THE TEMPERLEY-LIEB ALGEBRA AT ROOTS OF UNITY

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We present general techniques to determine the structure of Hecke algebras and similar algebras in the non-semisimple case. We apply these to give a complete description of the structure of the Temperley-Lieb algebras at a root of unity. Our description implies in particular that the representation of these algebras on tensor space $(\mathbf{C}^2)^{\otimes n}$ is faithful.

Introduction. We shall consider ascending sequences of finite dimensional algebras $A_1 \subset A_2 \subset \cdots$, given by generators and relations, where the relations depend on one or several parameters. Moreover, we also assume that the discriminants of these algebras are non-zero polynomials or rational functions in the parameters. This means the algebras are semisimple except for special values of the parameters. For applications (to the construction of topological invariants, the construction of subfactors, or in statistical mechanical models) these algebras are often needed at the critical values of the parameters; in such cases, interesting semisimple quotients have been constructed in [J1], [W2,4].

In this paper, we initiate a systematic study of the structure of such algebras at the critical values of the parameters, where they are not semisimple.

For the examples we have in mind such as Hecke algebras, Brauer algebras, etc., the structure is known in the semisimple case and can be described by the Bratteli diagram, which encodes how an irreducible representation of A_n, restricted to A_{n-1}, decomposes into irreducible representations of A_{n-1}. If all the multiplicities in the decompositions are 0 or 1, one can use this to define special path idempotents and matrix units (see e.g. [SV], [W1,2], [RW]), labelled by paths on the Bratteli diagram. These matrix units are only well defined for generic values of the parameters, for which the algebras are semisimple. Nevertheless, their (usually inductive) defining formulas carry a lot of information about the structure of the algebras which can also be exploited at the critical parameter values. In more detail, our main techniques are as follows:

1. Evaluation principle: We use representations of the algebras to obtain algebraic identities with the parameters viewed as indeterminates. Although these representations are not necessarily well defined at a critical value of the parameters, nevertheless the identities remain valid as long as the quantities appearing in them are well-defined at the critical value.

2. Spectral analysis for path idempotents: Let us assume for simplicity that the algebra A_n can be obtained from A_{n-1} by adding one more generator, say g_{n-1}. (We assume all the A_n's to be subalgebras of their inductive limit.) Moreover, assume the path idempotents for A_{n-1} are known. Then it is possible in interesting examples to obtain the path idempotents for A_n by determining the "spectral projections" of $p_s g_{n-1} p_s$, where p_s is a path idempotent of A_{n-1} (see e.g. [W1,2], [RW]). In these examples at generic parameter values, the characteristic polynomials of such elements have simple roots only; for the critical values, however, some of the characteristic values may coincide. This basically means that one can no longer "diagonalize" these elements; instead, one has to deal with Jordan forms, which yield nilpotent elements in the radical.

3. Regular idempotents: In our examples, although many path idempotents cease to be well defined at the critical parameter values, one still can find "good" path idempotents (or sometimes sums of path idempotents) for each component of the maximal semisimple quotient.

These principles have been stated quite generally. We expect that they can be applied to Iwahori-Hecke algebras at roots of unity, to Brauer algebras [Br, W3] and their q-deformations, the so-called Birman-Wenzl algebras [BW, W4] and presumably also to the group algebras of the symmetric group in positive characteristic.

As a first example, these techniques will be applied here to the Temperley-Lieb algebras, which are quotients of the Iwahori-Hecke algebras defined by generators e_i, $i = 1, 2, \ldots, n-1$, and relations $e_i e_{i\pm1} e_i = q/(1+q)^2 e_i$ and $e_i e_j = e_j e_i$ if $|i - j| \geq 2$. These algebras first appeared in work of H. Temperley and E. Lieb analyzing certain statistical mechanical models [TL, B]. They were rediscovered by V. Jones in connection with his study of von Neumann subfactors and used by him to define the Jones link invariant [J1,2].

It is well known that for q not a root of unity the Temperley-Lieb algebra $A_n(q)$ is isomorphic to the quotient of the group algebra of the symmetric group whose irreducible representations are labelled by

partitions, or Young diagrams, with at most two parts. In particular, the simple component labelled by the diagram λ is isomorphic to the f_λ-by-f_λ matrix ring, where f_λ is the number of standard tableaux of shape λ.

We can give a similar description of $A_n(q)$ for q a primitive lth root of unity, $l \geq 3$. We associate to a diagram $\lambda = [\lambda_1, \lambda_2]$ its weight $w(\lambda) = \lambda_1 - \lambda_2 + 1$. A diagram is called *critical* if its weight is divisible by l. Let the affine reflection group $A_1^{(1)}$ act on $\mathbf{Z}$ via reflections about the numbers ml, $m \in \mathbf{Z}$; this also defines a partial action on diagrams with a fixed size, via their weights. The results are then:

1. The dimension of $A_n(q)$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, as for generic values of q.

2. The blocks of $A_n(q)$ are labelled by the critical diagrams of size n and by the orbits of the action of $A_1^{(1)}$ on the non-critical diagrams of size n.

3. The blocks labelled by critical diagrams are simple with the same dimension as for generic values of q.

4. For a block labelled by a non-critical orbit, the simple components in the maximal semisimple quotient are again labelled by the individual diagrams in the orbit. If μ is a diagram with weight $ml < w(\mu) < (m+1)l$, then the dimension of the corresponding simple component is $(f_\mu^L)^2$, where f_μ^L is the number of standard tableaux of shape μ which (when regarded as increasing sequences of Young diagrams) have their last critical diagram of weight ml.

If the orbit $[\lambda]$ is $\{\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k)}\}$, where $\lambda^{(m)}$ has weight $ml < w(\lambda^{(m)}) < (m+1)l$, then one has the formula

$$f_{\lambda^{(m)}}^L = \sum_{j \geq m} (-1)^{j-m} f_{\lambda^{(j)}}.$$

5. The block corresponding to a non-critical orbit $[\lambda]$ contains an isomorphic lifting of the maximal semisimple quotient of the block, with minimal central idempotents z_μ^L for $\mu \in [\lambda]$. If the weight of μ is less than l, then $z_\mu^L A_n(q) z_\mu^L \cong \mathbf{M}_{f_\mu^L}$, where $\mathbf{M}_k$ denotes the k-by-k matrix ring. Otherwise $z_\mu^L A_n(q) z_\mu^L \cong \mathbf{M}_{f_\mu^L} \otimes \mathbf{C}[x]/(x^2)$.

The radical of the block is nilpotent of exponent 3, and is spanned by the spaces $z_\mu^L A_n(q) z_\nu^L$ for diagrams $\mu, \nu \in [\lambda]$ which are in adjacent fundamental intervals for the $A_1^{(1)}$ action, and by the algebra $\mathrm{rad}(z_\mu^L A_n(q) z_\mu^L)$. If μ and ν are not adjacent, then $z_\mu^L A_n(q) z_\nu^L = (0)$.

A complete description of the multiplication rules for the radical will be given at the end of §2.

6. An important corollary of this structure theorem is that the representation of the Temperley-Lieb algebra $A_n(q)$ in $\mathrm{End}((\mathbf{C}^2)^{\otimes n})$ is faithful, and the image has dimension $C_n = \frac{1}{n+1}\binom{2n}{n}$, also when q is a root of unity.

After this work was completed, we learned that P. Martin had obtained the same results independently [M]; however, our emphasis is somewhat different, and we believe that there are some advantages to our exposition. We would like to thank Martin for discussing his work with us.

0. Preliminaries.

0.1. *Finite dimensional semisimple algebras.* We first need some notation for the combinatorial description of increasing sequences of finite dimensional complex semisimple algebras. Let $\mathbf{M}_n$ denote the ring of all complex $n \times n$ matrices. If A and B are semisimple complex algebras, we can write them as $A = \bigoplus A_\lambda$ and $B = \bigoplus B_\mu$ with $A_\lambda \cong \mathbf{M}_{a_\lambda}$ and $B_\mu \cong \mathbf{M}_{b_\mu}$ for appropriate natural numbers a_λ and b_μ. The vector $\vec{b}$ whose components are the numbers b_μ is called the *dimension vector* of B.

If A is a subalgebra of B, any simple B_μ-module is an A-module. Let $g_{\lambda\mu}$ be the number of simple A_λ-modules in its decomposition into simple A-modules. The matrix $G = (g_{\lambda\mu})$ is called the inclusion matrix for $A \subset B$. The inclusion of A in B is conveniently described by an inclusion diagram, or Bratteli diagram, which is a bipartite graph in which one set of vertices label the simple direct summands A_λ of A, and the other set labels the simple summands B_μ of B. Then the vertex corresponding to A_λ is joined with the vertex corresponding to B_μ by $g_{\lambda\mu}$ edges. If A and B have the same identity, the dimension vectors of A and B are related by $\vec{b} = G^t \vec{a}$.

The inclusion matrix can also be interpreted in the following way: Let p_λ be a minimal idempotent of A_λ and let z_μ be a minimal central idempotent in B, the identity of B_μ. Then $g_{\lambda\mu}$ is the number of idempotents in any decomposition of $p_\lambda z_\mu$ into minimal idempotents in B_μ.

0.2. *The Temperley-Lieb algebra.* The (abstract) Temperley-Lieb algebra A_n is the $\mathbf{C}(x)$-algebra given by generators $e_1, e_2, \ldots, e_{n-1}$ and the following relations.

(B1)′ $e_i e_{i\pm1} e_i = x/(1+x)^2 e_i$ for all i.

(B2) $e_i e_j = e_j e_i$ for $|i - j| \geq 2$.

(H) $e_i^2 = e_i$ for all i.

The Temperley-Lieb algebra is a quotient of the Iwahori-Hecke algebra H_n of type A; for the Hecke algebra, (B1)$'$ is replaced by the more general relation

(B1) $e_i e_{i+1} e_i - x/(1+x)^2 e_i = e_{i+1} e_i e_{i+1} - x/(1+x)^2 e_{i+1}$ for $i = 1, 2, \ldots, n-2$.

For $q \in \mathbf{C}\backslash\{0, -1\}$, the complex Temperley-Lieb algebra $A_n(q)$ is the $\mathbf{C}$-algebra given by generators $e_1, e_2, \ldots, e_{n-1}$ and relations obtained by substituting q for x in the relations (B1)$'$, (B2), and (H).

The complex algebra $A_n(q)$ is an "evaluation" of the $\mathbf{C}(x)$-algebra A_n in the following sense. Let $\mathbf{C}[x]_{(x-q)}$ denote the localization of the polynomial ring $\mathbf{C}[x]$ at $(x - q)$, i.e. $\mathbf{C}[x]_{(x-q)}$ consists of all those rational functions whose denominators do not have a zero at q. Let $(A_n)_q$ be the $\mathbf{C}$-subalgebra of A_n consisting of all $\mathbf{C}[x]_{(x-q)}$-linear combinations of products of the generators. We show that there is an evaluation homomorphism from $(A_n)_q$ onto $A_n(q)$ mapping generators to generators. Consequently we refer to the elements of $(A_n)_q$ as *evaluable at q*.

PROPOSITION 0.1. *Fix $q \in \mathbf{C}\backslash\{0, -1\}$.*

(a) *There is a homomorphism of $(A_n)_q$ onto $A_n(q)$ mapping e_i to e_i for all i.*

(b) $\dim_{\mathbf{C}} A_n(q) = \dim_{\mathbf{C}(x)} A_n = \frac{1}{n+1}\binom{2n}{n}$.

For $u \in (A_n)_q$, let $u(q)$ denote its image in $A_n(q)$ under the evaluation homomorphism.

(c) (*Principle of constancy of dimension.*) *Let e and f be idempotents in $(A_n)_q$. Then*

$$\dim_{\mathbf{C}} e(q) A_n(q) f(q) = \dim_{\mathbf{C}(x)} e A_n f.$$

Proof. Consider the free $\mathbf{C}(x)$-algebra $P = \mathbf{C}(x)\langle y_1, \ldots, y_{n-1}\rangle$ and let I denote the ideal of P generated by

$$\{y_i y_{i\pm1} y_i - x/(1+x)^2 y_i\} \cup \{y_i y_j - y_j y_i \text{ for } |i - j| \geq 2\} \cup \{y_i^2 - y_i\}.$$

Thus $A_n \cong P/I$. Likewise, let $P_L = \mathbf{C}[x]_{(x-q)}\langle y_1, \ldots, y_{n-1}\rangle$, and let I_L denote the ideal of P_L generated by the same set of elements. Call a word in $y_1, \ldots, y_{n-1}$ reduced if it is not congruent modulo I to any shorter word. By the argument of [J1], 4.1.4, there is a family F_0 of reduced words such that $1 \in F_0$, the cardinality of F_0 is the

Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and such that any word is congruent modulo I_L to some element of F_0. Hence the image F of F_0 in A_n is a spanning set, and $\dim_{\mathbf{C}(x)} A_n \leq C_n$. On the other hand, by producing sufficiently many inequivalent irreducible representations of A_n ([J1], §5, and [W2]), one obtains that A_n is semisimple with $\dim_{\mathbf{C}(x)} A_n = C_n$. Thus F is a basis of A_n, and $\operatorname{span}(F_0) \cap I = (0)$. It follows that $I \cap P_L = I_L$. Now $(A_n)_q$ is the image of P_L under the quotient map, so $(A_n)_q \simeq P_L/(I \cap P_L) = P_L/I_L$. On the other hand, I_L is contained in the kernel of the natural homomorphism of P_L onto $A_n(q)$, so this homomorphism factors through $(A_n)_q$. This proves (a).

For (b), note first that the image of F in $A_n(q)$ spans $A_n(q)$. We show that it is a basis. For each i $(1 \leq i \leq n - 1)$ and for each $w \in F$ there is a unique $w' = w'(i, w) \in F$ and $\alpha_{iw}^{w'} \in \mathbf{C}[x]_{(x-q)}$ such that $e_i w = \alpha_{iw}^{w'} w'$. One can define a representation of $A_n(q)$ on the **C**-span of F in $(A_n)_q$ by $e_i(q)w = \alpha_{iw}^{w'}(q)w'$. For $w \in F$, one checks that $w(q)1 = w$. Consequently the image of F is linearly dependent.

To prove (c), take for each element of a basis of $e(q)A_n(q)f(q)$ a pre-image in $e(A_n)_q f$. If these elements were linearly dependent over $\mathbf{C}(x)$, then the linear dependence could be re-written with relatively prime coefficients in $\mathbf{C}[x]$, and evaluation at q would then give a non-trivial linear dependence of the basis elements of $e(q)A_n(q)f(q)$. This gives the inequality $\dim_{\mathbf{C}} e(q)A_n(q)f(q) \leq \dim_{\mathbf{C}(x)} eA_n f$. Similar inequalities hold with e replaced by $1 - e$ and/or f replaced by $1 - f$. If any of these inequalities were strict, then adding them would give $\dim_{\mathbf{C}} A_n(q) < \dim_{\mathbf{C}(x)} A_n$, contradicting (b).

Another proof of (b) consists in identifying $A_n(q)$ with the subalgebra of Brauer's centralizer algebra spanned by all diagrams with no intersecting edges (see [Br] and [L]). One can also give a proof of (a) using this idea.

The following observation, although completely evident, is essential to our method of obtaining algebraic identities for $A_n(q)$ via matrix representations of A_n which are *not* well defined for $A_n(q)$.

Evaluation principle. Any algebraic identity between evaluable elements in A_n with coefficients in $\mathbf{C}[x]_q$ yields by evaluation an identity in $A_n(q)$.

0.3. *Representations and path idempotents.* We shall now describe irreducible representations for the Hecke algebra, following [W2], and

identify those which factor through the Temperley-Lieb algebra. Recall that a Young (or Ferrers) diagram $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ is a left justified array of boxes with λ_i boxes in the ith row and $\lambda_i \geq \lambda_{i+1}$ for all i. For example,

$$[5, 3, 1] = \quad.$$

For $n \geq 0$ let Λ_n denote the set of Young diagrams with n boxes.

Young's lattice is the graph whose vertices are labelled by Young diagrams, with two vertices connected by an edge if the corresponding diagrams differ by only one box. It is well known that the simple components of $\mathbf{C}S_n$ are labelled by the elements of Λ_n and that the restriction to S_{n-1} of the representation labelled by $\lambda \in \Lambda_n$ decomposes into the direct sum of the representations of S_{n-1} labelled by all subdiagrams of λ with $n-1$ boxes. Hence the Bratteli diagram of the S_n's is Young's lattice, with the elements of Λ_n arranged on the nth level.

For $\lambda \in \Lambda_n$, a standard tableau t of shape λ is a filling of the diagram λ with the numbers $1, 2, \ldots, n$ so that the entries increase to the right in each row and downwards in each column. Let $T(\lambda)$ denote the set of standard tableaux of shape λ, and T_n the set of all standard tableaux with n boxes. Observe that $t \in T(\lambda)$ may be identified with a path on Young's lattice from $[1]$ to λ, i.e. an increasing sequence of Young diagrams

$$[1] = \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(n)} = \lambda,$$

where $\lambda^{(i)}$ is the diagram which consists of the boxes of t containing the numbers $1, 2, \ldots, i$. By a subtableau of t, we will always mean a standard subtableau,

$$[1] = \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(k)} \quad \text{for some } k \leq n.$$

We denote by t' the subtableau of size $n-1$.

For $t \in T_n$ and $1 \leq i \leq n-1$, define

$$d(t, i) = c(i) - c(i+1) + r(i+1) - r(i),$$

with $c(j)$ and $r(j)$ denoting column and row of the box containing the number j. Observe that $|d(t, i)|$ is the length of the shortest walk on the tableau t, with vertical and horizontal steps, between the cell containing i and the cell containing $i + 1$, and that the sign depends

on whether $i + 1$ is northeast or southwest of i in t. Define for $d \in \mathbf{N} \setminus \{0\}$ the rational function

$$a_d(x) = \frac{1 - x^{d+1}}{(1 + x)(1 - x^d)}.$$

In [W2] and [H] an irreducible representation π_λ of the Hecke algebra H_n is defined on the vector space V_λ with (orthonormal) basis labelled by $T(\lambda)$ in the following way. These representations are q-analogues of Young's normal representations. Let $\sigma_i(t)$ be the tableau obtained by interchanging the numbers i and $i + 1$ in the standard tableau t. Then define

$$\pi_\lambda(e_i) = a_d t + \sqrt{a_d a_{-d}} \, \sigma_i(t),$$

where $d = d(t, i)$. Observe that if t is a standard tableau, $\sigma_i(t)$ fails to be a standard tableau only if i and $i + 1$ are in the same row or in the same column of t. But in this case, $a_d a_{-d} = 0$, so the representation stays within standard tableaux. Note that $t \mapsto t'$ induces a linear isomorphism between V_λ and $\bigoplus_{\nu \subset \lambda, \nu \in \Lambda_{n-1}} V_\nu$, which is evidently an H_{n-1}-module isomorphism. Thus

$$(0.1) \qquad \pi_{\lambda | H_{n-1}} \cong \bigoplus_{\nu \subset \lambda, \nu \in \Lambda_{n-1}} \pi_\nu.$$

The proof of the irreducibility of the π_λ is inductive: since the π_ν for $\nu \in \Lambda_{n-1}$ can be assumed irreducible, it suffices to observe that $V_\nu e_{n-1} V_\mu \neq 0$ if both ν and μ are obtained by removing one box from λ, and this follows easily from the definition of π_λ. One can also show inductively that the representations π_λ for distinct λ are mutually inequivalent. Since $\sum_\lambda \dim(\pi_\lambda)^2 = n! = \dim H_n$, it follows that H_n is semisimple. Similarly, if q is not a root of unity, complex representations of $H_n(q)$ are defined by the same formulas, and they are likewise irreducible and mutually inequivalent, which shows that $H_n(q)$ is semisimple for such q. One can check directly that the representations π_λ factor through the Temperley-Lieb algebra A_n if λ has no more than two rows. Since the sum of the squares of the dimensions of these is the Catalan number C_n, this is a complete family of inequivalent irreducible representations of A_n. Thus one has the Bratteli diagram shown in Figure 0.1 for the inclusions of the Temperley-Lieb algebras A_n or $A_n(q)$ for q not a root of unity.

Fix $\lambda \in \Lambda_n$ and let z_λ denote the corresponding minimal central indempotent in H_n defined by $\ker(\pi_\lambda) = (1 - z_\lambda)H_n$. For each $t \in$
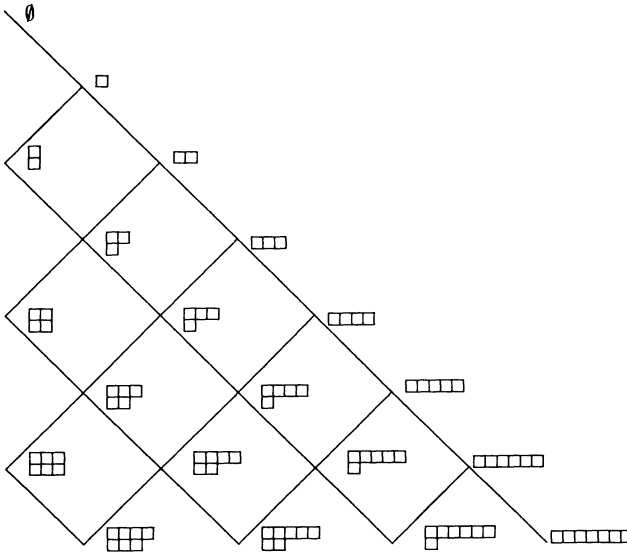
FIGURE 0.1. Bratteli diagram for A_n

$T(\lambda)$, let p_t be the orthogonal projection of V_λ onto $\mathbf{C}t$. Since π_λ is an isomorphism of $z_\lambda H_n$ onto $\mathrm{End}(V_\lambda)$, p_t may also be regarded as an element of $z_\lambda H_n$. The idempotent p_t can be evaluated inductively in the following way. Let $r \in T_{n-1}$. Using the definition of the representations π_λ and the restriction rule (0.1), one obtains

$$\pi_\lambda(p_r e_{n-1} p_r)t = \begin{cases} a_{d(t,n-1)}t & \text{if } t' = r, \\ 0 & \text{otherwise.} \end{cases}$$

Since one can show that $a_{d(t,n-1)} \neq a_{d(s,n-1)}$ for any two extensions $t \neq s$ of r, one can compute the p_t's as spectral idempotents of $p_r e_{n-1} p_r$; namely if $r = t'$, then

$$p_t = \prod_s \frac{p_r e_{n-1} p_r - a_{d(s,n-1)} p_r}{a_{d(t,n-1)} - a_{d(s,n-1)}},$$

where the product is over all $s \in T_n$ such that $s \neq t$ but $s' = t' = r$.

It is important to observe that for the Temperley-Lieb algebra one needs to take into consideration only tableaux belonging to diagrams with two rows at the most. In this case, there is at most one tableau s such that $s \neq t$ but $s' = t' = r$, so one obtains path idempotents p_t in the Temperley-Lieb algebra as follows:

$$p_t = p_r \qquad \text{if } r \text{ has only one extension, and}$$

$$p_t = \frac{p_r e_{n-1} p_r - a_{d(s,n-1)} p_r}{a_{d(t,n-1)} - a_{d(s,n-1)}} \qquad \text{if } s' = t' = r.$$

Furthermore, either $a_{d(t,n-1)} = 0$ or $a_{d(s,n-1)} = 0$, so the latter formula becomes

$$p_t = (a_{d(t,n-1)})^{-1} p_r e_{n-1} p_r \qquad \text{if } a_{d(t,n-1)} \neq 0,$$

$$p_t = p_r - (a_{d(s,n-1)})^{-1} p_r e_{n-1} p_r \qquad \text{otherwise.}$$

It is evident that these formulas are evaluable in $A_n(q)$ if q is not a root of unity. In the following section, we shall show that certain of the path idempotents are still evaluable when q is a root of unity, and moreover, even when the individual path idempotents are not evaluable, certain sums of them are still evaluable.

1. Path idempotents and nilpotents. In this section and the next, q will denote a fixed primitive lth root of unity ($l \geq 3$). We collect here various preliminary results needed for our analysis of the Temperley-Lieb algebras $A_n(q)$. Recall that an element of the generic Temperley-Lieb algebra A_n is called evaluable if it is a linear combination of words in the generators e_i with rational coefficients having no poles at q. Our method is to identify idempotents, matrix units, and nilpotents in $A_n(q)$ by analyzing evaluable elements in A_n.

With this in mind, we adopt the following convention: two evaluable idempotents e, f in A_n will be called equivalent *only if* there is an equivalence between them which is implemented by evaluable elements; that is, there exist evaluable u, v in A_n such that $e = uv$ and $f = vu$.

We will also need the following definitions:

DEFINITIONS. (a) A diagram λ is called *critical* if $w(\lambda) = \lambda_1 - \lambda_2 + 1$ is divisible by l. The mth *critical line* on the Bratteli diagram for the generic Temperley-Lieb algebra is the line containing the diagrams $\{\lambda : w(\lambda) = ml\}$ (see Figure 1.1). A tableau t of shape λ is called *critical* if λ is a critical diagram. We refer to the largest proper critical subtableau of t (if any) simply as the *critical subtableau of t*.

(b) If a tableau t has a proper critical subtableau, then the conjugate $\bar{t}$ of t is defined to be the tableau obtained by leaving the critical subtableau r of t invariant and by reflecting $t \backslash r$ in the critical line containing the endpoint of r (if this reflection exists); see Figure 1.2 on p. 318. For any tableau t, we define $p_{[t]} = p_t + p_{\bar{t}}$ if $\bar{t}$ exists; otherwise we set $p_{[t]} = p_t$.

(c) We say a critical tableau t is *evaluable* if p_t is evaluable. We say that a non-critical tableau is *evaluable* if $p_{[t]}$ is evaluable.
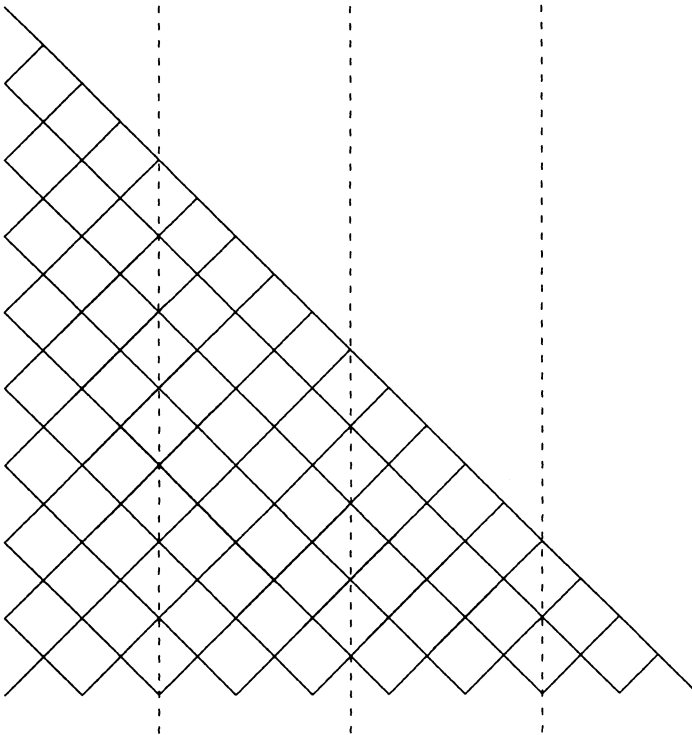
FIGURE 1.1. Critical lines

(d) A tableau t is called *regular* if any two successive critical diagrams on t are on different critical lines.

REMARKS. 1. We will see eventually that every regular tableau is evaluable.

2. There is one instance in which t has a proper critical subtableau but $\bar{t}$ does not exist, namely if the endpoint of the critical subtableau is on the first critical line, and the endpoint of t is on the second critical line.

LEMMA 1.1. *Let* $t = ([1], \lambda^{(2)}, \ldots, \lambda^{(k)} = \lambda)$ *be a tableau with no proper critical subtableau (i.e. such that* $w(\lambda^{(i)}) < l$ *for* $1 \leq i < k-1$*). Then* t *is evaluable. Furthermore, if* t *and* s *are two such tableaux of the same shape, then* p_t *is equivalent to* p_s*.*

Proof. The proof, which goes by induction on k, is the same as in the generic case, and is contained in [W2]. The interpolation formula for p_t from [W2] is $p_t = (a_d)^{-1} p_{t'} e_{k-1} p_{t'}$, or $p_t = p_{t'} - (a_d)^{-1} p_{t'} e_{k-1} p_{t'}$, where d satisfies $|d| \leq l-2$ or $d = 1 - l$. In each
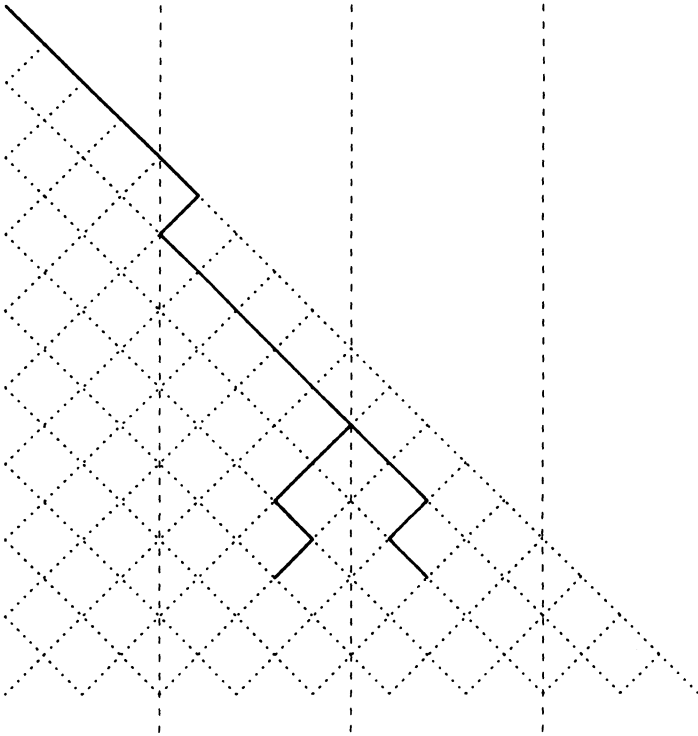
FIGURE 1.2. Conjugate tableaux

case, a_d is non-zero and non-singular at q, so if $p_{t'}$ is evaluable at q, then so is p_t. If t and s are two such tableaux of shape λ, then one can be obtained from the other by applying a sequence of adjacent transpositions σ_i, with $2 \le i \le k - 1$. Therefore it suffices to consider the case that $s = \sigma_i(t)$ for some such i. The equivalence is then implemented by $u = 1/(a_d a_{-d}) p_t e_i p_s$, $v = p_s e_i p_t$, with $d = d(t, i)$.

LEMMA 1.2 (*Little Diamond Lemma*). *Let r be an evaluable critical tableau of shape μ and length k. Set $\lambda = \mu + [1, 1]$. Let s denote the unique tableau of length $k + 2$ extending r and ending at λ for which $t = \sigma_k(s)$ is standard. (See Figure 1.3.) Then there exists a 3-by-3 system of evaluable matrix units in A_{k+2} with the diagonal units equal to p_t, $p_r e_{k+1}$ and $p_r(1 - e_{k+1}) z_\lambda$. In particular $p_r z_\lambda$ is evaluable.*

Proof. The three dimensional subspace V of V_λ spanned by s, $\bar{s}$, and t is invariant under the action of e_k, e_{k+1} and p_r. Set $u = (1 - p_r) e_k e_{k+1} p_r$ and $u^* = e_{k+1} p_r e_k (1 - p_r)$. Computation with the
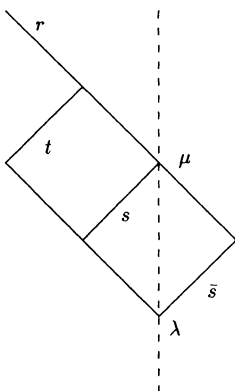
FIGURE 1.3. Little Diamond Lemma

matrix representatives on V show that

$$p_t = \frac{1}{a_{\pm ml}a_{ml\pm 1}a_{-(ml\pm 1)}}uu^*, \quad \text{while}$$

$$p_r e_{k+1} = \frac{1}{a_{\pm ml}a_{ml\pm 1}a_{-(ml\pm 1)}}u^*u.$$

Since $a_{\pm ml}$ has a simple pole at $x = q$ while the product $a_{ml\pm 1}a_{-(ml\pm 1)}$ has a simple zero, it follows that p_t is evaluable and equivalent to $p_r e_{k+1}$.

A similar computation shows that $p_r(1 - e_{k+1})z_\lambda$ is evaluable and equivalent to p_t. Finally, the sum $p_r z_\lambda = p_r e_{k+1} + p_r z_\lambda(1 - e_{k+1})$ is evaluable. $\qquad \square$

LEMMA 1.3 (*Interpolation Lemma*). *Let t be a tableau with evaluable critical subtableau r. Then:*

(a) *$p_{[t]}$ is evaluable. In particular, if r ends on the first critical line and t ends on the second critical line, then p_t is evaluable.*

(b) *Let s be another tableau with the same critical subtableau r and the same shape as t. Then $p_{[s]}$ and $p_{[t]}$ are equivalent.*

Proof. (a) Write $t = ([1], \lambda^{(2)}, \ldots, \lambda^{(n)})$, and let k denote the length of the critical subtableau r. Suppose that r ends on the mth critical line at the diagram μ. Also, suppose without loss of generality that $\lambda^{(n)}$ is weakly to the right of the mth critical line; that is, $w(\lambda^{(n)}) \geq ml$.

We proceed by induction on $n - k$. If $n - k = 1$, then $p_{[t]} = p_r$, which is evaluable by hypothesis. If $n - k = 2$, then $p_{[t]} = p_r z_{(\mu + [1, 1])}$

or $p_{[t]} = p_r(1 - z_{(\mu+[1,1])})$; in either case, $p_{[t]}$ is evaluable by Lemma 1.2. Suppose now that $n - k \geq 3$ and that $p_{[t']}$ is evaluable.

Case 1. $\lambda^{(n)} = \lambda^{(n-2)} + [1, 1]$.

Set $d = d(t, n-1)$ and $\bar{d} = d(\bar{t}, n-1)$. Then a_d and $a_{\bar{d}}$ are non-zero and non-singular at q, and $a_d(q) = a_{\bar{d}}(q)$. Let s be the unique tableau such that $s \neq t$ and $s' = t'$, and consider the space V with basis $\{\bar{s}, \bar{t}, t, s\}$. (If $m = 1$ and s ends on the second critical line, then $\bar{s}$ does not exit; in this case take instead $V = \text{span}\{\bar{t}, t, s\}$.) On V, the element $E = p_{[t']}e_{n-1}p_{[t']}$ has matrix $\text{diag}(0, a_{\bar{d}}, a_d, 0)$ (resp. $\text{diag}(a_{\bar{d}}, a_d, 0)$ in the special case). Moreover, observe that $p_{[t]}$ is given by the matrix $\text{diag}(0, 1, 1, 0)$ (resp. $\text{diag}(1, 1, 0)$). Hence it follows that

$$p_{[t]} = (a_d a_{\bar{d}})^{-1}((a_d + a_{\bar{d}})E - E^2),$$

so $p_{[t]}$ is evaluable.

Case 2. $\lambda^{(n)} = \lambda^{(n-2)} + [2, 0]$ or $\lambda(n) = \lambda^{(n-2)} + [0, 2]$.

With s defined as above, $p_{[s]}$ is evaluable by Case 1, and thus $p_{[t]} = p_{[t']} - p_{[s]}$ is evaluable. In the special case that $m = 1$ and t ends on the second critical line, observe that $\bar{t}$ is not defined but $p_t = p_{[t']} - p_{[s]}$, so p_t is evaluable.

(b) Our assumptions imply that $\bar{s}$ exists if and only if $\bar{t}$ exists; suppose this is so, the other case being similar. Now the proof is essentially the same as that of the equivalence statement in Lemma 1.1, except that we will have to adjust our initial choice u, u^* for elements implementing the equivalence because the representing matrices for u^*u and uu^* have two distinct non-zero eigenvalues.

The tableau s can be obtained from t by applying a sequence of adjacent transpositions. Therefore it suffices to suppose that $s = \sigma_i(t)$ for some i. Set $u^* = p_{[t]}e_i p_{[s]}$ and $u = p_{[s]}e_i p_{[t]}$. On the space with basis $\{\bar{s}, \bar{t}, t, s\}$, the element u^*u has matrix $\text{diag}(0, c_-, c_+, 0)$ and uu^* has matrix $\text{diag}(c_-, 0, 0, c_+)$, where the rational functions c_- and c_+ are non-zero and non-singular q. It follows that

$$(c_- c_+)^{-1}(u^*u - c_-)(u^*u - c_+)(p_{[s]} + p_{[t]}) = p_{[s]}.$$

Expanding this and solving for $p_{[t]}$ gives

$$p_{[t]} = u^*v \quad \text{and} \quad p_{[s]} = vu^*,$$
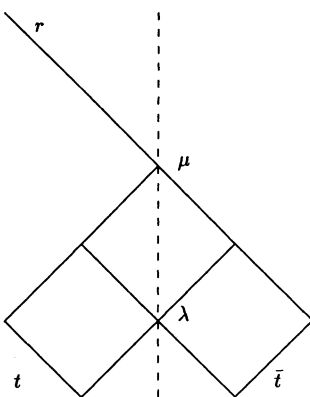
where $v = -(c_- c_+)^{-1}(uu^*u - (c_- + c_+)u)$.

FIGURE 1.4

COROLLARY 1.4. *Let* r *be an evaluable critical tableau of length* k *and shape* μ. *Set* $\lambda = \mu + [1, 1]$. *Consider the six extensions of* r *of length* $k + 3$ *which end at* $\mu + [2, 1]$ *or at* $\mu + [1, 2]$, *and let* t *denote the leftmost of these extensions in Figure* 1.4. *Then there is a 3-by-3 system of evaluable matrix units in* A_{k+3} *with diagonal matrix units* $p_{[t]}$, $p_r e_{k+1}$, *and* $p_r(1 - e_{k+1})z_\lambda$.

Proof. Observe that $p_r z_\lambda$ and $p_{[t]}$ are evaluable by Lemmas 1.2 and 1.3. One obtains the equivalences by the same computations as in Lemma 1.2, where one has to add a correction term for the matrix units as in Lemma 1.3(b).   □

The following lemma concerns a rational function which appears as the trace of a certain evaluable essential idempotent in the Temperley-Lieb algebra over $\mathbf{C}(x)$. We need to show that this quantity is non-zero and non-singular at $x = q$ in order to know that the evluation of the algebra element at $x = q$ is still an essential idempotent.

LEMMA 1.5. *Fix* $m \geq 1$. *The function*

$$T_k = (a_{ml-1}a_{1-ml}a_{-ml})^{k-1}a_{-ml} + (a_{ml+1}a_{-(ml+1)}a_{ml})^{k-1}a_{ml}$$

has a removable singularity at q, *and*

$$T_k(q) = \tau^k(q + 4 - 2k + q^{-1}),$$

where $\tau = q/(1 + q)^2$. *In particular* $T_k(q) \neq 0$ *for* $k \geq 3$.

Proof. It suffices to prove this for $m = 1$ since the lth root of unity q is in particular an mlth root of unity. Let t_k denote the

quantity $\tau^k(q + 4 - 2k + q^{-1})$. We find a recursion for T_k and for t_k. First $T_1 = a_l + a_{-l} = 1 = t_1$. Denote $D_- = a_{l-1}a_{1-l}a_{-l}$ and $D_+ = a_{l+1}a_{-(l+1)}a_l$. Then D_- and D_+ are non-singular at q with value $D_-(q) = D_+(q) = \tau$. We have

$$\begin{aligned}
T_{k+1} &= D_-^k a_{-l} + D_+^k a_l \\
&= D_-(D_-^{k-1}a_{-l} + D_+^{k-1}a_l) + D_+^{k-1}(D_+ - D_-)a_l \\
&= D_- T_k + D_+^{k-1}(D_+ - D_-)a_l.
\end{aligned}$$

Now one computes that

$$(D_+ - D_-)a_l(x) = \frac{x^2(1 + x^l)(1 - x)}{(1 + x)^4(x - x^l)},$$

so that $(D_+ - D_-)a_l(q) = -2\tau^2$. Assuming inductively that T_k is non-singular at q and that $T_k(q) = t_k$, we obtain that T_{k+1} is non-singular as well and

$$T_{k+1}(q) = \tau t_k - 2\tau^{k+1} = t_{k+1}.$$

Because $0 \leq |q + q^{-1}| < 2$, the quantity t_k is non-zero for all $k \geq 3$. $\square$

LEMMA 1.6 (*Big Diamond Lemma*). *Let t be a tableau such that both t and its critical subtableau r end on the same critical line. Suppose that p_r is evaluable. Then $p_{[t]}$ dominates an evaluable minimal idempotent f whose matrix coefficients on the space with basis $\{\bar{t}, t\}$ all have simple poles $x = q$.*

Proof. Suppose that r ends at the diagram μ of size k on the mth critical line and that t ends at $\mu + [n, n]$. We consider first instead of the original t the following zig-zag path s (see Figure 1.5):

$$s = (r, \mu^{(k+1)}, \ldots, \mu^{(k+2n)} = \lambda + [n, n]),$$

where

$$\begin{aligned}
\mu^{(k+2j)} &= \mu + [j + 1, j - 1] \quad \text{for } 1 \leq j \leq n - 1, \\
\mu^{(k+2j+1)} &= \mu + [j + 1, j] \qquad \text{for } 0 \leq j \leq n - 1.
\end{aligned}$$

We show that $p_{[s]}$ dominates an evaluable minimal idempotent p whose matrix coefficients on the space spanned by $\{s, \bar{s}\}$ have simple poles at $x = q$. Define

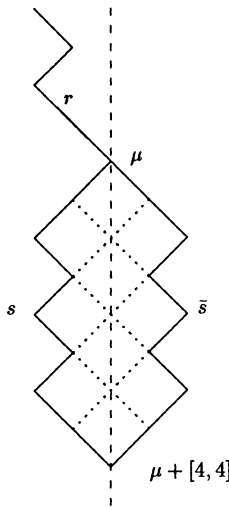$$w = (e_{k+1}e_{k+3} \cdots e_{k+2n-1})(e_{k+2}e_{k+4} \cdots e_{k+2n-2})p_{[s]}$$

FIGURE 1.5. Big Diamond Lemma

and
$$w^* = p_{[s]}(e_{k+2}e_{k+4}\cdots e_{k+2n-2})(e_{k+1}e_{k+3}\cdots e_{k+2n-1}).$$
The matrix of w^*w on the space with basis $\{\bar{s}, s\}$ is
$$\begin{bmatrix} (a_{ml-1}a_{1-ml}a_{-ml})^{n-1}a_{-ml} & \sqrt{} \\ \sqrt{} & (a_{ml+1}a_{-(ml+1)}a_{ml})^{n-1}a_{ml} \end{bmatrix},$$
where the off-diagonal entries are the square root of the product of the diagonal entries. The trace of this matrix is the rational function T_n of Lemma 1.5, which is non-zero and non-singular at q. Therefore $p := T_n^{-1}w^*w$ is a minimal idempotent with the desired properties. Rewrite the matrix of p as $\begin{bmatrix} b & \sqrt{} \\ \sqrt{} & 1-b \end{bmatrix}$.

It remains to observe that the equivalence between $p_{[s]}$ and $p_{[t]}$ which is given by Lemma 1.3(b) preserves the poles of the matrix representatives. For this, note that the matrices of the implementing elements u, u^* on the space spanned by $\{\bar{t}, \bar{s}, s, t\}$ are of the form
$$u \begin{bmatrix} 0 & \alpha_- & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & \alpha_+ & 0 \end{bmatrix} \quad \text{and} \quad u^* = \begin{bmatrix} 0 & 0 & & \\ \alpha_-^{-1} & 0 & & \\ & & 0 & \alpha_+^{-1} \\ & & 0 & 0 \end{bmatrix},$$
where $\alpha_\pm$ are rational functions which are non-zero and non-singular at q. A simple matrix computation shows that $f = upu^*$ has the desired properties. □

Several of our lemmas have hypothesized an evaluable critical tableau, but so far the only such tableaux which we have in hand are

those provided by Lemmas 1.1 and 1.3(a). We are now in the position to produce a sufficient family of evaluable critical tableaux of any critical shape. Recall that a tableau is called regular if its successive critical diagrams lie on different critical lines.

LEMMA 1.7. (a) *Every regular critical tableau is evaluable.*

(b) *If s and t are regular critical tableaux with the same critical subtableau and the same shape, then p_s and p_t are equivalent.*

Proof. The proof of (a) is by induction on the number of critical diagrams on the tableau t. If this number is one or two, the conclusion follows from Lemmas 1.1 and 1.3(a). So suppose that t has at least three critical diagrams and that every regular critical tableau with fewer critical diagrams is evaluable.

Consider the case that $t = ([1], \lambda^{(2)}, \dots, \lambda^{(k)} = \lambda)$ has its last three critical diagrams $\lambda^{(i)}$, $\lambda^{(j)}$, and $\lambda^{(k)} = \lambda$ on the mth, $(m \pm 1)$st, and mth critical lines. We can assume that $\bar{t}$ exists, since otherwise the desired conclusion follows from the induction hypothesis and Lemma 1.3(a). Set $s = \sigma_j(t)$. See Figure 1.6. By the induction hypothesis and Lemma 1.3, both $p_{[s]}$ and $p_{[t]}$ are evaluable and by Lemma 1.6, $p_{[s]}$ dominates an evaluable minimal idempotent f whose matrix coefficients on the space with basis $\{s, \bar{s}\}$ have simple poles at $x = q$.

Let b denote the matrix coefficient, $p_s f p_s = b p_s$. Define $u = f e_j p_{[t]}$ and $u^* = p_{[t]} e_j f$. Computation with the matrix representatives of f, e_j, and $p_{[t]}$ on the space with basis $\{\bar{s}, s, t, t\}$ shows that

$$p_t = \frac{1}{b a_{(m \pm 1)l \mp 1} a_{-((m \pm 1)l \mp 1)}} u^* u \quad \text{and} \quad f = \frac{1}{b a_{(m \pm 1)l \mp 1} a_{-((m \pm 1)l \mp 1)}} u u^*.$$

Since b has a simple pole at $x = q$ while the product $a_{(m \pm 1)l \mp 1} \times a_{-((m \pm 1)l \mp 1)}$ has a simple zero, it follows that p_t is evaluable and equivalent to f.

If the last three critical diagrams on t are all on different critical lines, then $\bar{t}$ is evaluable by the previous case, while $p_{[t]}$ is evaluable by the induction hypothesis and Lemma 1.3, so p_t is also evaluable.

For part (b), note that s can be obtained from t by applying a sequence of adjacent transpositions; the same proof as in Lemma 1.1 shows that p_s and p_t are equivalent.   □

LEMMA 1.8. *Fix $m \geq 1$ and let t be the unique tableau of shape $\lambda = [(m+1)l - 1, l]$ which passes through the diagram $[(m+1)l - 1]$.*
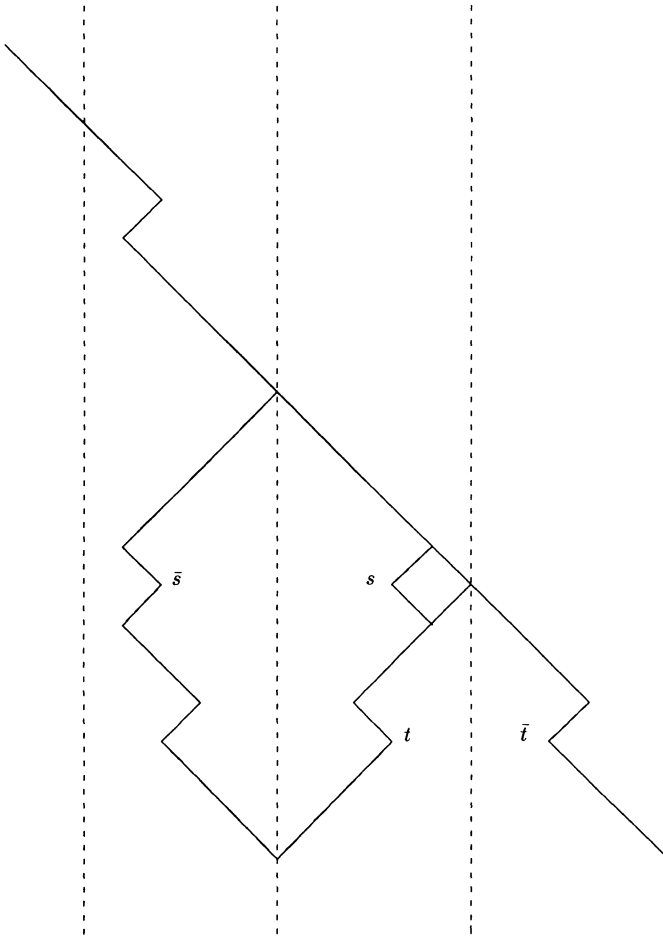
FIGURE 1.6

*Then p_t is equivalent to p_r where r is a tableau of shape λ with
critical subtableau of shape $[(m-1)l-1]$.*

Proof. Set $s = \sigma_{(m+1)l-1}(t)$ and $r = \sigma_{ml-1}(\bar{s})$. Both p_t and p_r are
evaluable by Lemma 1.7. Moreover, by the argument of Lemma 1.7,
both are equivalent to f, where f is the evaluable minimal idempo-
tent dominated by $p_{[s]}$ which is given by Lemma 1.6.   □

Let t be a non-critical tableau of length n with a critical tableau r
of length k. In the generic algebra, the element $p_{[t]}e_k p_{[t]}$ is an essen-
tial idempotent, in fact essentially p_t or $p_{\bar{t}}$. However, the evaluation
of such elements in $A_n(q)$ gives non-zero nilpotents of order 2 which
moreover are contained in the radical of $A_n(q)$.

LEMMA 1.9 (*Nilpotent elements*). *Let t be a non-critical tableau with critical subtableau of shape $[ml-1]$ for some $m \geq 1$. Assume that the endpoint of t is to the left of the mth critical line. Write $e = e_{ml-1}$, $s = \sigma_{ml-1}(t)$, and $n_{[t]} = p_{[t]}ep_{[t]}$.*

(a) $n_{[t]}(q) \neq 0$ *but* $n_{[t]}(q)^2 = 0$.

(b) $p_{[t]}A_np_{[t]}(q)$ *is two dimensional, isomorphic to* $\mathbf{C}[x]/(x^2)$.

(c) $p_{[t]}A_np_{[s]}(q)$ *and* $p_{[s]}A_np_{[t]}(q)$ *are one dimensional.*

(d) $p_{[t]}A_np_{[s]}A_np_{[t]}(q) = Cn_{[t]}(q)$ *and*

$$p_{[s]}A_np_{[t]}A_np_{[s]}(q) = \begin{cases} (0) & \text{if } m = 1 \\ Cn_{[s]}(q) & \text{otherwise.} \end{cases}$$

(e) $n_{[t]}p_{[t]}A_np_{[s]}(q) = p_{[s]}A_np_{[t]}n_{[t]}(q) = (0)$. *If $m > 1$, then*

$$p_{[t]}A_np_{[s]}n_{[s]}(q) = n_{[s]}p_{[s]}A_np_{[t]}(q) = (0).$$

Proof. The idempotents $p_{[t]}$ and $p_{[s]}$ are evaluable by Lemmas 1.7 and 1.3. For part (a), note that $p_{[t]}ep_{[t]} = p_tep_t = a_{ml-1}p_t$, so $(p_{[t]}ep_{[t]})^2 = a_{ml-1}p_{[t]}ep_{[t]}$. This last relation can be evaluated at q and gives $(p_{[t]}e_kp_{[t]})^2(q) = 0$. To see that $p_{[t]}ep_{[t]}(q) \neq 0$, let T be an extension of t ending on the mth critical line and also having critical subtableau of shape $[ml-1]$. Let f be an evaluable minimal idempotent dominated by $p_{[T]}$ whose matrix coefficients with respect to the basis $\{T, \overline{T}\}$ have simple poles at $x = q$; let b denote the matrix coefficient, $p_Tfp_T = bp_T$. Note that $f(p_{[t]}ep_{[t]})f = a_{ml-1}fp_tf = a_{ml-1}fp_Tf = a_{ml-1}bf$. Since $a_{ml-1}b$ is neither zero nor singular at q, $f(p_{[t]}e_kp_{[t]})f(q) = a_{ml-1}bf(q) \neq 0$.

(b) It follows from constancy of dimension that $p_{[t]}A_np_{[t]}(q)$ is two dimensional, so that $\{p_{[t]}, n_{[t]}\}$ is a basis.

(c) Follows from constancy of dimension.

(d) Set $u = p_{[s]}ep_{[t]}$ and $u^* = p_{[t]}ep_{[s]}$. By the calculation of part (a), we have $a_{ml-1}p_t = n_{[t]}$, so that $u^*u = p_tep_sep_t = a_{ml-1}a_{1-ml}p_t = a_{1-ml}n_{[t]}$. Now evaluating at $x = q$ gives $u^*u(q) = n_{[t]}(q)$. In particular, $u(q)$ and $u^*(q)$ are non-zero. If $m = 1$, then p_s is evaluable, and $uu^* = p_sep_tep_s = a_{ml-1}a_{2-ml}p_s$. Evaluating this at $x = q$ gives $uu^*(q) = 0$. If $m > 1$, then $uu^* = a_{ml-1}a_{1-ml}p_s = a_{1-ml}a_{ml-1}(a_{(m-1)l-1})^{-1}n_{[s]}$. Evaluating this at $x = q$ gives $uu^*(q) = n_{[s]}(q)$.

(e) It follows from calculations similar to those in part (d) that $n_{[t]}u^*(q) = un_{[t]}(q) = 0$, and, if $m > 1$, then also $u^*n_{[s]}(q) = n_{[s]}u(q) = 0$. □

2. The structure of $A_n(q)$. In this section we will complete the analysis of the structure of the Temperley-Lieb algebras $A_n(q)$ at a primitive lth root of unity q ($l \geq 3$). The first step is to identify idempotents which map to the minimal central idempotents in the maximal semisimple quotient of $A_n(q)$, and to analyze the reduction of $A_n(q)$ by these idempotents.

Let λ be a diagram between the mth and $m + 1$st critical line. Partition the set $T(\lambda)$ of tableaux of shape λ into the classes $L(\lambda)$ and $R(\lambda)$, which consist of those tableaux t whose critical subtableau ends in the mth (for $L(\lambda)$), or in the $m + 1$st critical line (for $R(\lambda)$). If $m = 0$, then $L(\lambda)$ consists of those tableaux of shape λ having no intersection with a critical line. Let f_λ, f_λ^R, and f_λ^L denote the cardinalities of $T(\lambda)$, $R(\lambda)$, and $L(\lambda)$ respectively. Define

$$z_\lambda^R = \sum \{p_{[t]} : t \in R(\lambda)\} \quad \text{and} \quad z_\lambda^L = \sum \{p_{[t]} : t \in L(\lambda)\}.$$

The summands in these expressions are not necessarily evaluable. Observe that if μ is the image of λ under the reflection in the $m + 1$st critical line, we have $z_\lambda^R = z_\mu^L$ and $f_\lambda^R = f_\mu^L$. The summation formula for f_λ^L given in the introduction can be obtained as follows: $f_\lambda^L = f_\lambda - f_\lambda^R = f_\lambda - f_\mu^L$; iterating this step gives the formula $f_\lambda^L = f_\lambda - f_\mu + \cdots$.

We are going to show that

$$\{z_\lambda^L(q) : \lambda \text{ non-critical and } |\lambda| = n\} \cup \{z_\lambda(q) : \lambda \text{ critical and } |\lambda| = n\}$$

are all the minimal central idempotents of $A_n(q)$ modulo its radical and that the corresponding simple components are isomorphic to $f_\lambda^L \times f_\lambda^L$ (resp. $f_\lambda \times f_\lambda$) matrix rings over $\mathbf{C}$.

We shall need some more notation for the inductive proof of this statement. Observe that any diagram λ with two rows has at most two subdiagrams with one box less. We use the notation $\alpha = \lambda - [1, 0]$ (*the left subdiagram*) and $\beta = \lambda - [0, 1]$ (*the right subdiagram*), whenever these diagrams exist. Obviously, we have the identities

$$f_\lambda = f_\alpha + f_\beta,$$

and, if both α and β are noncritical,

$$f_\lambda^L = f_\alpha^L + b_\beta^L, \qquad f_\lambda^R = f_\alpha^R + f_\beta^R.$$

If α is critical, then

$$f_\lambda^L = f_\alpha + f_\beta^L \quad \text{and} \quad f_\lambda^R = f_\beta^R,$$

and similarly if β is critical.

In the following proposition we will construct a semisimple subalgebra of $A_n(q)$ which will turn out to be an isomorphic lifting of the maximum semisimple quotient of $A_n(q)$. We do not work directly in $A_n(q)$ but rather with evaluable elements of A_n. To construct the semisimple subalgebra, we assume inductively that we have a partition of unity of mutually orthogonal evaluable idempotents of A_{n-1} and sufficiently many evaluable matrix units connecting them. Furthermore, we assume that each of the equivalence classes of idempotents contains an evaluable path idempotent p_t or $p_{[t]}$ corresponding to a regular path t. Then by "refining" these path idempotents and applying the lemmas of the previous section, we obtain a partition of unity in A_n with the same properties. Thus the inductive construction of the semisimple lifting is analogous to the proof of irreducibility of the representations π_λ which we sketched in §0.3.

We recall our convention that equivalence of evaluable idempotents in A_n means equivalence implemented by evaluable elements.

PROPOSITION 2.1. (a) *If λ is a critical diagram, then z_λ has a partition of unity consisting of f_λ mutually orthogonal, mutually equivalent evaluable idempotents, among them at least one p_t such that t is regular of shape λ. Moreover, for any two regular tableaux s, t of shape λ, p_s is equivalent to p_t.*

(b) *If λ is non-critical, then z_λ^L has a partition of unity consisting of f_λ^L mutually orthogonal, mutually equivalent evaluable idempotents, among them at least one $p_{[t]}$ for a regular $t \in L(\lambda)$. Moreover, if s and t are two such tableaux, then $p_{[s]}$ and $p_{[t]}$ are equivalent.*

Proof. The proof is by induction on the size of the diagram λ. The statements hold for all diagrams with one row by Lemmas 1.7(a) and 1.3(a), and for all diagrams to the left of the first critical line by Lemma 1.1. We assume that they hold for all diagrams of size less then $|\lambda|$, that λ has a non-zero second part, and that λ is on or to the right of the first critical line. It suffices to produce the required number of mutually orthogonal, mutually equivalent evaluable idempotents dominated by z_λ (resp. z_λ^L), including those of the specified types, since the sum of these is then necessarily equal to z_λ (resp. z_λ^L).

First consider a diagram λ on the mth critical line. Set $\mu = \lambda - [1, 1]$, and let r and t be tableaux of shape μ and λ respectively whose critical subtableaux are of shape $[(m-1)l-1]$ (or, if $m = 1$, which have no proper critical subtableaux). These are regular,

so evaluable by Lemma 1.1 or 1.7. By the induction assumption and Lemma 1.1 or Lemma 1.3(b), z_α^L has a partition of unity consisting of f_α^L mutually orthogonal, mutually equivalent evaluable idempotents in A_{n-1}, including $p_{[t']}$. From this we obtain the same number of mutually orthogonal, mutually equivalent idempotents in A_n, dominated by $z_\alpha^L z_\lambda$, and including p_t; namely, if $\{u_{ij}\}$ is a system of evaluable matrix units with $u_{11} = p_{[t']}$, then $\{v_{ij} = u_{i1} p_t u_{1j}\}$ is such a system with $v_{11} = p_t$. By Lemmas 1.7(b) and 1.2, p_t is equivalent to both $p_r e_{n-1}$ and $p_r(1 - e_{n-1}) z_\lambda$. Using this and the induction hypothesis for z_μ, we obtain $2 f_\mu = 2 f_\alpha^R$ mutually orthogonal, mutually equivalent evaluable idempotents in A_n dominated by $z_\alpha^R z_\lambda$, and all equivalent to p_t. If $\lambda_2 < l$, this suffices, since $f_\beta^R = 0$. Now suppose $\lambda_2 = l$; then $f_\beta^R = 1$, corresponding to the unique tableau s of shape λ with critical subtableau of shape $[(m + 1)l - 1]$. The idempotent p_s is evaluable by Lemma 1.7 and equivalent to p_t by Lemmas 1.7(b) and 1.8. Finally, suppose $\lambda_2 > l$. Using the induction assumption for z_β^R (which is the same as $z_{\bar\beta}^L$, where $\overline{\beta}$ is the reflection of β in the $(m + 1)$st critical line), and arguing as above, we obtain f_β^R mutually orthogonal, mutually equivalent evaluable idempotents dominated by $z_\beta^R z_\lambda$, and all equivalent to $p_{\tilde r} e_{n-1}$, where $\tilde r$ is some tableau of shape μ with critical subtableau of shape $[(m+1)l - 1]$. But p_r is equivalent to $p_{\tilde r}$ in A_{n-2} by the induction assumption, so $p_{\tilde r} e_{n-1}$ is equivalent to $p_r e_{n-1}$ and hence to p_t in A_n.

We now know that z_λ is evaluable and has the required partition of unity; it remains only to verify the statement regarding equivalence of idempotents corresponding to regular tableaux. For two regular tableaux s and t both coming from the left or both from the right, equivalence of p_s and p_t follows from the equivalence of $p_{[s']}$ and $p_{[t']}$, which is given by the induction hypothesis, together with the equalities $p_s = z_\lambda p_{[s']}$ and $p_t = z_\lambda p_{[t']}$ and the centrality of z_λ. On the other hand, we already showed the equivalence of p_s and p_t for two particular regular tableaux s and t, one coming from the left and one coming from the right.

Next we consider the case that λ is non-critical. In case the predecessor β of λ is critical, both the existence of the required partition of unity and the statement regarding equivalence of idempotents follow from the induction statement applied to z_α^L. Suppose next that α is critical. From the induction hypothesis applied to $z_\alpha = z_\lambda^L z_\alpha$, we obtain f_α mutually orthogonal mutually equivalent evaluable

idempotents dominated by $z_\lambda^L z_\alpha$. Furthermore, using the induction hypothesis, Lemma 1.2 and Corollary 1.4, we see that each of these is equivalent to $p_{[t]}$, where t is regular and has critical subtableau of shape $\alpha - [1, 1]$. From the induction hypothesis applied to z_β^L, we obtain a collection of f_β^L mutually equivalent, mutually orthogonal evaluable idempotents dominated by $z_\beta^L z_\lambda^L$, including $p_{[t]}$. This gives the required partition of unity and the evaluability of z_λ^L. Finally, the equivalence statement is obtained as before from the induction hypothesis, the centrality of z_λ^L in $(z_\alpha + z_\beta^L)A_n(z_\alpha + z_\beta^L)$, and the equivalence, known from Corollary 1.4, between $p_{[s]}$ and $p_{[t]}$ for two particular s and t, with $s' \in T(\alpha)$ and $t' \in L(\beta)$.

The last case to consider is that the diagrams λ, α, and β are all non-critical. Let t be any regular tableau of shape λ such that $t' \in L(\alpha)$, and t'' has shape $\mu = \lambda - [1, 1]$. Set $s = \sigma_{n-1}(t)$, so that $s' \in L(\beta)$. Then all of the idempotents $p_{[t]}$, $p_{[t']}$, $p_{[s]}$, and $p_{[s']}$ are evaluable by the induction hypothesis and Lemma 1.3(a), and $p_{[t]}$ and $p_{[s]}$ are equivalent by Lemma 1.3(b). The induction assumption gives f_α^L mutually equivalent, mutually orthogonal evaluable idempotents dominated by $z_\alpha^L z_\lambda^L$, including $p_{[t]}$, and f_β^L such idempotents dominated by $z_\beta^L z_\lambda^L$, including $p_{[s]}$. This gives the desired partition of unity and the evaluability of z_λ^L. The equivalence statement is proved as before. $\square$

THEOREM 2.2. (a) *If λ is a critical diagram, then z_λ is evaluable at q, and is a minimal central idempotent in $A_n(q)d$. Furthermore $z_\lambda A_n(q) \cong M_{f_\lambda}$.*

Suppose now that λ is a non-critical diagram. Then:

(b) *z_λ^L is evaluable at q.*

(c) *If λ is to the left of the first critical line, then $z_\lambda^L A_n z_\lambda^L(q) \cong M_{f_\lambda^L}$.*

(d) *Otherwise, $z_\lambda^L A_n z_\lambda^L(q) \cong \{\begin{bmatrix} A & 0 \\ B & A \end{bmatrix} : A, B \in M_{f_\lambda^L}\}$.*

Proof. From Proposition 2.1, we obtain for each critical diagram λ an f_λ-by-f_λ system of matrix units in $A_n(q)$ dominated by $z_\lambda(q)$, and for each non-critical diagram λ an f_λ^L-by-f_λ^L system dominated by $z_\lambda^L(q)$. If λ is critical, then $\dim(z_\lambda A_n(q)) = f_\lambda^2$ by the principle of constancy of dimension, and therefore $z_\lambda A_n(q) \cong M_{f_\lambda}$. Similarly, if λ is to the left of the first critical line, then $z_\lambda^L A_n z_\lambda^L(q) \cong M_{f_\lambda^L}$. If λ is between the mth and $(m+1)$st critical lines for $m \geq 1$, then $\dim(z_\lambda^L A_n z_\lambda^L(q)) = 2(f_\lambda^L)^2$, and the system of matrix units $\{u_{ij}\}$

in $z_\lambda^L A_n z_\lambda^L(q)$ can be chosen such that $u_{11} = p_{[t]}(q)$, where t is a (necessarily evaluable) tableau of shape λ with critical subtableau of shape $[ml - 1]$. Now $n := p_{[t]} e_{ml-1} p_{[t]}$ is a non-zero nilpotent of order 2 by Lemma 1.9. It follows that $N := \sum u_{i1} n u_{1i}$ is a non-zero nilpotent of order 2 commuting with all u_{ij}. Thus the subalgebra of $z_\lambda^L A_n z_\lambda^L(q)$ generated by N and the matrix units u_{ij} is isomorphic to $M_{f_\lambda^L} \otimes \mathbf{C}[x]/(x^2) \cong \{ \begin{bmatrix} A & 0 \\ B & A \end{bmatrix} : A, B \in M_{f_\lambda^L} \}$. Because of the dimension constraint, this is all of $z_\lambda^L A_n z_\lambda^L(q)$. $\qquad\square$

We can now finish the description of the blocks, the radical, and the maximal semisimple quotient of $A_n(q)$. We know that for a critical diagram λ of size n, $z_\lambda(q)$ is a minimal central idempotent in $A_n(q)$, and $z_\lambda A_n(q) \cong M_{f_\lambda^L}$. Now consider an orbit $[\lambda]$ of a non-critical diagram under the action of $A_1^{(1)}$. Note first that

$$z_{[\lambda]} := \sum_{\mu \in [\lambda]} z_\mu^L = \sum_{\mu \in [\lambda]} z_\mu$$

is an evaluable central idempotent. We will show that $z_{[\lambda]} A_n(q)$ is a block of $A_n(q)$. If $\mu, \nu \in [\lambda]$ are adjacent (that is, if they are in adjacent fundamental intervals for the $A_1^{(1)}$ action), then $\dim(z_\mu^L A_n z_\nu^L(q)) = f_\mu^L f_\nu^L$, while if μ and ν are not adjacent, then $z_\mu^L A_n z_\nu^L(q) = (0)$, by constancy of dimension. Furthermore, using Lemma 1.9 and the results of this section, one has for adjacent diagrams μ, ν,

$$z_\mu^L A_n z_\nu^L(q) \, \mathrm{rad}(z_\nu^L A_n z_\nu^L(q)) = \mathrm{rad}(z_\mu^L A_n z_\mu^L(q)) z_\mu^L A_n z_\nu^L(q) = (0),$$

and as a $M_{f_\mu^L}$-$M_{f_\nu^L}$ bimodule, $z_\mu^L A_n z_\nu^L(q)$ is isomorphic to the space of f_μ^L-by-f_ν^L matrices. Moreover,

$$z_\mu^L A_n z_\nu^L A_n z_\mu^L(q) = \mathrm{rad}(z_\mu^L A_n z_\mu^L(q)).$$

It follows that

$$R_{[\lambda]} := \bigoplus_{\substack{\mu, \nu \in [\lambda] \\ \mu, \nu \text{ adjacent}}} z_\mu^L A_n z_\nu^L(q) \oplus \bigoplus_{\mu \in [\lambda]} \mathrm{rad}(z_\mu^L A_n z_\mu^L(q))$$

is a nilpotent ideal in $z_{[\lambda]} A_n(q)$. But the quotient is semisimple:

$$z_{[\lambda]} A_n(q) / R_{[\lambda]} \cong \bigoplus_{\mu \in [\lambda]} z_\mu^L A_n z_\mu^L(q) / \mathrm{rad}(z_\mu^L A_n z_\mu^L(q)) \cong \bigoplus_{\mu \in [\lambda]} M_{f_\mu^L},$$

so in fact $R_{[\lambda]}$ is the radical of $z_{[\lambda]} A_n(q)$. Since the $z_\mu^L(q)$ are minimal central idempotents modulo the radical, it follows that any central

idempotent in $z_{[\lambda]}A_n(q)$ is a sum of certain of these plus an element of $R_{[\lambda]}$. But one easily checks that no such element is a central idempotent, other than $z_{[\lambda]}(q)$. To summarize:

THEOREM 2.3. *If λ is a non-critical diagram, then $z_{[\lambda]}(q)$ is a minimal central idempotent in $A_n(q)$. The radical of $z_{[\lambda]}A_n(q)$ is nilpotent of exponent 3 and is spanned by the spaces $z_\mu^L A_n z_\nu^L(q)$ for pairs of adjacent diagrams μ, ν in the orbit $[\lambda]$ and by the algebras $\mathrm{rad}(z_\mu^L A_n z_\mu^L(q))$ for $\mu \in [\lambda]$. The maximal semisimple quotient of $z_{[\lambda]}A_n(q)$ is isomorphic to $\bigoplus_{\mu \in [\lambda]} \mathrm{M}_{f_\mu^L}$.*

For each $\zeta \in \mathbf{C}\backslash\{-1\}$, one has a representation of $A_n(\zeta)$ on $(\mathbf{C}^2)^{\otimes n}$ defined as follows. Define $E(\zeta) \in \mathrm{End}(\mathbf{C}^2 \otimes \mathbf{C}^2)$ by

$$E(\zeta) = \frac{1}{1+\zeta}(\zeta\varepsilon_{11} \otimes \varepsilon_{22} + \varepsilon_{22} \otimes \varepsilon_{11} + \zeta\varepsilon_{12} \otimes \varepsilon_{21} + \varepsilon_{21} \otimes \varepsilon_{12}),$$

and for $1 \le i \le n-1$, let $E_i(\zeta) = I \otimes \cdots \otimes I \otimes E(\zeta) \otimes \cdots \otimes I$, where $E(\zeta)$ acts in the ith and $(i+1)$st tensor places. Then $\pi_\zeta: e_i \mapsto E_i$ defines a representation of $A_n(\zeta)$. This representation was introduced by Temperley and Lieb and was later rediscovered by Pimsner and Popa [PP] in their study of the Jones index. The operator $E(\zeta)$ is a spectral idempotent for the $\check{R}$ matrix for the quantum universal enveloping algebra of $\mathrm{sl}(2)$ in its vector representation.

THEOREM 2.4. *The representation π_ζ of $A_n(\zeta)$ on $(\mathbf{C}^2)^{\otimes n}$ is faithful for all $\zeta \neq -1$. The dimension of $\pi_\zeta(A_n(\zeta))$ is the Catalan number C_n.*

Proof. This is known for ζ not a root of unity, but we include a proof of this case as well. The faithfulness of the representation for $\zeta = 1$ follows from the representation theory of the symmetric group on tensor space. If p is any idempotent in the $\mathbf{C}(x)$-algebra A_n, then p is evaluable at all but finitely many complex numbers and $\zeta \mapsto \pi_\zeta(p)$ is a continuous family of idempotents in $\mathrm{End}((\mathbf{C}^2)^{\otimes n})$ on the complement of this finite set, hence of constant rank. For all Young diagrams λ with no more than two rows and for all ζ not a root of unity, z_λ is evaluable at ζ and $z_\lambda(\zeta)$ is a minimal central idempotent in the semisimple algebra $A_n(\zeta)$. Since $\pi_1(z_\lambda)$ is nonzero, so is $\pi_\zeta(z_\lambda)$ for all such ζ. Therefore π_ζ is faithful on $A_n(\zeta)$ for ζ not a root of unity.

Now consider the primitive lth root of unity q. If p is any idempotent in the $\mathbf{C}(x)$-algebra evaluable at q, then p is also evaluable at

ζ in a neighborhood of q. Because π_ζ is faithful and thus $\pi_\zeta(p) \neq 0$ for ζ in a deleted neighborhood of q, it follows by constancy of rank that $\pi_q(p) \neq 0$ as well. This implies that π_q is faithful on a lifting of the maximal semisimple quotient of $A_n(q)$.

Let λ be a non-critical diagram, t a regular tableau of shape λ, and $n_{[t]}(q)$ a non-zero nilpotent in the two dimensional algebra $p_{[t]}A_n p_{[t]}(q)$. The proof of Lemma 1.9 shows that there is an evaluable minimal idempotent f in A_N for some $N > n$ such that $f n_{[t]} f(q)$ is a non-zero multiple of $f(q)$. The representations π_q intertwine the usual imbeddings of $A_n(q)$ in $A_N(q)$ and of $\text{End}((\mathbf{C}^2)^{\otimes n})$ in $\text{End}((\mathbf{C}^2)^{\otimes N})$, so $\pi_q(f) \neq 0$ implies that $\pi_q(n_{[t]}) \neq 0$. Now for any non-zero element in $\text{rad}(A_n(q))$, the ideal generated by this element in $A_n(q)$ contains an element of the form $n_{[t]}$. Hence π_q is faithful on $\text{rad}(A_n(q))$.

The statement regarding dimensions results from Proposition 0.1.   □

It is also possible to give a short combinatorial proof of the faithfulness of the representation on tensor space which does not rely upon our structure theorem.

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