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Nonnegative bounded operators A and B on a Hilbert space $\mathscr{H}$ commute if $AB^n + B^n A \geq 0$ for $n = 1, 3, \ldots,$, or if $e^{tA} \leq e^{tA+sB} \leq e^{tA+s\|B\|}$ for every $s, t > 0$.

In this paper A and B represent (not necessarily bounded) selfadjoint operators with spectral families $\{E_\lambda\}$ and $\{F_\lambda\}$, respectively, on a Hilbert space $\mathscr{H}$. We study some conditions which imply that A and B commute.

1. In general, $AB + BA$ is not necessarily nonnegative for some nonnegative operators A and B (cf. [**3**]).

THEOREM 1. *Let A and B be nonnegative and bounded operators. Then $AB = BA$ if and only if*

$$0 \leq AB^n + B^n A \quad \text{for } n = 1, 2, \ldots.$$

To prove this theorem, we need the following:

LEMMA. *If a projection P satisfies $0 \leq AP + PA$, then $AP = PA$.*

Proof. For arbitrary vectors $x \in P\mathscr{H}$, $y \in (1 - P)\mathscr{H}$, and arbitrary complex numbers s and t, we have

$$\begin{aligned} 0 &\leq ((AP + PA)(tx + sy), (tx + sy)) \\ &= 2|t|^2(Ax, x) + 2\operatorname{Re} t\bar{s}(Ax, y), \end{aligned}$$

from which it follows that $0 = (Ax, y)$. Thus we get $AP = PA$.

Proof of Theorem 1. The "only if" part is clear, so we show the "if" part. We may assume that $\|B\| \leq 1$, which means $0 \leq B \leq 1$. Since $0 \leq AB^n + B^n A$, we get

(1) $$0 \leq A \exp(tB) + \exp(tB)A \quad \text{for every } t > 0,$$

from which it follows that

$$0 \leq \exp(-tB)A + A\exp(-tB).$$

Thus (1) is valid for $-\infty < t < \infty$. Since $0 \leq A\exp(tB)\exp(sB) + \exp(sB)\exp(tB)A$ for $-\infty < s$, $t < \infty$, we have

$$0 \leq \exp(-sB)A\exp(tB) + \exp(tB)A\exp(-sB).$$

By the Laplace transform relation

$$\text{(2)} \quad \int_0^\infty s^{n-1}\exp(-\lambda s)\exp(-sB)\,ds = (n-1)!(B+\lambda)^{-n} \quad \text{for } \lambda > 0,$$

we obtain

$$0 \leq (B+\lambda)^{-n}A\exp(tB) + \exp(tB)A(B+\lambda)^{-n} \quad \text{for } \lambda > 0,$$

which implies that

$$0 \leq A\exp(tB)(B+\lambda)^n + (B+\lambda)^n\exp(tB)A.$$

Since A and B are continuous, by letting $\lambda \to 0$, we get

$$\begin{aligned} 0 &\leq A\exp(tB)B^n + B^n\exp(tB)A \\ &= AB^n\exp(tB) + \exp(tB)B^nA \quad \text{for } -\infty < t < \infty. \end{aligned}$$

It is easy to show that

$$0 \leq \exp(-t(I-B))AB^n + B^nA\exp(-t(I-b)) \quad \text{for } t > 0,$$

from which, using (2) again, we obtain

$$0 \leq AB^n(1-B)^m + (1-B)^mB^nA \quad \text{for } m, n = 0, 1, 2, \ldots.$$

By Bernstein's theorem, each polynomial $p(x)$ which is positive on the interval $[0, 1]$ is a linear combination of polynomials of the form $x^n(1-x)^m$ with real nonnegative coefficients. Thus we have

$$0 \leq Ap(B) + p(B)A.$$

For each continuous function $f(x)$ which is > 0 on $[0, 1]$ we can select a sequence of polynomials as above which uniformly converges to $f(x)$. Therefore we have

$$0 \leq Af(B) + f(B)A.$$

It is easy to show that the latter inequality holds for any continuous function $f(x)$ which is ≥ 0 on $[0, 1]$, and hence that $0 \leq AF_\lambda + F_\lambda A$, where $\{F_\lambda\}$ is the spectral family corresponding to b. From the lemma we obtain $AF_\lambda = F_\lambda A$ and hence $AB = BA$. This concludes the proof.

Corollary 2. *Let A and B be nonnegative bounded operators. Then $AB = BA$ if $A^2 \leq (A+tB)^2$ for every $t > 0$ and $n = 1, 2, \ldots$.*

Proof. From the assumption, it follows that

$$0 \leq (AB^n + B^n A) + tB^{2n} \quad \text{for } t > 0.$$

Letting $t \to 0$, we get $0 \leq AB^n + B^n A$.

Corollary 3. *Let $0 \leq A$ and $0 \leq B$. Suppose B is bounded. Then $BA \subset AB$ if for $n = 1, 2, \ldots,$*

(3) $B\mathscr{D}(A) \subset \mathscr{D}(A)$ *and* $0 \leq ((AB^n + B^n A)x, x)$ *for every* $x \in \mathscr{D}(A)$.

Proof. For $t > 0$, $(t+A)^{-1}$ is bounded and nonnegative. From (3) it follows that $0 \leq (t+A)^{-1}B^n + B^n(t + A^{-1})$, which implies $(t+A)^{-1}B = B(t+A)^{-1}$ and hence $BA \subset AB$.

Corollary 4. *Let A be unbounded selfadjoint, and let B be selfadjoint and bounded from below. Then $E_\lambda F_\mu = F_\mu E_\lambda$ for every λ, μ if $0 \leq \exp(A)\exp(-nB) + \exp(-nB)\exp(A)$ for $n = 1, 2, \ldots,$ where the inequality should be interpreted like* (3).

Proof. Clearly $\exp(-B)$ is bounded and nonnegative. Since $\exp(-nB) = \{\exp(-B)\}^n$ (cf. §128 of **[9]**), we have

$$\exp(-B)\exp(A) \subset \exp(A)\exp(-B).$$

Since the spectral family corresponding to $\exp(A)$ is $\{E_{\log t}\}_{0<t<\infty}$, $\exp(-B)$ and E_λ commute. Thus we get $E_\lambda F_\mu = F_\mu E_\lambda$.

For a C^*-algebra $\mathscr{A}$, Ogasawara [7] showed that $\mathscr{A}$ is abelian if the condition $0 \leq a \leq b$, $a, b \in \mathscr{A}$ implies $a^2 \leq b^2$. In other words, $\mathscr{A}$ is abelian if $0 \leq ab + ba$ for every $0 \leq a$, $b \in \mathscr{A}$. Clearly Theorem 1 and Corollary 2 are true for nonnegative a, b in $\mathscr{A}$. Consequently we can consider them to be extensions of Ogasawara's theorem.

2. Let us recall that if A and B are unbounded, then $A \leq B$ means that $\mathscr{D}(B^{1/2}) \subset \mathscr{D}(A^{1/2})$ and $\|A^{1/2}x\| \leq \|B^{1/2}x\|$ for $x \in \mathscr{D}(B^{1/2})$. We have

(4) $$0 \leq A \leq B \Rightarrow 0 \leq B^{-1} \leq A^{-1}.$$

PROPOSITION 5. *Let A and B be bounded from below, and suppose $A \geq -\zeta$, $B \geq -\zeta$. Then the following are equivalent:*

(a) $(A+\zeta)^n \leq (B+\zeta)^n$ *for every* $n = 1, 2, \ldots$.
(b) $F_\lambda \leq E_\lambda$ *for every* λ.
(c) $\exp(tA) \leq \exp(tB)$ *for every* $t > 0$.
(d) $\exp(-tB) \leq \exp(-tA)$ *for every* $t > 0$.

Proof. Olson **[8]** (cf. **[12]**) showed that (a) and (b) are equivalent if A and B are bounded and $\zeta = 0$. We can easily apply his proof to this case. To show (a) $\Rightarrow$ (d), we need the following (cf. Chap. 9 of **[5]**):

$$\exp(-tA) = \lim_{m\to\infty} (I + t/mA)^{-m}. \tag{5}$$

If $m > t\zeta$, then each term in the right side is positive and bounded. From (a) we get

$$(1 + t/mA)^{-m} \geq (1 + t/mB)^{-m} \quad \text{for } m > t\zeta.$$

By using (5) we have (d). We show (d) $\Rightarrow$ (a). Since (d) is equivalent to

$$\exp(-t(B+\zeta)) \leq \exp(-t(A+\zeta)),$$

from (2) it follows that

$$(B+\zeta+\lambda)^{-n} \leq (A+\zeta+\lambda)^{-n} \quad \text{for } \lambda > 0,\ n = 1, 2, \ldots.$$

Thus for $x \in \mathscr{D}((A+\zeta)^{-n/2})$ we have

$$\|(B+\zeta+\lambda)^{-n/2}x\| \leq \|(A+\zeta+\lambda)^{-n/2}x\| \leq \|(A+\zeta)^{-n/2}x\|.$$

By using Fatou's lemma we obtain

$$\|(B+\zeta)^{-n/2}x\| \leq \varliminf_{\lambda\to 0} \|(B+\zeta+\lambda)^{-n/2}x\| \leq \|(A+\zeta)^{-n/2}x\|,$$

that is, $(B+\zeta)^{-n} \leq (A+\zeta)^{-n}$. Taking their inverses, we obtain (a). Now we have only to show (c) $\Leftrightarrow$ (d). But since

$$I = \exp(tA)\exp(-tA) \supset \exp(-tA)\exp(tA)$$

(cf. §128 of **[9]**), $\exp(tA)$ is the inverse of $\exp(-tA)$; by (4) we obtain it. This concludes the proof.

THEOREM 6. *Let A and B be unbounded selfadjoint operators with spectral families $\{E_\lambda\}$ and $\{F_\lambda\}$, respectively. Then the following are equivalent:*

(b) $F_\lambda \leq E_\lambda$ *for every* λ.
(c) $\exp(tA) \leq \exp(tB)$ *for every* $t > 0$.
(d) $\exp(-tB) \leq \exp(-tA)$ *for every* $t > 0$.

Proof. (b) implies that for every $\mu > 0$, $F_{\log \mu} \le E_{\log \mu}$. Since these operators are the spectral families corresponding to $\exp(B)$ and $\exp(A)$, respectively, by Proposition 5 we obtain

$$0 \le (\exp(A))^n \le (\exp(B))^n \quad \text{for } n = 1, 2, \ldots . \tag{6}$$

To see that the above inequalities hold for all $t > 0$, we use Heinz's inequality [**6**]. Since $\exp(tA) = (\exp(A))^t$, we have (c). Conversely, (c) implies (6). By using Proposition 5 again, we arrive at (b). (c) $\Leftrightarrow$ (d) is obvious. This concludes the proof.

THEOREM 7. *Let A be a (not necessarily bounded) selfadjoint operator. Let X be a bounded operator which is nonnegative. If there is a real number $\alpha \ge \|X\|$ such that*

$$\exp(tA) \le \exp(t(A + \varepsilon X)) \le \exp(t(A + \varepsilon\alpha I)) \quad \textit{for every } t, \varepsilon > 0, \tag{7}$$

then $XA \subset AX$.

Proof. Set $B = A + \varepsilon X$. Then B is selfadjoint and $\mathscr{D}(B) = \mathscr{D}(A)$. Now let us denote the spectral families corresponding A and B by $E(\lambda)$ and $F(\lambda)$, respectively. From Theorem 6, it follows that

$$E(\lambda - \varepsilon\alpha) \le F(\lambda) \le E(\lambda) \quad \text{for } -\infty < \lambda < \infty .$$

The above inequalities are equivalent to

$$E(\lambda)\mathscr{H} \subset F(\lambda + \varepsilon\alpha)\mathscr{H} \subset E(\lambda + \varepsilon\alpha)\mathscr{H} \quad \text{for } -\infty < \lambda < \infty .$$

Since $BE(\lambda)\mathscr{H} \subset BF(\lambda + \varepsilon\alpha)\mathscr{H} \subset F(\lambda + \varepsilon\alpha)\mathscr{H} \subset E(\lambda + \varepsilon\alpha)\mathscr{H}$, we have $XE(\lambda)\mathscr{H} \subset E(\lambda + \varepsilon\alpha)\mathscr{H}$. Since $E(\lambda)$ is continuous from the right, we obtain $XE(\lambda)\mathscr{H} \subset E(\lambda)\mathscr{H}$ and hence $XE(\lambda) = E(\lambda)X$, which implies $XA \subset AX$. Thus the proof is complete.

COROLLARY 8. *Let A and X be nonnegative operators. Suppose X is bounded. If there is a real number $\alpha \ge \|X\|$ such that*

$$A^n \le (A + \varepsilon X)^n \le (A + \varepsilon\alpha I)^n \quad \textit{for every } \varepsilon > 0, \ n = 1, 2, \ldots, \tag{8}$$

then $XA \subset AX$.

Proof. It is clear.

For finite matrices or compact operators, we can get better conditions than (7) or (8). From now on, A and B are nonnegative finite matrices or compact operators which are represented as $A = \sum \mu_i(A) e_i \otimes e_i$ and $B = \sum \mu_i(B) d_i \otimes d_i$, where $\{\mu_i(\cdot)\}$ is a decreasing sequence of eigenvalues. It is easy to see that, in this case, the condition (b) in Proposition 5 is equivalent to

(b′) $\mu_i(A) \leq \mu_i(B)$, and if $\mu_i(A) > \mu_j(B)$, then $e_i \perp d_j$.

PROPOSITION 9. *Let A be a nonnegative finite matrix. Set* $\delta(A) := \min\{|\lambda - \mu| : \lambda \neq \mu,\ \lambda, \mu \in \sigma_p(A)\}$.

(i) *If $0 \leq X < \delta(A)$, and $(A + X)^n \geq A^n$ for $n = 1, 2, \dots$, then $AX = XA$.*

(ii) *If $0 \leq X < \delta(A)$, and $A^n \geq (A - X)^n \geq 0$ for $n = 1, 2, \dots$, then $AX = XA$.*

Proof. (i) Set $B = A + X$ and suppose $\mu_1(A) = \cdots = \mu_i(A) > \mu_{i+1}(A)$. Then, by Ky Fan [**4**] (cf. [**10**]), we obtain

$$\mu_{i+1}(B) \leq \mu_{i+1}(A) + \mu_1(X) \leq \mu_{i+1}(A) + \delta(A) < \mu_i(A).$$

(b′) implies $\{e_1, \dots, e_i\} \perp \{d_{i+1}, d_{i+2}, \dots\}$ and hence the subspace $\{e_1, \dots, e_i\} = \{d_1, \dots, d_i\}$ reduces A and B. Since the reduced operator of A is constant, A and B commute there. Repeating this procedure in the same way to the other restrictions of A and B, we can derive $AB = BA$, which means $AX = XA$.

(ii) To prove this in the same way as (i), we need only to start with the smallest eigenvalue of A. Thus the proof is complete.

COROLLARY 10. *Let A be a selfadjoint finite matrix which is not necessarily nonnegative.*

(i) *If $0 \leq X < \delta(A)$, and* $\exp(tA) \leq \exp(t(A+X))$ *for every $t > 0$, then $AX = XA$.*

(ii) *If $0 \leq X < \delta(A)$, and* $\exp(t(A-X)) \leq \exp(tA)$ *for every $t > 0$, then $AX = XA$.*

Proof. (i) Take a real number $\zeta > 0$ so that $A + \zeta I \geq 0$. From $\exp(t(A+\zeta I)) \leq \exp(t(A+\zeta I+X))$, using Proposition 5.9. $AX = XA$ follows.

(ii) Take $\zeta > 0$ such that $A + \zeta I - X \geq 0$. Then we can derive $AX = XA$.

PROPOSITION 11. *Let A and X be nonnegative compact operators. If $A^n \leq (A+sX)^n$ for every $s > 0$ and $n = 1, 2, \dots$, then $AX = XA$.*

Proof. Suppose $\mu_1(A) = \cdots = \mu_j(A) > \mu_{i+1}(A)$ as in the proof of Proposition 7. Let us take s which satisfies $s\|X\| < \mu_i(A) - \mu_{i+1}(A)$. Then the subspace $\{e_1, \ldots, e_i\}$ reduces A and $A + sX$, where they commute. We have only to repeat this procedure to get $AXe_m = XAe_m$ for every m.

Let us end this paper by giving an example. Let A and B be nonnegative matrices. Set $V = \{rA + sB + tI; r, s, t > 0\}$. Then $AB = BA$ if

$$\exp(\tfrac{1}{2}(X + Y)) \leq \tfrac{1}{2}(\exp(X) + \exp(Y)) \quad \text{for every } X, Y \in V, \tag{9}$$

In fact, take $r > 0$ such that $A \leq rI \leq B + rI$. Then we have $\exp(tA) \leq \exp(t(B + rI))$ for every $t > 0$. From this and (9) it follows that

$$\exp\left(t(B + rI)(\tfrac{1}{2} + (\tfrac{1}{2})^2 + \cdots + (\tfrac{1}{2})^n) + t(\tfrac{1}{2})^n A\right) \leq \exp(t(B + rI)).$$

By Corollary 10(ii), we get $AB = BA$. This example shows that we cannot regard $\exp(\frac{1}{2}(X + Y))$ as the geometric mean of $\exp X$ and $\exp Y$ if they do not commute (cf. [**1**]).

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