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HORIZONTAL PATH SPACES AND CARNOT-CARATHÉODORY METRICS

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In this paper we study a class of sub-spaces of loop spaces which have appeared in the calculus of variations. Generalizing a result of Smale, we show that the space of loops tangent to a distribution satisfying Hörmander's condition is weakly homotopic to the space of all loops. If the distribution is fat, we resolve the end point map from the space of horizontal paths. This resolution has two applications: (1) the proof that the cut-locus on an analytic fat Carnot-Carathéodory manifold is sub-analytic; (2) a study of the singularity of the horizontal loop space. At the end we study the geometry of left-invariant Carnot-Carathéodory metrics on fact nilpotent groups.

0. Introduction. In this paper we will study a class of sub-spaces of loop spaces which have appeared in the calculus of variations, control theorem (cf. [7], [12], [28]).

Let M be a connected manifold, $H \subset TM$ a smooth distribution on M. We say that a H^1-curve $\gamma: [0, 1] \to M$ is *horizontal* if it is tangent to H almost everywhere. Let $\langle \cdot, \cdot \rangle$ be a fiberwise metric on H (i.e. a Carnot-Carathéodory metric, or a CC metric for short), then the Carnot-Carathéodory distance (or a CC distance for short) on M is defined to be

$$d(x, y) = \inf E(\gamma)^{1/2}, \quad \text{where } E(\gamma) = \int_0^1 \left\langle \frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt} \right\rangle dt.$$

Here γ runs over the space of horizontal paths connecting x and y. A classical result of Chow's [9] says that if H satisfies Hörmander's bracket generating condition, then every two points on M can be joined by a horizontal path, so the distance is finite. The interest in horizontal loop spaces in part lies in the fact that they play a similar role in CC metrics as ordinary loop spaces play in Riemannian geometry.

There are many applications of CC metrics to Riemannian geometry, hypoelliptic operators, control theory, physics, etc. see [4], [6], [7], [14], [17], [18], [27], [30], [31], [33], [36], [38], [39], [40].

Let $\Omega_H M(x_0, \cdot)$ (resp. $\Omega_H M(x_0, x_0)$) be the space of H^1-horizontal paths starting from x_0 (resp. H^1-horizontal loops based at x_0).

We first look at the topology of $\Omega_H M(x_0, x_0)$. Smale proved that if H is a contact distribution on a three dimensional manifold, then $\Omega_H(x_0, x_0)$ is weakly homotopic to the ordinary based loop space. In [12] we proved that $\pi_0(\Omega_H(x_0, x_0)) = \pi_1(M)$. In this paper we will prove

THEOREM 1. *Let M be a connected manifold with a bracket generating distribution H. Then the space of based horizontal loops, $\Omega_H(x_0, x_0)$ is weakly homotopic to the space $\Omega(x_0, x_0)$ of all loops: their homotopy groups are all equal. If, in addition the distribution is fat (see the definition in §1.1), then two spaces are homotopic.*

In spite of Theorem 1 the two spaces are very different geometrically; for example, $\Omega_H M(x_0, x_0)$ is smooth, as the end point map $e \colon \Omega_H(x_0, \cdot) \to M$, $e(\gamma) \to \gamma(1)$ is not a submersion at the constant loop. We will focus our attention on the case of fat distributions, see Weinstein [41]. (Fat distributions are also called "strongly bracket generating" or "of maximal co-rank".) The following result is inspired by a result of Milnor [25] in finite dimensions:

THEOREM 2. *If H is a fat distribution, then the complement of the intersection of $\Omega_H M(x_0, x_0)$ with a sufficiently small sphere in $\Omega_H M(x_0, \cdot)$ at the constant loop is the total space of fibration over S^{n-1} with contractible fibers, $n = \dim(M)$.*

In contrast to the Riemannian case in which cut points cannot come arbitrarily close to x_0 and the complement of a cut loss is contractible [21], we prove

THEOREM 3. *If M is a fat CC manifold of dimension n, then the complement of the cut locus (see the definition in §5.1) of x_0 within any ball $d(x, x_0) \leq T$ is homeomorphic to the complement of a k-plane in R^n. Here $k = n - m$ is the codimension of the distribution.*

Our final result concerns the structure of analytic CC manifolds.

THEOREM 4. *Let M be a fat analytic CC manifold. Then the cut locus of any point is sub-analytic within a sufficiently small neighborhood of that point.*

There is a stronger version of this result, which says the ratio $d(x, x_0)/d_1(x, x_0)$, where $d_1(x, x_0)$ is the CC distance function on the tangent cone, is sub-analytic near x_0 (cf. Theorem 5.5).

Sussmann [42] proved a result to which Theorem 4 is in a sense complementary. His result says that on a three-dimensional analytic CC manifold with a contact distribution, the cut locus of x_0 is subanalytic within any compact set of $M - \{x_0\}$.

Note that Theorem 3 and Theorem 4 together give us some information about how the cut-locus of x_0 within a small neighborhood of x_0 looks like, for example, that it is triangulable, and it has only one "branch". We conjecture that the cut locus within a sufficiently small neighborhood of x_0 can be stratified into analytic submanifolds of dimension l, where $k \leq l \leq n - 1$.

The main technical tool used to prove Theorems 2 and 4 was introduced in the author's previous paper [12] in this journal and is of interest in its own right. This is the resolution of the end-point map $e \colon \Omega_H M(x_0, \cdot) \to M$, i.e. a submersion

$$B_e \colon B\Omega_H M(x_0, \cdot) \to B_w M,$$

where $B\Omega_H M(x_0, \cdot)$ is the blow-up of $\Omega_H M(x_0, \cdot)$, $B_w M$ is the (weighted) blow-up of M at x_0.

This paper is organized as follows. In §1 we recall the notion of tangent cones as developed by Folland, Rothschild, Stein, et al. In §2 we prove Theorem 1. In §3 we develop the main technical tool, i.e. the resolution of e. In §4 we study the singularity of the horizontal loop space. In §5 we study the cut-locus. In §6 we study left-invariant CC metrices on fat nilpotent Lie groups; in particular, we show that the set of fat nilpotent Lie algebras with n, m fixed, $n - m = 2$, is connected.

After the first version of this paper was written in the fall of 1989, Professor H. Sussman kindly pointed out some gaps in the proof of Theorem 4 in that version; in this version we have taken this into account. In fact we found a simpler method to prove Theorem 4, but as the old method yields a stronger result (Theorem 5.5), we include both methods. Also Professor A. V. Sarychev kindly informed us that he has proved Theorem 1, which was announced in Dokl. Akad. Sci. USSR, v. 314(6), 1990.

referee for his many suggestions on how to improve the organization of this paper.

1. Preliminary facts.

1.1. *Tangent cones.* In this subsection we will recall some basic facts on Carnot-Carathéodory manifolds (or CC manifolds for short).

A Carnot-Carathéodory metric (or a CC metric) on a manifold M is a symmetric bilinear semi-positive form $Q(\cdot, \cdot) \colon T^*M \times T^*M \to R$. It induces a bundle homomorphism $L_Q \colon T^*M \to TM$ by

$$\langle L_Q(p), q \rangle_{TM, T^*M} = Q(p, q), \qquad p, q \in T^*M,$$

where $\langle \cdot, \cdot \rangle_{TM, T^*M}$ is the natural pairing between TM and T^*M. The image of T^*M under L_Q, $H = L_Q(T^*M)$ is called the horizontal bundle, on which there is an induced metric

$$(\cdot, \cdot) \colon H \times H \to R.$$

We always assume that H is smooth.

We say that H satisfies *Hörmander bracket generating condition* if we define

$$H_1 = H + [H, H], \ldots, H_s = H_{s-1} + [H_{s-1}, H_{s-1}], \ldots$$

then for s big enough, $H_s = TM$. H is s_0-steps bracket generating if s_0 is the smallest integer such that $H_{s_0} = TM$. From now on we assume that H satisfies Hörmander's condition unless specified otherwise.

A class of distributions of special interest to us is fat distributions. Recall that H is fat if for any $a \in C^\infty(H)$, $a \neq 0$, every vector can be locally written as $b_1 + [a, b_2]$ for some $b_1, b_2 \in C^\infty(H)$.

A rank m fat distribution on an n manifold must be a contact distribution if $m = n-1$; and m must be a multiple of 4 if $n-m > 1$ (cf. [32], Appendix 2).

LEMMA 1.1. *If H is fat, $x_0 \in M$, then there are local coordinates (x, y) at x_0 such that H is locally defined by $\Lambda_i = 0$, $i = 1, \ldots, k$, where*

$$\Lambda_i = dy_i - \sum \Gamma^i_{jk} x_j dx_k + R_i, \qquad \Gamma^i_{jk} = -\Gamma^i_{kj},$$

and R_i are 1-form satisfying

$$|R_i| \leq M(|x|^2 + |Y|).$$

A proof can be found in Goodman [14].

We now define a two-step nilpotent Lie algebra associated with H at x_0, which plays a similar role as tangent spaces play in Riemannian geometry.

DEFINITION 1.1. Using the notations in Lemma 1.1, we define a graded two-step nilpotent Lie algebra $n = V_1 \oplus V_2$, where $[V_1, V_1] \to V_2$ is given by

$$(1.1) \qquad [(x_1^1, \ldots, x_m^2), (x_1^2, \ldots, x_m^2)]$$
$$= \left(\sum \Gamma_{ij}^1 x_i^1 x_j^2, \ldots, \sum \Gamma_{ij}^k x_i^1 x_j^2 \right).$$

The simply connected Lie group of n is called the tangent cone of M at x_0.

Note that the bracket (1.1) can also be written in a coordinate-free way as follows

$$(1.2) \qquad H_{x_0} \times H_{x_0} \to T_{x_0} M / H_{x_0}, \qquad \Omega(v_1, v_2) = [v_1, v_2] \bmod(H).$$

Obviously H is fat if and only if the nilpotent algebra n has the property that for any $a \in V_1$, $a \neq 0$, $\mathrm{ad}_a \colon V_1 \to V_2$ is onto. We will call such two-step nilpotent Lie algebras *fat nilpotent*.

The construction of tangent cones can be generalized to higher order bracket generating distributions, due to Rothschild-Stein [33], see also Goodman [14].

1.2. *Partial connections.* To write down the geodesic equation for a CC metric, we will need the notation of partial connections.

DEFINITION 1.2. Given a projection onto H, $\pi \colon TM \to H$, a map

$$H_x \times C^\infty(H) \to C^\infty(H), \qquad (u, v) \to D_u V$$

depending smoothly on x, is a partial connection if
  (1) it is linear in $u \in H_x$, and for a function f on M,

$$D_u(fv) = u(f)v + f D_u v.$$

  (2)

$$D_{v_1} v_2 - D_{v_2} v_1 = \pi[v_1, v_2].$$

It is easy to prove that once π is given, the partial connection exists uniquely. In fact, the partial connection can be given in the following way. We extend the CC metric to a Riemannian metric g on M such that π becomes an orthogonal projection, and denote by D^g the Levi-Civita connection on g; then the partial connection is given by

$$D_u v = \pi D_u^g v.$$

Using these notations, we can easily write down the geodesic equations, which in local coordinates was derived in [**39**], p. 435, (12.1).

Suppose that a Riemannian metric g is an extension of the CC metric, so we have an orthogonal decomposition $TM = H \oplus H^{\perp}$. Let $\gamma \colon [0, 1] \to M$ be a CC geodesic; then it satisfies

(1.3) $$D_{\gamma'}\gamma' = \lambda \cdot (\gamma', \cdot),$$

where Ω is as in (1.2) and $\lambda \in H^{\perp}$ is the Lagrange multiplier, which satisfies

(1.4) $$(I - \pi)D_{\gamma}^{g}\lambda = T(\gamma', \gamma'),$$

where $T \colon H \times H \to H^{\perp}$ is a tensor, whose exact form is not our concern in this paper.

2. Horizontal path space and horizontal loop space.
In this section we will study the topology of the horizontal path space $\Omega_H M(x_0, \cdot)$ and the horizontal loop space $\Omega_H M(x_0, x_0)$.

LEMMA 2.1. $\Omega_H M(x_0, \cdot)$ *is contractible.*

LEMMA 2.2. $\Omega_H M(x_0, x_0)$ *in the C^0-norm is a Hopf space.*

These two lemmas can be proved as in the case of ordinary path spaces. Moreover, the above results are still true under the weaker hypothesis that every two points on M can be joined by a horizontal path.

If γ_1, γ_2 are two horizontal paths, $\gamma_1(1) = \gamma_2(0)$, then $\gamma_2 \cdot \gamma_1$ denotes the path defined by $\gamma(t) = \gamma_1(2t)$, $0 \le t \le 1$; $\gamma(t) = \gamma_2(2t - 1)$, $1/2 \le t \le 1$.

LEMMA 2.3. *Every $x_0 \in M$ has a neighborhood N_{x_0} such that for any x, $y \in N_{x_0}$, there is a horizontal path γ_{xy} within N_{x_0} which joins x to y and depends continuously on x, y.*

Proof. It suffices to prove that the following is true: for any $x \in N_{x_0}$, there is a horizontal path γ_x within N_{x_0}, joining x_0 to x and depending continuously on x. In fact, if the above is true, let $\gamma_{x_1 x_2} = \gamma_{x_2} \cdot \gamma_{x_1}^{-1}$, then $\gamma_{x_1 x_2}$ satisfies the requirement.

Suppose the vector fields $a_1, \ldots, a_m$ span H in a neighborhood of x_0, and, together with their commutators,

$$a_J = [a_{i_1}, \ldots, [a_{i_{r-1}}, a_{i_r}] \cdots], \quad J = (i_1, \ldots, i_r), \quad r \le s$$

span $T_{x_0}M$.

Choose k multiple indices $J_1, \dots, J_k$ such that

$$a_1(x_0), \dots, a_m(x_0), a_{J_1}(x_0), \dots, a_{J_k}(x_0)$$

are linearly independent and span $T_{x_0}M$. Denote $J_j = (i_{1,j}, i_{2,j}, \dots, i_{r_j,j})$ and $\phi_j(t) = \exp(ta_j)$, $j = 1, \dots, m$. We define inductively, for $J_j = (J_1, i_j)$,

$$\phi_{J_i}(t_1, \dots, t_j) = \phi_{J_1}(-|t_1|^{1/2}, \dots, -|t_{j-1}|^{1/2})\phi_{i_j}(-|t_j|^{1/2})$$
$$\phi_{J_1}(|t_1|^{1/2}, \dots, |t_{j-1}|^{1/2})\phi_{i_j}(|t_j|^{1/2}),$$

if each $t_0 > 0$; otherwise, the right-hand side above should be changed to

$$\phi_{i_j}(-|t_j|^{1/2})\phi_{J_1}(-|t_1|^{1/2}, \dots, -|t_{j-1}|)^{1/2}$$
$$\phi_{i_j}((|t|)^{1/2})\phi_{J_1}(|t_1|^{1/2}, \dots, |t_{j-1}|^{1/2}).$$

Then we have

(2.1) $d\phi_{J_i}(t, t, \dots, t)/dt|_{t=0} = a_{J_i}.$

Define

$$\Phi(t_1, \dots, t_n) = \phi_{J_k}(t_1, \dots, t_1)\phi_{J_{k-1}}(t_2, \dots, t_2)\cdots\phi_1(t_n)x_0.$$

It is easy to verify that Φ is differentiable at 0. Hence by the implicit function theorem and (2.1), Φ is a local diffeomorphism at x_0. We then define a horizontal path

$$\gamma_x(t) = \phi_{J_j}(0, \dots, 0, [r_j(nt - j + 1) - (r - 1)]t_j, t_j, \dots, t_j)$$
$$\times \phi_{J_{k-1}}(t_{j+1}, \dots, t_{j+1})\cdots\phi_1(t_n)x_0,$$

$$\frac{(j-1)}{n} \leq t \leq \frac{j}{n}, \quad \frac{(r-1)}{r_j} \leq nt - (j-1) \leq \frac{r}{r_j}, \quad j = 1, \dots, n;$$

where $(t_1, \dots, t_n) = \Phi^{-1}(x)$, then this path joins x_0 to x and depends continuously on x.

REMARK. Hörmander's condition is essential for the validity of Lemma 2.3. In fact, we can construct a distribution which does not satisfy Hörmander's condition, still any two points on M can be joined by a piecewise smooth horizontal path, but for which Lemma 2.3 is not true: let $M = R^3$, and H be spanned by a_1, a_2, where

$$a_1 = \begin{cases} \varphi(x_1)\partial/\partial x_1, & 0 \leq x_1; \\ \varphi(-x_1)(\partial/\partial x_2 - x_1\partial/\partial x_3), & x_1 \leq 0; \end{cases}$$

$$a_2 = \begin{cases} 0, & 0 \leq x_1; \\ \varphi(-x_1)(\partial/\partial x_1 + x_2\partial/\partial x_3), & x_1 \leq 0; \end{cases}$$

where $\varphi(x_1)$ is positive for $x_1 > 0$, and is zero for $x_1 \leq 0$.

THEOREM 2.4. $e: \Omega_H M(x_0, \cdot) \to M$ *has the homotopy lifting property for polyhedra.*

Proof. (We will use the notations in Lemma 2.3.) By Hurewicz's theorem, it suffices to prove that the map $e: e^{-1}(N_{x_0}) \to N_{x_0}$ has the lifting homotopy property for polyhedra (cf. [**35**]).

Let P be a polyhedron, $h_s: P \to N_{x_0}$, $s \in [0, 1]$ a given homotopy, $\mu: P \to e^{-1}(N_{x_0})$ a map covering h_0. We construct a homotopy lifting $\mu_s: P \to e^{-1}(N_{x_0})$ as follows.

Choose a continuous function ι, $\iota(0) = 0$, $\iota(s) > 0$ if $s > 0$, such that the path $t: [0, 1 - \iota(s)] \to (\mu(p))(t)$ is within N_{x_0} for all $p \in P$, $s \in [0, 1]$. Now define, for $p \in P$,

$$(\mu_s(p))(t) = \begin{cases} (\mu(p))(t), & 0 \leq t \leq 1 - \iota(s); \\ \gamma_{(\mu(p))(1-\iota(s)), h_s(p)}((t - 1 + \iota(s))\iota(s)^{-1}), & 1 - \iota(s)t \leq 1. \end{cases}$$

So $\mu_s(p)$ first joins $(\mu(p))(0)$ to $(\mu(p))(1 - \iota(s))$ along $\mu(p)$, then joins $(\mu(p))(1 - \iota(s))$ to $h_s(p)$ along the horizontal path constructed in Lemma 2.3. By Lemma 2.3, $\mu_s(p)$ depends on s, p continuously. Moreover, $e(\mu_s(p)) = h_s(p)$. Hence $e: e^{-1}(N_{x_0}) \to N_{x_0}$ is a fibration.

Proof of Theorem 1. Using the long exact sequence associated to the fibration e, we obtain $\pi_k(\Omega_H M(x_0, x_0)) = \pi_{k+1}(M)$.

If H is fat, it is proved in [**12**] that the energy functional E on $\Omega_H M(x_1, x_2)$ satisfies the Palais-Smale condition and is non-degenerate for generic x_2, so $\Omega_H M(x_0, x_0)$ is a CW-complex. By a theorem of Whitehead, a weak homotopy equivalence between two CW-complexes is a homotopy equivalence (cf. [**35**]), so $\Omega_H M(x_0, x_0)$ is homotopic to the ordinary based loop space.

As an application, we consider the case where M is the total space of a principal G-bundle over B: $G \to M \xrightarrow{\pi} B$ with a connection ω_0. Applying Theorem 1, we have

COROLLARY 2.5. *If at every point $x \in B$ the holonomy algebra is g (the Lie algebra of G), then the space of based loops on B with a given holonomy has the same homotopy groups as the based loop space does on M.*

3. Resolution of end point map.

Suppose that H is fat. Then x_0 is an isolated singularity of the end point map e, and hence

$\Omega_H M(x_0, x_0)$ is singular at the constant loop. In the next two sections we will study this singularity. In this section we will develop a resolution of e by blowing up the source manifold (i.e. the horizontal paths space) and the target manifold M simultaneously. We begin with the blow-up of M.

3.1. *Weighted blow-up of M*. We will use the notations in Lemma 1.1. We define the weighted blow-up of M at x_0 to be $B_w M = S_1 \times [0, \infty)$, where S_1 is the set of points $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ satisfying $x_1^4 + \cdots + x_m^4 + y_1^2 + \cdots + y_k^2 = 1$. The blow-up map $B_w\colon B_w M \to M$ is given by

$$(3.1) \qquad \begin{aligned} & B_w((x_1, \ldots, x_m, y_1, \ldots, y_k), s) \\ & = (s \cdot x_1, \ldots, s \cdot x_m, s^2 \cdot y_1, \ldots, s^2 \cdot y_k). \end{aligned}$$

It is easy to see that B_w is a diffeomorphism except at the boundary.

REMARK. A similar construction was developed by Kupka [44].

3.2. *Blow-up of $\Omega_H M(x_0, \cdot)$*. Let ΩR^m be the Hilbert space of H^1 paths on R^m starting from $0 \in R^m$. As we will see §3.3, the horizontal path space $\Omega_H M(x_0, \cdot)$ in a neighborhood of the constant loop can be identified with ΩR^m. From now on we will make this identification.

In view of this identification, we define the blow-up of $\Omega_H M(x_0, \cdot)$ to be $B\Omega_H M(x_0, \cdot) = S^\infty \times [0, \infty)$, where S^∞ is the space of curves γ on R^m through 0, $|\gamma|_{H^1} = 1$. The blow-up map $B\colon B\Omega_H M(x_0, \cdot) \to \Omega R^m$ is given by

$$(3.2) \qquad\qquad\qquad B(\gamma, s) = s\gamma.$$

As in §3.1, it is easy to see that B is a diffeomorphism except at the boundary.

3.3. *Resolution of e*. Having developed the blow-ups of the target and the source manifold of e, now we develop the resolution of e.

First we need a normal form of e. We begin with the simplest case: M is a fat nilpotent group N with the Lie algebra given by (1.1). In this case $\Omega_H M(x_0, \cdot)$ can be identified with the space of H^1-paths on R^m starting from 0, ΩR^m, and e can be written as

$$\gamma \to \left(\gamma(1), \int_\gamma \omega_1, \ldots, \int_\gamma \omega_k\right),$$

where

$$\omega_i = \sum \Gamma^i_{jr} x_j dx_r, \qquad i = 1, \ldots, k.$$

For a general fat distribution, we will approximate e using tangent cones. More precisely,

LEMMA 3.1. *Suppose that H is a fat distribution and N is the tangent cone at x_0. In a coordinate neighborhood of the constant loop, we can write e as $e: \Omega R^m \to R^n$,*

$$\gamma_1 \to \left(\gamma_1(1), \int_{\gamma_1} \omega_1 + T_1(\gamma_1), \dots, \int_{\gamma_1} \omega_k + T_k(\gamma_1) \right),$$

where

$$|T_j(\gamma_1)| \le C\|\gamma_1\|^3, \qquad j = 1, \dots, k.$$

Proof. Let $\gamma = (\gamma_1(t), y(t))$ be a horizontal path; then we have the following equations: $(\gamma_1(t) = (x_1(t), \dots, x_m(t)))$

$$(3.3) \quad \frac{d}{dt} y_i(t) = \sum \Gamma^i_{jr} x_j(t) \frac{d}{dt} x_i(t) + \overline{T}_i(x, y, \dot{x}, \dot{y}),$$

$$i = 1, \dots, k,$$

where $\overline{T}_i(x, y, z, w)$ is a smooth function

$$|\overline{T}_i(x, y, z, w)| \le C_0(|x|^3 + |y|^2),$$

where C_0 is a uniform constant for bounded z.

It is obvious that given $\gamma_1(t) \in \Omega R^m$, there is a unique $\gamma = (\gamma_1(t), y(t))$ satisfying (3.3) and vice-versa, and the solution can be written as

$$y_i(t) = \int_{\gamma_1} \omega_i + O(\|x\|^3 + \|y\|^2).$$

Hence the lemma follows.

Let gra(e) be the graph of the restriction of e

$$e: \Omega_H M(x_0, \cdot) - \Omega_H M(x_0, x_0) \to M - \{x_0\},$$

and

$$(B, B_w): B\Omega_H M(x_0, \cdot) \times B_w(M) \to \Omega_H M(x_0, \cdot) \times M$$

be the obvious map.

DEFINITION 3.1. The resolution

$$B_e: B\Omega_H M(x_0, \cdot) - B^{-1}(\Omega_H M(x_0, x_0)) \to B_w M$$

of e is determined by its graph, which is the closure of $(B, B_w)^{-1}(\text{gra}(e))$ in $B\Omega_H M(x_0, \cdot) \times B_w(M)$.

The relation between various blow-ups can be summerized in the following diagram

$$\begin{array}{ccc} B\Omega_H M(x_0, \cdot) - B^{-1}(\Omega_H M(x_0, x_0)) & \xrightarrow{\;B_e\;} & B_w M \\ \quad\big\downarrow{\scriptstyle B} & & \quad\big\downarrow{\scriptstyle B_w} \\ \Omega_H M(x_0, \cdot) - \Omega_H M(x_0, x_0) & \xrightarrow{\;e\;} & M \end{array}$$

(3.4)

LEMMA 3.2. B_e is a submersion everywhere.

Proof. By (3.4), since both B_w and B are diffeomorphisms except at the boundaries, we need only to prove that B_e is submersive at the boundary of $B\Omega_H M(x_0, \cdot)$.

Using Lemma 3.1 and (3.1), (3.2), we see that B_e can be written as

$$(3.5) \quad (\gamma_1, s) \to \left(\left(\frac{1}{b} s\gamma_1(1), \frac{1}{b^2} \left(\int_{s\gamma_1} \omega_1 + T_k(s\gamma_1) \right), \ldots, \right. \right.$$
$$\left. \left. \frac{1}{b^2} \left(\int_{s\gamma_1} \omega_k + T_k(s\gamma_1) \right) \right), b \right),$$

where

$$b = \left(s^4(x_1^4 + \cdots + x_m^4) + \left(\int_{s\gamma_1} \omega_1 + T_k(s\gamma_1) \right)^2 \right.$$
$$\left. + \cdots + \left(\int_{s\gamma_1} \omega_k + T_k(s\gamma_1) \right)^2 \right)^{1/4}$$

for $(x_1, \ldots, x_m) = \gamma_1(1)$. By (3.3), we can rewrite (3.5) as

$$B_e(\gamma_1, s) = \left(\left(\gamma_1(1), \int_{\gamma_1} \omega_1, \ldots, \int_{\gamma_1} \omega_k \right), \right.$$
$$\left. s \left(x_1^4 + \cdots + x_k^4 + \left(\int_{\gamma_1} \omega_1 \right)^2 + \cdots + \left(\int_{\gamma_1} \omega_k \right)^2 \right)^{1/4} \right) + o(s).$$

So it is a submersion.

4. Singularity of horizontal loop space. In this section we will study the singularity of the horizonal loop space, using the resolution of e.

4.1. *The case of nilpotent groups.* Suppose that M is a simple connected nilpotent Lie group N whose Lie algebra is

$$n = V_1 \oplus V_2 \oplus \cdots \oplus V_s, \quad [V_i, V_j] \subset V_{i+j}, \quad i + j \le s,$$
$$[V_i, V_j] = 0, \quad i + j > s,$$

and H is the left translation of V_1. As usual, we denote the space of H^1 horizontal paths starting from 0 by $\Omega_H N(0, \cdot)$.

R_+ (the multiplicative group of positive numbers) acts on $N - \{0\}$ as

$$t(a_1, a_2, \dots, a_n) = (a_1 t, \dots, a_m t, a_{m+1} t^2, \dots, a_{i_k} t^k, \dots, a_n t^s),$$

where $a_{i_k} \in V_k$. Let $P_w N$ be the quotient space. It is easy to prove that P_w is diffeomorphic to S^{n-1}, $n = \dim(N)$.

The action of R_+ on N induces an action on $\Omega_H N(0, \cdot)$, $\Omega_H N(0, 0)$ respectively. Let $P(\Omega_H N(0, \cdot) - \Omega_H N(0, 0))$ be the quotient of the space $\Omega_H N(0, \cdot) - \Omega_H N(0, 0)$ by this action. Then the end point map e induces a map

$$\{e\}: P(\Omega_H N(0, \cdot) - \Omega_H N(0, 0)) \to P_w N$$

in an obvious way.

THEOREM 4.1. *$\{e\}$ is a fibration over S^{n-1}.*

Proof. We only need to prove an equivariant version of Lemma 2.3, i.e. there is $\delta_0 > 0$, such that for any x, y, $d(x, y) \le \delta_0$, there is a horizontal curve $\gamma_{x,y}$ connecting x, y, depending continuously on x, y, and

$$\gamma_{sx, sy} = s \cdot \gamma_{x,y}$$

where $S \cdot \gamma$ denotes the induced action of R_+ on $\Omega_H N(0, \cdot)$.

In fact, there are left-invariant vector fields on N, $a_1, \dots, a_m \in \Gamma(H)$ spanning H, and, together with their commutators of length less than $(s + 1)$, $a_J = [a_{i_1}, \dots, [a_{i_{r-1}}, a_{i_r}]]$, $J = (i_1, \dots, i_r)$, $r \le s$ span TN. We then define Φ in the same way as in Lemma 2.3.

To construct $\gamma_{x,y}$, we need to prove that Φ is equivariant, i.e.

(4.1)
$$\Phi(s(t_1, \dots, t_n))(s(x_1, \dots, x_n))$$
$$= s \cdot \Phi(t_1, \dots, t_n)(x_1, \dots, x_n).$$

To this end, we need the notation of (weighted)-homogeneity.

A function f on N is homogeneous of degree k if

$$f(s(x_1, \dots, x_n)) = s^k f(x_1, \dots, x_n).$$

A vector field V on N is homogeneous of order 1 if for every homogeneous function f of degree k, $V(f)$ is homogeneous of degree $(k-1)$. It is easy to prove that $a_1, \ldots, a_m$ are homogeneous of degree 1, so (4.1) follows. The rest of the proof is the same as in Theorem 2.4.

REMARK 1. The fibre of the above fibration is contractible. In fact, the fibre is just $e^{-1}(x_0)$, $x_0 \neq 0$, which, by a result of Weinstein (cf. [12], Lemma 2.1), is homotopic to $e^{-1}(0)$, which is contractible.

REMARK 2. Let $\Omega R^n(0, \cdot)$ be the space of H^1-curves on R^n through 0, e_1 the end point map, and $\Omega R^n(0, 0) = e_1^{-1}(0)$. R_+ acts on $R^n - \{0\}$ in an obvious way and hence on $\Omega R^n(0, \cdot) - \Omega R^n(0, 0)$. Let $P_+ R^n$, $P_+(\Omega R^n(0, \cdot) - \Omega R^n(0, 0))$ be the quotient spaces respectively; then as in Theorem 4.1, the induced map $\{e_1\}$: $P_+(\Omega R^n(0, \cdot) - \Omega R^n(0, 0)) \to P_+ R^n$ is a fibration with contractible fiber. It would be interesting to compare this fibration with the fibration in Theorem 4.1.

4.2. *The case of general fat distributions.* In this subsection we will generalize the results in §4.1 to a general fat distribution H. This generalization is not straight forward, as the map e is not stable at the constant loop.

Let N be the tangent cone at $x_0 \in M$, and

$$B(\Omega_H M(x_0, \cdot) - \Omega_H M(x_0, x_0))$$
$$= B\Omega_H M(x_0, \cdot) - B^{-1}(\Omega_H M(x_0, x_0)),$$

which can be considered as the blow-up of $\Omega_H M(x_0, \cdot) - \Omega_H M(x_0, x_0)$. Note that the boundary of $B(\Omega_H M(x_0, \cdot) - \Omega_H M(x_0, x_0))$ is just $P(\Omega_H N(0, \cdot) - \Omega_H N(0, 0))$, which is a fibration over S^{n-1} by Theorem 4.1.

We define a Riemannian structure on ΩR^n by

$$(\gamma_1, \gamma)_1 = \int_0^1 \left(\frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt} \right) dt, \quad \|\gamma\| = (\gamma, \gamma)_1^{1/2},$$

where $(\cdot, \cdot)$ is the Euclid structure on R^m. Also denote

$$D^{-1}\gamma(t) = \int_0^t \gamma(s) \, ds.$$

To be precise about the sphere in Theorem 2, we need to introduce a function on ΩR^m.

Write $e = (e_1, \dots, e_n)$. Let de_i be the differential of the functional e_i. Then $D^{-2}de_i$ formally is the gradient of the functional e_i with respect to $(\cdot, \cdot)_1$.

We define a function ΩR^m

$$P(\gamma) = \det(D^{-2}de_i(\gamma)(t) - tD^{-2}de_i(\gamma)(1),$$
$$(D^{-2}de_i(\gamma)(t) - tD^{-2}de_i(\gamma)(1))).$$

Since H is fat, $P(\gamma) \neq 0$ if $\gamma \neq 0$.

Now we define a sphere

$$S_\varepsilon = \{\gamma \in \Omega R^m, \|\gamma\|^{1/2} + \|\gamma\|/P(\gamma) = \varepsilon\}.$$

Now we reformulate Theorem 2 as follows:

THEOREM 4.2. *Let* $S_e = \Omega_H M(x_0, x_0) \cap S_\varepsilon$. *Then the complement of* S_e *in* S_ε *is the total space of a fibration over* S^n *with contractible fiber for* ε *small enough.*

REMARK. We do not know if the theorem will remain true if we replace S_ε by the standard sphere.

4.3. The proof of Theorem 4.2. The basic idea of the proof is to study a vector field (which we call F) on $B\Omega_H M(x_0, \cdot)$. We will prove that its phase flow induces a homeomorphism from $S_\varepsilon - S_e$ to $P(\Omega_H N(0, \cdot) - \Omega_H N(0, 0))$, which is, by Theorem 4.1, a fibration over S^{n-1}.

We first construct a vector field on ΩR^m,

$$(4.2) \quad \frac{d\gamma_s(t)}{ds} = -\gamma_s(t) + \sum a_i(\gamma_s)(D^{-2}de_i(\gamma_s)(t) - D^{-2}de_i(\gamma_s)(1)),$$

where $a_1(\gamma), \dots, a_k(\gamma)$ satisfy

$$(4.3) \quad \sum a_i(\gamma)(D^{-2}de_i(\gamma) - tD^{-2}de_i(\gamma)(1),$$
$$D^{-2}de_j(\gamma)(t) - tD^{-2}de_j(\gamma)(1))_1$$
$$= (D^{-2}de_j(\gamma)(t) - t^2 de_j(\gamma)(1), \gamma)_1 - 2e_j(\gamma),$$
$$j = 1, \dots, k.$$

This condition is to assure that the vector field (4.2) leaves the set $e^{-1}(0)$ invariant. Note that when restricted to $e^{-1}(0)$, (4.2) is just the gradient flow of the functional $\|\gamma\|^2$ with respect to the Riemannian product $(\cdot, \cdot)_1$.

Solve $a_1, \dots, a_k$ from (4.3); then we can rewrite (4.2) as

$$(4.4) \qquad \frac{d\gamma(s, t)}{ds} = -\gamma(s, t) + \frac{V(\gamma_s)(t)}{P(\gamma_s)},$$

where V, P satisfy the following conditions (C will denote any constant)

(1) P is a smooth function on ΩR^m, $P(\gamma) = P_0(\gamma) + P_1(\gamma)$, where P_0 is a positive homogeneous functional of degree $2k$, and

$$(4.5) \qquad |P_1(\gamma)| \le C\|\gamma\|^{2k+1}.$$

(2) V is a smooth vector field on ΩR^m

$$(4.6) \qquad \|V(\gamma)\| \le C\|\gamma\|^{2k+2}.$$

We will use d/ds to denote the Lie derivative with respect to the vector field (4.2). Using (4.5), (4.6), by a direct computation we obtain

$$
\begin{aligned}
(4.7) \qquad \frac{d}{ds}\left(\frac{\|\gamma\|^{2k+1/2}}{P(\gamma)}\right) &\le -\frac{\|\gamma\|^{2k+1/2}}{2P} + C\frac{\|\gamma\|^{4k+3/2}}{P^2} \\
&\quad + C\frac{\|\gamma\|^{6k+3/2}}{P^3}(1 + \|\gamma\|).
\end{aligned}
$$

This will be our fundamental inequality. As we will see, if $\|\gamma\|$ and $\|\gamma\|^{2k+1/2}/P(\gamma)$ are sufficiently small, then they are fast decreasing.

LEMMA 4.3. *If*

$$(4.8) \qquad \frac{\|\gamma_0\|}{P(\gamma_0)}\left(\|\gamma_0\|^{1/2} + \frac{\|\gamma_0\|}{P(\gamma_0)}(1 + \|\gamma_0\|)\right) \le c \le 1,$$

where c is sufficiently small, then there is a constant c_1, $0 < c_1 < 1$,

$$(4.9) \qquad \frac{\|\gamma_s\|^{2k+1/2}}{P(\gamma_s)} \le \frac{\|\gamma_0\|^{2k+1/2}}{P(\gamma_0)} \exp((-1 + c_1)s).$$

Proof. Applying the Gronwall inequality to (4.7).

COROLLARY 4.4. *Under the same condition in Lemma 4.3, there is a constant c_2, $0 < c_2 < 1/6$, such that*

$$\|\gamma_s\| \le \exp(-s + sc_2)\|\gamma_0\|.$$

Proof. Using (4.4), we have

$$\frac{d}{ds}\|\gamma\| = -\|\gamma\| + \frac{(\gamma, V(\gamma))_1}{\|\gamma\|P(\gamma)} \le -\|\gamma\| + \frac{\|\gamma\|^{2k+1}}{\|\gamma\|P(\gamma)} \le -\|\gamma\|(1 - Cc^2).$$

Using the Gronwall inequality, we prove the lemma.

So far we have proved that the phase flow of (4.2) will approach 0 within S_ε very fast, so 0 is a singularity of this vector field. To resolve this singularity, we will pull back this vector field by B to $B\Omega R^m$ and then divide it by $|\gamma|$. The vector field F thus obtained will induce a homeomorphism $S_\varepsilon - S_e \to P(\Omega_H N(0, \cdot) - \Omega_H N(0, 0))$.

To be more precise, define $\beta_s = \gamma_s / \|\gamma_s\|$; then β_s satisfies the following equation:

$$(4.10) \qquad \frac{d\beta_s}{ds} = -\frac{(V(\gamma_s), \gamma_s)_1 \beta_s}{\|\gamma_s\|^2 P(\gamma_s)} + \frac{V(\gamma_s)}{\|\gamma_s\| P(\gamma_s)},$$

which, together with the equation for $\|\gamma_s\|$,

$$(4.11) \qquad \frac{d\|\gamma_s\|}{ds} = -\|\gamma_s\| + \frac{(V(\gamma_s), \gamma_s)}{\|\gamma_s\| P(\gamma_s)},$$

is the pull back of (4.2). Obviously this vector field vanishes on the boundary of $B\Omega R^m$.

Note that β_s converges as $s \to \infty$. To see this, we estimate $d\beta_s/ds$. From (4.10), we have

$$\left\| \frac{d}{ds} \beta_s \right\| \leq C \frac{\|\gamma_s\|^{2k+1}}{P(\gamma_s)},$$

then, using Lemma 4.3, Corollary 4.4, we have

$$(4.12) \qquad \|\beta_s - \beta_0\| \leq C \frac{\|\gamma_0\|^{2k+1}}{P(\gamma_0)}.$$

The vector field F is then obtained by dividing (4.10), (4.11) by $\|\gamma\|$,

$$(4.13) \qquad \frac{d\beta_s}{ds} = -\frac{(V(\gamma_s), \gamma_s)_1 \beta_s}{\|\gamma\|^3 P(\gamma_s)} + \frac{V(\gamma_s)}{\|\gamma_s\|^2 P(\gamma_s)},$$

$$(4.14) \qquad \frac{d\|\gamma_s\|}{ds} = -1 + \frac{(V(\gamma_s), \gamma_s)}{\|\gamma_s\|^2 P(\gamma_s)}.$$

Obviously F is a smooth vector field on $S^\infty \times [0, \infty)$, non-vanishing up to the boundary. Note that by (12), the phase flow of F determines a continuous map $S_e \to S^\infty$. Moreover, using a standard argument in the theory of smooth vector fields on manifolds and the fact that F is non-vanishing, we then prove that this map is in fact a homeomorphism, so

LEMMA 4.5. *The phase flow of F induces a homeomorphism $S_\varepsilon - S_e \to P(\Omega_H N(0, \cdot) - \Omega_H N(0, 0))$.*

5. Cut-locus of CC metrics. In this section we will study the cut locus of a CC metric. We will prove Theorem 3 in §5.1, Theorem 4 in §5.2.

5.1. *Geometry and definition of the cut locus.* Let M be a complete CC manifold. Geodesics and minimizing geodesics are defined as in Riemannian geometry. If H is fat, then the geodesics satisfy the geodesic equation ([12], compare [28] for the general case). $\exp_{x_0} : T^*_{x_0} M \to M$, or simply exp will denote the exponential map for the CC metric (cf. [12]). Thus exp is the composition of the time 1 map of the Hamiltonian flow on T^*M composed with the inclusion of $T^*_{x_0} M$ into T^*M, projected to M.

Fix a point $x_0 \in M$. Then any geodesic through x_0 has the form $\gamma(t) = \exp(tp)$. A "cut point" of x_0 along γ is a point $x_1 = \gamma(t_1)$ lying on γ such that $[x_0, x_1]$ is the largest subinterval of γ for which the geodesic is minimizing. In other words, if $t > t_1$ then $\gamma[0, t]$ is not a minimizing geodesic, and if $t \leq t_1$ then it is a minimizing geodesic. The set of all cut points (as γ varies) to x_0 is called the cut locus (of x_0).

Note that x_1 is a cut point of x_0 if and only if either (a) there is more than one minimizing geodesics connecting x_0 to x_1 or (b) we can write $x_1 = \exp(p)$ where p is a critical point of exp, and if $t < 1$ then tp is not a critical point. Note that (a) and (b) are not mutually exclusive. The points satisfying (b) are said to be in the "conjugate locus". Strictly speaking, the critical values of exp form the conjugate locus and the set of points satisfying (b) are said to be in the "first conjugate locus". For more on the cut locus and conjugate locus see Taylor [38], Lemma 5, or Kobayashi-Momizu [43], vol. 2, §8.7.

Strichartz [36] proved that cut points of x_0 occur arbitrarily near x_0. In fact conjugate points occur arbitrarily near x_0 (cf. [12]) and 0 is a critical point for exp with corresponding value x_0. For these reasons we now *define* x_0 to be in its own cut locus. Thus, for our purpose the cut locus is the usual cut locus, with x_0 included.

If H is strong bracket generating, a geodesic does not minimize past its first conjugate point. It follows from this that (after adding x_0 in!) the cut locus is closed. Taylor (see [38], Theorem 4) claimed that a geodesic does not minimize past its first conjugate point, but we find a counter-example, see the Remark 2 after the proof of Theorem 3.

Let S be the unit cylinder $Q(p, p) = 1$ in $T_{x_0}^* M$. Let $\mu = \mu_{x_0}: S \to R^+$ be the function which assigns to each $p \in S_x$ the largest extended real number $T \in [0, \infty]$ such that the geodesic $t \to \exp_{x_0}(tp)$ is minimizing for all $t \in (0, T]$. Taylor [38] proved that if μ is locally bounded away from zero and H is fat, then μ is continuous. He also conjectured that μ is continuous. Here we will prove his conjecture.

LEMMA 5.1. *μ is locally bounded from below away from zero.*

Proof. By the result of Rayner [32], for any CC geodesic $\gamma: [a, b] \to M$ without double points, there is a Riemannian metric which extends the CC metric, with respect to which γ is still a geodesic. Moreover, this Riemannian metric depends smoothly on γ. Obviously, if γ minimizes the Riemannian action then it also minimizes the CC action. Since for a family of smoothly varying Riemannian metrics the injectivity radii are locally bounded away from zero, so is μ.

Proof of Theorem 3. For $r > 0$, let B_r be the ball at x_0, $d(x, x_0) \leq r$. We want to prove that the complement of the cut locus within B_r is homeomorphic to the complement of a k-plane in R^n.

Let D be the set of points p in $T_{x_0}^* M$ such that the geodesic $t \to \gamma(t) = \exp(tp)$ is minimizing for $0 < t \leq 1$, $\exp(p) \in B_r$, and $\exp(p)$ is not a cut point to x_0. We first prove that D is homeomorphic to the complement of R^k in R^n. This follows from the following properties of D:

(1) D is open (since the cut-locus is closed).

(2) If $p \in D$, then $tp \in D$ for $0 < t < 1$. In fact, if $\exp_{x_0}(t_0 p)$ is a cut point for some $t_0 \in (0, 1)$, then there is another geodesic γ_0 joining $\exp_{x_0}(t_0 p)$ to x_0 with length less or equal to that of $t \to \exp_{x_0}(tp)$; hence if γ_1 denotes the curve $\gamma_1 = \gamma_0(2t)$, if $0 \leq t \leq 1/2$; $\gamma_1(t) = \exp_{x_0}((2(1 - t)t_0 + (2t - 1))p)$, if $1/2 \leq t \leq 1$, then γ_1 is also minimizing as it has the same length as γ_0, and hence must be smooth (as every minimizing geodesic is smooth), which is a contradiction.

(3) If $Q(p, p) \neq 0$, then for t sufficiently small, $tp \in D$. This follows from $\mu_{x_0} > 0$.

(4) If $Q(p, p) = 0$, then $\exp(p) = x_0$.

Let $C(x_0)$ be the cut locus of x_0. Then we prove that $\exp: D \to M - C(x_0) \cap B_r$ is a homeomorphism. Obviously $\exp_{x_0}: D \to M - C(x_0) \cap B_r$ is a local diffeomorphism. We need only to prove that it is one-to-one.

Suppose that p_1, $p_2 \in D$, $p_1 \neq p_2$, $\exp_{x_0}(p_1) = \exp_{x_0}(p_2)$. Choose $t_1 > 1$ such that $t_1 p_2 \in D$. Define $\gamma_3(t) = \exp(2tp_1)$, if $0 \leq t \leq 1/2$, and $\gamma_3(t) = \exp((((2t-1)t_1 + 2 - 2t)p_2))$ if $1/2 \leq t \leq 1$, then γ_3 is also minimizing (since it has the same length as the path $t\colon [0, t_1] \to \exp(tp_2)$), and hence must be smooth, which is a contradiction.

REMARK 1. There is no such simple relation between the topologies of the cut locus and M as in Riemannian geometry (cf. [21]).

REMARK 2. Theorem 3 is not true in general, as the following example shows. Take $M = R^5$, H spanned by

$$\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial y_2} + x_2 \frac{\partial}{\partial z_1},$$

and the metric

$$(dx_1)^2 + (dy_1)^2 + (dx_2 - y_2 dz_1)^2 + (dy_2 + x_2 dz_1)^2,$$

then the cut-locus of 0 is

$$\{(x_1, y_1, x_2, y_2, z_1) \in R^5, x_2 = y_2 = 0\}$$
$$- \{(x_1, y_1, 0, 0, 0), (x_1, y_1) \neq 0\}.$$

In particular, the cut locus is not closed. The geodesic $t \to (p_1 t, p_2 t, 0, 0, 0)$ is minimizing within $[0, \infty)$, though every point on it is a critical value of the exponential map.

5.2. *Cut-locus on fat analytic CC manifolds.* We will prove Theorem 4. Let M be a fat analytic CC manifold. We will need the following characterization of the cut-locus.

LEMMA 5.2. *The cut-locus of x_0 is the singular support of the function $d(x, x_0)$. (The singular support of a function is the set of points where the function is not analytic, cf. [37].)*

Proof. To see that this is in fact equivalent to our definition apply the inverse function theorem to exp to conclude that if x is not in the cut locus then it is not in this singular support of $d(\cdot, x_0)$. (Note that, according to our definition of cut locus, if x is not in the cut locus and if $\gamma = \exp(tp)$ is a minimizing geodesic joining x_0 to $x = \exp(p)$, then p is not a critical point of exp.) Conversely, if x is in the cut locus, then it must be in the singular support. Join x to x_0 by a geodesic γ as in the definition. Without loss of generality, we may take γ to be unit speed. Suppose that $x = \gamma(t_1)$. Now observe that

$d(\gamma(t), x_0) = t$ for $t \le t_1$ but that $d(\gamma(t), x_0) < t$ for $t > t_1$. Since γ is analytic it follows that d cannot be analytic at x.

Now it suffices to prove that the distance $d(x, x_0)$ is sub-analytic in x near x_0 since the singular support of a sub-analytic function is sub-analytic.

THEOREM 5.3. *Suppose that the CC metric is real analytic and H is fat; then there exists $T > 0$ such that $d(x, x_0)$ is sub-analytic within the ball $d(x, x_0) \le T$.*

We extend the CC metric to a Riemannian metric and will use the notations in §1.2. So in the following $\gamma = \gamma(t)$, $0 \le t \le 1$, $\gamma(0) = x_0$, will denote a geodesic for the CC metric with the Lagrangian multiplier $\lambda = \lambda(t)$, which satisfy equations (1.3), (1.4).

We define a map

$$\exp: H_{x_0} \times H_{x_0}^\perp \to M,$$
$$(\gamma'(0), \lambda(0)) \to \gamma(1)$$

which is obviously analytic. Note this is simply the old exponential map composed with the metric-induced identification on $H_{x_0} \times H_{x_0}^\perp = T_{x_0}M$ with $T_{x_0}^*M$.

LEMMA 5.4. *There exists $T > 0$ and a compact subset $F \subset H_{x_0} \times H_{x_0}^\perp$ (e.g. a closed ball) such that if $\gamma: [0, 1] \to M$, $\gamma(0) = x_0$, is minimizing and $d(\gamma(1), x_0) \le T$, then $(\gamma'(0), \lambda(0)) \in F$.*

Proof. By contradiction.

If it is not true, then there is a sequence of minimizing geodesics $\gamma_i: [0, 1] \to M$, $\gamma_i(0) = x_0$, $e(\gamma_i) := \gamma_i(1) \ne x_0$, with Lagrange multipliers λ_i satisfying (1.3), (1.4), such that

(5.1) $\gamma_i(1) \to x_0$, $\lambda_i(0) \to \infty$,   as $i \to \infty$.

So $l_i = \max_i |\lambda_i(t)| \to \infty$.

Now use the coordinates of Lemma 1.1 to identify a neighborhood of x_0 with $R^m \times R^k$, where 0 corresponds to x_0 and m is the distribution's rank. Then we have $\gamma_i = (\alpha_i, \beta_i)$, $\alpha_i(t) \in R^m$, $\beta_i(t) \in R^k$. Let

$$\delta_i = a(e(\gamma_i))$$

where

$$a(x) = (x_i^4 + \cdots + x_m^4 + y_1^2 + \cdots + y_k^2)^{1/4}.$$

Define

$$\overline{\alpha}_i = \frac{\alpha_i}{\delta_i}, \quad \overline{\beta}_i = \frac{\beta_i}{\delta_i^2}, \quad \overline{\gamma}_i = (\overline{\alpha}_i, \overline{\beta}_i).$$

From (1.3), we see that $|\gamma_i'(t)|^2$ is independent of t. Strichartz (cf. [36], Theorem 11.1) has shown that $E(\gamma_i)/\delta_i^2$ is uniformly bounded. It follows that $\overline{\gamma}_i$ has a weakly convergent sub-sequence (still denoted by $\overline{\gamma}_i$) in H^1, $\overline{\gamma}_i \to \overline{\gamma}_\infty$. In particular, $\overline{\gamma}_i \to \overline{\gamma}_\infty$ uniformly in C^0. It follows from the weak convergence of $\overline{\gamma}_i$ that $\overline{\gamma}_\infty$ is a horizontal path on the tangent cone. This can be seen as follows.

As we have seen in the proof of Lemma 3.1, a horizontal path (x, y) satisfies

$$\frac{d}{dt}y_i(t) = \sum \Gamma_{jr}^i x_j(t) \frac{d}{dt}x_i(t) + \overline{T}_i(x, y, \dot{x}, \dot{y}), \qquad i = 1, \ldots, k,$$

where $\overline{T}_i(x, y, z, w)$ is a smooth function

$$|\overline{T}_i(x, y, z, w)| \le C_0(|x|^3 + |y|^2).$$

So

$$\beta_i(t) = \int_{\alpha_i|[0, t]} \omega + O(\delta_i^3),$$

where $\omega = (\omega_1, \ldots, \omega_k)$, ω_i are as in §3.3. So it follows that

$$\overline{\beta}_i(t) = \int_{\overline{\alpha}_i|[0, t]} \omega + O(\delta_i)$$

i.e. the limit $(\overline{\alpha}_\infty, \overline{\beta}_\infty)$ is horizontal.

First we remark that $\overline{\gamma}_\infty$ is not constant, as

$$e(\overline{\gamma}_\infty) = \lim \frac{e(\gamma_i)}{\alpha(e(\gamma_i))} \ne 0.$$

From (1.4) we obtain

(5.2) $$|\lambda_i'| \le C_0(|\gamma_i'| |\lambda_i| + |\gamma_i'|^2).$$

Hence λ_i/l_i is uniformly bounded in C^0. So after passing to a subsequence, we assume that

$$\frac{\lambda_i}{l_i} \to b \ne 0 \text{ uniformly in } C^0.$$

Dividing (1.3) by δ_i, we obtain

(5.3) $$D_{\overline{\gamma}_i'}\overline{\gamma}_i' + \delta_i Q_1(\overline{\gamma}_i', \gamma_i) = \lambda_i \Omega(\overline{\gamma}_i' + \delta_i Q_2(\overline{\gamma}_i', \gamma_i), \cdot),$$

where Q_1 (resp. Q_2) is an analytic function, quadratic (resp. linear) in the first variable. Divide (5.3) by l_i, and then taking its inner product with a H^1-map V with compact support, we obtain

$$\frac{1}{l_i}\int_0^1 \{-(\overline{\gamma}_i', V') + (\delta_i Q_1(\overline{\gamma}_i', \gamma_i), V)\}\, dt$$

$$= \int_0^1 \frac{\lambda_i}{l_i}\Omega(\overline{\gamma}_i' + \delta_i Q_2(\overline{\gamma}_i', \gamma_i), V)\, dt\,.$$

Let $i \to \infty$ above and note that the integral on the right-hand side is uniformly bounded to conclude that

$$\int_0^1 b \cdot \Omega(\overline{\gamma}_\infty', V)\, dt = 0$$

for any compactly supported V. But b, $\overline{\gamma}_\infty' \neq 0$. This contradicts the fact that H is fat. (Observe that if H is fat, then so is every tangent cone.)

Proof of Theorem 4. Define two subsets
$$A = \{(x, t, \gamma'(0), \lambda(0)) \in M \times R \times F,$$

$$x = \exp(\gamma'(0), \lambda(0)),\ \ t = |\gamma'(0)|\},$$

$$B = \{(x, t, \gamma'(0), \lambda(0)) \in M \times R \times F,$$

$$x = \exp(\gamma'(0), \lambda(0)),\ \ 0 < |\gamma'(0)| < t\}\,.$$

Since these sets are defined by analytic equations and inequalities, they are sub-analytic.

Let $P: M \times R \times F \to M \times R$ be the projection; then by definition, the graph of the distance function $t = d(x, x_0)$ within $d(x, x_0) \leq T$ is just

$$P(A) - P(B)\,.$$

Since F is compact, we see that P is proper on the closures of A and B respectively. By the definition of sub-analytic sets, $P(A) - P(B)$ is sub-analytic. So we see that the graph of $d(x, x_0)$ is sub-analytic, and hence $d(x, x_0)$ is sub-analytic within $d(x, x_0) \leq T$.

5.3. *A stronger version of Theorem* 4. Here we will prove a stronger version of Theorem 4, using a more direct approach.

THEOREM 5.5. *Suppose H is fat and the CC metric on M is real analytic. Choose coordinates $x = (x_i, y_j)$ near x_0 as in Lemma* 1.1, *and define*

$$a(x) = (x_i^4 + \cdots + x_m^4 + y_1^2 + \cdots + y_k^2)^{1/4}\,,$$

then there is a neighborhood U of x_0 such that the closure of the graph of

$$d(x, x_0)/a(x)$$

on $U - \{x_0\}$ is sub-analytic within $U \times R$. Moreover, when pulled back to the weighted blow-up, $d(x, x_0)/a(x)$ is continuous and sub-analytic.

We will need a result of Tamm [37].

LEMMA 5.6. *Suppose that $\varphi \colon N \to M$ and $g \colon N \to R$ are sub-analytic and φ is proper. Then the function*

$$\varphi_g(x) = \inf\{g(u) : u \in \varphi^{-1}(x)\}$$

is sub-analytic.

Now $d(x, x_0)$ can be written as

$$d(x, x_0)/a(x) = \inf\{E(\gamma)/a(x),\, e(\gamma) = x\}.$$

But e is not proper, so Lemma 5.6 cannot be applied directly. The following result will enable us to overcome this difficulty.

LEMMA 5.7 ([17]). *Let B be a Hilbert space, $f(y, z) \colon B \times R^r \to R$ an analytic function satisfying*

$$(5.5) \qquad (D_y^2 f(0, 0)a, b) = ((I - A)a, b), \qquad a, b \in B,$$

where A is a linear compact operator of B, $B = \ker(I - A) \oplus V$. Then there are local origin preserving analytic transformations $T \colon B \times R^r \to B \times R^r$ and $h \colon \ker(I - A) \times R^r \to B$, defined in a neighborhood of $(0, 0) \in \ker(I - A) \times R^r$, such that f takes the form of

$$f \circ T((y_1, y_2)z) = ((I - A_1)y_2, y_2) + f(h(y_1, z), z),$$
$$y = (y_1, y_2) \in \ker(I - A) \oplus V,$$

where A_1 is the restriction of A to V.

In the proof of Theorem 5.5 we will use the resolution of e, as in Lemma 5.7 it requires that the projection is a submersion (recall that e is not a submersion).

Proof of Theorem 5.5. Write

$$(5.6) \quad \frac{1}{a(B_w(z))} d(B_w(z), x_0)$$

$$= \inf \left\{ \frac{B^* E(\gamma)^{1/2}}{a(B_w(z))}, \quad \gamma \in B(\Omega_H M(x_0, \cdot) - \Omega_H M(x_0, x_0)), \right.$$

$$\left. B_e(\gamma) = z \right\}.$$

We note that the functional in (5.6), $B^* E(\gamma)^{1/2}/a(B_w(z))$, has an analytic continuation to the boundary of the domain. In fact, we have

$$(5.7) \qquad E(a\overline{\gamma}) = a^2 E_0(\overline{\gamma}) + a^3 E_1(\overline{\gamma}, a),$$

where E_0 is the energy functional on the tangent cone, E_1 an analytic functional. So

$$\frac{B^* E(\gamma)}{a(B_w(z))^2} = E_0 \left(\frac{B(\gamma)}{a(B_w \circ B_e(\gamma))} \right)$$

$$+ a(B_w \circ B_e(\gamma)) E_1 \left(\frac{B(\gamma)}{a(B_w \circ B_e(\gamma))}, a(B_w \circ B_e(\gamma)) \right).$$

Since all the terms above are analytic, so the functional in (5.6) has an analytic continuation. Moreover, when $z \in B_w^{-1}(x_0)$, the minimizers of (5.6) come from the minimizing geodesics on the tangent cone N.

We first check that the condition in Lemma 5.7 is satisfied when $z \in B_w^{-1}(x_0)$. From (5.7), we see that the second derivative of E_0 is associated with the Jacobi's field on the tangent cone N, which is a 2nd order linear differential-integral equation (cf. [12], page 93)

$$D^2 - L = 0,$$

where L is a 1st order linear differential-integral operator. So

$$A = D^{-2} L,$$

and hence A is compact.

Consider the set of minimizing geodesics γ on the tangent cone, with $e(\gamma) \neq 0$. R_+ acts on this sets, induced from its action on N. Let S be the quotient of this set by the action of R_+, then S is compact. Then, by (5.7), the set of minimizers of (5.6) when $z \in B_w^{-1}(x_0)$ is just S. Thus every point in S has a neighborhood in $B(\Omega_H M(x_0, \cdot) - \Omega_H M(x_0, x_0))$ in which Lemma 5.6 is valid. Choose a cover of S by a finite number of such neighborhoods, denoted by

U_i (such a cover exists since S is compact). If we can prove that the minimizers of (5.6) are contained in a compact subset in $\bigcup U_i$ if z is sufficiently near x_0, then we can use Lemma 5.6 and Lemma 5.7 to prove the theorem.

Then we only need a corollary of Lemma 5.4 (whose proof is rather straight forward) to complete the proof.

COROLLARY 5.8. *There is an open set* $W \subset \bigcup U_i$ *such that* $B_e(W)$ *contains an open neighborhood of* x_0, *and if* $z \in B_e(W)$, *then the minimizers of* (5.6) *lie in* W.

REMARK. The cut-locus of a CC metric seems hopelessly complicated in general, despite Theorem 4. We don't even know what a cut-locus in a general three-dimensional CC manifold looks like *diffeomorphically*, while in Riemannian geometry, the generic cut-locus on manifolds of dimension ≤ 6 admits a complete classification up to diffeomorphisms (Bunchner [8]). Even the case of a left invariant CC metric on a nilpotent group is not trivial, as the energy functional here has non-isolated critical points (cf. §6). Montgomery has found complicated "secondary conjugate locus" on Heisenberg groups. We conjecture that if the cut locus of any point of a three-dimensional fat CC manifold is locally *diffeomorphic* to a line, then the metric is homogeneous, and for every point there is a S^1 isometric group which leave both this point and the cut-locus through this point invariant.

6. Left-invariant CC metrics on fat two-step nilpotent groups.

As we have shown in previous sections, the geometry of a CC manifold M is largely determined by the tangent cones, especially when H is fat. In this section we will study the CC geometry on fat nilpotent group.

6.1. *The case of codimension* 1. Let $n = V_1 \oplus V_2$ be a fat two-step nilpotent Lie algebra, $[V_1, V_1] = V_2$, $[V_1, V_2] = [V_2, V_2] = 0$, $\dim V_1 = m$, $\dim V_2 = k$, and N be the corresponding simply connected nilpotent group.

We begin with the case $k = 1$. The nilpotent Lie algebra then is Heisenberg. We first look at the case $n = 3$ which was studied in detail in [11]. Take x, y, z as the coordinates on N such that H is spanned by $\partial/\partial x - y\partial/\partial z$, $\partial/\partial y + x\partial/\partial z$, and the CC metric is (up to a constant) $Q = (dx - y\,dz)^2 + (dy + x\,dz)^2$. The geodesic flow on T^*H passing through $x = y = z = 0$ is

$(x(t), y(t), z(t), p(t), q(t), r(t))$, where

$$r(t) = r(0) = r,$$
$$x(t) = \sin(4rt)(p(0)\cos(4rt) + q(0)\sin(4rt))/2r,$$
$$y(t) = \sin(4rt)(-p(0)\sin(4rt) + q(0)\cos(4rt))/2r,$$
$$p(t) = \cos(4rt)(p(0)\cos(4rt) + q(0))\sin(4rt),$$
$$q(t) = \cos(4rt)(-p(0)\sin(4rt) + q(0))\cos(4rt),$$
$$z(t) = 2r^{-1}\left(t - \frac{\sin(8rt)}{8r}\right)(p^2(0) + q^2(0)).$$

The 1st conjugate locus of $(0, 0, 0)$ is the set of points $(0, 0, z)$. Moreover, the Morse index of $(0, 0, z)$ $(z \neq 0)$ along a prime geodesic (i.e. a geodesic which does not contain conjugate point except the end point) is 1. In this case, the 1st conjugate locus coincides with the cut locus.

If $n \geq 5$, the CC metric can be written as

$$\sum a_i((dx_i - y_i dz)^2 + (dy_i + x_i dz)^2),$$

where a_i are positive numbers. It is easy to prove the following result:

LEMMA 6.1. *A point $(0, 0, z)$ is a conjugate point of $(0, 0, 0)$. If $a_i \neq a_j$ for $i \neq j$, the Morse index of $(0, 0, z)$ $(z \neq 0)$ to $(0, \ldots, 0, 0)$ along a prime geodesic is 1, while if $a_1 = a_2 = \cdots = a_t$, the Morse index is $t(t-1)/2$ $(n = 2t + 1)$.*

In this case, if $a_1 = a_2 = \cdots = a_t$, then the cut-locus of 0 is the set $\{(0, \ldots, 0, z)\}$; otherwise the cut locus is rather complicated.

6.2. *The case $k = 2$.* In this case the fat two-step nilpotent Lie algebra is not unique for given n. Nevertheless, we have the following result.

LEMMA 6.2. *The set of fat two-step nilpotent Lie algebras with $m = n - 2$ is connected for given n.*

Thus the "moduli space" of fat CC metrics with $k = 2$ is connected. To prove this lemma, we need the following results.

LEMMA 6.3. *The set*

$$U = \{A \in \mathrm{GL}(n, R), J_n A = A^\mathsf{T} J_n A, \text{ has no real eigenvalues}\}$$

is connected. Here J_n is the matrix

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Proof. Any element A in U can be written as

(6.1) $$PAP^{-1} = \begin{pmatrix} B & 0 \\ 0 & \overline{B} \end{pmatrix},$$

where B is a complex matrix, $\overline{B}$ its complex conjugation, $B \neq \overline{B}$, P is an invertible matrix of the form of

$$P = \begin{pmatrix} P_1 & \overline{P}_1 \\ P_2 & \overline{P}_2 \end{pmatrix}.$$

From $JA = A^\mathsf{T} J$, we see that P must satisfy

$$P_1^\mathsf{T} \overline{P}_2 - P_2^\mathsf{T} \overline{P}_1 = 0,$$

i.e. $P_1^\mathsf{T} \overline{P}_2 \in u(t, C)$ (the space of $t \times t$ infinitesimally hermitian matrices, $n = 2t$). Such P is called admissible. We want to prove that the set of admissible P is connected.

(1) If P_1 is non-singular, then P can be connected to

$$\begin{pmatrix} I & I \\ P_2 P_1^{-1} & \overline{P}_2 \overline{P}_1^{-1} \end{pmatrix}$$

by a curve of admissible P_t

$$\begin{pmatrix} P_1 A_t & \overline{P}_1 \overline{A}_t \\ P_2 A_t & \overline{P}_2 \overline{A}_t \end{pmatrix}$$

where A_t is a matrix depending continuously on t, $A_0 = I$, $A_1 = P_1^{-1}$.

(2) If P_1 is singular, then there are C, $D \in \mathrm{GL}(t, C)$ such that $C^\mathsf{T} \overline{P}_2 \overline{D} \in u(t, C)$. Since for any D, $(P_1 D)^\mathsf{T} (\overline{P}_2 \overline{D}) \in u(t, C)$ whenever $P_1^\mathsf{T} \overline{P}_2 \in u(t, C)$, hence P can be connected to

$$\begin{pmatrix} W_1 & \overline{W}_1 \\ W_2 & \overline{W}_2 \end{pmatrix} = \begin{pmatrix} P_1 & \overline{P}_2 \\ P_2 & \overline{P}_2 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix},$$

and then to matrix

$$\begin{pmatrix} W_1 + tC & \overline{W}_1 + t\overline{C} \\ W_2 & \overline{W}_2 \end{pmatrix}.$$

Note that for t small enough $W_1 + tC$ is non-singular. Thus we have reduced the case (2) to the case (1).

So far we have proved that any P can be connected to a matrix of the form

$$\begin{pmatrix} I & I \\ F & \overline{F} \end{pmatrix},$$

where $F \in u(t, C)$, $\det(F - \overline{F}) \neq 0$. On the other hand, the set

$$\{F \in u(t, C), \det(F - \overline{F}) \neq 0\}$$

is contractible to the set of non-singular anti-symmetric matrices, which is connected. So, the set of admissible P is connected.

Having proved that the set of admissible P is connected, we can prove that the set U is connected. In fact, let A be any matrix of the form (6.1) with $B \neq \overline{B}$; then there is B_t such that $B_0 = B$, $B_1 = iI_t$, $\overline{B_t} \neq B_t$, and depends continuously on t. So any $A \in U$ can be connected in U to

$$P \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} P^{-1},$$

where P is admissible, which forms a connected set, since the set of admissible P is connected.

Proof of Lemma 6.2. Suppose that the two-step nilpotent Lie algebra is of the form of $R^m + R^2$ with $[R^m, R^m] = R^2$, $[R^m, R^2] = [R^2, R^2] = 0$. Write $[\cdot, \cdot]: R^m \times R^m \to R^2$ as $[a, b] = (a^T B_1 b, a^T B_2 b)$, where B_1, B_2 are skew-symmetric matrices. The Lie algebra is fat if and only if for every $(c_1, c_2) \in R^2 - \{0\}$,

(6.2) $\det(c_1 B_1 + c_2 B_2) \neq 0$.

We denote the set of such pairs (B_1, B_2) by V. Write B_1 as $B_1 = P^T J P$, with P invertible. Write $B_2 = P^T D P$; then (6.2) is equivalent to $\det(c_1 J + c_2 D) \neq 0$. Denote the set of skew-matrices D satisfying the above condition by U_1. Then V is connected if and only if U_1 is. So we need only to prove that U_1 is connected. But U_1 is precisely the set U in Lemma 6.3, since the map $U_1 \to U$ which sends D to DJ^{-1} is a homeomorphism.

6.3. *Generic cases.* We will study general left-invariant CC metrics when $N - m \geq 2$.

Given a fat nilpotent Lie algebra $V_1 + V_2$, $[V_1, V_1] = V_2$, $[V_1, V_2] = 0$, denote the V_2-valued anti-symmetric bilinear form on V_1, $J(a_1, a_2) = [a_1, a_2]$. Then the multiplication of the Lie group is $(a_1, b_1)(a_2, b_2)$

$= (a_1 + a_2, b_1 + b_2 + J(a_1, a_2)/2)$. Let L be the left translation. Then

$$TL_{(a_1, b_1)}(\delta a, \delta b) = (\delta a, \delta b + J(a_1, \delta a)/2),$$
$$L^*_{(a_1, b_1)}(da, db) = (da + J^*(a_1, db)/2, db),$$

where $J^*: V_1 \times V_2^* \to V_1^*$ is defined by

$$(J^*(a, b))(c) = b(J(a, c)), \qquad a, c \in V_1, \ b \in V_2^*.$$

For a metric Q_0 on V_1^*, the induced left-invariant CC metrices is $Q_0(da + J^*(a, db))$. We identify V_1 with V_1^* via Q_0. The Hamiltonian of the Carnot-Carathéodory metric is

$$(6.3) \qquad Q_0(p + J^*(a, r)), \qquad (a, b, p, r) \in T^*N.$$

Since (6.3) does not contain b explicitly, r is a V_2^*-valued first integral of the Hamiltonian (6.3).

Fixing $r_0 \in V_2^*$, define $J_{r_0}: V_1 \to V_1$ by $J_{r_0}(a) = J^*(a, r_0)$. Then the Hamiltonian system (6.3) can be written as

$$(6.4) \qquad \frac{d}{dt}\begin{pmatrix} a \\ p \end{pmatrix} = \begin{pmatrix} J_{r_0} & I \\ -J^*_{r_0}J_{r_0} & -J^*_{r_0} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Let P be an orthogonal matrix such that

$$J_{r_0} = P \ \ \mathrm{diag}(c_1 J_2, \ldots, c_t J_2)P^{-1},$$

where $\pm c_1 i, \pm c_2 i, \ldots, \pm c_t i$ are the eigenvalues of J_{r_0}, $c_i > 0$. Introduce $a = P\bar{a}$, $p = P^{-1}\bar{p}$; then (6.4) can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} \bar{a} \\ \bar{p} \end{pmatrix}\begin{pmatrix} \mathrm{diag}(c_1 J_2, \ldots, c_t J_2) & I \\ -\mathrm{diag}(c_1^2 I_2, \ldots, c_t^2 I_2) & -\mathrm{diag}(c_1 J_2, \ldots, c_t J_2) \end{pmatrix}\begin{pmatrix} \bar{a} \\ \bar{p} \end{pmatrix}.$$

Write $\bar{a}$ as $(\bar{a}_1, \ldots, \bar{a}_t)$ and similarly for $\bar{p}$. Then (6.4) can be written as

$$\frac{d}{dt}\begin{pmatrix} \bar{a}_i \\ \bar{p}_i \end{pmatrix} = \begin{pmatrix} c_i J_2 & I \\ -c_i^2 I_2 & -c_i I_2 \end{pmatrix}\begin{pmatrix} \bar{a}_i \\ \bar{p}_i \end{pmatrix}, \qquad i = 1, \ldots, t;$$

which has been solved in §6.1.

LEMMA 6.4. *If the eigenvalues of J_{r_0} are $\pm c_1 i, \ldots, \pm c_t i$; $c_i > 0$, $c_i \neq c_j$, for $i \neq j$, then the Morse index of $(0, z)$ to $(0, 0)$ along a prime geodesic c for which $r(c) = r_0$ is 1.*

REMARK 1. In general the cut-locus is rather complicated.

REMARK 2. It seems that a "generic" conjugate point has multiplicity 1. More precisely, we conjecture that the following is true: *Fixing m, n with $n \geq 4$, then given a positive definite bilinear form Q_0 on V_1^*, there is an open and dense set of nilpotent Lie algebras such that a generic conjugate point of $0 \in V_1 \times V_2$ of the induced fat left-invariant CC metric has multiplicity 1.*

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