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**ERRATA TO: “THE SET OF PRIMES DIVIDING THE LUCAS  
NUMBERS HAS DENSITY  $2/3$ ”**

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ERRATA TO:  
 THE SET OF PRIMES DIVIDING THE LUCAS  
 NUMBERS HAS DENSITY 2/3

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**Theorem C of my paper [2] states an incorrect density for the set of primes that divide the terms  $W_n$  of a recurrence of Laxton [3], due to a slip in the proof. A corrected statement and proof are given.**

The corrected version of Theorem C of [2] is:

**THEOREM C.** *Let  $W_n$  denote the recurrence defined by  $W_0 = 1$ ,  $W_1 = 2$  and  $W_n = 5W_{n-1} - 7W_{n-2}$ . Then the set*

*$S_W = \{p: p \text{ is prime and } p \text{ divides } W_n \text{ for some } n \geq 0\}$  has density  $3/4$ .*

The proof below proceeds along the general lines of §4 of [2].

*Proof.* One has

$$W_n = \left(\frac{3 + \sqrt{-3}}{6}\right) \left(\frac{5 + \sqrt{-3}}{2}\right)^n + \left(\frac{3 - \sqrt{-3}}{6}\right) \left(\frac{5 - \sqrt{-3}}{2}\right)^n.$$

If

$$\alpha = \frac{3 + \sqrt{-3}}{6} \quad \text{and} \quad \phi = \frac{5 + \sqrt{-3}}{5 - \sqrt{-3}} = \frac{11 + 5\sqrt{-3}}{14}$$

then

$$W_n \equiv 0 \pmod{p} \Leftrightarrow \phi^n \equiv -\frac{\bar{\alpha}}{\alpha} \pmod{(p)} \quad \text{in } \mathbb{Z} \left[ \frac{1 + \sqrt{-3}}{2} \right],$$

where  $-\frac{\bar{\alpha}}{\alpha} = \frac{-1 + \sqrt{-3}}{2}$  is a cube root of unity. Consequently

$$(1.1) \quad p \text{ divides } W_n \text{ for some } n \geq 0 \Leftrightarrow \text{ord}_{(p)} \phi \equiv 0 \pmod{3}.$$

The argument now depends on whether the prime ideal  $(p)$  splits or remains inert in the ring of integers  $\mathbb{Z} \left[ \frac{1 + \sqrt{-3}}{2} \right]$  of  $\mathbb{Q}(\sqrt{-3})$ .

*Case 1.*  $p \equiv 1 \pmod{3}$ , so that  $p = \pi\bar{\pi}$  in  $\mathbb{Z} \left[ \frac{1 + \sqrt{-3}}{2} \right]$ . Since  $\text{ord}_{(\pi)} \phi = \text{ord}_{(\bar{\pi})} \phi$ , one has

$$\text{ord}_{(p)} \phi \equiv 0 \pmod{3} \Leftrightarrow \text{ord}_{(\pi)} \phi \equiv 0 \pmod{3}.$$

Now suppose that  $3^j \nmid (p - 1)$ , in which case

$$(1.2) \quad \text{ord}_{(\pi)} \phi \not\equiv 0 \pmod{3} \Leftrightarrow \phi^{(p-1)/3^j} \equiv 1 \pmod{(\pi)}.$$

Set

$$\zeta_j := \exp\left(\frac{2\pi i}{3^j}\right), \quad \phi_j := \sqrt[3]{\phi},$$

and define the fields  $F_j = \mathbb{Q}(\zeta_j, \phi_j)$  and  $F_j^* = \mathbb{Q}(\zeta_{j+1}, \phi_j) = F_j(\zeta_{j+1})$ . The last equivalence holds since  $F_j$  and  $F_j^*$  are normal extensions of  $\mathbb{Q}$ . Both  $F_j$  and  $F_j^*$  are normal extensions of  $\mathbb{Q}$ , because  $\phi$  has norm one, so that the complex conjugate  $\bar{\phi} = \phi^{-1}$ , and  $\bar{\phi}_j = \phi_j^{-1} \in F_j$ . Now

$$(1.3) \quad \begin{aligned} &3^j \nmid p - 1 \text{ and } \phi^{\frac{p-1}{3^j}} \equiv 1 \pmod{(\pi)} \\ &\Leftrightarrow (\pi) \text{ splits completely in } F_j/\mathbb{Q}(\sqrt{-3}) \text{ and not completely in } F_j^*/\mathbb{Q}(\sqrt{-3}) \\ &\Leftrightarrow (p) \text{ splits completely in } F_j/\mathbb{Q} \text{ but not completely in } F_j^*/\mathbb{Q}. \end{aligned}$$

Applying the prime ideal theorem for the fields  $F_j$  and  $F_j^*$ , the density of primes such that (1.3) holds is

$$[F_j : \mathbb{Q}]^{-1} - [F_j^* : \mathbb{Q}]^{-1} = (2 \cdot 3^{2j-1})^{-1} - (2 \cdot 3^{2j})^{-1} = 3^{-2j}.$$

Hence the density of primes  $d_j$  having  $3^j \nmid p - 1$  and  $p \mid W_n$  for some  $n$ , which are those for which (1.3) doesn't hold, is  $d_j = 3^{-j} - 3^{-2j}$  and the total density of primes  $p \equiv 1 \pmod{3}$  dividing some  $W_n$  is  $D_1 = \sum_{j=1}^{\infty} d_j = \frac{3}{8}$ .

*Case 2.*  $p \equiv 2 \pmod{3}$ , so  $(p)$  is inert in  $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ . Since  $(p)$  is inert

$$\phi^{p^2-1} \equiv 1 \pmod{(p)}.$$

Assuming that  $3^j \nmid (p + 1)$ , one has

$$(1.4) \quad \text{ord}_{(p)} \phi \not\equiv 0 \pmod{3} \Leftrightarrow \phi^{\frac{p^2-1}{3^j}} \equiv 1 \pmod{(p)}.$$

Now for  $3^j \nmid (p + 1)$ ,

$$(1.5) \quad \begin{aligned} &\phi^{\frac{p^2-1}{3^j}} \equiv 1 \pmod{(p)} \\ &\Leftrightarrow \text{The inert prime ideal } (p) \text{ in } \mathbb{Q}(\sqrt{-3}) \text{ splits completely in } F_j \text{ but not completely in } F_j^*. \end{aligned}$$

This latter condition is characterized as exactly those primes whose Artin symbol  $[\frac{F_j^*/\mathbb{Q}}{(p)}]$  lies in certain conjugacy classes of the Galois

group  $G^* = \text{Gal}(F_j^*/\mathbb{Q})$ . (More generally such a characterization exists for any set of primes  $p$  determined by prime-splitting conditions on  $(p)$  in the subfields of a finite extension of  $\mathbb{Q}$ , see [1], Theorem 1.2.) To specify the conjugacy classes, we use the following facts. The group  $G^*$  is of order  $2 \cdot 3^j$  with generators  $\sigma_1, \sigma_2$  given by

$$\begin{aligned} \sigma_1(\zeta_{j+1}) &= \zeta_{j+1}^2, & \sigma_1(\phi_j) &= \bar{\phi}_j, & \sigma_1(\bar{\phi}_j) &= \phi_j, \\ \sigma_2(\zeta_{j+1}) &= \zeta_{j+1}, & \sigma_2(\phi_j) &= \zeta_j \phi_j, & \sigma_2(\bar{\phi}_j) &= \zeta_j^{-1} \bar{\phi}_j, \end{aligned}$$

where  $\bar{\phi}_j = \phi_j^{-1}$  is the complex conjugate of  $\phi_j$ . A general element of  $G^*$  is denoted  $[k, l]$  where  $\sigma = [k, l]$  acts by

$$\sigma(\zeta_{j+1}) = \zeta_{j+1}^{2^k}, \quad \sigma(\phi_j) = \zeta_j^l \phi_j^{(-1)^k}, \quad \sigma(\bar{\phi}_j) = \zeta_j^{-l} \phi_j^{(-1)^{k+1}}.$$

Here  $k$  is taken  $(\text{mod } 2 \cdot 3^j)$  and  $l (\text{mod } 3^j)$ , and the group law is

$$[k, l] \circ [k', l'] = [k + k', l(-1)^{k'} + l'2^k].$$

Note that  $\tau = \sigma_1^{3^j} = [3^j, 0]$  is complex conjugation. We claim that

$$(1.6) \quad 3^j \mid\mid (p + 1) \text{ and } \phi^{\frac{p^2-1}{3^j}} \equiv 1 \pmod{p}$$

$$\Leftrightarrow \text{The Artin symbol } \left[ \frac{F_j^*/\mathbb{Q}}{(p)} \right] \text{ is either } \langle \sigma_1^{3^{j-1}} \rangle \text{ or } \langle \sigma_1^{-3^{j-1}} \rangle.$$

One easily checks that the conjugacy classes containing  $\sigma_1^{3^{j-1}}$  and  $\sigma_1^{-3^{j-1}}$  each consist of one element. To prove the  $\Rightarrow$  implication in (1.6), note first that the condition that  $3^j \mid\mid (p + 1)$  implies that the Artin symbol  $\left[ \frac{F_j^*/\mathbb{Q}}{(p)} \right]$  contains only elements of  $G^*$  of the form  $\sigma_1^{\pm 3^{j+1}} \sigma_2^k$ . Indeed, consider the action of an automorphism  $\sigma$  in  $\left[ \frac{F_j^*/\mathbb{Q}}{(p)} \right]$  restricted to the subfield  $\mathbb{Q}(\zeta_{j+1})$ . Now  $\text{Gal}(\mathbb{Q}(\zeta_{j+1})/\mathbb{Q})$  is isomorphic to the subgroup generated by  $\sigma_1$  and the restriction map sends  $\sigma_1 \rightarrow \sigma_1$  and  $\sigma_2 \rightarrow (\text{identity})$ . Then  $3^j \mid\mid (p + 1)$  says that  $\sigma$  restricted to  $\mathbb{Q}(\zeta_j)$  is complex conjugation, but is not complex conjugation on  $\mathbb{Q}(\zeta_{j+1})$ . Hence  $\sigma = [\pm 3^{j-1}, l]$  for some  $l$ . Next, any element  $\sigma$  of  $\left[ \frac{F_j^*/\mathbb{Q}}{(p)} \right]$  when restricted to acting on the subfield  $F_j$  has order equal to the degree over  $\mathbb{Q}$  of the prime ideals in  $F_j$  lying over  $(p)$ , which is 2. The group  $G = \text{Gal}(F_j/\mathbb{Q})$  is isomorphic to the subgroup generated by  $\sigma_1^3$  and  $\sigma_2$ , with the restriction map  $\Omega: G^* \rightarrow G$  sending  $\sigma_1 \rightarrow \sigma_1^3$  and  $\sigma_2 \rightarrow \sigma_2$ . Thus  $\Omega(\sigma) = [3^j, l]$  for some  $l$ . However the group law gives

$$[3^j, l] \circ [3^j, l] = [0, -2l].$$

Thus  $[3^j, l]$  is of order 2 only if  $l = 0$ , and this proves the right

side of (1.6) holds. For the reverse direction, if  $\sigma = [\pm 3^{j-1}, 0]$ , then  $\sigma$  restricted to acting on  $F_j$  is  $\Omega(\sigma) = [3^j, 0]$ , which is complex conjugation  $\tau$ , hence of order 2, so that

$$x^{p^2} \equiv x^{\sigma^2} = x \pmod{\mathfrak{p}}$$

for all prime ideals  $\mathfrak{p}$  in  $F_j$  lying over  $(p)$ , for all algebraic integers  $x$  in  $F_j$ . Thus

$$x^{p^2-1} \equiv 1 \pmod{(\mathfrak{p})}$$

for all such  $x$ , such that  $(x, (p)) = 1$ , including  $\phi_j$ , and the left side of (1.6) holds.

Now the set of primes satisfying (1.6) has density  $2[F_j^* : \mathbb{Q}]^{-1} = 3^{-2j}$ , by the Chebotarev density theorem. The density of primes with  $p^j \parallel (p+1)$  and  $p|W_n$  for some  $n$  then is  $d_j^* = 3^{-j} - 3^{-2j}$ , and the total density of primes  $p \equiv 2 \pmod{3}$  with  $p$  dividing some  $W_n$  is

$$D_2 = \sum_{j=1}^{\infty} d_j = \frac{3}{8}.$$

Finally  $D_1 + D_2 = \frac{3}{4}$ , completing the proof.  $\square$

**REMARK.** Of the 1228 primes less than  $10^4$ , one finds:

$$\begin{aligned} \#\{p: p \equiv 1 \pmod{3}, p \text{ divides some } W_n\} &= 450, \\ \#\{p: p \equiv 2 \pmod{3}, p \text{ divides some } W_n\} &= 466, \\ \#\{p: p \text{ does not divide any } W_n\} &= 312. \end{aligned}$$

These give frequencies of 36.6%, 37.3%, 25.4%, which may be compared with the asymptotic densities  $3/8$ ,  $3/8$ ,  $1/4$ , respectively, predicted by the proof of Theorem C.

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