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We introduce some operators on the Bergman space A^2 on the unit ball that generalize the classical (big) Hankel operator. For such operators we prove boundedness, compactness, and Schatten-ideal property criteria. These extend known results. These new operators are defined in terms of a symbol. We prove in particular that for $2 \leq p < \infty$, these operators belong to the Schatten ideal S_p if and only if the symbol f is in the Besov space B_p. We also give several different characterizations of the norm on the Besov spaces B_p. In particular we prove that the Besov spaces are the mean oscillation spaces in the Bergman metric, for $1 \leq p < \infty$.

Consider the unit ball B in $\mathbf{C}^n$ and the Bergman space $A^2(B)$ of the holomorphic functions that are square integrable with respect to the (normalized) Lebesgue measure dm. The space A^2 admits a reproducing kernel $\mathscr{K}(z, w)$, the well-known Bergman kernel. Let $\mathscr{P}$ denote the orthogonal projection of $L^2(dm)$ onto A^2. The Hankel operator on A^2 with symbol f, H_f, is defined by

$$H_f g(z) = (I - \mathscr{P})(\overline{f} g)(z)$$
$$= \int_B \overline{(f(z) - f(w))} \mathscr{K}(z, w) g(w) \, dm(w).$$

In recent times, the Hankel operator H_f first appeared in the context of Hardy spaces on the unit circle. It has been extensively studied by now, even on the Bergman spaces on the disc and on the ball. Important papers in this context are [1], [3] for the case $n = 1$, and [2], [7], [19], and [21] and [23] for the case $n > 1$. It is known that H_f is bounded if and only if f is in the Bloch space (see §2) and it is compact if and only if f is in the little Bloch space. These results are due to Axler for the case of the unit disc, and to Arazy, Fisher, Janson, and Peetre, and independently to Zhu, for the case of the ball. The Schatten ideal properties of H_f have been studied too. Arazy, Fisher, and Peetre for $n = 1$, [3], and independently four (groups of) authors ([2], [7], [19] and [21]) have proved that for $c(n) < p < \infty$, the Hankel operator H_f is in the Schatten von Neumann class S_p

(see 2.5) if and only if f is in the Besov space B_p (see (4)). Here $c(n)$ is a value depending on the (complex) dimension n of the unit ball and it has been called "cut-off" in the literature. The cut-off $c(n)$ equals 1 for $n = 1$ and $2n$ for $n > 1$. If $0 < p \leq c(n)$ and H_f is in the Schatten class S_p, then $H_f = 0$ and f is constant.

An essential tool in proving the above-mentioned results is a characterization of the norm in the Besov spaces. Namely for $p > c(n)$ (in particular, for $n > 1$ if $p > 2n$) we have that the following are equivalent:

(i) $f \in B_p$;
(ii) $\int_B \int_B |f(z) - f(w)|^p |\mathscr{K}(z, w)|^2 \, dm(z) \, dm(w) < \infty$.

Moreover, the Besov spaces can be described in geometrical terms, that is, in terms of oscillations over balls in the Bergman metric. Let $\rho > 0$ be fixed, and for $\zeta \in B$, let E_ζ be the ball in the Bergman metric with center ζ and radius ρ. For g locally integrable, define the mean of g over E_ζ as

$$(1) \qquad g_{E_\zeta} = \frac{1}{|E_\zeta|} \int_{E_\zeta} g \, dm.$$

Here $|E_\zeta|$ denote the normalized Lebesgue measure of the ball E_ζ. Define the mean oscillation of g at ζ to be

$$(2) \qquad \mathrm{MO}_\rho \, g(\zeta) = \frac{1}{|E_\zeta|} \int_{E_\zeta} |g - g_{E_\zeta}| \, dm.$$

Then the following are equivalent for $p > 2n$ (see [22] for the case $n = 1$)

(i) $f \in B_p$;
(ii) $\mathrm{MO}_\rho \, f \in L^p(d\lambda)$.

It is also easy to see that if (ii) above holds and $0 < p \leq 2n$ then f has to be constant.

In this paper we investigate these phenomena, trying to explain what "happens" below the cut-off. We replace the expression $f(z) - f(w)$ by $f(w) - \sum_{|\alpha| < j} \partial^\alpha f(z)/(\alpha!)(w - z)^\alpha$ in all the different situations in which the former appears. The latter expression has a higher order of zero at $w = z$, that is, along the diagonal of $B \times B$. We introduce in this way what we call generalized Hankel operators of order j and with symbol f. We denote them by $H_{f;j}$ and $H'_{f;j}$, explicitly:

$$H'_{f;j}g(z) = \int_B \overline{\left(f(w) - \sum_{|\alpha| < j} \frac{\partial^\alpha f(z)}{\alpha!}(w - z)^\alpha \right)} \frac{g(w)}{(1 - z \cdot \overline{w})^{n+1}} \, dm(w);$$

and

$$H_{f;j}g(z) = \int_B -\overline{\left(f(z) - \sum_{|\alpha|<j} \frac{\partial^\alpha f(w)}{\alpha!}(z-w)^\alpha\right)} \frac{g(w)}{(1-z\cdot\overline{w})^{n+1}}\, dm(w).$$

These operators coincide with the classical one when the order j is 1. For these operators we prove boundedness and compactness criteria. In these cases the results are the same as in the classical case: the generalized Hankel operators are bounded if and only if the symbol is in the Bloch space, and they are compact if and only if the symbol is in the little Bloch space. For the range $2 \le p < \infty$, we prove that these new operators belong to S_p if and only if $b \in B_p$ when p is in the range $2n/j < p < \infty$. That is, we are able to lower the cut-off $2n$ by increasing the order of zero on the diagonal in the expression given by the symbol. In this process, we characterize the norm on the Besov spaces in terms of double integrals against the modulus squared of the Bergman kernel, extending the description mentioned earlier. As a consequence we also prove that, for $1 \le p < \infty$, the Besov spaces are the mean oscillation spaces in the Bergman metric.

The phenomenon of the cut-off for Hankel-type operators was first observed by Janson and Wolff in [10] in the context of commutators of singular integral operators. Later, Rochberg raised the question of finding Hankel operators with cut-off $1/j$, j a positive number. Recently, Janson, Peng, Rochberg, and Wu have studied some operators on the Bergman space on the unit disc, called intermediate or middle Hankel operators, with cut-off $1/j$, where j is the "order" of the operator. Their approach is different, and we refer the reader to the papers [9] and [13], and references therein.

The paper is organized as follows. In §1 we state the main results. Section 2 summarizes the known facts and definitions needed in the sequel. Section 3 contains the results about what we call higher moment conditions. This characterization of the Besov spaces is of its own interest, but also allows us to prove the theorem concerning the generalized Hankel operators. This is the content of §4. Section 5 contains the proof of one important estimate. Finally in §6 we prove that the Besov spaces can be characterized as mean oscillation spaces; we make few remarks and present some open questions.

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1. Statement of the main results. Throughout this paper we denote by B the unit ball in $\mathbf{C}^n$. For $z, w \in \mathbf{C}^n$ we write the inner product as

$$z \cdot \overline{w} = \sum_{i=1}^n z_i \overline{w}_i.$$

The space of holomorphic functions on B will be denoted by $\mathscr{H}(B)$ and the group of automorphisms of B by Aut B. The group Aut B consists of all biholomorphic self-maps of B onto itself. The group Aut B can be described as follows (see [14]). For any $\zeta \in B$ define φ_ζ by

$$(3) \qquad \varphi_\zeta = \frac{\zeta - P_\zeta z - (1 - |z|^2)^{1/2} Q_\zeta z}{1 - z \cdot \overline{\zeta}},$$

where P_ζ is the orthogonal projection onto the subspace generated by ζ and $Q_\zeta = I - P_\zeta$. Then $\varphi_\zeta \in$ Aut B and φ_ζ is an involution that interchanges ζ with the origin. Moreover,

$$\text{Aut } B = \{\varphi_\zeta \circ U : \zeta \in B, U \in \mathscr{U}\},$$

where $\mathscr{U}$ is the space of unitary transformations of $\mathbf{C}^n$.

The normalized Lebesgue measure on B will be denoted by dm, and the *invariant volume form* by $d\lambda$. Explicitly,

$$d\lambda(z) = \frac{dm(z)}{(1 - |z|^2)^{n+1}}$$

(see [14]).

The gradient of a holomorphic function f will be denoted by ∂f, that is,

$$\partial f = (\partial_1 f, \ldots, \partial_n f)$$

where

$$\partial_j f = \frac{\partial f}{\partial z_j}.$$

Moreover we will use the following notations. We write

$$\partial^\alpha f(z) = \frac{\partial^\alpha f}{\partial z^\alpha}(z),$$

where α is a multi-index. The radial derivative of $f \in \mathscr{H}(B)$ is defined to be

$$Rf(z) = z \cdot \partial f(z) = \sum_{i=1}^n z_i \partial_i f(z).$$

The *Bergman space* A^2 is the closed subspace of $L^2(dm)$ that consists of the holomorphic functions. We write $\mathcal{K}(z, w)$ to denote the Bergman kernel, that is, the reproducing kernel of $A^2(B)$, and $\mathcal{P}$ to denote the Bergman projection. Explicitly, for any $g \in L^2(dm)$,

$$\mathcal{P}g(z) = \int_B g(w) \frac{1}{(1 - z \cdot \overline{w})^{n+1}} \, dm(w).$$

The (*diagonal*) *Besov spaces* of analytic functions B_p, $0 < p < \infty$, are defined by

(4) $B_p = \{f \in \mathcal{H}(B) : (1 - |z|^2)^m R^m f(z) \in L^p(d\lambda)\},$

where $m > n/p$. We remark that the Besov spaces B_p defined here are the spaces $B_p^{n/p}$ (or $B_p^{n/p,\,1/p}$) of the scale of weighted Besov spaces (see [11] or [12]). We will not pursue the weighted case in this paper, and we consider only the spaces defined in (4).

DEFINITION 1.1. For any $f \in \mathcal{H}(B)$, j a positive integer, and for z, $w \in B$, we set

$$\Delta_j f(w, z) = f(w) - \sum_{|\alpha| < j} \frac{\partial^\alpha f(z)}{\alpha!} (w - z)^\alpha.$$

Notice the different roles played by the variables w and z. The first should be interpreted as the variable, and the second as the base point for the Taylor expansion.

We are ready to state our first main result.

THEOREM 1.2. *Let* $0 < p < \infty$, $n \geq 2$. *Let* j *be a positive integer. If* $j > 2n/p$ *the following are equivalent.*

(i) $\|f\|_{B_p} < \infty$.

(ii) $\int_B \int_B \frac{|\Delta_j f(w, z)|^p}{(1 - z \cdot \overline{w})^{2(n+1)}} \, dm(z)\, dm(w) < \infty$.

If $j \leq 2n/p$ *and* $f \in \mathcal{H}(B)$ *satisfies* (ii), *then* f *is a polynomial of degree at most* $j - 1$.

REMARK 1.3. A version of the above theorem holds for $n = 1$. In this case the integer j must satisfy the inequality $j > 1/p$. This phenomenon will carry over through the whole paper. The general reading key is that one should replace "$2n$" with "1" when $n = 1$. This is a well-known fact by now. It depends essentially on the shape of the balls in the Bergman metric.

Even if not stated explicitly, all the results proved for $n > 1$ *carry over to the case* $n = 1$, *with the convention mentioned above.*

REMARK 1.4. The above theorem is an important result. It gives an equivalent norm for all Besov spaces in terms of a double integral against the modulus squared of the Bergman kernel. It gives a natural extension to the case $j > 1$ and $0 < p \le 2n$ of the results in [2], [22], [19], and [21] where they treat the case $j = 1$.

We now introduce the generalized Hankel operators on the Bergman space $A^2(B)$.

DEFINITION 1.5. Let j be a positive integer. Let $f \in \mathscr{H}(B)$ and such that $\partial^\alpha f \in L^1(dm)$ for $|\alpha| < j$. For $g \in L^\infty \cap \mathscr{H}(B)$, we define the generalized Hankel operators of order j and with symbol f, $H_{f;j}$ and $H'_{f;j}$ by

$$H_{f;j}g(z) = \int_B \overline{-\Delta_j f(w,z)} \mathscr{K}(z,w) g(w) \, dm(w),$$

and

$$H'_{f;j}g(z) = \int_B \overline{\Delta_j f(z,w)} \mathscr{K}(z,w) g(w) \, dm(w).$$

If $H_{f;j}$ and $H'_{f;j}$ turn out to be bounded on A^2, we can extend the domain of the operators to all of A^2.

REMARK 1.6. Notice that if $j = 1$, $H_{f;1} = H'_{f;1} = H_f$, where H_f is the classical Hankel operator on $A^2(B)$.

For the generalized Hankel operators our main theorems are the following.

THEOREM 1.7. *Let* $f \in \mathscr{H}(B)$, j *be any positive integer. Then the following are equivalent.*

(i) $H_{f;j}$ *is bounded.*
(ii) $H'_{f;j}$ *is bounded.*
(iii) $f \in \mathscr{B}$.

THEOREM 1.8. *Let* $f \in \mathscr{H}(B)$, j *be any positive integer. Then the following are equivalent.*

(i) $H_{f;j}$ *is compact.*
(ii) $H'_{f;j}$ *is compact.*
(iii) $f \in \mathscr{B}_0$.

THEOREM 1.9. *Let* $n \ge 2$, $f \in \mathscr{H}(B)$, *and let* $2 \le p < \infty$. *Let* j *be any positive integer. If* $j > 2n/p$, *then the following are equivalent.*

(i) $H_{f;j} \in S_p$.

(ii) $H'_{f;j} \in S_p$.

(iii) $f \in B_p$.

If $j \le 2n/p$ and $H_{f;j}$ or $H'_{f;j} \in S_p$, then $H_{f;j} = H'_{f;j} = 0$ and f is a polynomial of degree $j - 1$ at most.

REMARK 1.10. The above theorem is an extension of the results about S_p-properties of the classical Hankel operators on the Bergman spaces, as proved in the already mentioned papers [3], [2], [7], [19], and [21]. Theorem 1.9 describes the phenomenon of the cut-off in some detail. We should interpret the order j as the order of zero along the diagonal of the term $\Delta_j f$ defined by the symbol f. As j grows we obtain better S_p-properties for the operators $H_{f;j}$ and $H'_{f;j}$.

We conclude this section by stating our theorems about the mean oscillation spaces in the Bergman metric.

Let $\rho > 0$, and let $E_\zeta = E(\zeta, \rho)$ be the ball in the Bergman metric with center ζ and radius ρ. Recall that in (1) we have defined the mean of a locally integrable function over E_ζ.

DEFINITION 1.11. Let $f \in \mathcal{H}(B)$ and let j be a positive integer. We define the mean oscillation of order j of f at ζ to be

$$\mathrm{MO}_\rho^j f(\zeta) = \frac{1}{|E_\zeta|} \int_{E_\zeta} |\Delta_j f(w, \zeta) - (\Delta_j f(\cdot, \zeta))_{E_\zeta}| \, dm(w).$$

THEOREM 1.12. *Let $1 \le p < \infty$, and let j be a positive integer. Let $f \in \mathcal{H}(B)$ and $\mathrm{MO}_\rho^j f$ be defined as in 1.11. Then there exists $\rho_0 > 0$ such that for $0 < \rho < \rho_0$ the following hold. For $j > 2n/p$ the following are equivalent.*

(i) $f \in B_p$.

(ii) $\mathrm{MO}_\rho^j f \in L^p(d\lambda)$.

If $j \le 2n/p$ and $\mathrm{MO}_\rho^j f \in L^p(d\lambda)$, then f is a polynomial of degree at most $j - 1$.

Again this result extends the previous results by Zhu to the case (of the ball, and) to the case $1 \le p \le 2n$ and $j > 1$.

2. Preliminaries. For a C^1-function g we denote its gradient by ∇g. We define the *invariant derivative* of g by setting

(5) $$\tilde{D}g(z) = |\nabla(g \circ \varphi_z)(0)|.$$

DEFINITION 2.1. For $i, j = 1, \ldots, n$ define the differential operators T_{ij} by

$$T_{ij} = \overline{z}_i \partial_j - \overline{z}_j \partial_i \,.$$

Then T_{ij} is tangential, in the sense that $T_{ij}(|z|^2) = 0$. For C^1-function g we define the tangential derivative $D_\tau g$ by

$$D_\tau g(z)^2 = \sum_{ij} |T_{ij} g(z)|^2 \,.$$

When $k \geq 2$ and $g \in C^k$ we set

$$D_\tau^k g(z)^2 = \sum_{i_1, j_1, \ldots, i_k, j_k} |T_{i_1 j_1} \cdots T_{i_k j_k} g(z)|^2 \,.$$

Then D_τ measures the size of ∇g in the complex tangential directions. For $f \in \mathcal{H}(B)$ we have the following identities (recall that we write ∂f for ∇f if f is holomorphic).

$$|z|^2 |\partial f(z)|^2 = \left| \sum_{i=1}^n z_i \partial_i f(z) \right|^2 + \sum_{i,j=1}^n (|z|^2 \delta_{ij} - z_i \overline{z}_j) \partial_i f(z) \overline{\partial_j f(z)}$$

$$= |Rf(z)|^2 + D_\tau f(z)^2 \,.$$

We can relate the radial and tangential derivative to the invariant derivative. Using Möbius invariance it is easy to see that for $f \in \mathcal{H}(B)$

$$\begin{aligned}
\widetilde{D} f(z)^2 &= |\partial (f \circ \varphi_z)(0)|^2 \\
&= (1 - |z|^2)(|\partial f(z)|^2 - |Rf(z)|^2) \\
&= \frac{1}{|z|^2}((1 - |z|^2)^2 |Rf(z)|^2 + (1 - |z|^2) D_\tau f(z)^2) \,.
\end{aligned}$$

An application of the classical Hardy's lemma (see [8], Theorem 330, or [2], Lemma 3.3) gives the next result. Set

$$(6) \qquad |\partial^k f(z)|^2 \equiv \sum_{|\alpha|=k} |\partial^\alpha f(z)|^2 \,.$$

LEMMA 2.2. *Let $f \in \mathcal{H}(B)$ be such that $\partial^\alpha f(0) = 0$ for $|\alpha| < k$. Then for $0 < p < \infty$, $s > -1$, there exists a constant $C > 0$ such that*

$$\int_B (1 - |z|^2)^s |f(z)| \, dm(z) \leq C \int_B (1 - |z|^2)^{s+kp} |\partial^k f(z)|^p \, dm(z) \,.$$

Proof. The case $k = 1$ is just Lemma 3.3 together with Remark 3.2 in [2]. Now it suffices to iterate the same argument using the fact that $\partial^\alpha f(0) = 0$ for $|\alpha| < k$. $\qquad \square$

The *Bloch space* $\mathscr{B}$ on B has been introduced by Timoney in [17] and [18], and then studied intensively by many authors. The space $\mathscr{B}$ is defined as the space of holomorphic functions f for which $\tilde{D}f$ is bounded in B. The little Bloch space $\mathscr{B}_0$ is defined to be the subspace of functions $f \in \mathscr{B}$ for which $\tilde{D}f(z) \to 0$ as $|z| \to 1^-$.

Recall that in (4) we have defined the Besov spaces B_p by

$$B_p = \left\{ f \in \mathscr{H}(B) : \int_B ((1 - |z|^2)^m |R^m f(z)|)^p \, d\lambda(z) < \infty \right\}.$$

Here m is any integer such that $mp > n$. It is well known that the definition of B_p is independent of m, and that one can replace $|R^m f(z)|$ with $|\partial^m f(z)|$ in the definition of the Besov spaces (recall the notation (6)). We want to describe the atomic decomposition of the Besov spaces B_p, as obtained by Coifman and Rochberg in [6] (see [12] for some details).

THEOREM 2.3 (*Coifman and Rochberg*). *Let* $0 < p < \infty$. *Let*

$$\nu > \max \left(\frac{n}{p}(p - 1), 0 \right).$$

Then there exists a positive number $\vartheta_0 = \vartheta_0(p, \nu, n)$ *and a constant* C *such that if the points* $\{\zeta_i\}$ *form a* ϑ-*lattice with* $\vartheta < \vartheta_0$ *then the following hold.*

(A) *If* $f \in B_p$ *there exist numbers* $\{c_i\}$ *such that*

$$f(z) = \sum_1^\infty c_i \frac{(1 - |\zeta|^2)^\nu}{(1 - z \cdot \overline{\zeta^i})^\nu}$$

and

$$\sum_1^\infty |c_i|^p \leq \|f\|_{B_p}^p.$$

(B) *If* $\sum_1^\infty |c_i|^p < \infty$ *then the function defined in* (A) *is in* B_p *and*

$$\|f\|_{B_p}^p \leq \sum_1^\infty |c_i|^p.$$

We recall the Forelli-Rudin estimate (see [14], 1.4.10).

PROPOSITION 2.4. *Let* $s > -1$, $t \in \mathbf{R}$. *Define*

$$I_t(z) = \int_B \frac{(1 - |w|^2)^s}{|1 - z \cdot \overline{w}|^{n+1+s+t}} \, dm(w).$$

Then, as $|z| \to 1^-$, I_t *is bounded if* $t < 0$, I_t *is asymptotic to* $\log(1 - |z|^2)^{-1}$ *if* $t = 0$, *and asymptotic to* $(1 - |z|^2)^{-t}$ *if* $t > 0$.

Finally, we define the Schatten ideal S_p.

DEFINITION 2.5. Let $1 < p < \infty$. Let H be a Hilbert space and let T be a compact operator on H. For any integer k define the $s(k)$-number of T by

$$s(k) = \inf\{\|T - F\| : \operatorname{rank}(F) \le k\}.$$

Then we say that $T \in S_p$ if

$$\sum_{k=1}^{\infty} s(k)^p < \infty.$$

In this case we set

$$\|T\|_{S_p} = \left(\sum_{k=1}^{\infty} s(k)^p\right)^{1/p}.$$

3. Higher moment conditions. Recall that in (1.1) we have defined the jth difference of f at z by

$$\Delta_j f(w, z) = f(w) - \sum_{|\alpha| < j} \frac{\partial^\alpha f(z)}{\alpha!}(w - z)^\alpha.$$

Now let $d\Lambda(z, w)$ be the measure on $B \times B$ defined by

$$(7) \qquad d\Lambda(z, w) = \frac{1}{|1 - z \cdot \overline{w}|^{2(n+1)}} \, dm(z) \, dm(w).$$

In the next proposition recall the note made in 1.3.

PROPOSITION 3.1. *Let* $n > 1$, $0 < p < \infty$, j *be an integer,* $j > 2n/p$. *Then there exists a constant* $C = C(j, p, n)$ *such that for all* $f \in B_p$ *we have*

$$\|\Delta_j f\|_{L^p(d\Lambda)} \le C\|f\|_{B_p}.$$

Although this result is essential in the sequel, the proof is rather long and (somewhat) technical. It is therefore postponed until §5. However, some of the techniques involved might be of interest of their own.

Recall that we have set

$$(8) \qquad |\partial_w^k g|^2 = \sum_{|\alpha|=k} \left|\frac{\partial^\alpha g}{\partial w^\alpha}\right|^2.$$

PROPOSITION 3.2. *Let* $n > 1$, $0 < q < \infty$ *and* j *be any positive integer. Then for* $f \in \mathscr{H}(B)$ *we have that*

$$(1 - |z|^2)^j |R^j f(z)| \leq C \left(\int_B |\Delta_j f(\varphi_z(w), z)|^q \, d(w) \right)^{1/q},$$

and

$$(1 - |z|^2)^j |R^j f(z)| \leq C \left(\int_B |\Delta_j f(z, \varphi_z(w))|^q \, dm(w) \right)^{1/q},$$

for some constant C *independent of* f.

Proof. Recall that $\Delta_j f(\varphi_z(w), z)$ vanishes of order j at $\varphi_z(w) = z$, i.e. at $w = 0$. Therefore taking j derivatives of $\Delta_j f(\varphi_z(\cdot), z)$ and evaluating at $w = 0$, in order to obtain non-zero terms, we need to land all derivatives on f, not on the terms coming from the chain rule. Moreover, we use the identity

$$\varphi_z'(0) = -(1 - |z|^2) P_z - (1 - |z|^2)^{1/2} Q_z,$$

as in [14], 2.2.2. Thus,

$$|\partial_w^j \Delta_j f(\varphi_z(w), z)|_{w=0}| \approx \sum_{i=0}^{j} (1 - |z|^2)^{j-i/2} |D_\tau^i R^{j-i} f(z)|.$$

In an analogous manner, it is easy to see that

$$\frac{\partial}{\partial w_k} \Delta_j f(z, \varphi_z(w)) = \frac{\partial \varphi_z}{\partial w_k}(w) \cdot \sum_{|\alpha|=j-1} \frac{\partial^\alpha \partial f}{\alpha!}(\varphi_z(w))(z - \varphi_z(w))^\alpha.$$

Therefore, applying the same observation as above, we see that

$$|\partial_w^j \Delta_j f(z, \varphi_z(w))|_{w=0}| \approx \sum_{i=0}^{j} (1 - |z|^2)^{j-i/2} |D_\tau^i R^{j-i} f(z)|.$$

Now, write $F(w, z)$ for any of the expressions $\Delta_j f(w, z)$ or $\Delta_j f(z, w)$. By the Cauchy estimates it follows that

$$(1 - |z|^2)^j |R^j f(z)| \leq |\partial_w^j F(\varphi_z(w), z)|_{w=0}|$$

$$\leq C \left(\int_B |F(\varphi_z(w), z)|^q \, dm(w) \right)^{1/q}. \qquad \square$$

THEOREM 3.3. *Let* $n > 1$, *and* $0 < p < \infty$. *Let* $0 < q \leq p$ *and* j *be a positive integer satisfying* $j > 2n/p$. *Then the following are equivalent for* $f \in \mathcal{H}(B)$.

(i) $\|\Delta_j f(\varphi_z(\cdot), z)\|_{L^q(dm)} \in L^p(d\lambda)$.

(ii) $\|\Delta_j f(z, \varphi_z(\cdot))\|_{L^q(dm)} \in L^p(d\lambda)$.

(iii) $\int_B \int_B |\Delta_j f(w, z)|^p |\mathcal{K}(z, w)|^2 \, dm(z) \, dm(w) < \infty$.

(iv) $f \in B_p$.

If $j \leq 2n/p$ *and* f *satisfies one of the conditions* (i)–(iii), *then* f *is a polynomial of degree at most* $j - 1$.

Proof. Again, write $F(w, z)$ for either expression $\Delta_j f(w, z)$ or $\Delta_j f(z, w)$. If $q \leq p$, by 3.2 and by Jensen inequality it follows that

$$\int_B ((1 - |z|^2)^j |R^j f(z)|)^p \, d\lambda(z) \leq c \int_B \|F(\varphi_z(\cdot), z)\|^p_{L^q(dm)} \, d\lambda(z)$$

$$= c \int_B \int_B |\Delta_j f(\eta, z)|^p \, d\Lambda(z, \eta),$$

where we have performed the change of variables $\eta = \varphi_z(w)$ in the inner integral. Recall that the Jacobian of the above transformation is

$$\left(\frac{1 - |z|^2}{|1 - \eta \cdot \overline{z}|^2} \right)^{n+1}$$

(see [14] 2.2.6). Thus, we have proved that

$$\text{(iii)} \Rightarrow \text{(i)} \Rightarrow \text{(iv)},$$

and

$$\text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv)}.$$

Now, 3.1 gives that (iv) $\Rightarrow$ (iii).

We only need to prove the statement when $j \leq 2n/p$. It suffices to notice that if

$$\|F(\varphi_z(\cdot), z)\|_{L^q(dm)} \in L^p(d\lambda)$$

then $\Delta_j f \equiv 0$. By subharmonicity and 3.2 again we have that

$$\|F(\varphi_z(\cdot), z)\|_{L^q(dm)} \geq |\partial_w^j F(\varphi_z(w), z)|_{w=0}|$$

$$\geq (1 - |z|^2)^{j/2} D_\tau^j f(z).$$

Thus,

$$\int_B (1 - |z|^2)^{jp/2} D_\tau^j f(z)^p \, d\lambda(z)$$

must be finite. If we assume (without loss of generality) that f is holomorphic across the boundary, it follows that

$$D_\tau^j f \equiv 0.$$

Simple considerations now show that f is a polynomial of degree less than j.   □

We want to prove a version of the previous theorem for the case $p = \infty$. We need few modifications.

THEOREM 3.4. *Let* $f \in \mathscr{H}(B)$, j *be a positive integer*, $0 < q < \infty$. *Then the following quantities are equivalent.*

$$\mathrm{I} := \sup_{\zeta \in B}(1 - |\zeta|^2)^j |R^j f(\zeta)|.$$

$$\mathrm{II} := \sup_{\zeta \in B}\left(\int_B |\Delta_j f(\varphi_\zeta(w), \zeta)|^q \, dm(w)\right)^{1/q}.$$

REMARK. The case $j = 1$ of the theorem is proved in [1]; similar results are proved in [16].

Proof. By 3.2 it follows that

$$(1 - |\zeta|^2)^j |R^j f(\zeta)| \le c \left(\int_B |\Delta_j f(\varphi_\zeta(w), \zeta)|^q \, dm(w)\right)^{1/q}.$$

Thus,

$$\mathrm{I} \lesssim \mathrm{II}.$$

Write F for $\Delta_j f$. Since $F(\varphi_\zeta(w), \zeta)$ has a zero of order j at $w = 0$, we can apply Hardy's Lemma 2.2.

$$\text{(9)} \quad \|F(\varphi_\zeta(\cdot), \zeta)\|_{L^p(dm)}^p$$

$$\le \int_B (1 - |w|^2)^{jp} |\partial_w^j F(\varphi_\zeta(w), \zeta)|^p \, dm(\zeta)$$

$$\le c \sup_{\xi \in B} \widetilde{D}(F(\varphi_\zeta(\cdot), \zeta))(\xi) = c \sup_{\xi \in B} \widetilde{D}(F(\cdot, \zeta))(\xi),$$

where we have used the equality

$$\sup_{\xi \in B} \widetilde{D}(g \circ \psi) = \sup_{\xi \in B} \widetilde{D}g.$$

Now it follows that

$$\|F(\,\cdot\,,\zeta)\|_{\mathscr{B}} \leq \sup_{z\in B}(1-|z|^2)|\partial_z(F(z\,,\zeta))| \leq c\|f\|_{\mathscr{B}}\,.$$

This shows that

$$\mathrm{II} \lesssim \mathrm{I}.$$

Hence, we are done. $\square$

COROLLARY 3.5. *Let* $f \in \mathscr{H}(B)$. *Let* j *be any positive integer,* $0 < p < \infty$, *and* $\rho > 0$. *Then the following are equivalent.*

(i) $f \in \mathscr{B}_0$.
(ii) $\lim_{|\zeta|\to 1}\|\Delta_j f(\varphi_\zeta(\,\cdot\,)\,,\zeta)\|_{L^p(dm)} = 0$.

Proof. This is actually a corollary of the proof of 3.4. Recall the notation there introduced. We have shown that

$$\mathrm{I} \lesssim \mathrm{II}.$$

Thus, (ii) $\Rightarrow$ (i). Assume that (i) holds, and let f be holomorphic across the boundary. Then (ii) holds as a consequence of the Lebesgue dominated convergence theorem. Now a density argument finishes the proof. $\square$

4. Generalized Hankel operators. In this section we prove the main theorems about the generalized Hankel operators. Recall that we have defined the operators $H_{f;j}$ and $H'_{f;j}$ on A^2 by setting

$$H_{f;j}g(z) = \int_B \overline{-\Delta_j f(w\,,z)}\mathscr{K}(z\,,w)g(w)\,dm(w)\,,$$

and

$$H'_{f;j}g(z) = \int_B \overline{\Delta_j f(z\,,w)}\mathscr{K}(z\,,w)g(w)\,dm(w)\,.$$

The operators $H_{f;j}$ and $H'_{f;j}$ share many important properties with the classical big Hankel operators. Some of these are illustrated in the propositions that follow. Write $(A^2)^\perp$ to indicate the orthogonal complement of A^2 in $L^2(dm)$.

LEMMA 4.1. *Let* j *be any positive integer. Then*

$$H_{f;j}\,,H'_{f;j}\colon A^2 \to (A^2)^\perp\,.$$

Proof. Same as in the classical case. $\square$

LEMMA 4.2. *Write $\mathscr{K}_\zeta(z) = \mathscr{K}(z, \zeta)$. Then for any positive integer j*

$$H_{f;j}\mathscr{K}_\zeta = \overline{-\Delta_j f(\zeta, \cdot)}\mathscr{K}_\zeta,$$

and

$$H'_{f;j}\mathscr{K}_\zeta = \overline{\Delta_j f(\cdot, \zeta)}\mathscr{K}_\zeta.$$

Proof. It suffices to recall that for $h \in A^2$, $\mathscr{P}(\overline{h}\mathscr{K}_\zeta) = \overline{h}(\zeta)\mathscr{K}_\zeta$. □

The next lemma is a generalization of Lemma 5 in [1].

LEMMA 4.3. *Let $f \in \mathscr{K}(B)$, j be a positive integer. Let $0 < p < 1$. Then*

$$\int_B |\Delta_j f(w, z)|^2 \frac{(1 - |w|^2)^{-p}}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w) \le c(1 - |z|^2)^{-p}\|f\|_{\mathscr{B}}^2.$$

Proof. Write $F(w, z)$ for $\Delta_j f(w, z)$. Using the identities in [14], 2.2.2 and the change of variables $w = \varphi_z(\zeta)$, we obtain

$$(10) \quad \int_B |F(w, z)|^2 \frac{(1 - |w|^2)^{-p}}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w)$$

$$= \int_B |F(\varphi_z(\zeta)), z|^2 \frac{(1 - |\varphi_z(\zeta)|^2)^{-p}}{|1 - z \cdot \overline{\varphi}_z(\zeta)|^{n+1}} \frac{(1 - |z|^2)^{n+1}}{|1 - \zeta \cdot \overline{z}|^{2(n+1)}} \, dm(\zeta)$$

$$= (1 - |z|^2)^{-p} \int_B |F(\varphi_z(\zeta), z)|^2 \frac{(1 - |\zeta|^2)^{-p}}{|1 - \zeta \cdot \overline{z}|^{n+1-2p}} \, dm(\zeta).$$

Let

$$1 < q' < \min\left(\frac{1}{p}, \frac{n+1}{n+1-p}\right),$$

and set $q = q'/(q'-1)$. Then applying Hölder's inequality to equation (10) we obtain

$$\int_B |F(w, z)|^2 \frac{(1 - |w|^2)^{-p}}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w)$$

$$\le (1 - |z|^2)^{-p} \left(\int_B |F(\varphi_z(\zeta), z)|^{2q} \, dm(\zeta)\right)^{1/q}$$

$$\times \left(\int_B \frac{(1 - |\zeta|^2)^{-pq'}}{|1 - \zeta \cdot \overline{z}|^{(n+1-2p)q'}} \, dm(\zeta)\right)^{1/q'}.$$

Because of our choice of q', it follows that $-pq' > -1$ and $(n + 1 - 2p)q' < n + 1 - pq'$. Now, 2.4 implies that the second integral in the above inequality is bounded by a constant. Therefore applying 3.4 we obtain the result. □

Now we are ready to prove one of the main results of the paper.

THEOREM 4.4. *Let $f \in \mathscr{H}(B)$, j be any positive integer. Let k_ζ be the normalized reproducing kernel; i.e. $\mathscr{K}_\zeta / \|\mathscr{K}_\zeta\|$. Then the following quantities are equivalent.*

$$I := \sup_{\zeta \in B} (1 - |\zeta|^2)^j |R^j f(\zeta)|.$$

$$II' := \sup_{\zeta \in B} \|\Delta_j f(\zeta, \varphi_\zeta(\cdot))\|_{L^2(dm)}.$$

$$III := \sup_{\zeta \in B} \|H_{f;j} k_\zeta\|.$$

$$IV := \|H_{f;j}\|.$$

Proof. By 3.4 we know that

$$I \lesssim II'.$$

By 4.2 it follows that

$$|H_{f;j} k_\zeta(z)| = |\Delta_j f(\zeta, z) k_\zeta(z)|$$

so that

$$\|H_{f;j} k_\zeta\|^2 = \int_B |\Delta_j f(\zeta, z)|^2 |k_\zeta|^2 \, dm(z)$$
$$= \|\Delta_j f(\zeta, \varphi_\zeta(\cdot))\|^2_{L^2(dm)},$$

i.e.

$$II' = III.$$

It is trivial that $III \leq IV$. We now prove that $IV \lesssim I$. Let $g \in A^2$. Then applying 4.3 we obtain

$$|H_{f;j} g(z)|^2$$
$$\leq \left(\int_B |\Delta_j f(w, z)| \frac{|g(w)|}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w) \right)^2$$
$$\leq \int_B |\Delta_j f(w, z)|^2 \frac{(1 - |w|^2)^{-\rho}}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w)$$
$$\times \int_B |g(w)|^2 \frac{(1 - |w|^2)^\rho}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w)$$
$$\leq c(1 - |z|^2)^{-\rho} \|f\|^2_{\mathscr{B}} \int_B |g(z)|^2 \frac{(1 - |w|^2)^\rho}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w).$$

Thus,

$$\|H_{f;j} g\|^2 \leq \|f\|^2_{\mathscr{B}} \int_B (1 - |w|^2)^\rho |g(w)|^2 \int_B \frac{(1 - |z|^2)^{-\rho}}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(z) \, dm(w)$$
$$\leq c \|f\|^2_{\mathscr{B}} \|g\|^2,$$

by 2.4.   □

COROLLARY 4.5. *Let* $f \in \mathcal{H}(B)$, j *be any positive integer. Then the following are equivalent.*

$$\mathrm{I} := \sup_{\zeta \in B}(1 - |\zeta|^2)^j |R^j f(\zeta)|.$$

$$\mathrm{II}' := \sup_{\zeta \in B} \left(\int_B |\Delta_j f(\zeta, \varphi_\zeta(w))|^2 \, dm(w) \right)^{1/2}.$$

Moreover, let $0 < \rho < 1$. *Then*

$$\int_B |\Delta_j f(z, w)|^2 \frac{(1 - |w|^2)^{-\rho}}{|1 - z \cdot \overline{w}|^{n+1}} \, dm(w) lec(1 - |z|^2)^{-\rho} \|f\|_{\mathscr{B}}^2.$$

Proof. The equivalence of II and II$'$ is contained in 4.4. With this, the same proof as 4.3 gives the second part of the statement. $\quad\square$

The following is the companion of 4.4.

THEOREM 4.6. *Let* $f \in \mathcal{H}(B)$, j *be any positive integer. Then the following are equivalent.*

$$\mathrm{I} := \sup_{\zeta \in B}(1 - |\zeta|^2)^j |R^j f(\zeta)|.$$

$$\mathrm{II} := \sup_{\zeta \in B} \|\Delta_j f(\varphi_\zeta(\cdot), \zeta)\|_{L^2(dm)}.$$

$$\mathrm{III}' := \sup_{\zeta \in B} \|H'_{f;j} k_\zeta\|.$$

$$\mathrm{IV}' := \|H'_{f;j}\|.$$

Proof. As in the previous proof we can see that II $=$ III$'$. Thus, we know that

$$\mathrm{I} \leq \mathrm{II} = \mathrm{III}' \leq \mathrm{IV}'.$$

Now, that IV$' \leq$ I follows from 4.3. $\quad\square$

Proof of 1.7. It follows immediately from 4.4 and 4.6. $\quad\square$

Proof of 1.8. Let $f \in \mathscr{B}_0$. Write T for any of the generalized Hankel operators $H_{f;j}$ and $H'_{f;j}$, and $F(z, w)$ for either $-\Delta_j f(w, z)$ or $\Delta_j f(z, w)$. Let $0 < \rho < 1$. For $g \in A^2$ we have

$$Tg(z) = \chi_{\{|z| \leq \rho\}} \int_B \overline{F(z, w)} \mathscr{K}(z, w) \, dm(w)$$

$$+ \chi_{\{\rho < |z| < 1\}} \int_B \overline{F(z, w)} \mathscr{K}(z, w) \, dm(w)$$

$$\equiv T_1 g(z) + T_2 g(z).$$

We first show that T_1 is compact. Let $\{g_l\}$ be a sequence weakly converging to 0. Then,

$$|T_1 g_l|^2 \leq \int_{|z|\leq\rho} \int_B |F(\varphi_z(w), z)|^2 |\mathscr{K}(z, w)|(1-|w|^2)^{-\varepsilon} \, dm(w)$$

$$\times \int_B |g_l(w)|^2 |\mathscr{K}(z, w)|(1-|w|^2)^{\varepsilon} \, dm(w)\, dm(z)$$

$$\leq c(1-\rho^2)^{-\varepsilon} \|f\|_{\mathscr{B}} \int_B |g_l(w)|^2 (1-|w|^2)^{-\varepsilon} \log(1-|w|)^2 dm(w),$$

which tends to 0 as l tends to ∞. Next, by the same argument as in 4.3 it follows that for any given $\varepsilon > 0$ we can choose ρ close enough to 1 so that

$$\|T_2\| \leq \sup_{\rho<\zeta<1} \widetilde{D}f(\zeta)^2 \leq c\varepsilon.$$

Since T can be approximated by compact operators in norm, it follows that T is compact.

If T is compact then $\|Tk_\zeta\| \to 0$ as $|\zeta| \to 1^-$, which implies that $f \in \mathscr{B}_0$ by 3.5 and 4.5. This concludes the proof.   □

We now turn to the Schatten-ideal properties of the generalized Hankel operators. It turns out that the cut-off depends in an essential way on the order of zero of (the expression defined by) the symbol f. We remark that our result contains as particular case (when $j = 1$) the unweighted case of the theorems in [2], [7], [19], and [21].

Proof of 1.9. Write T_j to indicate any of the generalized Hankel operators $H_{f;j}$ and $H'_{f;j}$. By 1.2 it follows that

$$\|T_j\|_{S_2} \approx \|f\|_{B_2}$$

for $j > n$. For $j \leq n$, if $\|T_j\|_{B_2}$ is finite, then f is a polynomial of degree at most $j - 1$, i.e. $T_j = 0$.

Let $2 < p < \infty$. We use Lemma 3.6 of [2], which is an improvement of a result of Russo, [15]. Let $j > 2n/p$. Then,

$$\int_B \int_B |\Delta_j f(w, z)|^p |\mathscr{K}(z, w)|^2 \, dm(z)\, dm(w) \leq c\|f\|_{B_p}^p < \infty$$

implies that $T_j \in S_p$ with S_p-norm bounded by the B_p-norm of f.

Conversely, if T is any bounded operator from A^2 into a Hilbert space H, one can see that

$$\|T\|_{S_2}^2 = \int_B \|T\mathscr{K}_\zeta\|_H^2 \, dm(\zeta) = \int_B \|Tk_\zeta\|_H^2 \, d\lambda(\zeta).$$

Using interpolation with the obvious case $p = \infty$ it follows that for $2 \le p \le \infty$, and $T \in S_p$,

$$\| \|Tk_\zeta\|_H \|_{L^p(d\lambda)} \le \|T\|_{S_p}.$$

Next, let T_j be either $H_{f;j}$ or $H'_{f;j}$. By 4.2 $\|T_j k_\zeta\| = \|F(\zeta, \cdot)k_\zeta\|_{A_2}$, where F means the appropriate expression between $\Delta_j f(w, \zeta)$ and $\Delta_j f(\zeta, w)$. Now we apply 3.3 with $q = 2$. It follows that

$$\|\Delta_j f\|_{L^p(d\Lambda)} \le \|T_j\|_{S_p}.$$

This implies that if $T \in S_p$ and $j > 2n/p$, then $f \in B_p$. If $T \in S_p$ and $j \le 2n/p$, then f is a polynomial of degree less than j (and $T_j = 0$) as a consequence of 3.3. again.   □

REMARK. It is the author's belief that the result of 1.9 should hold for the range $0 < p < 2$ too. In this case a more detailed analysis seems to be needed.

5. Proof of 3.1. In this section we prove 3.1. The proof of the case $j = 1$, using 2.2 and the Möbius invariance, does not give the general result. Hence, a different approach is needed. We split the proof in the cases $0 < p \le 1$ and $1 < p < \infty$. When $0 < p \le 1$ we use the atomic decomposition of the Besov spaces. For the case $1 < p < \infty$ we use an integration by parts and Hölder's inequality. However, in both cases we need an estimate of Forelli-Rudin type (see 2.4) for certain integrals. These integrals come from the expression for $\Delta_j K_\zeta$ where K_ζ is a suitable power of the reproducing kernel.

LEMMA 5.1. *Let* j, ν *be positive integers, and* $\zeta \in B$. *Let* K_ζ *be defined by*

$$K_\zeta(z) = \frac{1}{(1 - z \cdot \overline{\zeta})^\nu}.$$

Moreover, let

$$M_i(w, z, \zeta) = \frac{((z - w) \cdot \overline{\zeta})^{j+i}}{(1 - w \cdot \overline{\zeta})^\nu (1 - z \cdot \overline{\zeta})^{j+i}}.$$

Then

$$\Delta_j K_\zeta(w, z) = \sum_{i=0}^{\nu-1} d_i M_i(w, z, \zeta),$$

for some positive constants d_i.

Proof. By rotational invariance it suffices to prove the lemma for $\zeta = \zeta_1 e_1$, where e_1 is the base vector $(1, 0, \ldots, 0)$. Write g for $K_{\zeta_1 e_1}$. We first show that

(11) $\Delta_j K_\zeta(w, z)$

$$= \frac{1}{(1 - w_1 \overline{\zeta}_1)^\nu (1 - z_1 \overline{\zeta}_1)^{\nu+j-1}}$$

$$\times \left((1 - z_1 \overline{\zeta}_1)^{\nu+j-1} \right.$$

$$\left. - \sum_{i=0}^{j-1} \frac{1}{i!} \left(\frac{d}{dz_1} \right)^i (1 - z_1 \overline{\zeta}_1)^{\nu+j-1} |_{z_1 = w_1} (w_1 - z_1)^i \right).$$

Indeed we have,

(12) $\Delta_j K_\zeta(w, z)$

$$= \frac{1}{(1 - z_1 \overline{\zeta}_1)^{\nu+j-1}}$$

$$\times \left(\frac{(1 - z_1 \overline{\zeta}_1)^{\nu+j-1}}{(1 - w_1 \overline{\zeta}_1)^\nu} \right.$$

$$\left. - \sum_{i=0}^{j-1} \binom{\nu+i-1}{i} \overline{\zeta}_1^i (w_1 - z_1)^i (1 - z_1 \overline{\zeta}_1)^{j-i-1} \right).$$

Now write

(13) $(1 - z_1)^{j-i-1}$

$$= ((1 - w_1 \overline{\zeta}_1) - (w_1 - z_1) \overline{\zeta}_1)^{j-i-1}$$

$$= \sum_{l=0}^{j-i-1} \binom{j-i-1}{l} (1 - w_1 \overline{\zeta}_1)^i ((w_1 - z_1) \overline{\zeta}_1)^{j-i-1-l},$$

and substitute (13) into (12). Using the identity

$$\sum_{l=0}^{j-i-1} \binom{\nu+l-1}{l} \binom{j-l-1}{i} = \binom{\nu+j-1}{j-i-1}$$

we obtain (11). Finally,

$$(1 - z_1\overline{\zeta}_1)^{\nu+j-1}_{i=0} - \sum_{i=0}^{j-1} \binom{\nu+j-1}{i}(1 - w_1\overline{\zeta}_1)^{\nu+j-1-i}((w_1 - z_1)\overline{\zeta}_1)^i$$

$$= ((w_1 - z_1)\overline{\zeta}_1)^j \sum_{i=0}^{\nu-1} c_i(1 - w_1\overline{\zeta}_1)^{\nu-1-i}((w_1 - z_1)\overline{\zeta}_1)^i$$

$$= ((w_1 - z_1)\overline{\zeta}_1)^j \sum_{l=0}^{\nu-1} d_l(1 - z_1\overline{\zeta}_1)^{\nu-1-l}((w_1 - z_1)\overline{\zeta}_1)^l.$$

This, together with (11), concludes the proof.   □

LEMMA 5.2. *Let* $n \geq 1$. *Let* $d > 2n$ *if* $n > 1$, *and* $d > 1$ *if* $n = 1$. *Let* $0 \leq \varepsilon < 1$. *Then there exists an absolute constant* C_0 *such that*

$$\int_B \int_B \frac{|(z_1 - w_1)\overline{\zeta}_1|^d}{|1 - z_1\overline{\zeta}_1|^d|1 - w_1\overline{\zeta}_1|^d} \frac{(1 - |z_1|^2)^{-\varepsilon}(1 - |w_1|^2)^{-\varepsilon}}{|1 - z_1\overline{w}_1|^{2(n+1-\varepsilon)}} \, dm(z) \, dm(w)$$

$$\leq \frac{C_0}{(1 - |\zeta_1|^2)^d}.$$

Proof. It suffices to notice that, if $\varphi_{\zeta_1} \equiv \varphi_{\zeta_1 e_1}$ is defined as in (3), and $v_1(z) = z_1$ is the coordinate function, then

$$\frac{|(z_1 - w_1)|(1 - |\zeta_1|^2)}{|1 - w_1\overline{\zeta}_1||1 - z_1\overline{\zeta}_1|} = |v_1 \circ \varphi_{\zeta_1}(w) - v_1 \circ \varphi_{\zeta_1}(z)|.$$

Then applying [3] Theorem 1, when $n > 1$, or the results in [4] or in [23] when $n = 1$, we obtain that

$$\int_B \int_B \frac{|(z_1 - w_1)\overline{\zeta}_1|^d}{|1 - z_1\overline{\zeta}_1|^d|1 - w_1\overline{\zeta}_1|^d} \frac{(1 - |z_1|^2)^{-\varepsilon}(1 - |w_1|^2)^{-\varepsilon}}{|1 - z_1\overline{w}_1|^{2(n+1-\varepsilon)}} \, dm(z) \, dm(w)$$

$$\leq \frac{C}{(1 - |\zeta_1|^2)^d} \int_B \int_B |v_1 \circ \varphi_{\zeta_1 e_1}(z) - v_1 \circ \varphi_{\zeta_1, e_1}(w)|^d$$

$$\times \frac{(1 - |z_1|^2)^{-\varepsilon}(1 - |w_1|^2)^{-\varepsilon}}{|1 - z_1\overline{w}_1|^{2(n+1-\varepsilon)}} \, dm(z) \, dm(w)$$

$$\leq C \frac{\|v_1\|_{B_d}^d}{(1 - |\zeta_1|^2)^d},$$

where C in (another) constant, independent of ζ.   □

LEMMA 5.3. *Let* $f \in \mathcal{H}(B)$. *Let* $1 \leq p < \infty$. *Then there exists* $c_0 > 0$ *such that for* $0 \leq \varepsilon \leq \varepsilon_0$

$$\int_B \int_B |f \circ \varphi_{\zeta_1 e_1}(z) - f \circ \varphi_{\zeta_1 e_1}(w)|^p \frac{(1 - |w|^2)^{-\varepsilon}}{|1 - z_1 \overline{w}_1|^{2(n+1)}} \, dm(z) \, dm(w)$$

$$\leq \frac{C}{(1 - |\zeta|^2)^\varepsilon} \int_B |\widetilde{D}f(z)|^p (1 - |z|^2)^{-(n+1+\varepsilon)} \, dm(z).$$

REMARK. Notice that this is an extension to the case $\varepsilon > 0$ of one of the equivalences contained in [9] Theorem 1. Also recall that $\widetilde{D}f$ is the invariant derivative introduced in (5).

Proof. We change the variable $z = \varphi_w(\xi)$ in the first integral. Recall the formulas 2.2.2 and 2.2.6 in [14]. We also use 2.4, and Lemma 3.2 and Remark 3.2 of [3]. We obtain

$$\int_B \int_B |f \circ \varphi_\zeta(z) - f \circ \varphi_\zeta(w)|^p \frac{dm(z)}{|1 - z \cdot \overline{w}|^{2(n+1)}} (1 - |w|^2)^{-\varepsilon} \, dm(w)$$

$$= \int_B \int_B |f \circ \varphi_\zeta \circ \varphi_w(\xi) - f \circ \varphi_\zeta \circ \varphi_w(0)|^p \, dm(\xi)$$
$$\times (1 - |w|^2)^{-(n+1+\varepsilon)} \, dm(w)$$

$$\leq C \int_B \int_B |(\widetilde{D}f)(\varphi_{\varphi_\zeta(w)}(\xi))|^p \, dm(\xi)(1 - |w|^2)^{-(n+1+\varepsilon)} \, dm(w)$$

$$= \int_B \int_B |(\widetilde{D}f)(z)|^p \frac{(1 - |\varphi_\zeta(w)|^2)^{n+1}}{|1 - z \cdot \overline{\varphi}_\zeta(w)|^{2(n+1)}} \, dm(z)$$
$$\times (1 - |w|^2)^{-(n+1+\varepsilon)} \, dm(w)$$

$$= \int_B |(\widetilde{D}f)(z)|^p \int_B \frac{(1 - |\eta|^2)^{n+1}(1 - |\varphi_\zeta(\eta)|^2)^{-(n+1+\varepsilon)}}{|1 - z \cdot \overline{\eta}|^{2(n+1)}}$$
$$\times \frac{(1 - |\zeta|^2)^{n+1}}{|1 - \eta \cdot \overline{\zeta}|^{2(n+1)}} \, dm(\eta) \, dm(z)$$

$$= \int_B |(\widetilde{D}f)(z)|^p \int_B \frac{(1 - |\eta|^2)^{-\varepsilon}(1 - |\zeta|^2)^{-\varepsilon}}{|1 - \zeta \cdot \overline{\eta}|^{-2\varepsilon}} \, dm(\eta) \frac{dm(z)}{|1 - z \cdot \overline{\zeta}|^{2(n+1)}}$$

$$\leq \frac{1}{(1 - |\zeta|^2)^\varepsilon} \int_B |(\widetilde{D}f)(z)|^p \int_B \frac{(1 - |\eta|^2)^{-\varepsilon}}{|1 - \eta \cdot \overline{\zeta}|^{2(n+1)}} \, dm(\eta) \, dm(z)$$

$$\leq \frac{C}{(1 - |\zeta|^2)^\varepsilon} \int_B |(\widetilde{D}f)(z)|^p (1 - |z|^2)^{-(n+1+\varepsilon)} \, dm(z).$$

PROPOSITION 5.4. *Let* $c(n) = 2n$ *if* $n > 1$, *and* $c(n) = 1$ *if* $n = 1$. *Let* $a, b \in \mathbf{R}$, $a, b > c(n)$. *Let* $0 \leq 2\varepsilon < \min(a, b) - c(n)$. *Then*

there exists an absolute constant C_0 such that

$$\int_B \int_B \frac{|(z-w)\cdot\bar{\zeta}|^a}{|1-w\cdot\bar{\zeta}|^b|1-z\cdot\bar{\zeta}|^a} \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\cdot\bar{w}|^{2(n+1)}} \, dm(z)\, dm(w)$$
$$\leq \frac{C_0}{(1-|\zeta|^2)^{b+\varepsilon}}.$$

Proof. We give the proof in the case $n > 1$. If $n = 1$ the proof follows the same pattern as in the case $n > 1$, but some step has to be considered void. We leave the details to the reader.

By rotational invariance we may assume that $\zeta = \zeta_1 e_1$ where $e_1 = (1, 0, \dots, 0)$ is the base vector. We break the integral into two over the regions Ω and Ω'. The region Ω is defined by

$$\Omega = \{(z, w) \in B \times B : |z_1 - w_1| \leq \tfrac{1}{4}((1-|z_1|^2) + (1-|w_1|^2))\}.$$

Notice that in this case

$$|1 - z_1\bar{\zeta}_1| \leq 3|1 - w_1\bar{\zeta}_1| \leq 9|1 - z_1\bar{\zeta}_1|.$$

Thus, if $d = \min(a, b)$,

$$\frac{|(z-w)\cdot\bar{\zeta}|^a}{|1-w\cdot\bar{\zeta}|^b|1-z\cdot\bar{\zeta}|^a} \leq \frac{C}{(1-|\zeta|^2)^{b-d}} \frac{|(z-w)\cdot\bar{\zeta}|^d}{|1-w\cdot\bar{\zeta}|^d|1-z\cdot\bar{\zeta}|^d}.$$

Let v_1 be the coordinate function $v_1(z) = z_1$. Thus, the integral that we want to estimate is less or equal to a constant times

$$\frac{1}{(1-|\zeta_1|^2)^b} \int_B \int_B |v_1 \circ \varphi_{\zeta_1 e_1}(z) - v_1 \circ \varphi_{\zeta_1 e_1}(w)|^d$$
$$\times \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\cdot\bar{w}|^{2(n+1)}} \, dm(z)\, dm(w).$$

Now we apply 5.3. Because of our choice of d and ε the integral

$$\int_B |\tilde{D}v_1|^p (1-|z|^2)^{-(n+1+\varepsilon)} \, dm(z)$$

is finite. This gives the estimate for Ω.

The region Ω' is defined by

$$\Omega' = \{(z, w) \in B \times B : |z_1 - w_1| \geq \tfrac{1}{4}((1-|z_1|^2) + (1-|w_1|^2))\}.$$

Write $z = (z_1, z')$ and $w = (w_1, w')$. Now we integrate out.

$$\iint_{\Omega'} \frac{|(z-w)\cdot\overline{\zeta}|^a}{|1-w\cdot\overline{\zeta}|^b|1-z\cdot\overline{\zeta}|^a} \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\cdot\overline{w}|^{2(n+1)}} \, dm(z)\,dm(w)$$

$$\leq \iint_{\Omega'\cap D\times D} \frac{|(z-w)\cdot\overline{\zeta}|^a}{|1-w\cdot\overline{\zeta}|^b|1-z\cdot\overline{\zeta}|^a} \int_{|z'|<(1-|z_1|^2)^{1/2}} \int_{|w'|<(1-|w_1|^2)^{1/2}}$$

$$\times \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\cdot\overline{w}|^{2(n+1)}} \, dm(w')\,dm(z')\,dm(w_1)\,dm(z_1).$$

Consider the two inner integrals:

$$\int_{|z'|<(1-|z_1|^2)^{1/2}} \int_{|w'|<(1-|w_1|^2)^{1/2}} \frac{(1-|w|^2)^{-\varepsilon}}{|1-z\cdot\overline{w}|^{2(n+1)}} \, dm(w')\,dm(z')$$

$$= \int_{|z'|<(1-|z_1|^2)^{1/2}} \int_{|w'|<(1-|w_1|^2)^{1/2}} \frac{((1-|w_1|^2)-|w'|^2)^{-\varepsilon}}{|1-z_1\overline{w}_1 - z'\cdot\overline{w}'|^{2(n+1)}}$$

$$\times dm(w')\,dm(z').$$

Now we perform the change of variables:

$$u = \frac{z'}{(1-|z_1|^2)^{1/2}}, \quad \text{and} \quad v = \frac{w'}{(1-|w_1|^2)^{1/2}}.$$

Thus, the above integrals equal

$$\frac{(1-|z_1|^2)^{n-1}(1-|w_1|^2)^{n-1-\varepsilon}}{|1-z_1\overline{w}_1|^{2(n+1)}}$$

$$\times \int_{|u|<1} \int_{|v|<1} \frac{(1-|v|^2)^{-\varepsilon}}{|1-Qu\cdot\overline{v}|^{2(n+1)}} \, dm(u)\,dm(v),$$

where we have set

$$Q = \frac{(1-|z_1|^2)^{1/2}(1-|w_1|^2)^{1/2}}{(1-z_1\overline{w}_1)}.$$

Next, 2.4 gives (recall that $u, v \in \mathbf{C}^{n-1}$)

$$\int_{|u|<1} \int_{|v|<1} \frac{(1-|v|^2)^{-\varepsilon}}{|1-Qu\cdot\overline{v}|^{2(n+1)}} \, dm(u)\,dm(v)$$

$$\leq \int_{|u|<1} \frac{1}{(1-|Qu|^2)^{n+2+\varepsilon}} \, dm(u) \leq \frac{C}{(1-|Q|^2)^{n+1+\varepsilon}}.$$

Now notice that, since $(z, w) \in \Omega'$,

$$\frac{1}{(1-|Q|^2)} = \frac{|1-z_1\overline{w}_1|^2}{|z_1-w_1|^2} < 25.$$

Therefore, the integral over the region Ω' is less or equal to a constant times

$$
\int_D \int_D \frac{|(z_1 - w_1) \cdot \bar\zeta_1|^a}{|1 - w_1 \bar\zeta_1|^b |1 - z_1 \bar\zeta_1|^a}
$$
$$
\times \frac{(1 - |z_1|^2)^{n-1}(1 - |w_1|^2)^{n-1-\varepsilon}}{|1 - z_1 \overline{w}_1|^{2(n+1)}} \, dm(z_1) \, dm(w_1)
$$
$$
\leq \frac{C}{(1 - |\zeta_1|^2)^{b-d}} \int_D \int_D \frac{|(z_1 - w_1) \cdot \bar\zeta_1|^d}{|1 - w_1 \bar\zeta_1|^d |1 - z_1 \bar\zeta_1|^d}
$$
$$
\times \frac{(1 - |w_1|^2)^{-\varepsilon}}{|1 - z_1 \overline{w}_1|^4} \, dm(z_1) \, dm(w_1),
$$

where $d = \min(a, b)$. Now we apply 5.3 with $f(z) = z_1$. This finishes the proof. □

Proof of 3.1. Case $0 < p \leq 1$. This follows easily from the atomic decomposition and 5.4. Let $f \in B_p$, let $\{\zeta^i\}$ be a ϑ-lattice. Then there exist constants $\{c_i\}$ such that

$$
f(z) = \sum_{i=1}^\infty c_i \frac{(1 - |\zeta^i|^2)^\nu}{(1 - z \cdot \bar\zeta_i)^\nu},
$$

and

$$
\|f\|_{B_p}^p \approx \sum |c_i|^p .
$$

Hence,

$$
|\Delta_j f(w, z)|^p \leq \left(\sum |c_i| (1 - |\zeta^i|^2)^\nu |\Delta_j K_{\zeta^i}(w, z)| \right)^p
$$
$$
\leq \sum |c_i|^p (1 - |\zeta^i|^2)^{\nu p} |\Delta_j K_{\zeta^i}|^p .
$$

Therefore, for $j > 2n/p$, by 5.4 (with $\varepsilon = 0$), it follows that

$$
\|\Delta_j f\|_{L^p(d\Lambda)}^p \leq \sum |c_i|^p \|\Delta_j K_{\zeta^i}\|_{L^p(d\Lambda)}^p (1 - |\zeta^i|^2)^{\nu p}
$$
$$
\leq C_0 \sum |c_i|^p \leq C_0 \|f\|_{B_p}^p .
$$

Case $1 < p < \infty$. In this case we will use an integration by parts and Hölder's inequality. Given p, let j and ν be integers satisfying the following conditions. We let $j > 2n/p$. Pick ε, $0 < \varepsilon < (jp - 2n)(p - 1)$. Finally, let $2n < \nu < 2(n+1) + \varepsilon$.

Recall the notation introduced in 5.1. Notice that for $f \in \mathscr{H}(B)$ and $K_\zeta = (1 - z \cdot \bar\zeta)^{-\nu}$, the weighted reproducing kernels (see [14]

7.1.1) give

$$\Delta_j f(w\,,z) = c_\nu \int_B f(\zeta)\Delta_j K_\zeta(w\,,z)(1-|\zeta|^2)^{\nu-n-1}\,dm(\zeta)$$

$$= c_\nu \sum_{i=0}^{\nu-1} \int_B f(\zeta)M_i(w\,,z\,,\zeta)(1-|\zeta|^2)^{\nu-n-1}\,dm(\zeta).$$

Now let $\mathscr{R}_{n+1}^{-m}$ be the integral operator defined for $h \in \mathscr{H}(B)$ by

$$\mathscr{R}_{n+1}^{-m}h(z) = c_{n+1} \int_B \frac{h(w)}{(1-z\cdot\overline{w})^{2(n+1)}}(1-|w|^2)^{n+1+m}\,dm(w).$$

Its inverse is the differential operator on $\mathscr{H}(B)$ with constant coefficients (give [12] §3 for details)

$$\mathscr{R}_{n+1}^{m}h(z) = c_m \int_B \frac{h(w)}{(1-z\cdot\overline{w})^{2(n+1)+m}}(1-|w|^2)^{n+1}\,dm(w).$$

Here c_{n+1} and c_m are suitable positive constants of normalization. Now we apply an integration by parts, using Lemma 1.1 in [5]. Notice that the lemma holds with radial weights since normal derivatives of holomorphic functions can be written in terms of tangential derivatives. Then, applying the integration by parts near the boundary, the derivatives do not fall on the weight. We obtain:

$$\int_B f(\zeta)M_i(z\,,w\,,\zeta)(1-|\zeta|^2)^{\nu-n-1}\,dm(\zeta)$$

$$= \int_B \mathscr{L}^m f(\zeta) \int_B \frac{M_i(z\,,w\,,\eta)(1-|\eta|^2)^{n+1+m}}{(1-\zeta\cdot\overline{\eta})^{2(n+1)}}\,dm(\eta)$$

$$\times (1-|\zeta|^2)^{\nu-n-1}\,dm(\zeta),$$

where $\mathscr{L}^m$ is a differential operator of order m with coefficients C^∞ up the boundary. Now notice that, by Fubini's theorem and 2.4, if $n+\varepsilon < \nu < 2(n+1)+\varepsilon$,

$$(1-|w|^2)^\varepsilon \int_B \int_B \frac{(1-|\zeta|^2)^{\nu-n-1-\varepsilon}(1-|\eta|^2)^{n+1}}{|1-w\cdot\overline{\eta}|^\nu |1-\zeta\cdot\overline{\eta}|^{2(n+1)}}\,dm(\zeta)\,dm(\eta) \le c_0$$

where c_0 is a constant independent of w. Therefore, applying Jensen

inequality twice, we obtain

$$\left| \int_B f(\zeta) M_\iota(z,w,\zeta)(1-|\zeta|^2)^{\nu-n-1}\,dm(\zeta) \right|^p$$

$$\leq \int_B |\mathscr{L}^m f(\zeta)|^p$$

$$\times \left(\int_B \frac{|(z-w)\cdot\overline{\eta}|^{j+\iota}(1-|\eta|^2)^{n+1+m}(1-|\zeta|^2)^{\nu-n-1}}{|1-z\cdot\overline{\eta}|^{j+i}|1-w\cdot\overline{\eta}|^\nu|1-\zeta\cdot\overline{\eta}|^{2(n+1)}}\,dm(\eta) \right)^p dm(\zeta)$$

$$\leq C \int_B |\mathscr{L}^m f(\zeta)|^p \int_B \frac{|(z-w)\cdot\overline{\eta}|^{(j+i)p}}{|1-z\cdot\overline{\eta}|^{(j+i)p}} \frac{(1-|\eta|^2)^{mp}}{(1-|w|^2)^{\varepsilon p}}(1-|\zeta|^2)^{\varepsilon p}$$

$$\times \frac{(1-|w|^2)^\varepsilon(1-|\eta|^2)^{n+1}(1-|\zeta|^2)^{\nu-n-1-\varepsilon}}{|1-w\cdot\overline{\eta}|^\nu|1-\zeta\cdot\overline{\eta}|^{2(n+1)}}\,dm(\eta)\,dm(\zeta)$$

$$\leq \frac{C}{(1-|w|^2)^{\varepsilon(p-1)}} \int_B |\mathscr{L}^m f(\zeta)|^p (1-|\zeta|^2)^{\varepsilon p+\nu-(n+1+\varepsilon)}$$

$$\times \int_B \frac{|(z-w)\cdot\overline{\eta}|^{(j+i)p}}{|1-z\cdot\overline{\eta}|^{(j+i)p}} \frac{(1-|\eta|^2)^{n+1+mp}}{|1-w\cdot\overline{\eta}|^\nu}$$

$$\times \frac{1}{|1-\zeta\cdot\overline{\eta}|^{2(n+1)}}\,dm(\eta)\,dm(\zeta).$$

Therefore, since $\nu > 2n$,

$$\|\Delta_j f\|_{L^p(d\Lambda)}^p$$

$$\leq \sum_{i=0}^{\nu-1} \int_B \int_B |\mathscr{L}^m f(\zeta)|^p \frac{(1-|\zeta|^2)^{\varepsilon p+\nu-(n+1+\varepsilon)}(1-|\eta|^2)^{n+1+mp}}{|1-\zeta\cdot\overline{\eta}|^{2(n+1)}}$$

$$\times \int_B \int_B \frac{|(z-w)\cdot\overline{\eta}|^{(j+1)p}}{|1-z\cdot\overline{\eta}|^{(j+1)p}} \frac{(1-|w|^2)^{-\varepsilon(p-1)}}{|1-w\cdot\overline{\eta}|^\nu}$$

$$\times \frac{dm(z)\,dm(w)}{|1-z\cdot\overline{w}|^{2(n+1)}}\,dm(\eta)\,dm(\zeta)$$

$$\leq C \int_B |\mathscr{L}^m f(\zeta)|^p (1-|\zeta|^2)^{\varepsilon(p-1)+\nu-(n+1)}$$

$$\times \int_B \frac{(1-|\eta|^2)^{n+1+mp-\nu-\varepsilon(p-1)}}{|1-\zeta\cdot\overline{\eta}|^{2(n+1)}}\,dm(\eta)\,dm(\zeta)$$

$$\leq C \int_B |\mathscr{L}^m f(\zeta)|^p \frac{(1-|\zeta|^2)^{\varepsilon(p-1)+\nu-(n+1)}}{(1-|\zeta|^2)^{-mp+\nu+\varepsilon(p-1)}}\,dm(\zeta),$$

if m is chosen so that $\nu > mp > n$. The last integral is now less or equal to a constant times

$$\int_B |R^m f(\zeta)|^p (1-|\zeta|^2)^{mp-(n+1)}\,dm(\zeta) \equiv \|f\|_{B_p}^p.$$

This concludes the proof.   □

6. Mean oscillation spaces. We begin by proving that the Besov spaces can be characterized as the mean oscillation spaces in the Bergman metric. We then conclude with some remarks and open questions. Recall that in (2) we have defined the mean oscillation of order j of $f \in \mathscr{H}(B)$.

Proof of 1.12. Recall that for $w \in E_\zeta$, $|E_\zeta|^{-1} \approx |\det \varphi_\zeta'(w)|^2$. Let $\rho < \rho_0$ as in the proof of 1.2 (then ρ depends on the size of the lattice in the atomic decomposition). Then for $f \in \mathscr{H}(B)$ we have

$$\mathrm{MO}_\rho^j f(\zeta) \leq \frac{2}{|E_\zeta|} \int_{E_\zeta} |\Delta_j f(w, \zeta)| \, dm(w)$$

$$\leq c \int_{E_\zeta} |\Delta_j f(w, \zeta)| \, |\det \varphi_\zeta'(w)|^2 \, dm(w)$$

$$\leq c \int_{E_0} |\Delta_j f(\varphi_\zeta(w), \zeta)| \, dm(w)$$

$$\leq c \int_B |\Delta_j f(\varphi_\zeta(w), \zeta)| \, dm(w).$$

Thus, (i) $\Rightarrow$ (ii) follows from 3.1. Conversely, let C_ζ be a function that depends only on ζ. By 3.2 and Cauchy estimates we have that

$$(14) \quad (1 - |\zeta|^2)^j |R^j f(\zeta)| \leq c \int_{E_0} |\Delta_j f(\varphi_\zeta(w), \zeta) - C_\zeta| \, dm(w)$$

$$= c \int_{E_\zeta} |\Delta_j f(w, \zeta) - C_\zeta| \, |\det \varphi_\zeta'(w)|^2 \, dm(w)$$

$$\leq \frac{c}{|E_\zeta|} \int_{E_\zeta} |\Delta_j f(w, \zeta) - C_\zeta| \, dm(w).$$

We choose C_ζ to be equal to $(\Delta_j f(\cdot, \zeta))_{E_\zeta}$, and we integrate both sides of (14) with respect to $d\lambda$. This concludes the proof. □

Final remarks. Some comments and open questions are in order.

First of all, we expect the result in 1.9 to hold for all p's, that is, also for $0 < p < 2$. In this case more delicate techniques seem to be needed.

The results about the generalized Hankel operators could be extended to the weighted Bergman spaces. The same techniques as in the present paper should enable one to prove the corresponding theorems.

The characterization of Besov spaces as mean oscillation spaces in the Bergman metric probably holds also for p in the range $0 < p < 1$. Moreover a version of Theorem 1.12 should hold possible for the weighted Besov spaces.

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