In particular, if  $F: \mathbf{R}^4 \rightarrow \mathbf{R}^4$  is reflection in any hyperplane, then  $n(F \circ X) = -n(X)$ . We will use this fact in our later constructions.

We also mention a theorem of Whitney [Wh] which we will not use in the proof but which will play a role in the analysis of the examples: If  $X$  is a smooth embedding of an orientable surface into  $\mathbf{R}^4$ , with no self-intersection points, then  $n(X) = 0$ . In the case that  $X$  is an immersion with a finite number of transverse intersections, then  $n(X)$  is given by the algebraic number of intersection points. In particular, if there is only one such self-intersection point, then  $n(X) = \pm 2$ .

2D. *Elementary examples.* A surface immersed in  $X: M^2 \rightarrow \mathbf{R}^3$  is automatically immersed in  $\mathbf{R}^4$ , and the projection into the hyperplane  $\mathbf{R}^3$  is just  $X$  itself so it has no singularities. Thus the normal Euler number is 0, even though the tangential Euler number will not be zero for any orientable surface other than the torus. Thus in general  $\chi(M) \neq n(X)$ .

To measure the difference between the tangential and normal Euler numbers, we appeal to the “complex points” of  $X(M)$  in  $\mathbf{R}^3$  thought of as a hyperplane in  $\mathbf{C}^2$ . A *complex line* is the graph of a complex linear mapping,  $z = x + yi \rightarrow w = (a + bi)z + (c + di) = (a + bi)(x + yi) + (c + di) = (ax - by + c) + (bx + ay + d)i$ . In real coordinates, this graph is given by  $(x, y, ax - by + c, bx + ay + d)$ . The only plane in  $\mathbf{R}^3$  that can be represented in this way must have  $bx + ay + d = 0$  for all  $x, y$  so  $a = b = d = 0$ . Thus the only complex lines in  $\mathbf{R}^3$  are the horizontal planes  $(x, y, c, 0)$ . We already know that the tangential Euler number of  $M$  can be expressed as the sum of indices associated to the points where the tangent plane is horizontal, perpendicular to  $\mathbf{e}_3$ , and in the case of an immersion to  $\mathbf{R}^3$ , these are exactly the complex points, where the tangent plane is a complex line. We shall see that this relationship holds even when an immersion of  $M$  into  $\mathbf{R}^4$  is not contained in a hyperplane.

To see when the tangent plane at a point is a complex line, it is convenient to use the properties of a linear transformation of  $\mathbf{R}^4$



known as the *J homomorphism*. Multiplication by  $i$  in  $\mathbf{C}^2$  sends the vector  $(x + yi, u + vi)$  to  $(-y + xi, -v + ui)$ . This motivates the definition  $J(x, y, u, v) = (-y, x, -v, u)$ . We observe that if a two-dimensional plane through the origin is given by the graph of a linear function  $(x, y, mx + ny, px + qy)$ , then this plane will be preserved under the action of  $J$  if and only if

$$J(x, y, mx + ny, px + qy) = (-y, x, -px - qy, mx + ny)$$

satisfies the same linear equations, namely  $-px - qy = m(-y) + nx$  and  $mx + ny = p(-y) + qx$  for all  $x$  and  $y$ . This will happen only when  $n = -p$  and  $m = q$ , so the mapping is  $(x, y, mx + ny, -nx + my)$ . This map corresponds in complex notation to the graph of the complex linear function  $z \rightarrow z(m - ni)$ .

Another important example of an orientable surface in 4-space is the *flat torus*  $T^2$ , obtained by mapping the point  $(s, t)$  in the square domain  $(0 \leq s \leq 2\pi, 0 \leq t \leq 2\pi)$  to the point  $X(s, t) = (\cos(s), \sin(s), \cos(t), \sin(t))$ . The plane through the origin parallel to the tangent plane at a point  $X(s, t)$  is given by linear combinations of the partial derivative vectors  $X_s(s, t) = (-\sin(s), \cos(s), 0, 0)$  and  $X_t(s, t) = (0, 0, -\sin(t), \cos(t))$ . But then  $JX_s(s, t) = (-\cos(s), -\sin(s), 0, 0)$ , which is perpendicular to both  $X_s(s, t)$  and  $X_t(s, t)$ , so the tangent plane is not preserved under the homomorphism  $J$  at any point of the flat torus. It follows that there are no complex points on this surface. On the other hand, the Euler number  $\chi(T^2)$  is zero and so is the normal Euler number, by Whitney's theorem.

2E. *An abstract approach to the theorem.* We consider briefly how the complex structure of  $\mathbf{R}^4 = \mathbf{C}^2$  is involved in relating the normal and tangential Euler classes. Following an argument of Webster [W2], we compute the normal Euler number of an immersion of a surface  $M$  by finding the algebraic number of zeros of a particular normal vector field. We start with a constant vector  $\mathbf{v}$ . By taking the tangential part of  $\mathbf{v}$  at each point  $p$ , we obtain a tangential vector field  $T_p(\mathbf{v})$  along  $M$ . We then move this vector by the linear transformation  $J$  to get  $JT_p(\mathbf{v})$ , and then take the normal part of the result to obtain  $N_p(J(T_p(\mathbf{v})))$ . This normal vector field will have a zero when either  $T_p(\mathbf{v}) = 0$ , or when  $T_p(\mathbf{v}) \neq 0$  and  $J(T_p(\mathbf{v}))$  is also a tangent vector. But the homomorphism  $J$  has a special property, namely, that if the image of one non-zero vector of a plane lies in the plane, so does the image of every vector in the plane. Thus the set of zeros of the normal vector field  $N_p(J(T_p(\mathbf{v})))$  consists exactly of the zeros of  $T_p(\mathbf{v})$

together with the complex points of the immersion.

It follows then that a set of points used to count the normal Euler class is identical with a union of the set of complex points and a set used to count the tangential Euler class. Formalizing this as a proof of the theorem amounts to finding the correct way to assign indices to these points. To see that this may be non-trivial, observe that composing the maps in the other order and counting the zeros of  $T_p(J(N_p(\mathbf{v})))$  switches the complex points to the other side of the equation. How do we account for this difference in indexing? Rather than address this difficulty directly, we present a very different proof, where the geometric content is more apparent.

A difficulty with the argument given above is that while the zeros of the vector fields  $T_p(\mathbf{v})$  and  $N_p(\mathbf{v})$  are exactly the singularities of projections of a vector  $\mathbf{v}$  into the tangent planes or normal planes of an immersed surface, it is not so easy to interpret the geometric meaning of the zeros of the vector fields  $N_pJT_p(\mathbf{v})$  and  $T_pJN_p(\mathbf{v})$ . To find a proof of the theorem which relates to singularities of projections, we turn to the Gauss map of the immersion and therefore, first, to a Grassmannian manifold.

**3. The Grassmannian manifold of oriented two-planes in Euclidean four-space.** For an immersion  $X$  of an oriented two-dimensional surface in  $\mathbf{R}^4$ , the Gauss map takes a point of the surface to the oriented plane through the origin which is parallel to the tangent plane at  $X(p)$ . The image of this map then is the set of oriented two-planes in  $\mathbf{R}^4$ , which we call  $G(2, 4)$  or simply  $G$  when there is no danger of confusion. In this section, we coordinatize  $G$ , and analyze the structure it inherits through an identification  $\mathbf{R}^4 = \mathbf{C}^2$ . We then examine some structures associated with the proof to follow.

3A. *Plücker coordinates.* In  $\mathbf{R}^4$  with oriented basis given by  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ , we can completely determine an oriented plane by giving the (algebraic) areas of the projections of a unit square in that plane into the six coordinate planes. If the edges of the unit square are given by the ordered pair of perpendicular unit vectors  $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 + c_4\mathbf{e}_4$  and  $d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{e}_3 + d_4\mathbf{e}_4$ , then the projection of this square to the 1-2-plane is the parallelogram with edges  $c_1\mathbf{e}_1 + c_2\mathbf{e}_2$  and  $d_1\mathbf{e}_1 + d_2\mathbf{e}_2$ . This has algebraic area  $c_1d_2 - c_2d_1$ , which we define to be the 1-2-Plücker coordinate  $p_{12}$ . Similarly we find the six Plücker coordinates  $p_{ij}$ .

In terms of exterior algebra, we may write the 2-plane as

$$\omega = p_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + p_{23}\mathbf{e}_2 \wedge \mathbf{e}_3 + p_{31}\mathbf{e}_3 \wedge \mathbf{e}_1 + p_{34}\mathbf{e}_3 \wedge \mathbf{e}_4 + p_{14}\mathbf{e}_1 \wedge \mathbf{e}_4 + p_{24}\mathbf{e}_2 \wedge \mathbf{e}_4.$$

The Plücker coordinates are independent of the choice of basis used to define them—any other oriented pair of perpendicular unit vectors gives the same six coordinates  $p_{ij}$ .

The set of two-planes in four-space is not six-dimensional since there are some relations among these Plücker coordinates. Since we started with an orthonormal basis for the plane, by direct calculation we may show that  $\sum p_{ij}^2 = 1$ . This expresses an extension of the Pythagorean theorem, saying that the square of the area of a parallelogram is the sum of the squares of the areas of its projections to the six coordinate planes. This makes it possible to obtain the Plücker coordinates of an oriented two-plane by taking the wedge product of any pair of oriented basis vectors and then dividing by the square root of the sum of the squares of the six components. We will use this in our later computations concerning the tangent spaces of surfaces in  $\mathbf{R}^4$ .

Similarly we may show by direct calculation that  $2p_{12}p_{34} + 2p_{31}p_{24} + 2p_{14}p_{23} = 0$ . This relation is somewhat less intuitive geometrically. In terms of exterior algebra, this relation follows immediately from the fact that if the 2-form  $\omega$  represents a plane, then the exterior power  $\omega \wedge \omega$  must be 0. (Some readers may recall that a 2-vector can fail to be decomposable, as in the case of the dual of a symplectic form.)

As in [C-S], we will reorganize the six coefficients which determine a two-plane in  $\mathbf{R}^4$  in such a way that it becomes possible to view  $G(2, 4)$  as the product of two 2-spheres. Just as the locus  $2xy = 0$  in the plane can be rotated to the hyperbola  $x^2 - y^2 = 0$ , we may rotate (modulo a factor of the square root of 2) in three 2-planes in 6-space to obtain new coordinates

$$\begin{aligned} a_1 &= p_{12} + p_{34}, & a_2 &= p_{23} + p_{14}, & a_3 &= p_{31} + p_{24}, \\ b_1 &= p_{12} - p_{34}, & b_2 &= p_{23} - p_{14}, & b_3 &= p_{31} - p_{24}. \end{aligned}$$

The two algebraic conditions on the Plücker coordinates then imply that

$$a_1^2 + a_2^2 + a_3^2 = 1 = b_1^2 + b_2^2 + b_3^2.$$

Thus the vectors  $\mathbf{a} = (a_1, a_2, a_3)$  and  $\mathbf{b} = (b_1, b_2, b_3)$  are unit vectors in three-space, and the product of two two-dimensional spheres is therefore in one-to-one correspondence with  $G(2, 4)$ . We call these the *sphere parameters* for a plane in  $G(2, 4)$ , and from here on, we

will identify a plane with its sphere parameters. For reasons explained below, it will be convenient to think of the  $a_1$ - and  $b_1$ -axes as the polar axes of the spheres, so when we refer to the equators of these spheres we mean the great circles in the  $a_2$ - $a_3$ - and  $b_2$ - $b_3$ -planes.

By the *orthogonal complement* of an oriented plane  $\omega$  in  $\mathbf{R}^4$ , we mean the plane of all vectors orthogonal to the given plane, suitably oriented so that the exterior product of an oriented orthonormal basis of  $\omega$  with an oriented orthonormal basis of its orthogonal complement gives the standard orientation on  $\mathbf{R}^4$ . As in [C-S], we will use the easily-proved fact that if  $(\mathbf{a}, \mathbf{b})$  are the sphere coordinates of a plane, then  $(\mathbf{a}, -\mathbf{b})$  are the sphere coordinates of its orthogonal complement.

We also note here that if  $(\mathbf{a}, \mathbf{b})$  represents a given oriented two-plane, then  $(-\mathbf{a}, -\mathbf{b})$  represents the same plane with the opposite orientation. If we reflect an oriented plane through the hyperplane  $e_4 = 0$ , then the new oriented 2-plane has sphere coordinates  $(\mathbf{b}, \mathbf{a})$ .

3B. *Complex structures.* The usual identification of  $\mathbf{R}^4$  with  $\mathbf{C}^2$  leads to the complex structure homomorphism  $J$  mentioned earlier, which corresponds to scalar multiplication by  $i$ . Formally,  $J$  is identified by the property  $J^2 = -I$ , where  $I$  represents the identity transformation. Other complex structures, corresponding to other identifications of  $\mathbf{R}^4$  with  $\mathbf{C}^2$ , are in one-to-one correspondence with the various possible choices of a homomorphism with this property. We will mostly be satisfied with one complex structure map, given in coordinates by

$$J\mathbf{e}_1 = \mathbf{e}_2, \quad J\mathbf{e}_2 = -\mathbf{e}_1, \quad J\mathbf{e}_3 = \mathbf{e}_4, \quad \text{and} \quad J\mathbf{e}_4 = -\mathbf{e}_3.$$

We note that this complex structure map is compatible with the Euclidean structure on  $\mathbf{R}^4$ , in that  $\langle X, Y \rangle = \langle JX, JY \rangle$  for vectors  $X$  and  $Y$ , where  $\langle X, Y \rangle$  stands for the ordinary inner product.

It is easy to verify that  $J$  induces a well-defined map from  $G(2, 4)$  to itself; if  $\omega = X \wedge Y$  then  $J\omega = JX \wedge JY$  is independent of the choice of  $X$  and  $Y$ . A simple computation in sphere coordinates shows that if  $\omega = (\mathbf{a}, \mathbf{b})$  in sphere coordinates, then the sphere coordinates of  $J\omega$  are  $((a_1, -a_2, -a_3), (b_1, b_2, b_3))$ . Thus the action of  $J$  on  $G(2, 4)$ , is a rotation of  $\pi$  radians about the  $a_1$ -axis in the first factor while the second factor is fixed.

This enables us to identify the complex lines of  $\mathbf{C}^2$ , or equivalently, the complex planes in  $G(2, 4)$  as the planes invariant under

$J$ . The set of complex planes,  $C$ , consists of two spheres  $C = \{((\pm 1, 0, 0), \mathbf{b})\}$ .

The sphere with  $a_1 = 1$  corresponds to the complex planes oriented by a wedge product of the form  $X \wedge JX$ , while the planes parametrized by the other sphere have the opposite orientation.

Note that rotating the first factor sphere about any of its other axes corresponds to the action of a different complex structure on  $\mathbf{R}^4$ . Since this is not crucial for our results, we merely mention here that the first factor sphere can be identified with the unit pure quaternions in a natural way: for each pure unit quaternion,  $q$ , there is a complex structure on  $\mathbf{R}^4$  corresponding to multiplication by  $q$ . For the standard complex structure we use  $q = i$ .

There is another family of complex structures on  $\mathbf{R}^4$  corresponding to rotations of the second factor sphere. The simplest one is given in coordinates by

$$Ke_1 = e_2, \quad Ke_2 = -e_1, \quad Ke_3 = -e_4, \quad Ke_4 = e_3.$$

It rotates in the same direction  $J$  does in the plane  $e_1 \wedge e_2$  while rotating the opposite way from  $J$  in the plane  $e_3 \wedge e_4$ .

This complex structure map acts on  $G(2, 4)$  in sphere parameters by fixing the first factor sphere while rotating the other about the  $b_1$ -axis through  $\pi$  radians. Other complex structure maps correspond to rotating the second factor sphere about its various axes. We state without proof that these complex structures are the only ones compatible with the Euclidean structure of  $\mathbf{R}^4$ , in the sense defined above. (In the sense of Gromov [G], the complex structures in the component connected to  $J$  are those tamed by the standard symplectic form.)

3C. *Subsets related to projections.* When we compute the normal or tangential Euler number of a surface in  $\mathbf{R}^4$ , we can choose any normal or tangential vector field to do so. Now we once and for all choose the normal and tangential parts of the constant  $e_4$  vector field to use in the computation. In this way the question of the vanishing of the normal or tangential vector field can be viewed in a simple way in terms of the image of the Gauss map of the surface.

With this in mind, we identify the subsets of  $G(2, 4)$  in which  $e_4$  has vanishing normal part or vanishing tangential part. Call  $N = \{\omega | e_4 \text{ is contained in } \omega\}$  and  $T = \{\omega | e_4 \perp \omega\}$ . Thus if  $T_p M$  belongs to  $N$ , the normal part of  $e_4$  is zero, while if it belongs to  $T$ , the tangential part is zero. To coordinatize both  $N$  and  $T$  with one computation, we compute the effect on 2-vectors of  $p_{123}$ , the orthogonal

projection onto the 3-space perpendicular to  $\mathbf{e}_4$ .

$$\begin{aligned} p_{123}(\mathbf{a}, \mathbf{b}) &= p_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + p_{23}\mathbf{e}_2 \wedge \mathbf{e}_3 + p_{31}\mathbf{e}_3 \wedge \mathbf{e}_1 \\ &= ((a_1 + b_1)\mathbf{e}_1 \wedge \mathbf{e}_2 + (a_2 + b_2)\mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3 + b_3)\mathbf{e}_3 \wedge \mathbf{e}_1)/2. \end{aligned}$$

Therefore  $T = \{(\mathbf{a}, \mathbf{a})\}$ , the diagonal of the product, while  $N = \{(\mathbf{a}, -\mathbf{a})\}$ , the twisted diagonal. Thus the sets  $N$  and  $T$  are each diffeomorphic to the two-sphere. They do not intersect one another, but each intersects  $C$ , the set of complex planes, in two points.

**4. Examples.** We now present a set of examples to show the relationship of the Gauss map of surfaces to the sets  $N$ ,  $T$  and  $C$  in the Grassmannian.

4A. *An ordinary sphere.* We construct a family of embeddings of the 2-sphere into  $\mathbf{R}^4$ , all congruent to the usual two-sphere in  $\mathbf{R}^3$ .  $X(u, v) = (\cos u, \sin u \cos v, \sin u \sin v, 0)$ , where  $0 \leq u \leq \pi$  and  $0 \leq v \leq 2\pi$ . We tilt this sphere into  $\mathbf{R}^4$  by defining

$$X^b(u, v) = (\cos b \cos u, \sin u \cos v, \sin u \sin v, \sin b \cos u)$$

where the parameter  $b$  represents the angle between the standard  $\mathbf{R}^3$  and the three-dimensional subspace in which the sphere lies; when  $b$  is 0, the sphere sits in the standard  $\mathbf{R}^3$ .

The sphere coordinates of the tangent plane are then

$$\begin{aligned} X_u^b \wedge X_v^b &= ((\sin u \sin(v + b), \cos u, \sin u \cos(v + b)), \\ &\quad (\sin u \sin(v - b), \cos u, \sin u \cos(v - b))). \end{aligned}$$

There are exactly two points where  $GX$  meets  $T$ , the two poles  $u = 0$  and  $\pi$ . The index of each of these is 1, so we obtain tangential Euler number 2. There are exactly two complex points:  $u = \pi/2$ ,  $v = \pi/2 - b$ ,  $3\pi/2 - b$  and at each point, the sign of the intersection with  $C$  is 1, as can be computed using the techniques of 5D below. The theorem in this case gives the equation  $2 + 0 = 2$ .

4B. *A sphere that intersects itself in exactly one point.* In this section, we will consider two immersions of the sphere, and one closely related embedding. Consider first the immersion of the sphere into  $\mathbf{R}^4$  defined over the domain  $0 \leq u \leq 2\pi$ ,  $-\pi/2 \leq v \leq \pi/2$ :

$$\begin{aligned} X(u, v) &= (\cos(u) \cos(v), -\cos(u) \sin(v) \cos(v), \\ &\quad \sin(u) \cos(v), -\sin(u) \cos(v) \sin(v)). \end{aligned}$$

Here we are thinking of  $v$  as giving the latitude angle of the sphere and  $u$  as given the longitude. The only double points occur when  $\cos(v) =$

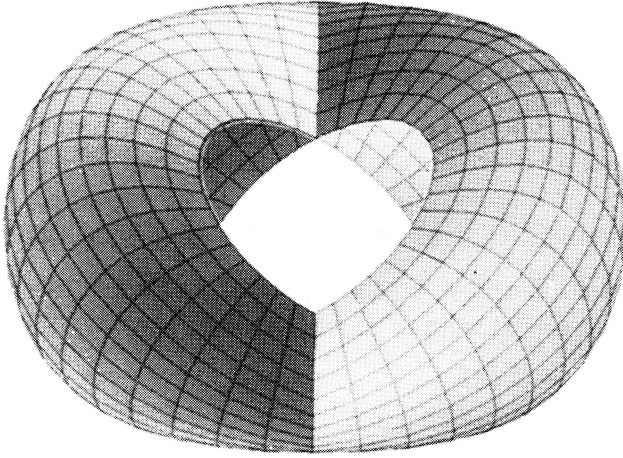


FIGURE 2

0 i.e. at  $v = \pm\pi/2$ , at the “poles” of the parametrization of the sphere, which are both mapped to the origin in  $\mathbf{R}^4$ . The immersion  $X$  is thus analogous to the “figure eight” immersion of the circle into the plane.

We can now appeal to Whitney’s theorem to calculate the normal Euler number by computing the index of the double point. Although the parametrization is singular at the poles, the surface has a well-defined tangent plane at each of these points. Near  $v = \pi/2$  the surface is approximately  $x = -y$ ,  $z = -w$  while near  $v = -\pi/2$ , the surface is approximated by the plane  $x = y$ ,  $z = w$ . In terms of Plücker coordinates, the tangent plane at  $(0, \pi/2)$  is  $(\mathbf{e}_1 - \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_4) = -(\mathbf{e}_1 \wedge \mathbf{e}_4) - (\mathbf{e}_2 \wedge \mathbf{e}_3) + (\mathbf{e}_1 \wedge \mathbf{e}_3) + (\mathbf{e}_2 \wedge \mathbf{e}_4)$  while the tangent plane at  $(0, -\pi/2)$  is  $(\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 + \mathbf{e}_4) = (\mathbf{e}_1 \wedge \mathbf{e}_4) + (\mathbf{e}_2 \wedge \mathbf{e}_3) + (\mathbf{e}_1 \wedge \mathbf{e}_3) + (\mathbf{e}_2 \wedge \mathbf{e}_4)$ . Taking the wedge product of these two two-vectors gives  $-4$  times the four-vector  $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$  so we may give to this double point the index  $-1$ . By Whitney’s theorem, the normal Euler number of this immersion will be  $-2$ . The projection into 3-space will have a pinch point when  $\sin(v) = 0$  and  $\cos(u) = 0$ , so  $v = 0$  and  $u = \pm\pi/2$ . The images of these pinch points will be the points  $(0, 0, \pm 1)$ . (See Figure 2.)

A straightforward computation shows that the Gauss mapping in sphere coordinates is given by normalizing the following vectors:

$$\begin{aligned} a_1 &= 0, & b_1 &= \sin(2u) \cos^2(v), \\ a_2 &= \cos^2(v) - 2 \sin^2(v), & b_2 &= \cos(2u) \cos^2(v), \\ a_3 &= 2 \sin(v) \cos(v), & b_3 &= -2 \sin^3(v). \end{aligned}$$

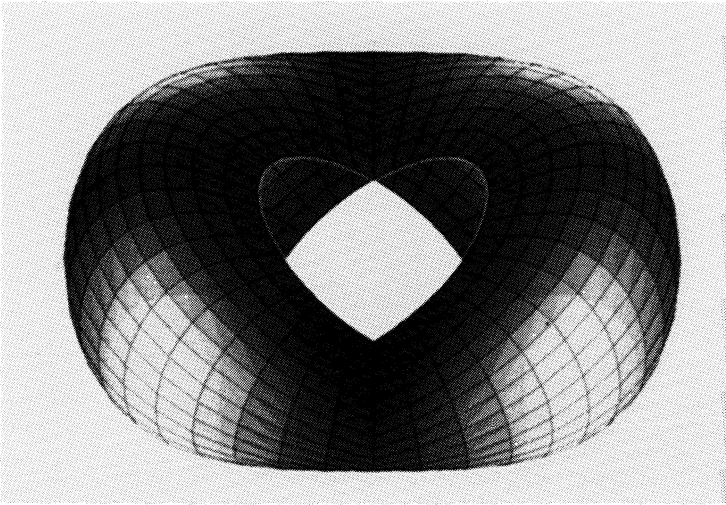


FIGURE 3

Therefore there are no complex points. The normal Euler number of the immersion is the negative of the tangential Euler number and the theorem reads  $2 + (-2) = 0$ .

If we negate the fourth coordinate of  $X$  to obtain a surface

$$Y(u, v) = (\cos(u) \cos(v), -\cos(u) \sin(v) \cos(v), \\ \sin(u) \cos(v), \sin(u) \cos(v) \sin(v)),$$

then the normal Euler number of the immersion  $Y$  is the negative of that of  $X$ , namely 2. The Gauss map of the surface  $Y$  is obtained by interchanging the  $\mathbf{a}$  and  $\mathbf{b}$  vectors above. We obtain complex points when  $a_2 = 0 = a_3$ , so  $\sin v = 0$  and  $\cos(2u) = 0$ , leading to four complex points, with  $v = 0$  and  $u = \pi/4, 3\pi/4, 5\pi/4,$  and  $7\pi/4$ . In order to balance the contribution of the tangential and normal Euler numbers, the index of each of these complex points should be 1 as can be found by using the local methods in the next section. (See Figure 3.)

It is instructive to compare the above example with the surface

$$W(u, v) = (\cos(u) \cos(v), -\cos(u) \sin(v) \cos(v), \\ \sin(u) \cos(v), -\sin(v)).$$

This surface differs from  $X$  and  $Y$  only in the last coordinate, so it has the same projection into three-space, with two pinch points. In this case however, the mapping is an embedding, since the first, third, and fourth coordinates describe a standard two-dimensional sphere, already embedded in a hyperplane. Thus by Whitney's theorem, the normal Euler number will be zero. The illustration indicates that the



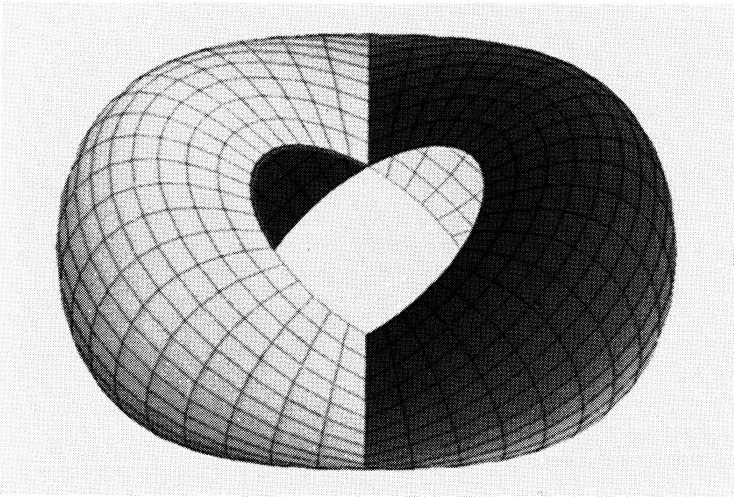


FIGURE 4

two pinch points will have opposite indices, corroborating this fact. (See Figure 4.)

The first set of sphere coordinates for the tangent plane are obtained by normalizing the vectors

$$\begin{aligned} a_1 &= \cos u \cos v (-1 + \sin u \cos v), \\ a_2 &= \sin u \cos v - \sin^2 v + \cos^2 u \cos^2 v, \\ a_3 &= -\sin v (1 + \sin u \cos v). \end{aligned}$$

The points with complex tangent planes will have  $a_2 = 0 = a_3$ , so  $\sin v = 0$ ,  $\cos(v) = 1$ , and  $\sin u + \cos^2 u = 0 = -\sin^2 u + \sin u + 1$  so there are two solutions for  $u$  with  $\sin u = (1 - \sqrt{5})/2$ . (See Figure 5.) These two complex points give a total contribution of 2 to balance the equation in the theorem, since the tangential Euler number is 2 and the normal Euler number of any embedding is 0.

4C. *Gauss maps for function graphs.* To motivate our definitions of the indices of intersections of the image of the Gauss map with the sets  $T$ ,  $N$ , and  $C$ , we present examples that demonstrate typical behavior. This gallery of Gauss maps consists of function graphs of quadratic surfaces that intersect the various sets in each of the possible typical ways. After presenting the examples, we will be more precise about orientations; for intuitive purposes we indicate here that we will always orient the set  $T$  using the “outward” unit normals, while  $N$  will be oriented using the inward unit normal in the first factor and the outward unit normal in the second.  $C$  is oriented using the inward unit normal.

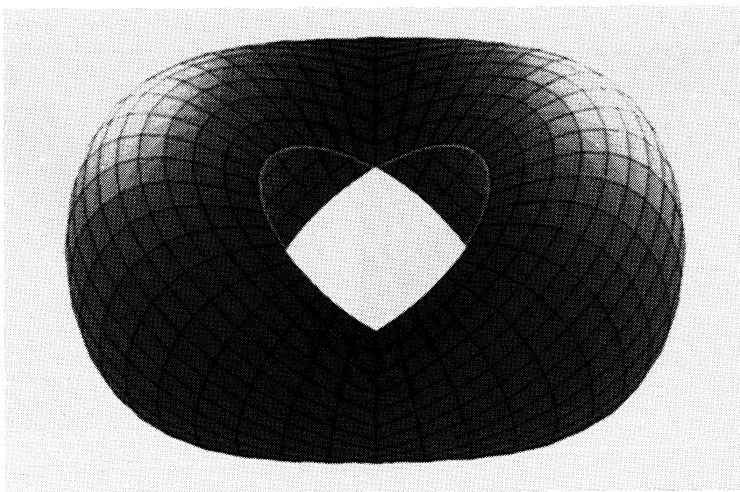


FIGURE 5

For typical intersections of a surface with the set  $T$ , consider the functions

$$X(u, v) = (0, u, v, (u^2 \pm v^2)/2).$$

It will turn out that the choice of sign in the fourth coordinate is exactly what determines the sign of the associated intersection with  $T$ . This is not surprising, as the plus sign corresponds to a minimum of the fourth coordinate function, while the minus sign is a saddle. This matches the discussion of the tangential Euler class in §2.

In sphere coordinates, the Gauss map is given by

$$GX(u, v) = ((-u, 1, \pm v), (u, 1, \mp v))/(1 + u^2 + v^2)^{1/2}$$

so that  $u = v = 0$  gives an isolated intersection of the image of  $G$  with the set  $T$ . This Gauss map may be pictured, in the case of the plus sign, as preserving the orientation of the  $u$ - $v$  plane as it maps to each factor sphere, and reversing the orientation in each in the case of the minus sign.

Our definition of index will involve the relative orientations of the image of the differential of the Gauss map and the plane tangent to the set in question, whether  $T$ ,  $N$ , or  $C$ . To make this explicit, in the case of the plus sign, we compute

$$GX * (\mathbf{f}_1) = -\mathbf{e}_1 + \mathbf{e}_4, \quad GX * (\mathbf{f}_2) = \mathbf{e}_3 - \mathbf{e}_6$$

(where  $\mathbf{f}_1$  and  $\mathbf{f}_2$  the vectors in the  $u$  and  $v$  directions), while the tangent space to  $T$  at the point of intersection is spanned by  $\mathbf{v}_1 = \mathbf{e}_3 + \mathbf{e}_6$  and  $\mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_4$ . The wedge product of the four vectors in the order given is  $4\mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_6 \wedge \mathbf{e}_4$ , which we will agree later to be the positive orientation of the tangent space of  $G(2, 4)$ .

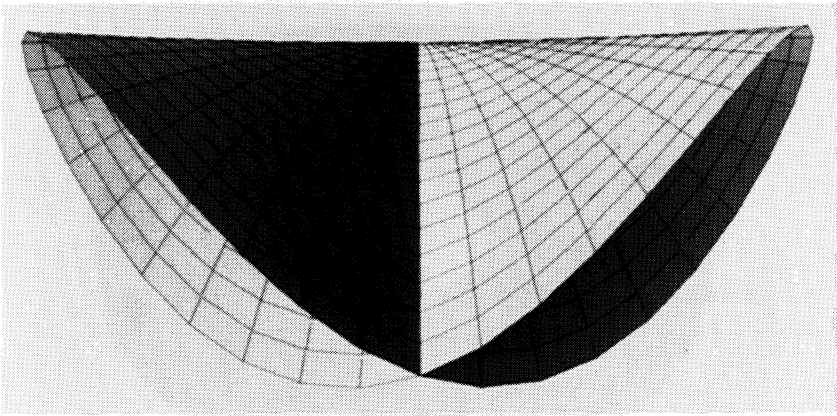


FIGURE 6

Selecting the minus sign in the definition of  $X$  reverses the sign of the final wedge product. To summarize what we have seen in this example, a positive intersection is one for which the wedge product of the two-vector  $GX * (\mathbf{f}_1) \wedge GX * (\mathbf{f}_2)$  with the two-vector representing a positive orientation of the tangent space to  $T$ , gives the previously-chosen orientation of the tangent space of  $G(2, 4)$ .

The other examples are similar. For a typical intersection with  $N$ , we offer

$$X(u, v) = (u, uv, v^2, v),$$

which projects down to  $\mathbf{R}^3$  as the Whitney umbrella (Figure 6). The Gauss map, before normalization,

$$GX(u, v) = ((u, 1 + 2v^2, -v), (u, -1 + 2v^2, -3v))$$

shows that the origin  $u = v = 0$  gives an isolated intersection with the set  $N$ , corresponding to the isolated pinch point of the Whitney umbrella.

This time the orientation of  $N$  at the point of intersection is given by  $(\mathbf{e}_1 - \mathbf{e}_4) \wedge (\mathbf{e}_3 - \mathbf{e}_6)$  (picture the inward orientation in the first factor and therefore the outward one on the second). When we wedge with  $GX * (\mathbf{f}_1) \wedge GX * (\mathbf{f}_2)$  the result is negative. This is a typical example of a negative pinch point.

When the sign of the second coordinate is switched, the sign of the intersection is reversed in this sense.

In choosing an example to illustrate the indexing of complex points, it was necessary to avoid the complex points belonging also to  $T$  and  $N$ . Our example is as follows:

$$X(u, v) = (u, v + (u^2 \pm v^2)/2, -u, -v + (u^2 \pm v^2)/2),$$

with Gauss map

$$GX(u, v) = ((1, \pm v, -u), (\pm v, 1, u))/(1 + u^2 + v^2)^{1/2}.$$

The origin  $u = v = 0$  is again an isolated intersection, this time with  $C$ . The tangent space to  $C$  in  $G(2, 4)$  is oriented by  $e_4 \wedge e_6$  and the sign of the wedge of this orientation with  $GX * (\mathbf{f}_1) \wedge GX * (\mathbf{f}_2)$  is the opposite of the sign of  $v$ .

**5. Constructing the explicit homology.** We construct our explicit homology in the Grassmannian in several steps. We first describe the “normal sweeps” leading to the construction of the three-dimensional chain with boundary given by  $T$ ,  $N$ , and  $C$ . We then describe the indexing of critical points and complete the proof of the theorem.

*5A. Normal sweeps.* In our attempt to relate the normal and tangential Euler classes, we considered the extent to which  $T$  and  $N$  failed to be homologous in  $G(2, 4)$ . Of course, since  $T$  can be viewed as the graph of the identity map of the sphere to itself, and  $N$  is the graph of the antipodal map of the sphere,  $T$  and  $N$  are simply not homologous. However, further investigation led to the construction of a three-cell in  $G(2, 4)$  which narrowly misses being a homotopy from  $T$  to  $N$ . In fact, it is a homotopy of  $T$  minus two points to  $N$  minus two points, the two points in each case being the two complex planes in the respective sets. We view this construction as explaining the presence of the term in the formula that counts complex points in the theorem.

We first note that whenever a plane belongs to  $T$ , its normal plane belongs to  $N$ . It turns out that the complex structure map  $J$  gives a reasonably natural way to rotate any plane into its normal plane, unless the plane is a complex one. This rotation can be carried out smoothly, resulting in a one-parameter family of planes beginning with any given non-complex plane, passing through its normal plane, and returning to the original one. We call these *normal sweeps*, since they sweep from a given plane to its normal plane.

The first relevant definition is that of a quantity we call the *reality angle* of a plane. Suppose we have a plane and wish to discover the extent to which it fails to be complex. If  $X$  is in the plane, then certainly  $JX$  will be perpendicular to  $X$ , but the angle between  $JX$  and other vectors in the plane can vary greatly. At the two extremes are planes for which  $JX$  belongs to the normal plane, and planes for which  $JX$  belongs to the original plane. If  $\omega = X \wedge Y$ , with  $X$  and  $Y$  orthonormal, then it is easily checked that  $\langle JX, Y \rangle$  and  $\langle -JY, X \rangle$

are independent of the particular choice of  $X$  and  $Y$ , depending only on the plane  $\omega$ . Since this inner product is between 1 and  $-1$ , we can call it  $\cos a$ , noting that  $a$  is zero if and only if  $\omega$  is a complex plane oriented by  $X \wedge JX$ , while  $a$  is  $\pi$  for complex planes oriented the other way. It is easy to show that in sphere coordinates, we have  $\cos a = a_1$ . (Planes with reality angle intermediate between these two have been called *totally real* by some analysts, while others restrict this term to the planes with reality angle  $\pi/2$ . The reality angle has also been called the “angle of holomorphy”.)

Now we use the reality angle to tilt a non-complex plane into its normal plane. If a plane has reality angle  $\pi/2$ , the complex structure map carries it into its normal plane; for other non-complex planes we simply modify  $J$  to accomplish this. If  $\omega = X \wedge Y$ , with  $X$  and  $Y$  orthonormal, a straightforward computation shows that

$$nX = (JX - \cos aY) / \sin a$$

is well-defined as a map from  $\omega$  to its normal plane, as long as  $\sin a$  is different from zero. The effect of this mapping on  $Y$ , as the second vector in the orthonormal pair, is easily computed:

$$nY = (JY + \cos aX) / \sin a.$$

With this choice of definition of the map  $n$ , the plane with the desired orientation orthonormal to  $\omega$  is  $nY \wedge nX$ . In effect, this is because the standard orientation of  $\mathbf{R}^4$  is  $X \wedge JX \wedge Y \wedge JY$  if  $X$  and  $Y$  belong to disjoint complex planes. Note that  $X \wedge Y \wedge nX \wedge nY$  is the negative of  $X \wedge nX \wedge Y \wedge nY$ .

We now simply fill in the gap between a plane and its normal by a rotation: if  $\omega = X \wedge Y$ , then we define

$$n_f \omega = (\cos fX + \sin fnX) \wedge (\cos fY - \sin fnY).$$

Note that this starts at  $X \wedge Y$ , passes through  $nY \wedge nX$  when  $f$  is  $\pi/2$ , and returns to  $X \wedge Y$  when  $f$  is  $\pi$ .

The 3-cell described above consists mostly of the collection of planes sweeping from the non-complex points of  $T$  through an angle of  $\pi/2$ , thus arriving at  $N$ . However, the set of complex planes actually belongs to the boundary of this set of normal sweeps, as will be made clear in the following analytic definition.

5B. *An analytic definition of the 3-cell.* Let  $f(\mathbf{a}, \mathbf{b}) = -a_1 \mathbf{a} \cdot \mathbf{b} + b_1$ . After motivating the choice of the function  $f$ , we will show that the set  $H = \{f = 0\}$  is locally a 3-manifold except at four singular points.

Then we will introduce a second function  $g$ , so that the sign of  $g$  divides  $H$  into three parts,  $H^+$ ,  $H^-$ , and  $H^0$ . Here  $H^0$  consists exactly of  $C$ ,  $T$ , and  $N$ , while  $H^+$  is an open 3-manifold with  $H^0$  as its boundary.

We designed  $f$  to vanish on the normal sweeps beginning in  $T$ . Computation shows that these all look essentially like the example described below: the point in the first factor remains fixed while the point in the second factor travels around the great circle passing through the starting vector  $\mathbf{a}$  and the point on the equator given by the cross-product of  $\mathbf{a}$  with  $(1, 0, 0)$ . Thus everywhere on the normal sweep,  $\mathbf{b}$  belongs to the plane determined by these two vectors. The function  $f$  is simply the dot product of  $\mathbf{b}$  and the normal vector of this plane.

**PROPOSITION.** *On the set  $H = \{f = 0\}$  the differential of  $f$  vanishes only at the poles  $\{((\pm 1, 0, 0), (\pm 1, 0, 0))\}$ . Therefore  $H$  is locally a 3-manifold except at these four points.*

*Proof.* Consider  $f$  as a function of six variables and compute

$$df = \mathbf{a} \cdot \mathbf{b} da_1 + a_1(b_1 da_1 + b_2 da_2 + b_3 da_3 + a_1 db_1 + a_2 db_2 + a_3 db_3) - db_1.$$

We examine the differential on two parts of  $H$ . First assume  $a_2^2 + a_3^2$  is non-zero and consider a point of the form  $(\mathbf{a}, (b_1, b_2, 0))$  that belongs to  $H$ . We exhibit a curve in  $G$  starting at this point and show that  $df$  gives a non-zero result when evaluated on its initial tangent vector. Let  $g(t) = (\mathbf{a}, (\cos tb_1 - \sin tb_2, \cos tb_2 + \sin tb_1, 0))$ . Then

$$df(g')|_{t=0} = a_1 a_2 b_1 + (a_2^2 + a_3^2) b_2.$$

But if  $f(\mathbf{a}, \mathbf{b}) = 0$ , we have  $-(a_2^2 + a_3^2)b_1 + a_1 a_2 b_2 = 0$ , and these two quantities cannot both vanish. That  $b_3$  was assumed to be zero is immaterial.

Now if  $a_1 = \pm 1$ ,  $df$  reduces to  $b_2 da_2 + b_3 da_3$ , since  $da_1$  is identically zero at such points. This vanishes only if  $b_2 = b_3 = 0$ . Hence, we obtain four singular points. This completes the proof.

Finally we introduce  $g(\mathbf{a}, \mathbf{b}) = a_2 b_3 - a_3 b_2 = \sin s$ , where  $s$  is the angle between the projections of  $\mathbf{a}$  and  $\mathbf{b}$  into the equatorial plane. Simple computation shows that  $\{g = 0\} \cap \{f = 0\}$  consists precisely of  $C$ ,  $T$ , and  $N$ , so  $g$  partitions  $H$  as claimed.

### 5C. *Indexing the critical points.*

DEFINITION. If  $S_1$  and  $S_2$  are local two-dimensional oriented submanifolds of an oriented four-manifold which intersect in exactly one point  $p$ , then the index of  $p$  is plus or minus one, depending on whether the wedge product of the orientations of  $S_1$  and  $S_2$  at  $p$  is the same as the orientation of the ambient manifold or the opposite.

Note that since the wedge of two two-vectors is independent of the order of wedging, the roles of  $S_1$  and  $S_2$  may be interchanged without changing the index of the intersection.

To apply this definition, we must assign orientations to  $T$ ,  $N$ , and  $C$ . On the one hand, these orientations are induced by the fact that they form the boundary of an oriented three-cell. Orienting  $H^+$  as the boundary of the set  $\{f > 0\}$  induces the outward orientation on  $T$ , the inward orientation on  $C$ , and an orientation on  $N$  that is inward on the first factor and outward on the second. On the other hand, it can be checked that the indices given by these orientations coincide with the classical indexing procedures used to compute the Euler numbers. For example, for the tangential Euler class, it is straightforward to compute the tangential component of  $e_4$  in the neighborhood of a critical point of a surface presented as a graph. The quantity used to compute the index is similar to the Hessian determinant mentioned in §2C. Explicit computations for the normal Euler number follow the same pattern.

5D. *The proof of theorem.* We now offer our constructive proof of the theorem:

THEOREM. *If  $X: M^2 \rightarrow \mathbf{R}^4$  is an immersion of an oriented two-surface in  $\mathbf{R}^4$ , denote the normal Euler number by  $n$  and the tangential Euler number by  $t$ . Let  $C$  denote the set of points at which the tangent plane is a complex line and for each  $p$  in  $C$ , let  $i(p)$  denote the index of  $p$ . Then*

$$n + t + \sum i(p) = 0,$$

where the sum is taken over all  $p$  in  $C$ .

Note that our indexing of complex points is the negative of the choice made in [W1] and [W2] so they appear on the other side of the equation.

This proof assumes various genericity hypotheses, to insure that intersections look simple. These are not necessary for the truth of the theorem; only for our illustration of it.

We now proceed to define the homology. First, recall from above the sets  $T$ ,  $N$  and  $C$  in  $G(2, 4)$ :  $T$  is the set of planes perpendicular to  $e_4$ ,  $N$  is the set of planes containing  $e_4$ , and  $C$  is the set of complex planes. The points where the image of the Gauss map,  $GX$ , meets  $T$  are those where the tangential component of  $e_4$  vanishes, and are hence those used to count the tangential Euler number. Those where this image meets  $N$  are those used to count the normal Euler number, while the points where the image meets  $C$  are the complex points. Thus we define

$$S_T = \{p \in M \mid GX(p) \in T\}, \quad S_N = \{p \in M \mid GX(p) \in N\},$$

and

$$S_C = \{p \in M \mid GX(p) \in C\}.$$

We assume, as is generically the case, that these three sets are disjoint. This amounts to assuming that the image of the Gauss map avoids the points where the three-cell  $H$  is singular.

Because the image of the Gauss map is generically two-dimensional, while our cell  $H^+$  is three-dimensional,  $GX(M)$  will meet  $H^+$  in a finite collection of curves, whose endpoints consist exactly of the union of  $S_T$ ,  $S_N$ , and  $S_C$ . Symbolically,

$$\{p \in M \mid fGX(p) = 0 \text{ and } gGX(p) \geq 0\} = C_1 \cup C_2 \cup \cdots \cup C_k,$$

where the  $C_i$ 's are disjoint curves on  $M$  with  $\bigcup \text{bd } C_i = S_T \cup S_N \cup S_C$ .

This already shows that the 0-chain consisting of the points in  $S_T \cup S_N \cup S_C$  is homologous to 0 mod 2. For instance, it would be impossible to embed the sphere in  $\mathbf{R}^4$  with only one complex point, for an odd number of points cannot be homologous to 0 mod 2.

But more is true. With appropriate orientation, the curves begin at points with index 1 and end at points with index  $-1$ , so the sum of all the indices is zero. Recall however that we reverse the orientation of  $C$ , so the index of complex points will appear on the other side of the equation. Thus  $[S_T] + [S_N] = [S_C]$  where the brackets indicate the sums of the indices of points in the 0-chain.

To orient the curves, we say that a vector  $Z$  tangent to the image of  $C_i$  goes in the direction of  $C_i$  if  $Z \wedge Y$  orients  $X(M)$ , where  $Yf > 0$ . It is easy to check that with these conventions, when a curve enters  $H^+$ , the index of the intersection is 1, while when it



exits, the index is  $-1$ . Using this indexing procedure for the 0-chain and the corresponding orientation for the 1-chain, we have an explicit homology relating the 0-chains that give the tangential Euler class, the normal Euler class, and the complex points of the immersion, and thus we have our constructive proof of the theorem.

The computer generated illustrations for this article were produced by David Kaplan on a Sun workstation using the program Fnoord developed at Brown University.

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