Pacific Journal of Mathematics

Volume 163 No. 2 April 1994

CONTENTS

Vol. 163, No. 2 April 1994

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

A NOTE ON INTERMEDIATE SUBFACTORS

DIETMAR BISCH

In this note we prove that if $N \subset M \subset P$ is an inclusion of II_1 factors with finite Jones index such that $N \subset P$ has finite depth, then $N \subset M$ and $M \subset P$ have finite depth. We show this result by studying the iterated basic constructions for $M \subset P$ and $N \subset P$. In particular our proof gives detailed information about the graphs for $N \subset M$ resp. $M \subset P$. Furthermore, we give an abstract characterization of intermediate subfactors in terms of Jones projections in $N' \cap P_1$, where $N \subset P \subset P_1$ is the basic construction for $N \subset P$ and give examples showing that if $N \subset M$ and $M \subset P$ have finite depth, then $N \subset P$ does not necessarily have finite depth.

1. Introduction. The problem of classifying subfactors of the hyperfinite II_1 factor is one of the most challenging problems in operator algebras. Starting with an inclusion $N \subset M$ of hyperfinite II_1 factors with finite Jones index $[M : N] < \infty$, one constructs the associated Jones tower of factors $N \subset M \subset M_1 \subset M_2 \subset \dots$, where M_{i+1} is the II_1 factor obtained from the *Jones basic construction* for $M_{i-1} \subset M_i$ (see **[Jo1]**). The centralizer algebras $\{M_i' \cap M_j\}_{i \leq j}$ are finite dimensional C^*-algebras sitting in the envelopping II_1 factor $M_\infty = \overline{\bigcup M_k}^w$. Furthermore, inclusions of four such algebras

$$\begin{array}{ccccccc} M_i' & \cap & M_k & \subset & M_i' & \cap & M_{k+1} \\ & \cup & & & & \cup & \\ M_{i+1}' & \cap & M_k & \subset & M_{i+1}' & \cap & M_{k+1} \end{array}$$

satisfy certain symmetry conditions: they form what is called a *commuting square* (**[Po2]**, see also **[GHJ]**). All the information contained in this double sequence of finite dimensional algebras is actually contained in the following sequence of commuting squares

$$\begin{array}{cccccccc} M' & \cap & M_k & \subset & M' & \cap & M_{k+1} & \subset \cdots \\ & \cup & & & & \cup & & \\ M_1' & \cap & M_k & \subset & M_1' & \cap & M_{k+1} & \subset \cdots \end{array}$$

which is an invariant for the inclusion $N \subset M$, called the *standard invariant* (**[Po4]** or *paragroup* **[Oc1]**). From this sequence one can form the inclusion $\overline{\bigcup_k M' \cap M_k}^w \subset \overline{\bigcup_k M_1' \cap M_k}^w$ of hyperfinite II_1 von Neumann algebras and ask if these algebras form a model for $N \subset M$, i.e. are (anti-)isomorphic to the inclusion $N \subset M$. Popa introduced recently a concept of *amenability* for inclusions $N \subset M$ (**[Po3]**, **[Po4]**) and showed that precisely the amenable subfactors of R, the hyperfinite II_1 factor, are classified by their standard invariant. A particular, but important class of amenable subfactors of R are the *finite depth* subfactors, referring to the condition $\sup_k \dim Z(M' \cap M_k) < \infty$, where $Z(M' \cap M_k)$ denotes the center of $M' \cap M_k$. Equivalently, this condition expresses the fact that the width of the Bratteli diagram describing the inclusions $\mathbb{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \dots$ is bounded from a certain point on. Popa showed in (**[Po2]**, see also **[Oc1]**) that finite depth subfactors N of the hyperfinite II_1 factor M are classified by an initial commuting square

$$\begin{array}{ccccccc} M' & \cap & M_{k_0} & \subset & M' & \cap & M_{k_0+1} \\ & \cup & & & & \cup & \\ M_1' & \cap & M_{k_0} & \subset & M_1' & \cap & M_{k_0+1} \end{array}$$

for k_0 large enough (which can be made precise). Subfactors of index < 4 are automatically of finite depth and the associated commuting squares can be classified in terms of graphs of Coxeter-Dynkin type A, D, E and certain connections on them (**[B-N]**, **[I1]**, **[I2]**, **[Jo1]**, **[Ka]**, **[Oc1]**, **[Oc2]**, **[SV]**). Wenzl constructed interesting series of finite depth subfactors via braid group representations, generalizing Jones' original construction of subfactors of the hyperfinite II_1 factor. It is by now well-known that Jones' discovery of certain remarkable braid group representations in the higher relative commutants of every finite index subfactor lead him to the construction of his link invariant, the Jones polynomial. Similarly, Wenzl's subfactors carry representations of the braid group in their higher relative commutants which can be used to obtain the HOMFLY and Kauffman polynomials using the same method as Jones' original construction of his link invariant (**[Jo1]**, **[Jo2]**, **[We1]**, **[We2]**). The simplest finite depth subfactors are obtained by letting a finite group G act by properly outer automorphisms of R and considering the inclusion $R \subset R \rtimes G$. The canonical (classifying) commuting square of this inclusion contains all the information on G and its representation theory: G can be completely recovered from the inclusion. Similarly, if $H \subset G$ is a subgroup of G of finite index, then $R \rtimes H \subset R \rtimes G$ is again a finite depth inclusion and the associated canonical commuting square can be described explicitly in terms of induced representations (for details of all this and more examples coming from groups, see **[Bi2]**, **[KY]**).

It is a well-known theorem in the theory of extensions of von Neumann algebras that if G is a countable discrete group of outer automorphisms on the II_1 factor N and P is a subfactor with $N \subset P \subset N \rtimes G$, then there is a subgroup $H \subset G$ such that $P = N \rtimes H$ (**[NT]**, **[Su]**). This result is quite apparent for a finite group G and it is natural to ask if a similar result holds for finite depth subfactors, where the role of the group is played by the more general object, the canonical commuting square or the paragroup. In other words, given an inclusion of II_1 factors $N \subset M \subset P$ such that $N \subset P$ has finite depth, does this force the finite depth condition on $N \subset M$ and $M \subset P$? We prove that this statement is indeed true, more precisely we show the following theorem:

THEOREM. *Let $N \subset M \subset P$ be an inclusion of* II_1 *factors with $[P : N] < \infty$ and assume $N \subset P$ has finite depth. Then $N \subset M$ and $M \subset P$ have finite depth.*

This theorem will follow from a detailed study of the basic construction for $N \subset P$ and $M \subset P$. We are able to describe the higher relative commutants of $M \subset P$ completely in terms of the higher relative commutants of the inclusion $N \subset P$. In particular we obtain information on the graphs for $N \subset M$ and $M \subset P$ and our proof provides an algorithm for computing these graphs from the graphs for $N \subset P$.

In §2 we collect for the convenience of the reader some facts about the basic construction, fix the notation and prove some useful lemmas. We proceed then with the proof of our theorem. Furthermore, we give some examples showing that the converse of our theorem does not hold: if $N \subset M$ and $M \subset P$ have finite depth, then $N \subset M$ need not have finite depth, in fact $N \subset M$ need not even be amenable in the sense of Popa (**[Po4]**).

In §3 we give an abstract characterization of intermediate subfactors M of a given irreducible inclusion $N \subset P$ in terms of Jones projections in $N' \cap P_1$, where $N \subset P \subset P_1$ is the basic construction for $N \subset P$. This allows us to recognize intermediate subfactors by looking at the projections in $N' \cap P_1$ and reconstruct the subfactor from these projections.[1]

2. The proof of the theorem. Let $N \subset M$ be an inclusion of II_1 factors with finite Jones index $[M : N]$. We denote by $L^2(M, \mathrm{tr}_M)$

[1] Y. Kawahigashi informed us that A. Ocneanu has a characterization of intermediate subfactors using his Fourier transform.

the closure of M in the Hilbert norm $\|x\|_2 = \mathrm{tr}_M(x^*x)^{1/2}$ induced by the unique trace tr_M on M. Let $e_N^M : L^2(M, \mathrm{tr}_M) \to L^2(N, \mathrm{tr}_N)$ be the orthogonal projection and let $J_M : L^2(M, \mathrm{tr}_M) \to L^2(M, \mathrm{tr}_M)$ be the canonical conjugation defined by $J_M(x) = x^*$, $x \in M$ viewed as a vector in $L^2(M, \mathrm{tr}_M)$. The algebra $M_1 = vN(M, e_N^M) = \langle M, e_N^M \rangle$, i.e. the von Neumann algebra generated by M and e_N^M in $B(L^2(M, \mathrm{tr}_M))$ is called the *basic construction* for $N \subset M$ (**[Jo1]**). We recall (**[Jo1]**):

(1) $e_N^M x e_N^M = E_N^M(x) e_N^M$, where $E_N^M : M \to N$ is the unique trace preserving conditional expectation from M onto N.
(2) $N = \{e_N^M\}' \cap M$.
(3) $e_N^M \langle M, e_N^M \rangle e_N^M = N e_N^M \simeq N$.
(4) $\langle M, e_N^M \rangle = J_M N' J_M$.
(5) $[J_M, e_N^M] = 0$.
(6) There is a unique trace tr_{M_1} on M_1 such that $\mathrm{tr}_{M_1}(x e_N^M) = [M : N]^{-1} \mathrm{tr}_M(x)$ for all $x \in M$.
(7) $M = \overline{\mathrm{span}\, N e_N^M N}^w$ (**[PiPo1]**).

It is easy to see that $N \subset M$ has finite depth iff $M \subset M_1$ has finite depth iff $\sup_k \dim Z(M' \cap M_k) < \infty$ iff $\sup_k \dim Z(M' \cap M_{2k}) < \infty$ iff $\sup_k \dim Z(M' \cap M_{2k+1}) < \infty$ iff $\sup_k \dim Z(N' \cap M_{2k}) < \infty$ iff $\sup_k \dim Z(N' \cap M_{2k+1}) < \infty$ (see for instance **[Po2]** for a proof). We will use at various instances the following simple abstract characterization of the basic construction (**[PiPo2]**): Let $N \subset M$ be an inclusion of II_1 factors with $[M : N] < \infty$ and let P be a II_1 factor containing M and a projection p such that $[p, N] = 0$ and $E_M^P(p) = [P : M]^{-1} 1_M = [M : N]^{-1} 1_M$, then P is the basic construction for $N \subset M$, i.e. there is an isomorphism from M_1 onto P leaving M pointwise fixed and carrying e_N^M to p.

It will be useful for the proof of our theorem to study the basic construction for certain reduced algebras. This is done in the next few lemmas.

LEMMA 2.1. *Let $N \subset M$ be II_1 factors, $[M : N] < \infty$. Let $p \in N$, $q \in N' \cap M$ be projections and consider the inclusion $qpNpq \subset pqMpq$ with trace $\mathrm{tr}_{pqMpq}(qpxpq) = \mathrm{tr}_M(p)^{-1} \mathrm{tr}_N(q)^{-1} \mathrm{tr}_M(pqxpq)$, $x \in M$. Then the unique trace preserving conditional expectation $E_{pqNpq}^{pqMpq} : pqMpq \to pqNpq$ is given by*

$$E_{pqNpq}^{pqMpq}(pqxpq) = \mathrm{tr}_M(q)^{-1} p E_N^M(qxq) pq, \qquad x \in M.$$

Proof. The proof is a straightforward computation and left to the reader. □

Recall that $N \subset M$ is called *extremal* iff $\mathrm{tr}_M|_{N' \cap M} = \mathrm{tr}_{N'}|_{N' \cap M}$ ([**PiPo1**]), which happens for instance whenever $N \subset M$ has finite depth.

LEMMA 2.2. *Let $N \subset M$ be an extremal inclusion of II_1 factors, $[M : N] < \infty$ and let $N \subset M \subset M_1 = \langle M, e_N^M \rangle$ be the basic construction. Suppose we have projections $p \in N' \cap M$ and $q \in M' \cap M_1$ with $\mathrm{tr}_M(p) = \mathrm{tr}_{M_1}(q)$ such that $\mathrm{tr}_{M_1}(pqe_N^M) = [M : N]^{-1}\,\mathrm{tr}_M(p)$. Then*

$$Npq \subset (pMp)q \subset pqM_1pq \tag{1}$$

is the basic construction for the pair $(Npq \subset (pMp)q) \simeq (Np \subset pMp)$.

Proof. Note that

$$[qpMpq : Npq] = [pqM_1pq : pqMpq] = \mathrm{tr}_M(p)^2[M : N]$$

since $N \subset M$ is extremal ([**PiPo1**]).

Consider $e := \mathrm{tr}_M(p)^{-1} pqe_N^M pq \in pqM_1pq$, then

$$\begin{aligned} \mathrm{tr}_{pqM_1pq}(e) &= \mathrm{tr}_M(p)^{-2}\,\mathrm{tr}_{M_1}(e) = \mathrm{tr}_M(p)^{-3}\,\mathrm{tr}_{M_1}(pqe_N^M) \\ &= \mathrm{tr}_M(p)^{-2}[M : N]^{-1} = [pqM_1pq : pqMpq]^{-1}. \end{aligned}$$

Let m be the unique element in M satisfying $me_N^M = qe_N^M$, i.e. $m = [M : N]E_M^{M_1}(qe_N^M)$. Then $m \in N' \cap M$. We have $e^2 = \mathrm{tr}_M(p)^{-2}E_N^M(pm)pqe_N^M pq = e$, since by hypothesis

$$\begin{aligned} [M : N]^{-1}\,\mathrm{tr}_M(p) &= \mathrm{tr}_{M_1}(pqe_N^M) = \mathrm{tr}_{M_1}(pme_N^M) \\ &= \mathrm{tr}_{M_1}(E_N^M(pm)e_N^M) = E_N^M(pm)[M : N]^{-1}, \end{aligned}$$

which implies $E_N^M(pm) = \mathrm{tr}_M(p)$. In order to see that (1) is indeed the basic construction we use the above-mentioned characterization of the basic construction ([**PiPo1**]). Since e clearly commutes with Npq, we only need to check that

$$E_{pqMpq}^{pqM_1pq}(e) = \mathrm{tr}_M(p)^{-1}[M : N]^{-1}pq. \tag{2}$$

By Lemma 2.1 we know that

$$\begin{aligned} E_{pqMpq}^{pqM_1pq}(e) &= \mathrm{tr}_M(p)^{-2}pE_M^{M_1}(qe_N^M q)pq \\ &= \mathrm{tr}_M(p)^{-2}pm[M : N]^{-1}m^*pq \\ &= \mathrm{tr}_M(p)^{-2}[M : N]^{-1}pmm^*pq. \end{aligned} \tag{3}$$

Computing traces on both sides of (3) gives

$$\mathrm{tr}_M(p)^2 = \mathrm{tr}_{M_1}(pmm^*pq) = \|pm\|_2^2\,\mathrm{tr}_{M_1}(q)$$

and hence

$$\|pm - p\|_2^2 = \|pm\|_2^2 - \mathrm{tr}_M(p) = 0\,,$$

i.e. $pm = p$. Note that this implies in particular that $pqe_N^M = pe_N^M$ (the condition $pm = p$ is actually equivalent to the condition on the traces in the statement of the lemma). Thus

$$E_{pqMpq}^{pqM_1pq}(e) = \mathrm{tr}_M(p)^{-1}[M:N]^{-1}pq\,,$$

which completes the proof. □

REMARK 2.3. Note that if $N \subset M$ is extremal, given a projection p as in the lemma, we can always find a projection $q \in M' \cap M_1$ such that $\mathrm{tr}_{M_1}(pqe_N^M) = \mathrm{tr}_M(p)[M:N]^{-1}$, $\mathrm{tr}_{M_1}(q) = \mathrm{tr}_M(p)$. Namely, let $q := J_M p J_M \in J_M(N' \cap M)J_M = M' \cap M_1$, where J_M denotes as usual the canonical conjugation on $L^2(M, \mathrm{tr}_M)$. We have then clearly $\mathrm{tr}_{M_1}(q) = \mathrm{tr}_M(p)$ (extremality) and $\mathrm{tr}_{M_1}(pJ_MpJ_Me_N^M) = [M:N]^{-1}\,\mathrm{tr}_M(p)$ since $pJ_MpJ_Me_N^M = pe_N^M$.

The proof of the following lemma is trivial.

LEMMA 2.4. *Let* $N \subset M$ *be* II_1 *factors,* $[M:N] < \infty$, $p \in N$ *a projection and* $N \subset M \subset M_1 \subset \dots$ *the basic construction. Then*

$$pNp \subset pMp \subset pM_1p \subset \dots$$

is that basic construction for $pNp \subset pMp$.

We describe now the construction which will be used to prove the theorem. Let

$$M \subset P \subset Q_1 \subset Q_2 \subset \dots \tag{4}$$

be the Jones tower of factors obtained by iterating the basic construction for $M \subset P$. Similarly, let

$$N \subset P \subset P_1 \subset P_2 \subset \dots \tag{5}$$

be the tower for $N \subset P$. Note that $N \subset M \subset P \subset Q_1 \subset P_1 \subset B(L^2(P, \mathrm{tr}_P))$. Let $\tilde{Q}_2 := \langle P_1, e_{Q_1}^{P_1}\rangle$ be the basic construction for $Q_1 \subset P_1$, then $P \subset Q_1 \subset P_1 \subset \tilde{Q}_2 \subset P_2 \subset B(L^2(P_1, \mathrm{tr}_{P_1}))$. Continuing this construction we obtain

$$N \subset M \subset P \subset Q_1 \subset P_1 \subset \tilde{Q}_2 \subset P_2 \subset \tilde{Q}_3 \subset P_3 \subset \dots, \tag{6}$$

where $\tilde{Q}_{i-1} \subset P_i \subset \tilde{Q}_i = \langle P_{i-1}, e_{\tilde{Q}_{i-1}}^{P_{i-1}} \rangle$ is the basic construction. If we set $\alpha := [M : N]$, $\beta := [P : M]$, then $\alpha\beta = [P : N]$ and the indices of the various inclusions are indicated as follows

$$N \subset^{\alpha} M \subset^{\beta} P \subset^{\beta} Q_1 \subset^{\alpha} P_1 \subset^{\alpha} \tilde{Q}_2 \subset^{\beta} P_2 \subset^{\beta} \tilde{Q}_3 \subset \dots, \tag{7}$$

and

$$\begin{aligned} \dots \subset^{\alpha} \tilde{Q}_{2k-2} \subset^{\beta} P_{2k-2} \subset^{\beta} \tilde{Q}_{2k-1} \subset^{\alpha} P_{2k-1} \subset^{\alpha} \tilde{Q}_{2k} \\ \subset^{\beta} P_{2k} \subset^{\beta} \tilde{Q}_{2k+1} \subset^{\alpha} P_{2k+1} \subset^{\alpha} \dots . \end{aligned} \tag{8}$$

We will denote the Jones' projections in the following way: for instance $e_{\tilde{Q}_{2k-1}}^{P_{2k-1}}$ denotes the Jones projection which implements the conditional expectation from P_{2k-1} onto $\tilde{Q}_{2k-1}$.

The computation in the next lemma will be used in the theorem.

LEMMA 2.5. *With the notation as above we have*
(1) $E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = \alpha^{-1} e_{\tilde{Q}_{2k}}^{P_{2k}}$ *and hence* $e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k-1}}^{P_{2k}} e_{P_{2k}}^{P_{2k+1}} = \alpha^{-1} e_{\tilde{Q}_{2k}}^{P_{2k}} e_{P_{2k}}^{P_{2k+1}}$, $k \geq 1$.
(2) $E_{\tilde{Q}_{2k}}^{P_{2k}}(e_{P_{2k-2}}^{P_{2k-1}}) = \beta^{-1} e_{\tilde{Q}_{2k-1}}^{P_{2k-1}}$ *and hence* $e_{\tilde{Q}_{2k}}^{P_{2k}} e_{P_{2k-2}}^{P_{2k-1}} e_{P_{2k-1}}^{P_{2k}} = \beta^{-1} e_{\tilde{Q}_{2k-1}}^{P_{2k-1}} e_{P_{2k-1}}^{P_{2k}}$, $k \geq 1$, $(\tilde{Q}_1 := Q_1)$.

Proof. The proof of (1) and (2) are identical, so we prove only (1). Since $e_{\tilde{Q}_{2k}}^{P_{2k}} \in \tilde{Q}_{2k+1}$, we have that

$$e_{\tilde{Q}_{2k}}^{P_{2k}} E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{\tilde{Q}_{2k}}^{P_{2k}} e_{P_{2k-1}}^{P_{2k}}) = E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}).$$

By **[PiPo1]** we know that there is a unique element $m \in P_{2k}$ such that $E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) e_{\tilde{Q}_{2k}}^{P_{2k}} = m e_{\tilde{Q}_{2k}}^{P_{2k}}$. Applying $E_{P_{2k}}^{\tilde{Q}_{2k+1}}$ to both sides of the equation gives

$$m = \operatorname{tr}_{P_{2k+1}}(e_{\tilde{Q}_{2k}}^{P_{2k}})^{-1} E_{P_{2k}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = \beta [P : N]^{-1} 1 = \alpha^{-1} 1.$$

Since

$$\begin{aligned} e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k-1}}^{P_{2k}} e_{P_{2k}}^{P_{2k+1}} &= e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k-1}}^{P_{2k}} e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k}}^{P_{2k+1}} \\ &= E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) e_{P_{2k}}^{P_{2k+1}}, \end{aligned}$$

the second part also follows. □

We restate now the theorem and give then the proof.

THEOREM 2.6. *Let $N \subset M \subset P$ be an inclusion of* II_1 *factors with* $[P : N] < \infty$ *and assume $N \subset P$ has finite depth. Then $N \subset M$ and $M \subset P$ have finite depth.*

Proof. (a) We show first that $M \subset P$ has finite depth. We actually prove the following statement by induction:

$$(9) \qquad \begin{aligned}&(P \subset Q_2 \subset Q_4 \subset \cdots \subset Q_{2k})\\ &\quad \simeq (Pf_{k-1} \subset f_{k-1}P_2f_{k-1} \subset \cdots \subset f_{k-1}P_{2k}f_{k-1}),\end{aligned}$$

where $f_k := e^{P_1}_{Q_1} e^{P_3}_{\tilde{Q}_3} \cdots e^{P_{2k+1}}_{\tilde{Q}_{2k+1}}$ (note that f_k is a projection since all the $e^{P_r}_{\tilde{Q}_r}$'s commute). First we show

$$(10) \qquad (P \subset Q_1 \subset Q_2) \simeq (P \simeq Pe^{P_1}_{Q_1} \subset e^{P_1}_{Q_1}\tilde{Q}_2 e^{P_1}_{Q_1} \subset e^{P_1}_{Q_1} P_2 e^{P_1}_{Q_1}).$$

Since $\tilde{Q}_2 = \langle P_1, e^{P_1}_{Q_1}\rangle$ we have $e^{P_1}_{Q_1}\tilde{Q}_2 e^{P_1}_{Q_1} = Q_1 e^{P_1}_{Q_1}$ and therefore $(P \subset Q_1) \simeq (Pe^{P_1}_{Q_1} \subset Q_1 e^{P_1}_{Q_1})$ and hence

$$(11) \qquad \begin{aligned}(P \subset Q_1 \subset Q_2) &\simeq (Pe^{P_1}_{Q_1} \subset Q_1 e^{P_1}_{Q_1} \subset \langle Q_1 e^{P_1}_{Q_1}, Pe^{P_1}_{Q_1}\rangle)\\ &\simeq (Pe^{P_1}_{Q_1} \subset e^{P_1}_{Q_1}\tilde{Q}_2 e^{P_1}_{Q_1} \subset e^{P_1}_{Q_1} P_2 e^{P_1}_{Q_1}),\end{aligned}$$

where the last isomorphism is checked by using again the abstract characterization of the basic construction (**[PiPo2]**): Set $e := e^{P_1}_{Q_1} e^{P_1}_{P} e^{P_1}_{Q_1}$ and note that actually $e^{P_1}_{Q_1} e^{P_1}_{P} = e^{P_1}_{P}$. Thus e is a projection in $e^{P_1}_{Q_1} P_2 e^{P_1}_{Q_1}$ of the right trace, namely β. Clearly $[e, Pe^{P_1}_{Q_1}] = 0$ and it remains to show that

$$(12) \qquad E^{e^{P_1}_{Q_1} P_2 e^{P_1}_{Q_1}}_{e^{P_1}_{Q_1}\tilde{Q}_2 e^{P_1}_{Q_1}}(e) = \beta^{-1} e^{P_1}_{Q_1}.$$

To simplify the notation we set $A := e^{P_1}_{Q_1} P_2 e^{P_1}_{Q_1}$ and we recall that $\tilde{Q}_2 = \overline{\operatorname{span} P_1 e^{P_1}_{Q_1} P_1}^{w}$. We need to check that

$$\operatorname{tr}_A(e^{P_1}_{Q_1} x e^{P_1}_{Q_1} e) = \beta^{-1} \operatorname{tr}_A(e^{P_1}_{Q_1} x e^{P_1}_{Q_1}),$$

for all $x \in \tilde{Q}_2$. Let $y \in Q_1$ with $e^{P_1}_{Q_1} x e^{P_1}_{Q_1} = y e^{P_1}_{Q_1}$, then

$$\begin{aligned}\operatorname{tr}_A(e^{P_1}_{Q_1} x e^{P_1}_{Q_1} e^{P_1}_{P} e^{P_1}_{Q_1}) &= [P_1 : Q_1] \operatorname{tr}_{P_2}(y e^{P_1}_{Q_1} e^{P_1}_{P}) = \alpha \operatorname{tr}_{P_2}(y e^{P_1}_{P})\\ &= [P_1 : P]^{-1} \alpha \operatorname{tr}_{P_2}(y) = (\beta)^{-1} \alpha \operatorname{tr}_{P_2}(y e^{P_1}_{Q_1})\\ &= \beta^{-1} \operatorname{tr}_A(e^{P_1}_{Q_1} x e^{P_1}_{Q_1}).\end{aligned}$$

This proves (12). In particular we have $(P \subset Q_2) \simeq (Pe_{Q_1}^{P_1} \subset e_{Q_1}^{P_1} P_2 e_{Q_1}^{P_1})$ (note that this implies already that $P \subset Q_2$ and hence $M \subset P$ have finite depth since $P \subset P_2$ does (assuming we know that reduced subfactors of finite depth subfactors have finite depth). Since we want to get an explicit description of the higher relative commutants, we want to prove more, namely (9)).

For clarity of exposition let us also do the next step of the induction. Since $P \subset P_2 \subset P_4$ is the basic construction, $e_{Q_1}^{P_1} \in P' \cap P_2$, $e_{\tilde{Q}_3}^{P_3} \in P_2' \cap P_4$, $\mathrm{tr}_{P_2}(e_{Q_1}^{P_1}) = \mathrm{tr}_{P_4}(e_{\tilde{Q}_3}^{P_3}) = \alpha^{-1}$, we only need to check

$$(13) \qquad \mathrm{tr}_{P_4}(e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} e_P^{P_2}) = [P_2 : P]^{-1} \mathrm{tr}_{P_2}(e_{Q_1}^{P_1}) = [P : N]^{-2}\alpha^{-1}$$

in order to be able to apply Lemma 2.2. By **[PiPo2]** we know that $e_P^{P_2} = [P : N] e_{P_1}^{P_2} e_P^{P_1} e_{P_2}^{P_3} e_{P_1}^{P_2}$ and hence

$$\begin{aligned}
\mathrm{tr}_{P_4}(e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} e_P^{P_2}) &= [P : N]\, \mathrm{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_P^{P_1} e_{P_2}^{P_3} e_{P_1}^{P_2} e_{\tilde{Q}_3}^{P_3}) \\
&= [P : N]\alpha^{-1}\, \mathrm{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_P^{P_1} e_{P_2}^{P_3} e_{\tilde{Q}_2}^{P_2}) \qquad \text{(Lemma 2.5)} \\
&= \alpha^{-1}\, \mathrm{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_P^{P_1} e_{\tilde{Q}_2}^{P_2}) \\
&= \alpha^{-1}\beta^{-1}\, \mathrm{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_{Q_1}^{P_1}) \qquad \text{(Lemma 2.5)} \\
&= [P : N]^{-2}\, \mathrm{tr}_{P_4}(e_{Q_1}^{P_1}) = [P : N]^{-2}\alpha^{-1}.
\end{aligned}$$

Thus we can apply Lemma 2.2 to $(P \subset Q_2) \simeq (Pe_{Q_1}^{P_1} \subset e_{Q_1}^{P_1} P_2 e_{Q_1}^{P_1})$ $\simeq (Pe_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} \subset e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} P_2 e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3})$ and obtain

$$(P \subset Q_2 \subset Q_4) \simeq (Pf_1 \subset f_1 P_2 f_1 \subset f_1 P_4 f_1).$$

This shows (9) for $k = 1, 2$. Now suppose (9) holds for k, and we will show it for $k + 1$. To this end it is enough to show that

$$(14) \quad (Q_{2k-2} \subset Q_{2k} \subset Q_{2k+2}) \simeq (f_k P_{2k-2} f_k \subset f_k P_{2k} f_k \subset f_k P_{2k+2} f_k).$$

Note that $f_{k-2} \in P_{2k-2}$, thus by Lemma 2.4 (and **[PiPo2]**) we know that $f_{k-2}P_{2k-2}f_{k-2} \subset f_{k-2}P_{2k}f_{k-2} \subset f_{k-2}P_{2k+2}f_{k-2}$ is the basic construction. We want to apply Lemma 2.2 with $N \leftrightarrow f_{k-2}P_{2k-2}f_{k-2}$, $M \leftrightarrow f_{k-2}P_{2k}f_{k-2}$, $M_1 \leftrightarrow f_{k-2}P_{2k+2}f_{k-2}$, $p \leftrightarrow f_{k-1}$ and $q \leftrightarrow e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} f_{k-2}$. Since $f_{k-2}P_{2k-2}f_{k-2} \subset f_{k-2}P_{2k}f_{k-2}$ is clearly extremal (because $P_{2k-2} \subset P_{2k}$ has finite depth) and $\mathrm{tr}_{f_{k-2}P_{2k+2}f_{k-2}}(e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} f_{k-2}) =$

$\operatorname{tr}_{f_{k-2}P_{2k}f_{k-2}}(f_{k-1}) = \alpha^{-1}$, we only need to check the trace condition in Lemma 2.2, i.e.

$$\operatorname{tr}_{f_{k-2}P_{2k+2}f_{k-2}}(f_k e^{f_{k-2}P_{2k}f_{k-2}}_{f_{k-2}P_{2k-2}f_{k-2}}) = \alpha^{-1}[P:N]^{-2}. \tag{15}$$

But using **[PiPo2]** and Lemma 2.5 we compute

$$\begin{aligned}
\operatorname{tr}_{f_{k-2}P_{2k+2}f_{k-2}}(f_k e^{f_{k-2}P_{2k}f_{k-2}}_{f_{k-2}P_{2k-2}f_{k-2}}) &= \alpha^{k-1}\operatorname{tr}_{P_{2k+2}}(f_k e^{P_{2k}}_{P_{2k-2}}) \\
&= \alpha^{k-1}[P:N]\operatorname{tr}_{P_{2k+2}}(f_k e^{P_{2k}}_{P_{2k-1}} e^{P_{2k-1}}_{P_{2k-2}} e^{P_{2k+1}}_{P_{2k}} e^{P_{2k}}_{P_{2k-1}}) \\
&= \alpha^{k-1}[P:N]\alpha^{-1}\operatorname{tr}_{P_{2k+2}}(f_{k-1} e^{P_{2k}}_{P_{2k-1}} e^{P_{2k-1}}_{P_{2k-2}} e^{P_{2k+1}}_{P_{2k}} e^{P_{2k}}_{\tilde{Q}_{2k}}) \\
&= \alpha^{k-2}\operatorname{tr}_{P_{2k+2}}(f_{k-1} e^{P_{2k}}_{P_{2k-1}} e^{P_{2k-1}}_{P_{2k-2}} e^{P_{2k}}_{\tilde{Q}_{2k}}) \\
&= \alpha^{k-1}\alpha^{-1}\beta^{-1}\operatorname{tr}_{P_{2k+2}}(f_{k-1} e^{P_{2k}}_{P_{2k-1}}) \\
&= [P:N]^{-2}\alpha^{k-1}\operatorname{tr}_{P_{2k}}(f_{k-1}) = [P:N]^{-2}\alpha^{-1}.
\end{aligned}$$

Applying Lemma 2.2 gives (14) and completes the induction, i.e. (9) holds. Therefore we proved that

$$P' \cap Q_{2r} \simeq f_{k-1}(P' \cap P_{2r})f_{k-1}, \qquad 1 \le r \le k, \tag{16}$$

which implies that $P \subset Q_2$ and hence $M \subset P$ have finite depth since $P \subset P_2$ does by assumption ($P \subset P_2$ has finite depth iff $N \subset P$ has finite depth, **[Po2]**).

(b) The fact that $N \subset M$ has finite depth follows now from a simple duality argument. We can choose P_{-1}, M_{-1} such that $P_{-1} \subset M_{-1} \subset N \subset M \subset P$ and $P_{-1} \subset N \subset P$ and $M_{-1} \subset N \subset M$ are basic constructions. $P_{-1} \subset N$ has finite depth since $N \subset P$ does by hypothesis and hence $M_{-1} \subset N$ has finite depth by what we just proved. But $M_{-1} \subset N$ has finite depth iff $N \subset M$ has finite depth, which completes the proof of the theorem. □

The main motivation for giving a detailed proof of the theorem is the fact that we want to obtain information on the principal graphs (see **[GHJ]** for terminology) for $N \subset M$ and $M \subset P$ in terms of the principal graphs for $N \subset P$. Some information can indeed be obtained by looking at the Bratteli diagrams of the inclusions of higher relative commutants associated to $N \subset P$. We summarize in the next corollary what can be read off the above proof.

COROLLARY 2.7. *Let $N \subset M \subset P$ be as in the theorem. Then the Bratteli diagram of $N \subset M$ (from 2 to 2 steps) is obtained as a* *subdiagram of the Bratteli diagram for* $N \subset P$ *(from* 2 *to* 2 *steps). Similarly, the Bratteli diagram for* $P \subset Q_1$ *(from* 2 *to* 2 *steps), which is the "dual" Bratteli diagram for* $M \subset P$*, is obtained as a subdiagram of the Bratteli diagram for* $P \subset P_1$ *(from* 2 *to* 2 *steps), which is the "dual" Bratteli diagram for* $N \subset P$ *(from* 2 *to* 2 *steps). Furthermore, the method in the above proof gives an explicit algorithm to compute these Bratteli diagrams (see* (9), (16)).

Note that the corollary generalizes what happens in the situation $N \subset N \rtimes H \subset N \rtimes G$, where $H \subset G$ are finite groups and H is a subgroup of finite index of G (see for instance **[KY]**). Even for these subfactors it is impossible to find a general and more explicit relation between the principal graphs of the "big" inclusion and the ones of the two "smaller" inclusions.

Let us also remark that since finite depth subfactors are classified by their canonical commuting squares or paragroups (**[Po1]**, **[Oc1]**), our theorem can be viewed as defining a *quotient* of the canonical paragroup associated to $N \subset P$ by the one associated to $N \subset M$: the result is again a paragroup, namely the one associated to $M \subset P$. As pointed out previously by Ocneanu, the quotient G/H of two groups (viewed as paragroups) $H \subset G$ with $[G : H] < \infty$ is always a paragroup. We intend to explore these ideas further in a future paper.

We mention that Popa has shown independently the analogous statement of the theorem with "finite depth subfactor" replaced by "amenable subfactor", which does not imply our theorem.

Finally we give some examples of finite depth subfactors $N \subset M$, $M \subset P$ such that $N \subset P$ is not of finite depth and/or amenable in the sense of Popa (**[Po3]**, **[Po4]**). Let N be the hyperfinite II_1 factor and consider $N \subset N \rtimes_\alpha \mathbb{Z}_2 \subset (N \rtimes_\alpha \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2 =: P$, with α and β outer actions of $\mathbb{Z}_2$ on N such that $\mathrm{period}(\alpha\beta) = \infty$. Then $N \subset P$ has standard graph D_∞, i.e. is not of finite depth (**[Po3]**). Haagerup showed in **[Ha]** that if there are subfactors $N \subset M \subset P$ of the hyperfinite II_1 factor where $N \subset M$ and $M \subset P$ have index 2 resp. $4\cos^2 \pi/5$ (hence are of finite depth), then $N \subset P$ cannot be amenable. Another such example was mentioned to us by V.F.R. Jones: take $\mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ with generators α and β. Let α and β act on the hyperfinite II_1 factor by properly outer automorphisms such that the action of $\mathrm{PSL}(2, \mathbb{Z})$ is ergodic on central sequences (**[Jo3]**) and consider the inclusion $N^\alpha \subset N \subset N \rtimes_\beta \mathbb{Z}_3$ of index $2 \cdot 3 = 6$, which cannot be amenable since all the central sequences for P contained in the subfactor are trivial (**[Bi1]**).

Given an inclusion of II_1 factors $N \subset P$, $[P:N] < \infty$, we would like to determine all intermediate subfactors of $N \subset P$ from looking only at this given inclusion. This can indeed be done and we settle this problem in the following section.

3. Abstract characterzation of intermediate subfactors. Consider II_1 factors $N \subset P$, $[P:N] < \infty$, not necessarily of finite depth and let $N \subset P \subset P_1$ be the basic construction. If there is an intermediate subfactor $N \subset M \subset P$, then the Jones' projection $e_M^P \in N' \cap P_1$ can be abstractly characterized among the projections in $N' \cap P_1$. Let us first collect the properties of e_M^P.

Proposition 3.1. *Let $N \subset M \subset P$ be II_1 factors with $[P:N] < \infty$. Let $N_1 \subset N \subset P \subset P_1$, $M_{-1} \subset N \subset M$ and $M \subset P \subset Q_1$ be basic constructions with $N \subset M \subset P \subset Q_1 \subset P_1$ and $N_1 \subset M_{-1} \subset N \subset M \subset P$. Then*

(1) $e_M^P \in N' \cap P_1$.
(2) $e_M^P e_N^P = e_N^P$.
(3) $E_{P'}^{P_1}(e_M^P) = [P:M]^{-1} 1_P$.
(4) $e_M^P P e_M^P = M e_M^P \subset P e_M^P$.
(5) $e_M^P e_{N_1}^N e_N^P = E_M^P(e_{N_1}^N) e_N^P$ *and* $E_M^P(e_{N_1}^N) = [P:M]^{-1} e_{M_{-1}}^N$.

Proof. (1)–(4) follow from properties of Jones projections, (5) is proved using the method of Lemma 2.5. □

It will turn out that (1)–(5) characterize the Jones projections coming from intermediate subfactors, but that actually not all these properties are needed to give this characterization.

Consider the basic construction $N \subset P \subset P_1$, choose a subfactor $N_1 \subset N$ such that $N_1 \subset N \subset P$ is the basic construction and and define the set

$\mathrm{IS}(N, P) := \{q \in N' \cap P_1$ projection such that

(1) $q e_N^P = e_N^P$.

(2) $E_{P'}^{P_1}(q) \in \mathbb{C}$.

(3) Let m be the unique element in P satisfying

$q e_{N_1}^N e_N^P = m e_N^P$, i.e. $m = E_{P'}^{P_1}(q e_{N_1}^N e_N^P)[P:N]$.

Assume that m is a scalar multiple of a projection.$\}$

Note that (2) is equivalent to requiring that $E_{P'}^{P_1}(q) = \mathrm{tr}_{P_1}(q) 1_P$ and that (3) does not depend on the choice of the subfactor $N_1 \subset N$

(two different choices are conjugate by a unitary in N). Furthermore, it is clear that we always have 1, $e_N^P \in \mathrm{IS}(N, P)$ and that every Jones projection e_M^P coming from an intermediate subfactor M lies in $\mathrm{IS}(N, P)$. Conversely, we prove that any projection in $\mathrm{IS}(N, P)$ is a Jones projection coming from an intermediate subalgebra $N \subset M \subset P$.

THEOREM 3.2. *Let $N \subset P$ be II_1 factors with $[P : N] < \infty$. Then every projection $q \in \mathrm{IS}(N, P)$ implements a conditional expectation from P onto the intermediate subalgebra $M := \{q\}' \cap P$. If M is a factor, then its index $[P : M]$ in P is equal to $\mathrm{tr}_{P_1}(q)^{-1}$. In particular, $q = 1$ corresponds to the subfactor P and $q = e_N^P$ to N.*

Proof. Let $q \in \mathrm{IS}(N, P)$ and $m = E_P^{P_1}(qe_{N_1}^N e_N^P)[P : N] = \lambda p$ for some $\lambda \in \mathbb{C}$ and a projection $p \in P$. We show first that q implements a conditional expectation from $\overline{P \text{ onto } M := \{q\}' \cap P, \text{ a finite}}$ von Neumann algebra. Since $qPq = \overline{\mathrm{span}\, qNe_{N_1}^N Nq} = \overline{\mathrm{span}\, Nqe_{N_1}^N qN}$, we need to determine $qe_{N_1}^N q$. But

$$qe_{N_1}^N q = [P : N]qe_{N_1}^N e_N^P e_{N_1}^N q = [P : N]\lambda^2 pe_N^P p.$$

We first compute λ:

$$\mathrm{tr}_P(p) = \lambda^{-1}\,\mathrm{tr}_{P_1}(qe_{N_1}^N e_N^P)[P : N] = \lambda^{-1}[P : N]^{-1}.$$

Since $me_{N_1}^N = [P : N]E_P^{P_1}(qe_{N_1}^N e_N^P e_{N_1}^N) = \mathrm{tr}_{P_1}(q)e_{N_1}^N$, we get that $pe_{N_1}^N = e_{N_1}^N p$ is a projection. But $pe_{N_1}^N = \lambda^{-1}\,\mathrm{tr}_{P_1}(q)e_{N_1}^N$, thus $\lambda = \mathrm{tr}_{P_1}(q)$.

We show now that $pe_N^P p = \mathrm{tr}_P(p)pq$. It is easy to see that $E_N^P(m) = [P : N]^{-1}1_N$, hence $E_N^P(p) = \mathrm{tr}_{P_1}(q)^{-1}[P : N]^{-1}1_N = \mathrm{tr}_P(p)$. Thus

$$\begin{aligned}
&\|pe_N^P p - \mathrm{tr}_P(p)pq\|_2^2 \\
&\quad = \mathrm{tr}_{P_1}(e_N^P pe_N^P p) - 2\,\mathrm{tr}_P(p)\,\mathrm{tr}_{P_1}(qpe_N^P p) + \mathrm{tr}_P(p)^3\,\mathrm{tr}_{P_1}(q) \\
&\quad = \mathrm{tr}_P(p)^2[P : N]^{-1} - 2\,\mathrm{tr}_P(p)\,\mathrm{tr}_{P_1}(e_N^P p) + \mathrm{tr}_P(p)^2[P : N]^{-1} = 0.
\end{aligned}$$

This implies that $qe_{N_1}^N q = [P : N]\,\mathrm{tr}_{P_1}(q)^2\,\mathrm{tr}_P(p)pq = \mathrm{tr}_{P_1}(q)pq$. We have therefore $qPq = \overline{(\mathrm{span}\, NpN)}q$, in particular $qPq \subset Pq$. This allows us to define explicitly the desired conditional expectation. Let $x \in P$, $y \in P$ with $qxq = yq$, then $y = E(x) := \mathrm{tr}_{P_1}(q)^{-1}E_P^{P_1}(qxq)$. Suppose $x = x^* \in P$, then $yq = qxq = (qxq)^* = (yq)^* = qy$, which shows that $E(x) \in M$ for all $x \in P$. If $x \in M$, then $E(x) = \mathrm{tr}(q)^{-1}E_P^{P_1}(xq) = \mathrm{tr}(q)^{-1}xE_P^{P_1}(q) = x$. Furthermore, if we let tr_M

be the trace on M induced from the trace on P, then $\mathrm{tr}_M(E(x)) = \mathrm{tr}(q)^{-1}\mathrm{tr}_{P_1}(qxq) = \mathrm{tr}(q)^{-1}\mathrm{tr}(q)\mathrm{tr}_P(x) = \mathrm{tr}_P(x)$, i.e. E is indeed the unique trace preserving conditional expectation from P onto M with $qxq = E(x)q$, for all $x \in P$. Hence $qPq = Mq = \overline{(\mathrm{span}\,NpN)q}$, which implies $M = \overline{\mathrm{span}\,NpN}$. Note that factoriality of M does not follow automatically. Using the Pimsner-Popa estimate ([**PiPo1**]) it is now easy to see that $\lambda(M, N)^{-1} = \mathrm{tr}_P(p)^{-1}$ ($\lambda(M, N)$ denotes the generalized index for non-factors ([**PiPo1**])) and hence $\lambda(P, M)^{-1} = [P : N]\mathrm{tr}(p) = \mathrm{tr}_{P_1}(q)^{-1}$, which says $[P : M] = \mathrm{tr}_{P_1}(q)^{-1}$ if M is a factor. □

The following corollary gives the desired abstract characterization of Jones projections coming from intermediate subfactors of an irreducible inclusion $N \subset P$.

COROLLARY 3.3. *Let $N \subset P$ be II_1 factors with $[P : N] < \infty$ and suppose $N' \cap P = \mathbb{C}$. Then $\mathrm{IS}(N, P)$ is precisely the set of Jones projections coming from intermediate subfactors $N \subset M \subset P$ and gives therefore a complete description of the intermediate subfactors of $N \subset P$.*

Proof. Apply Proposition 3.1 and Theorem 3.3. □

REMARK 3.4. (1) Property (3) used in the definition of the set $\mathrm{IS}(N, P)$ can be replaced by the following condition: (3)′ $qPq \subset Pq$. One can then show that (1), (2), (3)′ are equivalent to (1), (2) and (3), thus giving an alternative definition of the set $\mathrm{IS}(N, P)$.

(2) Conditions (1)–(3) in the definition of $\mathrm{IS}(N, P)$ do not insure factoriality of the intermediate subalgebra M obtained from $q \in \mathrm{IS}(N, P)$ in general. Of course, if $N \subset P$ is irreducible, then all intermediate subalgebras are factors. Condition (2) will imply factoriality in many cases: if M is not a factor, then $E_{P}^{P_1}(e_M^P) = E_{M}^{P_1}(e_M^P)$ is a central element, i.e. of the form $E_{P}^{P_1}(e_M^P) = \sum_{i=1}^{r} \alpha_i p_i$, where $Z(M) = \bigoplus_{i=1}^{r} \mathbb{C}p_i$, $\sum_{i=1}^{r} p_i = 1_P = 1_M$. Then $E_{P}^{P_1}(e_M^P)$ will be a scalar iff $\alpha_i \equiv$ const., $1 \le i \le r$. Whether this happens or not will depend on the traces of the minimal central projections p_i. For example, if $N \subset M \subset P$ is an intermediate subfactor, then $M \vee (M' \cap P)$ is an intermediate subalgebra, which will not be a factor in general, however the Jones projection corresponding to it may a priori be in $\mathrm{IS}(N, P)$. Conversely, if M is an intermediate subalgebra (not necessarily of factor) of $N \subset P$, then it is not clear whether (3) holds in general or not.

(3) Since we are interested mainly in irreducible subfactors, the corollary gives the desired description of intermediate subfactors in terms of information just coming from $N \subset P$. Furthermore, the Jones projections coming from all intermediate *factors* are contained in the set $\mathrm{IS}(N, P)$, which will be enough information in many concrete examples.

Subfactors $N \subset P$ with intermediate subfactors as in the corollary are of course easily obtained from group actions, i.e. $N := R \subset P := R \rtimes G$, G a finite (for instance non-simple) group acting properly outer on the hyperfinite II_1 factor R. We define

DEFINITION 3.5. Let $N \subset P$ be II_1 factors, $[P : N] < \infty$, then the inclusion $N \subset P$ is called *maximal* if there is no subfactor M of P such that $N \subset M \subset P$ other than N and P themselves. Equivalently, $N \subset P$, $N' \cap P = \mathbb{C}$, is maximal iff $\mathrm{IS}(N, P) = \{1, e_N^P\}$.

Note that clearly $R \subset R \rtimes G$ is maximal iff the group G has only the trivial group as a subgroup. Since an inclusion $N \subset P$ can only be non-maximal if the index is a product of two indices, we see that all inclusions of index < 4 and those with index $\in (4, 8\cos^2 \frac{\pi}{5})$ are certainly maximal. We gave above examples of non-maximal inclusions at index 4, index $8\cos^2 \frac{\pi}{5}$ and index 6.

REFERENCES

[Bi1] D. Bisch, *On the existence of central sequences in subfactors*, Trans. Amer. Math. Soc., **321** (1990), 117–128.

[Bi2] ——, *Subfactors arising from groups and their representations*, unpublished manuscript.

[B-N] J. Bion-Nadal, *An example of a subfactor of the hyperfinite II_1 factor whose principal graph invariant is the Coxeter graph E_6*, Current Topics in Operator Algebras, World Scientific Publishing, 1991, pp. 104–113.

[GHJ] F. Goodman, P. de la Harpe and V. F. R. Jones, *Coxeter graphs and towers of algebras*, MSRI publications 14, Springer, 1989.

[Ha] U. Haagerup, in preparation.

[I1] M. Izumi, *Application of fusion rules to classification of subfactors*, Publ. RIMS Kyoto Univ., **27** (1991), 953–994.

[I2] ——, *On flatness of the Coxeter graph E_8*, preprint 1992.

[Jo1] V. F. R. Jones, *Index of subfactors*, Invent. Math., **72** (1983), 1–25.

[Jo2] ——, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math., **126** (1987), 335–388.

[Jo3] ——, *A converse to Ocneanu's theorem*, J. Operator Theory, **10** (1983), 61–63.

[Ka] Y. Kawahigashi, *On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors*, preprint, 1990 .

[KY] H. Kosaki and S. Yamagami, *Irreducible bimodules associated with crossed product algebras*, preprint, 1992.

[NT] N. Nakamura and Z. Takeda, *On some elementary properties of the crossed products of von Neumann algebras*, Proc. Japan Acad., **34** (1958), 489–494.

[Oc1] A. Ocneanu, *Quantized group string algebras and Galois theory for operator algebras*, in Operator Algebras and Applications 2, London Math. Soc. Lect. Notes Series, **136** (1988), 119–172.

[Oc2] ——, *Quantum symmetry, differential geometry of finite graphs and classification of subfactors*, University of Tokyo Seminar Notes 45, 1991 (Notes recorded by Y. Kawahigashi).

[PiPo1] M. Pimsner and S. Popa, *Entropy and index of subfactors*, Ann. Scient. Ec. Norm. Sup., **19** (1986), 57–106.

[PiPo2] ——, *Iterating the basic construction*, Trans. Amer. Math. Soc., **310** (1988), 127–133.

[Po1] S. Popa, *Sousfacteurs, actions des groupes et cohomologie*, C. R. Acad. Sci. Paris, Série I, **309** (1989), 771–776.

[Po2] ——, *Classification of subfactors: reduction to commuting squares*, Invent. Math., **101** (1990), 19–43.

[Po3] ——, *Sur la classification des sous-facteurs d'indice fini du facteur hyperfini*, C. R. Acad. Sci. Paris, Série I, **311** (1990), 95–100.

[Po4] ——, *Classification of amenable subfactors and their automorphisms*, preprint, 1991.

[SV] V. S. Sunder and A. K. Vijayaran, *On the non-occurrence of the Coxeter graphs E_7, D_{2n+1} as principal graphs of an inclusion of* II_1 *factors*, preprint, 1991.

[Su] N. Suzuki, *Crossed products of rings of operators*, Tohoku Math. J., **11** (1959), 113–124.

[We1] H. Wenzl, *Hecke algebras of type A and subfactors*, Invent. Math., **92** (1988), 345–383.

[We2] ——, *Quantum groups and subfactors of type B, C and D*, Comm. Math. Phys., **133** (1990), 383–432.

Received April 15, 1992 and in revised form August 16, 1992. Supported by an MSRI postdoctoral fellowship through NSF grant DMS 8505550.

MATHEMATICAL SCIENCES RESEARCH INSTITUTE
1000 CENTENNIAL DRIVE
BERKELEY, CA 94720

Current address: Department of Mathematics
University of California, Berkeley
Berkeley, CA 94720

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

TENT SPACES OVER GENERAL APPROACH REGIONS AND POINTWISE ESTIMATES

MARÍA J. CARRO AND JAVIER SORIA

We consider the study of the tent spaces over general (possibly tangential) approach regions and their atomic decomposition. As a consequence, we obtain some pointwise estimates for a class of operators, using the duality properties of a certain type of Carleson measures. In particular, we can get the boundedness of a family of bilinear operators defined on the product of L^q and some space of measures, into a Lipschitz space; we give yet another proof of the pointwise boundedness for the Fourier transform of distributions in H^p and we improve and generalize the Féjer-Riesz inequality for harmonic extensions of H^p functions.

Several authors have studied the boundedness of maximal operators defined by means of general subsets. For example, in **[8]**, a Hardy-Littlewood type operator is associated with a collection of subsets $\Omega_x \subset \mathbf{R}_+^{n+1}$, $x \in \mathbf{R}^n$. The natural way to define the balls for these sets is to take the subset of Ω_x at level t, that is, the set of points $z \in \mathbf{R}^n$ so that $(z, t) \in \Omega_x$. Our idea is to also replace the cone $\Gamma(x) = \{(y, t) \in \mathbf{R}_+^{n+1} : |x-y| < t\}$ in the definition of the tent spaces (see **[2]**), by a more general family of subsets of $\mathbf{R}_+^{n+1}$. As an application, we look at a family of integral operators (e.g. the Fourier transform) as the action of continuous linear forms, and using the duality established between certain spaces, we obtain pointwise estimates that will allow us to give another proof of well-known bounds for the Fourier transform of H^p functions (see **[4]**, **[12]**). We can also improve the Féjer-Riesz inequality for harmonic extensions (see **[5]**) and we find a generalization considering Hardy spaces defined in terms of arbitrary kernels (see **[14]**). Our main tool will be given by the properties that the tent spaces satisfy (see **[2]**, **[1]**, **[10]**), and in particular their relation with a class of Carleson measures, for which we find a suitable atomic decomposition. We begin by giving some basic definitions.

DEFINITION 1. Let $\Omega = \{\Omega_x\}_{x \in \mathbf{R}^n}$ be a collection of measurable subsets, where $\Omega_x \subset \mathbf{R}_+^{n+1}$. For a measurable function f in $\mathbf{R}_+^{n+1}$ we define the maximal function of f with respect to Ω as

$$A_{\Omega}^{\infty}(f)(x) = \sup_{(y,t)\in\Omega_x} |f(y,t)|.$$

We will always assume that Ω is chosen so that $A_{\Omega}^{\infty}(f)$ is a measurable function. We also define

$$T_{\Omega}^{p} = T_{\infty,\Omega}^{p} = \{f : A_{\Omega}^{\infty}(f) \in L^p(\mathbf{R}^n)\},$$

with $\|f\|_{T_{\Omega}^{p}} = \|A_{\Omega}^{\infty}(f)\|_{L^p(\mathbf{R}^n)}$.

REMARK 2. It is clear that if $\Omega_x = \Gamma(x)$ then T_{Ω}^{p} is precisely the tent space T_{∞}^{p} of [2]. If $\Omega_x = \{(x,t) : t > 0\}$ then $A_{\Omega}^{\infty}(f)$ is the radial maximal function of f.

DEFINITION 3. Suppose $\Omega = \{\Omega_x\}_{x\in\mathbf{R}^n}$ is as above and F is any subset of $\mathbf{R}^n$. We define the tent over F, with respect to Ω, as

$$\widehat{F_{\Omega}} = \mathbf{R}_{+}^{n+1} \setminus \bigcup_{x\notin F} \Omega_x.$$

We also set $\Omega_x(t) = \{y \in \mathbf{R}^n : (y,t) \in \Omega_x\}$.

For a measure μ in $\mathbf{R}_{+}^{n+1}$ we say that μ is an (Ω, β)-Carleson measure $(\beta \geq 1)$ and write $\mu \in V_{\Omega}^{\beta}$ if

$$\|\mu\|_{V_{\Omega}^{\beta}} = \sup_{Q} \frac{|\mu|(\widehat{Q_{\Omega}})}{|Q|^{\beta}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$.

REMARK 4. If $\Omega_x = \Gamma(x)$ then $\widehat{F_{\Omega}} = \widehat{F}$, the usual tent over F. If we choose $\Omega_x = \{(x,t) : t > 0\}$ then $\widehat{F_{\Omega}} = F \times \mathbf{R}^{+}$ and it is denoted by $C(F)$.

LEMMA 5. *Suppose* $F \subset \mathbf{R}^n$ *and* $\Omega = \{\Omega_x\}_{x\in\mathbf{R}^n}$ *are as above. Then*

(i) $A_{\Omega}^{\infty}(\chi_{\widehat{F_{\Omega}}})(x) \leq \chi_F(x)$ *for all* $x \in \mathbf{R}^n$.

(ii) $A_{\Omega}^{\infty}(\chi_{\widehat{F_{\Omega}}})(x) = \chi_F(x)$ *if and only if* $\Omega_x \cap \widehat{F_{\Omega}} \neq \varnothing$ *for all* $x \in F$.

(iii) *If* Ω *is a symmetric family* (*that is, if* $x \in \Omega_y(t)$ *then* $y \in \Omega_x(t)$), *we have that*

$$\widehat{F_{\Omega}} = \{(y,t) \in \mathbf{R}_{+}^{n+1} : \Omega_y(t) \subset F\}.$$

In particular if $\Omega_x = x + \Omega$, *for a fixed* $\Omega \subset \mathbf{R}_{+}^{n+1}$, *the symmetric condition holds if and only if* $\Omega(t) = -\Omega(t)$, *for all* $t > 0$.

Proof. (i) Observe that

$$(1)\qquad \chi_{\widehat{F_\Omega}}(y, t) = \begin{cases} 1, & \text{if } (y, t) \notin \Omega_z, \text{ for all } z \notin F, \\ 0, & \text{otherwise}. \end{cases}$$

Suppose $x \notin F$. Then if $(y, t) \in \Omega_x$ we have that $\chi_{\widehat{F_\Omega}}(y, t) = 0$ (by (1)), and this shows (i).

(ii) $A_\Omega^\infty(\chi_{\widehat{F_\Omega}})(x) = \chi_F(x)$ if and only if for all $x \in F$, $A_\Omega^\infty(\chi_{\widehat{F_\Omega}})(x) = 1$ if and only if there exists $(y, t) \in \Omega_x$ such that $(y, t) \in \widehat{F_\Omega}$ if and only if $\Omega_x \cap \widehat{F_\Omega} \neq \varnothing$.

(iii) That $(y, t) \in \widehat{F_\Omega}$ means that $y \notin \Omega_x(t)$, for all $x \notin F$, which, by symmetry, is equivalent to saying that for all $x \notin F$, $x \notin \Omega_y(t)$; that is, $\Omega_y(t) \subset F$. □

A simple example of a symmetric family of sets of the form $x + \Omega$ can be found in the comments previous to Lemma 11. Another example, for a general family of sets $\{\Omega_x\}$, is given by defining $\Omega_n(t) = (-n, -n+1)$, if $n \in \mathbf{Z}$, and $\Omega_x(t) = (-n-1, -n+1)$, if $n < x < n+1$.

DEFINITION 6. We say that a measurable function $a\colon \mathbf{R}_+^{n+1} \to \mathbf{C}$ is an (Ω, p)-atom if there exists a cube $Q \subset \mathbf{R}^n$ such that $\operatorname{supp} a \subset \widehat{Q_\Omega}$, and $\|a\|_\infty \le |Q|^{-1/p}$.

We now give the proof of the atomic decomposition for the tent space T_Ω^p. We restrict ourselves to the case $n = 1$, but a similar proof also works in any other dimension. A related result is given in **[6]**.

THEOREM 7. *If $\Omega = \{\Omega_x\}_{x \in \mathbf{R}}$ is a symmetric family of sets (as in Lemma 5-(iii)), such that $\Omega_x(t)$ is an interval, for all $(x, t) \in \mathbf{R}_+^2$, then, for $0 < p \le 1$, $f \in T_\Omega^p$ if and only if*

$$(2)\qquad f \equiv \sum_j \lambda_j a_j,$$

where a_j is an (Ω, p)-atom and $\sum_j |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{T_\Omega^p} \approx \inf\left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} \right\},$$

where the infimum is taken over all sequences satisfying (2).

Proof. We first show the easy part, for which we will not make use of the extra hypotheses on Ω. The only thing to observe is that $\|\cdot\|_{T_\Omega^p}$

is always a p-norm, for $0 < p \leq 1$ and hence, if $f \equiv \sum_j \lambda_j a_j$, then $\|f\|^p_{T^p_\Omega} \leq \sum_j |\lambda_j|^p \|a_j\|^p_{T^p_\Omega}$. But, by (i) of the previous lemma:

$$\begin{aligned}\|a_j\|^p_{T^p_\Omega} &= \int_{\mathbf{R}} (A^\infty_\Omega(a_j)(x))^p\, dx \\ &\leq \int_{\mathbf{R}} \|a_j\|^p_\infty (A^\infty_\Omega(\chi_{\widehat{Q_{j,\Omega}}})(x))^p\, dx \leq \|a_j\|^p_\infty \int_{\mathbf{R}} \chi_{Q_j}(x)\, dx \leq 1,\end{aligned}$$

and hence, $\|f\|^p_{T^p_\Omega} \leq \sum_j |\lambda_j|^p$.

For the converse we need the following observation: if $f \in T^p_\Omega$ and $\lambda > 0$ then $\{x \in \mathbf{R}: A^\infty_\Omega(f)(x) > \lambda\}$ is an open set. In fact, if $A^\infty_\Omega(f)(x) > \lambda$, then there exists a point $(z, t) \in \Omega_x$ so that $|f(z, t)| > \lambda$. By hypotheses, we conclude that $x \in \Omega_z(t)$ and there exists an $\varepsilon > 0$ such that if $|x - y| < \varepsilon$ then $y \in \Omega_z(t)$. Again, by symmetry, $(z, t) \in \Omega_y$ and so $A^\infty_\Omega(f)(y) > \lambda$ if $|x - y| < \varepsilon$. Set now $M_k = \{x \in \mathbf{R} : A^\infty_\Omega(f)(x) > 2^k\}$, and write $M_k = \bigcup_{j\in\mathbf{Z}} I^k_j$, where I^k_j is an open interval and $I^k_j \cap I^k_{j'} = \varnothing$ if $j \neq j'$. Since $f \in T^p_\Omega$, I^k_j is bounded for all $j, k \in \mathbf{Z}$. Set

$$a_{j,k} \equiv \lambda^{-1}_{j,k} f\left(\chi_{\widehat{I^k_{j,\Omega}}} - \sum_{I^{k+1}_l \subset I^k_j} \chi_{\widehat{I^{k+1}_{l,\Omega}}}\right),$$

where $\lambda_{j,k} = 2^{k+1}|I^k_j|^{1/p}$. It is clear that $\operatorname{supp} a_{j,k} \subset \widehat{I^k_{j,\Omega}}$ and

$$\sum_{j,k} |\lambda_{j,k}|^p = \sum_k 2^{p(k+1)}|M_k| \leq C\|f\|^p_{T^p_\Omega} < \infty,$$

and so it remains to show that $f \equiv \sum_{j,k} \lambda_{j,k} a_{j,k}$ and $\|a_{j,k}\|_\infty \leq |I^k_j|^{-1/p}$. Let $(x, t) \in \widehat{I^k_{j,\Omega}}$ and suppose $|f(x, t)| > 2^{k+1}$. Let $y \in \Omega_x(t)$. Then $(x, t) \in \Omega_y$ and hence $y \in M_{k+1}$. Therefore $\Omega_x(t) \subset M_{k+1}$ and there exists a unique $l \in \mathbf{Z}$ so that $\Omega_x(t) \subset I^{k+1}_l$. Since $\Omega_x(t) \subset I^k_j$ then $I^{k+1}_l \subset I^k_j$. But if $I^{k+1}_{l'} \subset I^k_j$ and $l \neq l'$ then $\widehat{I^{k+1}_{l,\Omega}} \cap \widehat{I^{k+1}_{l',\Omega}} \neq \varnothing$. In fact, if $(z, s) \in \widehat{I^{k+1}_{l,\Omega}} \cap \widehat{I^{k+1}_{l',\Omega}}$ then $\Omega_z(s) \subset I^{k+1}_l \cap I^{k+1}_{l'}$, which is a contradiction. Thus,

$$\chi_{\widehat{I^k_{j,\Omega}}}(x, t) - \sum_{I^{k+1}_r \subset I^k_j} \chi_{\widehat{I^{k+1}_{r,\Omega}}}(x, t) = 0.$$

Therefore, for all $(x, t) \in \widehat{I^k_{j,\Omega}}$,

$$|a_{j,k}(x, t)| \leq 2^{-(k+1)}|I^k_j|^{-1/p} 2^{k+1} = |I^k_j|^{-1/p}.$$

Finally, if $(x, t) \in \mathbf{R}^2_+$ and $2^l < |f(x, t)| \leq 2^{l+1}$ then $\Omega_x(t) \subset M_l$. Let $K \in \mathbf{Z}$ be the greatest integer satisfying $\Omega_x(t) \subset M_K$ (it is clear that we can find such a number since $A^\infty_\Omega(f)(x) < \infty$, a.e. $x \in \mathbf{R}$). Let $s \in \mathbf{Z}$ so that $\Omega_x(t) \subset I^K_s$. We want to show that if

$$g_{j,k}(x, t) = \chi_{\widehat{I^k_{j,\Omega}}}(x, t) - \sum_{I^{k+1}_r \subset I^k_j} \chi_{\widehat{I^{k+1}_{r,\Omega}}}(x, t),$$

then $\sum_{j,k} g_{j,k}(x, t) = 1$. If $\Omega_x(t) \subset I^k_j$ then $k \leq K$. Suppose that $k < K$ and $(x, t) \in \widehat{I^k_{j,\Omega}}$, then $I^K_s \subset I^{k+1}_r \subset I^k_j$ for some $r \in \mathbf{Z}$ and hence $g_{j,k}(x, t) = 0$. If $(x, t) \in \widehat{I^K_{j,\Omega}}$ then clearly $j = s$ and $g_{K,s}(x, t) = 1$. □

We observe that in the previous proof, we obtained the atomic decomposition for all $0 < p < \infty$. An immediate application of this theorem is given by the following duality result. We first recall that for the case when Ω_x is the cone $\Gamma(x)$, it was proved in [2] and [1] that the space of Carleson measures of order $1/p$ $(0 < p \leq 1)$ could be identified as the dual of the tent space T^p_∞ (see Theorem 16). For the general case we are considering, we restrict our study only to the inclusion needed in order to obtain the estimates we mention below.

THEOREM 8. *Suppose Ω is a family of sets satisfying the hypotheses of the previous theorem and $0 < p \leq 1$. Then, for all $f \in T^p_\Omega$ and $\mu \in V^{1/p}_\Omega$,*

$$\left| \int_{\mathbf{R}^2_+} f(x, t)\, d\mu(x, t) \right| \leq \|f\|_{T^p_\Omega} \|\mu\|_{V^{1/p}_\Omega}.$$

That is, $V^{1/p}_\Omega \hookrightarrow (T^p_\Omega)^*$.

Proof. Let $f \in T^p_\Omega$ and $\mu \in V^{1/p}_\Omega$, and write $f \equiv \sum_j \lambda_j a_j$, as in Theorem 7. Then,

$$\begin{aligned}
\left| \int_{\mathbf{R}^2_+} f(x, t)\, d\mu(x, t) \right| &\leq \sum_j |\lambda_j| \int_{\widehat{I_{j,\Omega}}} |a_j(x, t)|\, d|\mu|(x, t) \\
&\leq \sum_j |\lambda_j|\, \|a_j\|_\infty |\mu|(\widehat{I_{j,\Omega}}) \leq \sum_j |\lambda_j|\, |I_j|^{-1/p} \|\mu\|_{V^{1/p}_\Omega} |I_j|^{1/p} \\
&\leq \left(\sum_j |\lambda_j|^p \right)^{1/p} \|\mu\|_{V^{1/p}_\Omega}.
\end{aligned}$$

□

REMARK 9. (i) In the proof of the previous theorem, if $p = 1$, we can give a direct argument without using the atomic decomposition. In fact, if $f \in T^1_\Omega$ and if we consider the set $F^\lambda = \{y \in \mathbf{R} : A^\infty_\Omega(f)(y) > \lambda\}$, then

$$\{(x, t) \in \mathbf{R}^2_+ : |f(x, t)| > \lambda\} \subset \widehat{F^\lambda_\Omega}. \tag{3}$$

In fact, if $|f(x, t)| > \lambda$, $A^\infty_\Omega(f)(z) \le \lambda$, implies that $(x, t) \notin \Omega_z$ and, hence,

$$(x, t) \in \mathbf{R}^2_+ \setminus \left(\bigcup_{z \notin F^\lambda} \Omega_z \right) = \widehat{F^\lambda_\Omega}.$$

As we saw before, F^λ is an open set and hence $F^\lambda = \bigcup_j I_j$. Moreover, by symmetry, $\widehat{F^\lambda_\Omega} \subset \bigcup_j \widehat{I_{j,\Omega}}$, and hence, for $\mu \in V^1_\Omega$, we have

$$\begin{aligned}
&\left| \int_{\mathbf{R}^2_+} f(x, t)\, d\mu(x, t) \right| \\
&\quad \le \int_0^\infty |\mu|(\{(x, t) \in \mathbf{R}^2_+ : |f(x, t)| > \lambda\})\, d\lambda \qquad \text{(by (3))} \\
&\quad \le \int_0^\infty |\mu|(\widehat{F^\lambda_\Omega})\, d\lambda \le \sum_j \int_0^\infty |\mu|(\widehat{I_{j,\Omega}})\, d\lambda \\
&\quad \le \|\mu\|_{V^1_\Omega} \int_0^\infty \left| \bigcup_j I_j \right| d\lambda = \|\mu\|_{V^1_\Omega} \|f\|_{T^1_\Omega}.
\end{aligned}$$

(ii) If Ω satisfies that for every compact $K \subset \mathbf{R}^2_+$, the set $\{x \in \mathbf{R} : \Omega_x \cap K \neq \varnothing\}$ has finite measure, then using the ideas of **[2]**, it is easy to show that in fact equality holds; namely $V^{1/p}_\Omega = (T^p_\Omega)^*$. We do not know what happens in the general case.

As was proved in **[4]** the non-tangential maximal function and the radial maximal function of Poisson integrals of functions (distributions) in the Hardy space $H^p(\mathbf{R^n})$ have an equivalent L^p-"norm", $p > 0$. This leads us to consider how this result could be extended for all functions in the tent spaces T^p_∞ relative to both cones $\Gamma(x)$ and lines $\{(x, t) : t > 0\}$. From the point of view of the dual spaces we see that the latter is a much bigger space than the former. We give the details in what follows.

EXAMPLE 10. If $\Omega_x = \{(x, t) : t > 0\}$ then $\widehat{O_\Omega} = C(O) = O \times \mathbf{R}^+$. Let us denote $V^\alpha_{\mathrm{rad}} = V^\alpha_\Omega$, where Ω_x is the vertical line above x.

First suppose that $0 < \alpha \leq 1$, $f \in L^{1/(1-\alpha)}(\mathbf{R}^n)$ and σ is a positive finite measure in $\mathbf{R}^+$. Then

$$d\mu(x, t) = f(x)\, dx\, d\sigma(t) \in V^{\alpha}_{\mathrm{rad}}.$$

In fact, if $O \subset \mathbf{R}^n$ then

$$\begin{aligned}\left|\int_{C(O)} d\mu(x, t)\right| &\leq \left(\int_O |f(x, t)|\, dx\right)\left(\int_0^\infty d\sigma(t)\right)\\ &\leq \|\sigma\|\, \|f\|_{L^{1/(1-\alpha)}} |O|^{\alpha}.\end{aligned}$$

An example of a measure that is in V^α but not in $V^{\alpha}_{\mathrm{rad}}$ is the Dirac mass at the point $(x_0, t_0) \in \mathbf{R}^{n+1}_+$. This follows by considering a collection of cubes converging to x_0.

However, for the case $\alpha > 1$ we get that

$$V^{\alpha}_{\mathrm{rad}} = \{0\}.$$

To show this fix a cube $Q \subset \mathbf{R}^n$ and $N \in \mathbf{Z}^+$. Decompose Q in 2^{nN} subcubes Q_i such that $\mathring{Q}_i \cap \mathring{Q}_j = \varnothing$, $i \neq j$, $Q = \bigcup_i Q_i$ and $|Q_i| = |Q|/2^{nN}$. Now, if $\mu \in V^{\alpha}_{\mathrm{rad}}$ we have

$$\begin{aligned}|\mu|(C(Q)) &\leq |\mu|\left(\bigcup_i C(Q_i)\right) \leq \sum_i |\mu|(C(Q_i)) \leq C_\mu \sum_i |Q_i|^\alpha\\ &= C_\mu \sum_{i=1}^{2^{nN}} \frac{|Q|^\alpha}{2^{\alpha n N}} = C_\mu |Q|^\alpha 2^{nN(1-\alpha)} \to 0 \quad \text{as } N \to \infty.\end{aligned}$$

Hence $\mu \equiv 0$.

Our first application of the duality result, deals with pointwise estimates for the Fourier transform of functions satisfying an H^p-type condition. Consider an increasing function $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$, ψ a C^1 change of variables. Define the sets $\Omega_x = \{(y, t) \in \mathbf{R}^2_+ : |x - y| < \psi(t)\}$. It is clear that Ω_x satisfies the hypotheses of Theorem 7. Observe that

$$\widehat{I_\Omega} = \{(y, t) \in \mathbf{R}^2_+ : d(y, \mathbf{R}\backslash I) \geq \psi(t)\}. \tag{4}$$

We say that a function f belongs to H^p_Ω if $PI(f)(x, t) = P_t * f(x)$ belongs to the space T^p_Ω, where P is the Poisson kernel in $\mathbf{R}$.

LEMMA 11. *Let ψ and Ω be as before, and suppose $0 < p < 1$. Consider the function $\varphi(t) = \psi^{1/p-2}(t)\psi'(t)$. Then, if $g \in L^\infty$ and*

$d\mu(y,t) = g(y)\varphi(t)\,dy\,dt$, we have that $\mu \in V_\Omega^{1/p}$ and $\|\mu\|_{V_\Omega^{1/p}} \le C_p\|g\|_\infty$.

Proof. Let $I = (a, b)$. Then, by (4):

$$\begin{aligned}|\mu|(\widehat{I_\Omega}) &\le \int_a^b \int_0^{\psi^{-1}(d(y,\mathbf{R}\setminus I))} |g(y)|\varphi(t)\,dt\,dy \\ &\le \|g\|_\infty \left(\int_a^{(a+b)/2} \int_0^{\psi^{-1}(y-a)} \psi^{1/p-2}(t)\psi'(t)\,dt\,dy \right. \\ &\qquad \left. + \int_{(a+b)/2}^b \int_0^{\psi^{-1}(b-y)} \psi^{1/p-2}(t)\psi'(t)\,dt\,dy \right).\end{aligned}$$

But,

$$\int_0^{\psi^{-1}(r)} \psi^{1/p-2}(t)\psi'(t)\,dt = \frac{p}{1-p} r^{1/p-1},$$

and hence,

$$\begin{aligned}|\mu|(\widehat{I_\Omega}) &\le C_p\|g\|_\infty \left(\int_a^{(a+b)/2} (y-a)^{1/p-1}\,dy \right. \\ &\qquad \left. + \int_{(a+b)/2}^b (b-y)^{1/p-1}\,dy \right) \\ &\le C_p\|g\|_\infty (b-a)^{1/p}. \qquad \square\end{aligned}$$

PROPOSITION 12. *Suppose ψ, Ω, φ and $0 < p < 1$ are as in the previous lemma. Then, for $f \in H_\Omega^p$,*

$$|\hat{f}(x)| \le C_p\|f\|_{H_\Omega^p} \left(\int_0^\infty e^{-2\pi|x|t}\varphi(t)\,dt \right)^{-1}.$$

Proof. Fix $0 < \varepsilon < 1$ and set $\varphi_\varepsilon(t) = \varphi(t)\chi_{(\varepsilon, 1/\varepsilon)}(t)$. If we define $d\mu_\varepsilon(y,t) = e^{-ixy}\varphi_\varepsilon(t)\,dy\,dt$, by Lemma 11, we have that $\|\mu_\varepsilon\|_{V_\Omega^{1/p}} \le C_p$. Now, if $f \in H_\Omega^p$ then $P_t * f \in T_\Omega^p$, and by Theorem 8,

$$\left| \int_{\mathbf{R}_+^2} P_t * f(y)\,d\mu_\varepsilon(y,t) \right| \le C_p\|f\|_{H_\Omega^p}.$$

But,

$$\left| \int_{\mathbf{R}_+^2} P_t * f(y)\,d\mu_\varepsilon(y,t) \right| = |\hat{f}(x)| \int_\varepsilon^{1/\varepsilon} e^{-2\pi|x|t}\varphi(t)\,dt. \qquad \square$$

EXAMPLE 13. (i) If $\psi(t) = t$ in the previous result, we get the classical estimate for the Fourier transform of functions in H^p:

$$|\hat{f}(x)| \le C_p |x|^{1/p-1}.$$

We will give more details about this result in Corollary 20.

(ii) If for example $\psi(t) = e^t - 1$, so that Ω_x is a domain containing the cone $\Gamma(x)$, then $\varphi(t) = (e^t - 1)^{1/p-2} e^t$, and the integral $\int_0^\infty e^{-2\pi|x|t}\varphi(t)\,dt$ converges if and only if $|x| > (1-p)/(2\pi p)$. Hence, $\hat{f}(x) = 0$ if $|x| \le (1-p)/(2\pi p)$ and $f \in H^p_\Omega$. Therefore, since $f_r(x) = f(rx) \in H^p_\Omega$, if $f \in H^p_\Omega$, one finds that $\hat{f}(x) = 0$, for all $x \in \mathbf{R}$, and so $H^p_\Omega = 0$.

(iii) The above calculations show that, in fact, a necessary condition for H^p_Ω to be nontrivial is that the Laplace transform of φ,

$$\mathscr{L}\varphi(x) = \int_0^\infty e^{-xt}\varphi(t)\,dt < \infty,$$

for all $x \ne 0$, which, for example, happens if for all $s > 0$, there exists a constant $C_s > 0$ such that $\psi(t) \le C_s e^{st}$, for all $t > 0$.

We give now a characterization of the class of Carleson measures in terms of the boundedness of the mean operator. Some related questions can be found in [7] and [9]. Given a symmetric family Ω such that $\Omega_x(t)$ is an open interval and for all intervals $I \subset \mathbf{R}$ there exists $(x, t) \in \mathbf{R}^2_+$ with $\Omega_x(t) = I$ (these conditions hold if, for example, Ω is given by a function ψ as in Lemma 11), we define the following mean operator:

$$T_\Omega f(x, t) = \frac{1}{|\Omega_x(t)|}\int_{\Omega_x(t)} f(y)\,dy.$$

We extend the notion of Carleson measure to consider the case of weights simply by saying that the pair $(\mu, u) \in V^\alpha_\Omega$ if

$$|\mu|(\widehat{I_\Omega}) \le C(u(I))^\alpha, \tag{5}$$

where u is a positive and locally integrable function in $\mathbf{R}$ and $u(I) = \int_I u(x)\,dx$. Thus, in our previous notation, $\mu \in V^\alpha_\Omega$ means that $(\mu, 1) \in V^\alpha_\Omega$. Recall that A_p denotes the class of Muckenhoupt's weights (see [5]).

THEOREM 14. (i) *If $\alpha \ge 1$, $p > 0$ and $T_\Omega\colon L^p(\mathbf{R}, u) \to L^{\alpha p}(\mathbf{R}^2_+, d\mu)$ is a bounded operator, then $(\mu, u) \in V^\alpha_\Omega$, and $\|\mu\| \le \|T_\Omega\|^{\alpha p}$, where $\|\mu\|$ is the best constant in* (5).

(ii) *If* $u \in A_p$, $p > 1$ *and* $(\mu, u) \in V_\Omega^\alpha$, $\alpha \geq 1$, *then* $T_\Omega \colon L^p(\mathbf{R}, u) \to L^{\alpha p}(\mathbf{R}_+^2, d\mu)$ *is a bounded operator, and* $\|T_\Omega\| \leq C\|\mu\|^{1/(\alpha p)}$.

(iii) *Fix* $1 < p < \infty$. *Then,* $\mu \in V_\Omega^\alpha$ *if and only if* $T_\Omega \colon L^p(\mathbf{R}) \to L^{\alpha p}(\mathbf{R}_+^2, d\mu)$ *is a bounded operator.*

(iv) *Let* $\delta_{(x_0, t_0)}$ *denote the Dirac delta at* $(x_0, t_0) \in \mathbf{R}_+^2$. *Then the operator* $T_\Omega \colon L^p(\mathbf{R}, u) \to L^p(\mathbf{R}_+^2, \delta_{(x_0, t_0)})$ *is bounded, for all* $(x_0, t_0) \in \mathbf{R}_+^2$, *and* $\|T_\Omega\| \leq C_p(u(\Omega_{x_0}(t_0)))^{-1/p}$, *if and only if* $u \in A_p$.

Proof. (i) Evaluate $T_\Omega f$, if $f = \chi_I$, to get

$$T_\Omega \chi_I(x, t) = \frac{|\Omega_x(t) \cap I|}{|\Omega_x(t)|} \geq \chi_{\widehat{I_\Omega}}(x, t),$$

and hence,

$$\mu(\widehat{I_\Omega})^{1/(\alpha p)} \leq \|T_\Omega \chi_I\|_{L^{\alpha p}(d\mu)} \leq \|T_\Omega\| \, \|\chi_I\|_{L^p(u)} = \|T_\Omega\| u(I)^{1/p}.$$

(ii) As we saw in Remark 9, if $F^t = \{y \in \mathbf{R} : A_\Omega^\infty(T_\Omega f)(y) > t\}$, then

$$\{(x, s) \in \mathbf{R}_+^2 : T_\Omega f(x, s) > t\} \subset \widehat{F_\Omega^t}.$$

If M denotes the Hardy-Littlewood maximal function, it is clear that by symmetry, $A_\Omega^\infty f(y) \leq Mf(y)$, and hence,

$$\begin{aligned}\mu(\{(x, s) \in \mathbf{R}_+^2 : T_\Omega f(x, s) > t\}) &\leq \mu(\widehat{F_\Omega^t}) \\ &\leq \|\mu\|(u(F^t))^\alpha \leq \|\mu\|(u(\{Mf > t\}))^\alpha.\end{aligned}$$

Using now that $L^p(u) \subset L^{p, \alpha p}(u)$, the classical Lorentz space,

$$\begin{aligned}&\|T_\Omega f\|_{L^{\alpha p}(d\mu)} \\ &\quad\leq C\left(\int_0^\infty t^{\alpha p - 1}\mu(\{(x, s) \in \mathbf{R}_+^2 : T_\Omega f(x, s) > t\})\, dt\right)^{1/(\alpha p)} \\ &\quad\leq C\|\mu\|^{1/(\alpha p)}\left(\int_0^\infty t^{\alpha p - 1}(u(\{Mf > t\}))^\alpha\, dt\right)^{1/(\alpha p)} \\ &\quad= C\|\mu\|^{1/(\alpha p)}\|Mf\|_{L^{p, \alpha p}(u)} \leq C\|\mu\|^{1/(\alpha p)}\|f\|_{L^p(u)}.\end{aligned}$$

(iii) It is a trivial consequence of (i) and (ii).

(iv) We first observe that for all $u \in L^1_{\text{loc}}$, $(\delta, u) \in V_\Omega^\alpha$, and $\|\delta\| \leq (u(\Omega_{x_0}(t_0)))^{-\alpha}$. Hence, if $u \in A_p$, we get the boundedness of T_Ω, by (ii). Conversely, if $f \in L^p(u)$,

$$\begin{aligned}\|T_\Omega f\|_{L^p(\delta)} &= \frac{1}{|\Omega_{x_0}(t_0)|}\int_{\Omega_{x_0}(t_0)} u^{-1}(x) f(x) u(x)\, dx \\ &\leq C(u(\Omega_{x_0}(t_0)))^{-1/p}\|f\|_{L^p(u)}.\end{aligned}$$

Taking the supremum when $\|f\|_{L^p(u)} \le 1$,

$$\frac{1}{|\Omega_{x_0}(t_0)|}\left(\int_{\Omega_{x_0}(t_0)} u^{-p'+1}(x)\,dx\right)^{1/p'} \le \left(\int_{\Omega_{x_0}(t_0)} u(x)\,dx\right)^{-1/p}.$$

Hence,

$$\left(\frac{1}{|\Omega_{x_0}(t_0)|}\int_{\Omega_{x_0}(t_0)} u(x)\,dx\right)\left(\frac{1}{|\Omega_{x_0}(t_0)|}\int_{\Omega_{x_0}(t_0)} u^{-p'+1}(x)\,dx\right)^{p-1} \le C,$$

and by the hypotheses on Ω, this implies $u \in A_p$. □

We consider now the usual case when Ω_x is a cone, to obtain some results in the classical theory of Hardy spaces.

DEFINITION 15. Suppose σ is a Borel measure in $\mathbf{R}^+$. We say that σ is a measure of order β, with $\beta \ge 0$, if there exists a constant $C > 0$ such that

$$\int_0^t d|\sigma| \le Ct^\beta, \quad \text{for all } t > 0. \tag{6}$$

In this case, we write $\sigma \in M^\beta$ and also $\|\sigma\|_{M^\beta} = \inf\{C\colon C$ satisfies $(6)\}$.

The following result corresponds to Theorem 8.

THEOREM 16 (see [2], [1]). *For $0 < p \le 1$, the pairing $(f, d\mu) \to \int_{\mathbf{R}^{n+1}_+} f(x, t)\,d\mu(x, t)$, with $f \in T^p_\infty$ and $\mu \in V^{1/p}$, realizes the duality of T^p_∞ with $V^{1/p}$.*

For our next result, we need to introduce a densely defined bilinear functional. We will restrict the action of this operator, when considering distributions in the Hardy space $H^p(\mathbf{R}^n)$, to the dense subspace $\mathscr{S}_0$ of those functions in the class $\mathscr{S}$ with mean zero.

DEFINITION 17. Fix $1 \le q \le \infty$. Suppose $F\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is a measurable function such that if we set $F_z(x) = F(z, x)$, $z, x \in \mathbf{R}^n$; then $F_z \in L^q(\mathbf{R}^n)$. Let $\alpha \ge 0$. For $g \in \mathscr{S}_0$, set

$$R_F(g)(x, z) = \int_{\mathbf{R}^n} g(y)F(z, y + x)\,dy.$$

We define, for $\sigma \in M^\alpha$,

$$T_F(g, \sigma)(z) = \int_0^\infty (R_F(g)(\cdot, z) * P_t)(0)\,d\sigma(t),$$

where $P(x) = c_n(1+|x|^2)^{-(n+1)/2}$ is the Poisson kernel in $\mathbf{R}^n$, and $P_t(x) = t^{-n}P(x/t)$.

EXAMPLE 18. Suppose $q=\infty$ and $F(z, x) = e^{-ixz}$. Then $\|F\|_\infty = 1$ and if $g \in \mathcal{S}_0$ we have that

$$R_F(g)(x, z) = \int_{\mathbf{R}^n} g(y)e^{-i(x+y)z}\,dy = e^{-ixz}\hat{g}(z).$$

Hence,

$$(R_F(g)(\cdot\,, z) * P_t)(0) = \int_{\mathbf{R}^n} e^{-ixz}\hat{g}(z)P_t(x)\,dx = \hat{g}(z)\widehat{P}_t(z).$$

If $0 < p < 1$ and we consider the measure $d\sigma(t) = t^{n(1/p-1)-1}\,dt$, then we have that $\sigma \in M^{n(1/p-1)}$, since

$$\int_0^t d|\sigma|(t) = \frac{t^{n(1/p-1)}}{n(1/p-1)},$$

and so,

$$\|\sigma\|_{M^{n(1/p-1)}} = \frac{1}{n(1/p-1)}.$$

Therefore,

$$\begin{aligned} T_F(g, \sigma)(z) &= \int_0^\infty \hat{g}(z)\widehat{P}_t(z)t^{n(1/p-1)-1}\,dt \\ &= c_n\hat{g}(z)\int_0^\infty e^{-2\pi t|z|}t^{n(1/p-1)-1}\,dt \end{aligned}$$

and the integral is finite since $n(1/p-1) > 0$.

THEOREM 19. *Suppose* $1 \le q \le \infty$, $\alpha \ge n/q$ *and* $1/p = \alpha/n + 1/q'$, *so that* $0 < p < 1$. *Then*

$$|T_F(g, \sigma)(z)| \le c_n\|\sigma\|_{M^\alpha}\|F_z\|_{L^q(\mathbf{R}^n)}\|g\|_{H^p(\mathbf{R}^n)},$$

for all $\sigma \in M^\alpha$ *and* $g \in \mathcal{S}_0$.

Proof. The proof is a consequence of the nontangential maximal characterization of $H^p(\mathbf{R}^n)$ (see [4]): $\|g\|_{H^p(\mathbf{R}^n)} \approx \|PI(g)\|_{T^p_\infty}$, where $PI(g)(x, t) = (P_t * g)(x)$. To estimate this quantity we use Theorem

16, $(T^p_\infty)^* = V^{1/p}$, $0 < p \le 1$:

$$\begin{aligned} T_F(g, \sigma)(z) &= \int_0^\infty \left(\int_{\mathbf{R}^n} P_t(u) R_F(g)(u, z)\, du \right) d\sigma(t) \\ &= \int_{\mathbf{R}^{n+1}_+} g(y) \left(\int_{\mathbf{R}^n} P_t(u) F(z, y+u)\, du \right) dy\, d\sigma(t) \\ &= \int_{\mathbf{R}^{n+1}_+} g(y) \left(\int_{\mathbf{R}^n} P_t(v-y) F(z, v)\, dv \right) dy\, d\sigma(t) \\ &= \int_{\mathbf{R}^{n+1}_+} PI(g)(v, t) F(z, v)\, dv\, d\sigma(t). \end{aligned}$$

For a fixed z, consider the measure

$$d\mu(v, t) = F_z(v)\, dv\, d\sigma(t).$$

Then, we claim that $\mu \in V^{1/p}$ and $\|\mu\|_{V^{1/p}} \le \|\sigma\|_{M^\alpha} \|F_z\|_{L^q}$. Thus,

$$\begin{aligned} |T_F(g, \sigma)(z)| &\le \int_{\mathbf{R}^{n+1}_+} |PI(g)(v, t)|\, d|\mu|(v, t) \\ &\le \|PI(g)\|_{T^p_\infty} \|\mu\|_{V^{1/p}} \le c_n \|\sigma\|_{M^\alpha} \|F_z\|_{L^q} \|g\|_{H^p}. \end{aligned}$$

To prove the claim, it suffices to show that if $f \in L^q(\mathbf{R}^n)$, $1 \le q \le \infty$, $\sigma \in M^\alpha$, with $\beta = 1/q' + \alpha/n \ge 1$ and we set $d\mu(x, t) = f(x)\, dx\, d\sigma(t)$, then $\mu \in V^\beta$ and $\|\mu\|_{V^\beta} \le \|\sigma\|_{M^\alpha} \|f\|_{L^q}$. Now, for a cube $Q \subset \mathbf{R}^n$,

$$\begin{aligned} |\mu|(\widehat{Q}) &\le \left(\int_Q |f(x)|\, dx \right) \left(\int_0^{|Q|^{1/n}} d|\sigma|(t) \right) \\ &\le \|f\|_{L^q} |Q|^{1/q'} \|\sigma\|_{M^\alpha} |Q|^{\alpha/n} = \|f\|_{L^q} \|\sigma\|_{M^\alpha} |Q|^\beta, \end{aligned}$$

and so, $\|\mu\|_{V^\beta} \le \|f\|_{L^q} \|\sigma\|_{M^\alpha}$. □

COROLLARY 20. *If* $0 < p \le 1$ *and* $g \in \mathscr{S}_0(\mathbf{R}^n)$, *then*

$$|\hat{g}(z)| \le C_{n,p} |z|^{n(1/p-1)} \|g\|_{H^p},$$

for all $z \in \mathbf{R}^n$.

Proof. It suffices to consider the case $0 < p < 1$ and $z \ne 0$. We recall that by Example 18 we have

$$T_F(g, \sigma)(z) = c_n \hat{g}(z) \int_0^\infty e^{-2\pi t|z|} t^{n(1/p-1)-1}\, dt.$$

But,

$$\int_0^\infty e^{-2\pi t|z|} t^{n(1/p-1)-1}\, dt$$
$$= C|z|^{-n(1/p-1)} \int_0^\infty e^{-2\pi u} u^{n(1/p-1)-1}\, du = C_{n,p}|z|^{-n(1/p-1)}.$$

Hence, by the theorem,

$$|T_F(g, \sigma)(z)| \leq c_n \|\sigma\|_{M^\alpha} \|F_z\|_\infty \|g\|_{H^p(\mathbf{R}^n)};$$

that is,

$$C_{n,p}|\hat{g}(z)|\,|z|^{-n(1/p-1)} \leq \frac{c_n}{n(1/p-1)} \|g\|_{H^p},$$

which gives the result. □

REMARK 21. Corollary 20 was first proved in [**4**], using a different approach. Later in [**12**], it was also proved using the atomic characterization of H^p. We want to give yet another simple proof using now the duality of the H^p spaces. In [**3**] it is shown that $(H^p(\mathbf{R}^n))^* = \dot{B}_\infty^{n(1/p-1),\infty}$, $0 < p < 1$, where the norm on this Besov space coincides with the Lipschitz norm of order $n(1/p-1)$ (see [**11**]); namely,

$$\|f\|_{\dot{B}_\infty^{n(1/p-1),\infty}} = \sup_{\substack{x\in\mathbf{R}^n \\ h\in\mathbf{R}^n\setminus\{0\}}} \frac{|(\Delta_h^k f)(x)|}{|h|^{n(1/p-1)}},$$

where, $k \in \mathbf{N}$, $k > n(1/p-1)$ and

$$(\Delta_h^k f)(x) = \sum_{r=0}^k \binom{k}{r} (-1)^r f(x+rh),$$

is the kth order difference operator. Now, we have the following

LEMMA 22. *Fix $y \in \mathbf{R}^n$ and $\alpha > 0$. Then*

$$\|e^{-iy\delta}\|_{\dot{B}_\infty^{\alpha,\infty}} \approx |y|^\alpha.$$

Proof. Let $k \in \mathbf{N}$, $k > \alpha$ and suppose $y \in \mathbf{R}^n\setminus\{0\}$. Then, for $h \in \mathbf{R}^n$

$$(\Delta_h^k e^{-iy\cdot})(x) = \sum_{r=0}^k \binom{k}{r} (-1)^r e^{-iy(x+rh)}$$
$$= e^{-iyx} \sum_{r=0}^k \binom{k}{r} (-1)^r e^{-iryh} = e^{-iyx}(1-e^{-iyh})^k.$$

Hence,

$$|(\Delta_h^k e^{-iy\cdot})(x)|^2 = (2 - 2\cos(yh))^k .$$

Thus,

$$\begin{aligned}\sup_{\substack{x\in\mathbf{R}^n\\ h\in\mathbf{R}^n\setminus\{0\}}} \frac{|(\Delta_h^k e^{-iy\cdot})(x)|}{|h|^\alpha} &= \sup_{h\in\mathbf{R}^n\setminus\{0\}} 2^{k/2}\frac{(1-\cos(yh))^{k/2}}{|h|^\alpha}\\ &\le C_k|y|^\alpha \sup_{u\in\mathbf{R}^+}\frac{(1-\cos u)^{k/2}}{u^\alpha}\\ &\le C_k \sup_{u\in\mathbf{R}^+}\frac{(1-\cos u)^{\alpha/2}}{u^\alpha}(1-\cos u)^{(k-\alpha)/2}|y|^\alpha\\ &\le C_{k,\alpha}|y|^\alpha ,\end{aligned}$$

since $k > \alpha$. Conversely, we want to show that for any $y \in \mathbf{R}^n\setminus\{0\}$, there exists an $h \in \mathbf{R}^n\setminus\{0\}$ such that $|y| = |h|^{-1}$ and $1 - \cos(yh) = 1 - \cos(1) > 0$. In fact, if $h = y/|y|^2$ then trivially $|y| = |h|^{-1}$ and $y\cdot h = 1$. Hence

$$\|e^{-iy\cdot}\|_{\dot B_\infty^{\alpha,\infty}} \ge 2^{k/2}(1-\cos 1)^{k/2}|y|^\alpha . \qquad \square$$

Thus, by the duality between H^p and $\dot B_\infty^{n(1/p-1),\infty}$, $0 < p < 1$, and using this lemma, we find that if $g \in \mathscr{S}_0$

$$\begin{aligned}|\hat g(y)| = \left|\int_{\mathbf{R}^n} g(x)e^{-iyx}\,dx\right| &\le \|g\|_{H^p}\|e^{-iy\cdot}\|_{\dot B_\infty^{n(1/p-1),\infty}}\\ &\le C_{n,p}|y|^{n(1/p-1)}\|g\|_{H^p} .\end{aligned}$$

As a curiosity, and from the proof of Corollary 20, we see that

$$\|e^{-iy\cdot}\|_{\dot B_\infty^{\alpha,\infty}} \approx \left(\int_0^\infty \widehat{P}_t(y)t^{\alpha-1}\,dt\right)^{-1}, \qquad \alpha > 0 .$$

One can also get very easily that, for $s > 0$, $1 < q \le \infty$ we have for the Besov space $\dot B_\infty^{s,q}$, $\|e^{-iy\cdot}\|_{\dot B_\infty^{s,q}} \approx |y|^s$. Hence (see **[13]**), since

$$(\dot B_p^{s,q})^* = \dot B_\infty^{-s+n(1/p-1),q'},$$
$$0 < p \le 1,\ 0 < q < \infty,\ 0 < s < n(1/p-1),$$

and

$$(\dot F_p^{s,q})^* = \dot B_\infty^{-s+n(1/p-1),\infty},$$
$$0 < p < 1,\ 0 < q < \infty,\ 0 < s < n(1/p-1),$$

where $q' = \infty$ if $0 < q \leq 1$, and $\dot{F}_p^{s,q}$ is a Triebel-Lizorkin space (see **[13]**), then, by a similar argument as above, we obtain

$$|\hat{f}(y)| \leq C|y|^{-s+n(1/p-1)}\|f\|_{\dot{B}_p^{s,q}},$$
$$0<p\leq 1,\ 0<q<\infty,\ 0<s<n(1/p-1),$$

and

$$|\hat{f}(y)| \leq C|y|^{-s+n(1/p-1)}\|f\|_{\dot{F}_p^{s,q}},$$
$$0<p<1,\ 0<q<\infty,\ 0<s<n(1/p-1).$$

The following result gives the regularity of a harmonic extension in the x-variable, when integrated against an M^α measure on t.

COROLLARY 23. *Suppose* $1 \leq q \leq \infty$, $\alpha \geq n/q$ *and* $1/p = \alpha/n + 1/q'$. *For a function* $f \in L^q(\mathbf{R^n})$ *and* $\sigma \in M^\alpha$ *define*

$$K(f, \sigma)(y) = \int_0^\infty (P_t * f)(y)\, d\sigma(t).$$

(i) *If* $0 < p < 1$ *then,*

$$K\colon L^q(\mathbf{R^n}) \times M^\alpha \to \dot{B}_\infty^{n(1/p-1),\infty},$$

and

$$\|K(f, \sigma)\|_{\dot{B}_\infty^{n(1/p-1),\infty}} \leq C_n \|\sigma\|_{M^\alpha} \|f\|_{L^q(\mathbf{R^n})}.$$

(ii) *If* $p = 1$, *then*

$$K\colon L^q(\mathbf{R^n}) \times M^\alpha \to \mathrm{BMO},$$

and

$$\|K(f, \sigma)\|_{\mathrm{BMO}} \leq C_n \|\sigma\|_{M^\alpha} \|f\|_{L^q(\mathbf{R^n})}.$$

Proof. We will only show (i), because the proof of (ii) follows similarly. Since $(H^p(\mathbf{R^n}))^* = \dot{B}_\infty^{n(1/p-1),\infty}$, then to show that $K(f, \sigma) \in \dot{B}_\infty^{n(1/p-1),\infty}$ we only need to see that

$$\left|\int_{\mathbf{R^n}} g(y)K(f, \sigma)(y)\, dy\right| \leq C_n \|\sigma\|_{M^\alpha} \|f\|_{L^q} \|g\|_{H^p},$$

for all $g \in \mathscr{S}_0$. Set $F(z, x) = f(x)$, for all $z \in \mathbf{R^n}$. Then,

$$\int_{\mathbf{R^n}} g(y)K(f, \sigma)(y)\, dy = \int_{\mathbf{R^n}} g(y) \int_0^\infty (P_t * F_z)(y)\, d\sigma(t)\, dy$$
$$= T_F(g, \sigma)(z),$$

for all $z \in \mathbf{R}^n$. Hence, by Theorem 19,

$$\left| \int_{\mathbf{R}^n} g(y) K(f, \sigma)(y)\, dy \right| \le C_n \|\sigma\|_{M^\alpha} \|f\|_{L^q} \|g\|_{H^p}. \qquad \square$$

We now give another application of our duality techniques to estimate harmonic extensions to $\mathbf{R}^{n+1}_+$ of functions in H^p. The next theorem gives, as a particular case, a generalization to higher dimensions of the Féjer-Riesz inequality (see **[5]** Theorems I-4.5 and III-7.57, for the case $p = 1$), and shows that it can also be proved in all cases $0 < p \le 1$. Moreover, in the previous theorems, the authors work with the atomic characterization of H^1 and some extra conditions on the kernel are required, that will not be needed in our proof. This inequality gives the behaviour in the vertical t-direction for the extension $\varphi_t * f(x)$, relative to a kernel φ, with $f \in \mathcal{S}_0$, instead of the well-known growth on the x-direction for the harmonic extension $u \equiv PI(f)$; namely,

$$\sup_{t>0} \int_{\mathbf{R}^n} |u(x, t)|^p\, dx \le C \|f\|^p_{H^p}.$$

The proof is based in finding the right pairing for an appropriate Carleson measure.

THEOREM 24. *If* $0 < p \le 1$, $F \in T^p_\infty$ *and* $\sigma \in M^{n/p}$, *then*

$$\sup_{x \in \mathbf{R}^n} \int_0^\infty |F(x, t)|\, d|\sigma|(t) \le \|\sigma\|_{M^{n/p}} \|F\|_{T^p_\infty}.$$

Proof. Fix $x \in \mathbf{R}^n$ and set $d\mu(y, t) = \delta_x(y)\, d\sigma(t)$, where δ_x is the Dirac mass in $\mathbf{R}^n$ at the point x. Then $\mu \in V^{1/p}$ and $\|\mu\|_{V^{1/p}} \le \|\sigma\|_{M^{n/p}}$. In fact, since $p \le 1$, then if Q is a cube in $\mathbf{R}^n$ we have that

$$|\mu|(\widehat{Q}) \le \left(\int_Q \delta_x(y) \right) \left(\int_0^{|Q|^{1/n}} d|\sigma|(t) \right) \le |Q|^{1/p} \|\sigma\|_{M^{n/p}}.$$

Therefore, since $(T^p_\infty)^* = V^{1/p}$, we get that

$$\begin{aligned} \int_0^\infty |F(x, t)|\, d|\sigma|(t) &\le \int_{\mathbf{R}^{n+1}_+} |F(y, t)|\, d|\mu|(y, t) \\ &\le \|F\|_{T^p_\infty} \|\mu\|_{V^{1/p}} \le \|\sigma\|_{M^{n/p}} \|F\|_{T^p_\infty}. \qquad \square \end{aligned}$$

For the next result we introduce the following notation (see **[14]**): if $f \in \mathcal{S}_0$, $0 < p \le 1$ and we choose $\varphi \in L^1 \cap L^\infty$, $\int_{\mathbf{R}^n} \varphi(x)\, dx \ne 0$ then we say that $f \in H^p_\varphi$ if $\|f\|_{H^p_\varphi} = \|\varphi_t * f\|_{T^p_\infty} < \infty$.

COROLLARY 25. *Let φ be as above,* $0 < p \leq 1$.

(i) (*Féjer-Riesz inequality, if φ is the Poisson kernel.*) *If $f \in H^p_\varphi$, then*

$$\sup_{x\in\mathbf{R}^n} \int_0^\infty |(\varphi_t * f)(x)| t^{n/p-1}\, dt \leq C_{n,p} \|f\|_{H^p_\varphi}.$$

(ii) *With more generality, if $p \leq q \leq 1$, then for $f \in H^p_\varphi$ we have*

$$\sup_{x\in\mathbf{R}^n} \int_0^\infty |(\varphi_t * f)(x)|^q t^{qn/p-1}\, dt \leq C_{n,p} \|f\|^q_{H^p_\varphi}.$$

Proof. (i) Consider the function $F(x, t) = (\varphi_t * f)(x)$ and the measure $d\sigma(t) = t^{n/p-1}\, dt$. Then $F \in T^p_\infty$ and $\sigma \in M^{n/p}$. Hence, by the previous theorem,

$$\sup_{x\in\mathbf{R}^n} \int_0^\infty |(\varphi_t * f)(x)| t^{n/p-1}\, dt$$
$$= \sup_{x\in\mathbf{R}^n} \int_0^\infty |F(x, t)|\, d|\sigma|(t) \leq C_{n,p} \|f\|_{H^p_\varphi}.$$

(ii) Let $p \leq q \leq 1$ and consider now the function

$$F(x, t) = |(\varphi_t * f)(x)|^q.$$

Then $F \in T^{p/q}_\infty$ with $\|F\|_{T^{p/q}_\infty} = \|f\|^q_{H^p_\varphi}$. Also, if we set $d\sigma(t) = t^{qn/p-1}\, dt$ then $\sigma \in M^{qn/p}$ and hence, since $p/q \leq 1$,

$$\sup_{x\in\mathbf{R}^n} \int_0^\infty |(\varphi_t * f)(x)|^q t^{qn/p-1}\, dt \leq C_{n,p} \|F\|_{T^{p/q}_\infty} = C_{n,p} \|f\|^q_{H^p_\varphi}. \quad \square$$

REFERENCES

[1] J. Alvarez and M. Milman, *Spaces of Carleson measures: duality and interpolation*, Ark. Mat., **25** (2) (1987), 155–174.

[2] R. Coifman, Y. Meyer and E. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal., **62** (1985), 304–335.

[3] P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on H^p spaces with $0 < p < 1$*, J. Reine Angew. Math., **238** (1969), 32–60.

[4] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137–193.

[5] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, Mathematical Studies, vol. 116, North-Holland, 1985.

[6] Han Yongsheng, *Certain Hardy-type spaces that can be characterized by maximal functions and variations of the square functions*, Ph.D. Thesis, Washington University (1984).

[7] R. Johnson, *Applications of Carleson Measures to Partial Differential Equations and Fourier Multiplier Problems*, Proc. Conf. Cortona, Lecture Notes in Math., vol. 992, (1982), pp. 16–72.
[8] A. Nagel and E. Stein, *On certain maximal functions and approach regions*, Adv. Math., **54** (1984), 83–106.
[9] F. J. Ruiz and J. L. Torrea, *A unified approach to Carleson measures and A_p weights*. II, Pacific J. Math., **120** (1985), 189–197.
[10] J. Soria, *Weighted Tent Spaces*, Math. Nachr., **155** (1992), 231–256.
[11] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
[12] M. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Astérisque, **77** (1980), 67–149.
[13] H. Triebel, *Theory of Function Spaces*, Birkhäuser-Verlag, Basel, Boston and Stuttgart, 1983.
[14] G. Weiss, *Some Problems in the Theory of Hardy Spaces*, Proc. Sympos. Pure Math., vol. 35, Amer. Math. Soc., Providence, RI, 1979, pp. 189–200.

Received March 17, 1992 and in revised form September 15, 1992.

UNIVERSITAT AUTÒNOMA DE BARCELONA
08193 BELLATERA BARCELONA, SPAIN

AND

UNIVERSITAT DE BARCELONA
08071 BARCELONA, SPAIN

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

ON SIEVED ORTHOGONAL POLYNOMIALS X: GENERAL BLOCKS OF RECURRENCE RELATIONS

JAIRO A. CHARRIS, MOURAD E. H. ISMAIL, AND SERGIO MONSALVE

Orthogonal polynomials defined by general blocks of recurrence relations are examined. The connection with polynomial mappings is established, and applications are given to sieved orthogonal polynomials. This work extends earlier work on symmetric sieved polynomials to the case when the polynomials are not necessarily symmetric.

1. Introduction. We study in this paper systems $\{p_n(x)\}$ of orthogonal polynomials defined by general blocks of recurrence relations of the type

$$
\begin{aligned}
(1.1)\quad (x-b_n^{(0)})p_{nk}(x) &= p_{nk+1}(x)+a_n^{(0)}p_{nk-1}(x)\,,\\
&\ \ \vdots\\
(x-b_n^{(j)})p_{nk+j}(x) &= p_{nk+j+1}(x)+a_n^{(j)}p_{nk+j-1}(x)\,,\\
&\ \ \vdots\\
(x-b_n^{(k-1)})p_{(n+1)k-1}(x) &= p_{(n+1)k}(x)+a_n^{(k-1)}p_{(n+1)k-2}(x)\,,
\end{aligned}
$$

$0\le j\le k-1$, $n\ge 0$, and satisfying initial conditions

$$
(1.2)\qquad p_{-1}(x)=0,\quad p_0(x)=1.
$$

We shall assume $a_n^{(j)}>0$, $j=0,1,\dots,k-1$, $n\ge 0$ and also that $k\ge 2$. Observe that the p_n's do not depend on $a_0^{(0)}$, so we make the convenient choice $a_0^{(0)}=1$. Clearly $\{p_n(x)\}$ is a system of monic orthogonal polynomials.

The case of $b_n^{(j)}=0$, $n\ge 0$, $0\le j\le k-1$, has been treated in a previous paper **[9]** by Charris and Ismail, where they also assumed that the determinants

$$
(1.3)\ \Delta_n(2,k-1)=\begin{vmatrix}
x & -1 & 0 & 0 & \cdots & 0 & 0\\
-a_n^{(2)} & x & -1 & 0 & \cdots & 0 & 0\\
0 & -a_n^{(3)} & x & -1 & \cdots & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\
0 & 0 & 0 & 0 & \cdots & a_n^{(k-1)} & x
\end{vmatrix},\qquad n\ge 0,
$$

are independent of n, that is $\Delta_n(2, k-1) = \Delta_0(2, k-1)$, $n \geq 0$. These two assumptions were motivated by the desire of the authors of **[9]** to provide a unified approach to symmetric sieved orthogonal polynomials.

Here we remove those two assumptions. Having done this, now (1.1) covers, of course, all monic three-term recurrence relations defining orthogonal polynomials. However, the separation in blocks is again naturally motivated by general sieved orthogonal polynomials and, as we shall see, also arises naturally when considering systems of polynomials obtained via polynomial mappings. In both cases $\Delta_n(2, k-1)$ (with x changed to $x - b_n^{(1)}, x - b_n^{(2)}, \ldots, x - b_n^{(k-1)}$ in descending order along the main diagonal) is independent of n. This is clearly the case for sieved polynomials of the first kind where $a_n^{(j)} = 1/4$, $b_n^{(1)} = b_n^{(j)} = 0$, $n \geq 0$, $2 \leq j \leq k-1$, but it is not so clear for polynomials obtained by means of polynomial mappings. In fact to prove that the modified determinant $\Delta_n(2, k-1)$ is independent of n in the case of polynomials obtained via a polynomial mapping, we needed to apply results where $\Delta_n(2, k-1)$ may depend on n. This is done in §4.

This paper not only represents a further contribution to the understanding of general sieved orthogonal polynomials and systems determined by polynomial mappings, but it also covers more general systems which are not determined by polynomial mappings. As a matter of fact, orthogonal polynomials defined through blocks of recurrence relations, which are not necessarily sieved orthogonal polynomials and do not originate—a priori—in conjunction with polynomial mappings, have continued to appear in the literature, mainly in connection with problems in physics and chemistry (see, for example, **[6]**, **[10]**, **[20]**, **[21]**).

The paper is organized as follows. Section 2 contains basic relationships and preliminaries while §3 describes the link polynomials which tie together the different blocks. Section 3 also exhibits the fundamental recurrence relationships satisfied by the link polynomials. These fundamental recurrence relations will enable us to express the polynomials under consideration in terms of the link polynomials. Section 4 studies the connection with polynomial mappings, and §5 deals with sieved polynomials.

The evaluation of the Stieltjes transform of the orthogonality measures of the polynomials $\{p_n(x)\}$ and their associated families are included in §3. Recall that if $\{p_n(x)\}$ is a system of monic polynomials which are orthogonal with respect to a unique measure μ with total mass 1, then the Stieltjes transform of μ is

$$(1.4)\qquad X(x)=\int_{-\infty}^{+\infty}\frac{d\mu(t)}{x-t},\qquad x\in\mathbb{C}-\mathbb{R},$$

and the literature on the moment problem (see [4], [11], [19]) ensures that

$$(1.5)\qquad X(x)=\lim_{n\to\infty}\frac{p_{n-1}^{(1)}(x)}{p_n(x)},\qquad x\in\mathbb{C}-\mathbb{R},$$

where $\{p_n^{(1)}(x)\}$ is the system of associated polynomials of order 1 of $\{p_n(x)\}$ (see §2 for the definition of $\{p_n^{(1)}(x)\}$). Hence, if $\{p_n(x)\}$ is given a priori by a recurrence relation such as (1.1), and it is known in advance that they are orthogonal with respect to a unique measure μ with total mass 1, then μ can be determined from $X(x)$, as given by (1.5), via the Perron-Stieltjes inversion formula ([7], [5], [14]),

$$(1.6)\quad \int_{-\infty}^{+\infty} f\,d\mu=\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\{X(x-i\varepsilon)-X(x+i\varepsilon)\}f(x)\,dx,$$

which holds for any bounded and continuous numerical function f on $\mathbb{R}$ provided that the support of $d\mu$ is contained in a half line. The existence of a unique measure μ as above can be guaranteed from properties of the coefficients $a_n^{(j)}$ in (1.1). This is the case, for example, if there is a constant $M>0$ such that

$$(1.7)\qquad 0<a_n^{(j)}<M,\qquad 0\le j\le k-1,\ n\ge 0.$$

In what follows, we will assume that conditions such as (1.7) are given which guarantee the uniqueness of μ. This is expressed by saying that the Hamburger moment problem for $\{p_n(x)\}$ is determined.

The notation

$$(1.8)\qquad (a)_n=\begin{cases}1, & n=0,\\ a(a+1)(a+2)\cdots(a+n-1), & n\ge 1,\end{cases}$$

for shifted factorials will be used throughout. If $a\neq 0,-1,-2,\dots$, then the shifted factorial is

$$(1.9)\qquad (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},$$

where Γ stands for the Gamma Function ([18]). The series

$$(1.10)\qquad {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\,x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,\qquad |x|<1,$$

is the hypergeometric series. We recall the binomial formula ([**18**])

$$(1.11) \qquad (1-x)^{-a} = {}_2F_1\left(\begin{matrix} a, & 1 \\ & 1 \end{matrix}\,\middle|\, x\right), \qquad |x| < 1,$$

and the Euler integral representation ([**18**])

$$(1.12) \qquad \begin{aligned} &\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt \\ &\quad = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, x\right), \end{aligned}$$

which holds for $|x| < 1$, when $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. Since the right-hand side of (1.12) is meaningful as long as $b > 0$ and c and $c-b$ are not integers ≤ 0, we can define

$$(1.13) \qquad \begin{aligned} &\int_0^{\overline{1|}} t^c(1-xt)^{-A}(1-t)^{-B}\,dt \\ &\quad = \frac{\Gamma(c+1)\Gamma(-B+1)}{\Gamma(-B+c+2)}\,{}_2F_1\left(\begin{matrix} A, & c+1 \\ & -B+c+2 \end{matrix}\,\middle|\, x\right) \end{aligned}$$

whenever $c > -1$, $|x| < 1$ and B is not an integer ≥ 1. The integral in (1.13) is called a Hadamard integral and will be used in §5. Details about the theory of Hadamard singular integrals can be found in [**4**], [**8**], [**17**].

2. Basic results. The results in this section and the next section follow closely those of §§2, 3 in [**9**], so our treatment will be rather sketchy.

The system of equations (1.1) can be written in matrix form as

$$(2.1) \qquad A\begin{bmatrix} p_{nk+1} \\ p_{nk+2} \\ p_{nk+3} \\ \vdots \\ p_{nk+k-1} \\ p_{nk-1} \end{bmatrix} = \begin{bmatrix} (x-b_n^{(0)})p_{nk} \\ a_n^{(1)}p_{nk} \\ 0 \\ \vdots \\ 0 \\ p_{nk+k} \end{bmatrix},$$

where A is the $k \times k$ matrix

(2.2)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_n^{(0)} \\ x-b_n^{(1)} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -a_n^{(2)} & x-b_n^{(2)} & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -a_n^{(3)} & x-b_n^{(3)} & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -a_n^{(k-2)} & x-b_n^{(k-2)} & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a_n^{(k-1)} & x-b_n^{(k-1)} & 0 \end{bmatrix}.$$

We will also write

$$\Delta_n(i,j) = \begin{cases} 0, & j < i-2, \\ 1, & j = i-2, \\ x - b_n^{(i-1)}, & j = i-1, \end{cases} \tag{2.3}$$

and

$$\Delta_n(i,j) = \begin{vmatrix} x-b_n^{(i-1)} & -1 & 0 & 0 & \cdots & 0 & 0 \\ -a_n^{(i)} & x-b_n^{(i)} & -1 & 0 & \cdots & 0 & 0 \\ 0 & -a_n^{(i+1)} & x-b_n^{(i+1)} & -1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -a_n^{(j)} & x-b_n^{(j)} \end{vmatrix} \tag{2.4}$$

for $n \geq 0$ and $j \geq i \geq 1$.

We now solve (2.1) for p_{nk+j} in terms of p_{nk} and p_{nk+k} by Cramer's rule and obtain the recursion

$$\begin{aligned} \Delta_n(2,k-1)p_{nk+j} &= \Delta_n(2,j-1)p_{nk+k} \\ &\quad + a_n^{(1)}a_n^{(2)}\cdots a_n^{(j)}\Delta_n(j+2,k-1)p_{nk}, \\ &\qquad n \geq 0,\ j = 1,\dots,k-1. \end{aligned} \tag{2.5}$$

Furthermore,

$$\begin{aligned} &a_n^{(0)}\Delta_n(2,k-1)p_{nk-1} \\ &\quad = [(x-b_n^{(0)})\Delta_n(2,k-1) - a_n^{(1)}\Delta_n(3,k-1)]p_{nk} - p_{nk+k}, \\ &\qquad n \geq 0. \end{aligned} \tag{2.6}$$

In particular (we assume $p_{-j}(x) = 0$, $j = 1,2,\dots$),

$$p_k(x) = (x-b_0^{(0)})\Delta_0(2,k-1) - a_0^{(1)}\Delta_0(3,k-1) = \Delta_0(1,k-1). \tag{2.7}$$

For $i = 1,2,\dots,k-1$, the associated polynomials of order i, $\{p_n^{(i)}(x)\}$, of $\{p_n(x)\}$ are defined recursively by

$$(x-b_n^{(j)})p_{nk-i+j}^{(i)} = p_{nk-i+j+1}^{(i)} + a_n^{(j)}p_{nk-i+j-1}^{(i)}, \qquad 0 \leq j \leq k-1, \tag{2.8}$$

$$p_{-1}^{(i)}(x) = 0, \quad p_0^{(i)}(x) = 1.$$

Writing (2.8) in matrix form and solving for $p_{nk-i+j}^{(i)}$ in terms of $p_{(n+1)k-i}^{(i)}$ and $p_{nk-i}^{(i)}$ gives, exactly as in **[9]**, the following results

$$\text{(2.9)} \quad \Delta_n(2, k-1)p_{nk-i+j}^{(i)} = \Delta_n(2, j-1)p_{(n+1)k-i}^{(i)} + a_n^{(1)} \cdots a_n^{(j)}\Delta_n(j+2, k-1)p_{nk-i}^{(i)},$$

and

$$\text{(2.10)} \quad a_n^{(0)}\Delta_n(2, k-1)p_{nk-i-1}^{(i)} = -p_{(n+1)k-i}^{(i)} + [(x-b_n^{(0)})\Delta_n(2, k-1) - a_n^{(1)}\Delta_n(3, k-1)]p_{nk-i}^{(i)} \quad \text{for } n \geq 1.$$

Let $\widetilde{\Delta}_n(i+1, k-1)$ be the matrix whose determinant is (2.4) with $j = k-1$ (so that $\operatorname{Det}(\widetilde{\Delta}_n(i+1, k-1)) = \Delta_n(i+1, k-1))$. Then the relationship

$$\text{(2.11)} \qquad \widetilde{\Delta}_0(i+1, k-1)\begin{bmatrix} p_0^{(i)} \\ p_1^{(i)} \\ \vdots \\ p_{k-i-2}^{(i)} \\ p_{k-i-1}^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p_{k-i}^{(i)} \end{bmatrix},$$

the initial condition $p_0^{(i)}(x) = 1$ and Cramer's rule give

$$\text{(2.12)} \qquad p_j^{(i)} = \Delta_0(i+1, j+i-1), \qquad j = 0, 1, \ldots, k-i.$$

In particular we find

$$\text{(2.13)} \qquad p_{k-1}^{(1)}(x) = \Delta_0(2, k-1).$$

The associated polynomials of higher order $\{p_n^{(lk+i)}(x)\}$, $l \geq 0$, $i = 0, 1, 2, \ldots, k-1$, are defined by

$$\text{(2.14)} \quad (x - b_{n+l}^{(j)})p_{nk-i+j}^{(lk+i)}(x) = p_{nk-i+j+1}^{(lk+i)}(x) + a_{n+l}^{(j)}p_{nk-i+j-1}^{(lk+i)}(x),$$
$$p_{-1}^{(lk+i)}(x) := 0, \quad p_0^{(lk+i)}(x) := 1, \qquad j = 0, 1, \ldots, k-1.$$

Thus,

$$\text{(2.15)} \qquad \Delta_{n+l}(2, k-1)p_{nk-i+j}^{(lk+i)} = \Delta_{n+l}(2, j-1)p_{(n+1)k-i}^{(lk+i)} + a_{n+l}^{(1)} \cdots a_{n+l}^{(j)}\Delta_{n+l}(j+2, k-1)p_{nk-i}^{(lk+i)},$$

$j = 0, 1, \dots, k-1$, $n \geq 1$, and

$$
\begin{aligned}
(2.16) \qquad & a^{(0)}_{n+1}\Delta_{n+l}(2, k-1)p^{(lk+i)}_{nk-i-1} \\
&= -p^{(lk+i)}_{(n+1)k-i} + [(x - b^{(0)}_{n+l})\Delta_{n+l}(2, k-1) \\
&\qquad - a^{(1)}_{n+l}\Delta_{n+l}(3, k-1)]p^{(lk+i)}_{nk-i},
\end{aligned}
$$

$n \geq 1$. Also,

$$
(2.17) \qquad p^{(lk+i)}_j(x) = \Delta_l(i+1, j+i-1), \qquad 0 \leq j \leq k-i.
$$

3. The link polynomials. Denote with $\{P^{(l)}_n(x)\}$ the system of polynomials defined for $l \geq 0$ by

$$
\begin{aligned}
(3.1) \quad & [(x - b^{(0)}_{n+l})\Delta_{n+l}(2, k-1)\Delta_{n+l-2}(2, k-1) \\
& - a^{(1)}_{n+l}\Delta_{n+l}(3, k-1)\Delta_{n+l-1}(2, k-1) \\
& - a^{(0)}_{n+l}\Delta_{n+l}(2, k-1)\Delta_{n+l-1}(2, k-2)]P^{(l)}_n(x) \\
&= \Delta_{n+l-1}(2, k-1)P^{(l)}_{n+1}(x) \\
&\quad + a^{(0)}_{n+l}a^{(1)}_{n+l-1}\cdots a^{(k-1)}_{n+l-1}\Delta_{n+l}(2, k-1)P^{(l)}_{n-1}(x), \qquad n \geq 0,
\end{aligned}
$$

and the initial conditions

$$
(3.2) \qquad P^{(l)}_{-1}(x) = 0, \quad P^{(l)}_0(x) = 1.
$$

We adopt the convention

$$
(3.3) \qquad \Delta_{-1}(2, k-2) := 0, \quad \Delta_{-1}(2, k-1) := 1.
$$

In (2.5) replace $n-1$ by n and take $j = k-1$ to find

$$
\begin{aligned}
(3.4) \quad \Delta_{n-1}(2, k-1)p_{nk-1} &= \Delta_{n-1}(2, k-2)p_{nk} \\
&\quad + a^{(1)}_{n-1}a^{(2)}_{n-1}\cdots a^{(k-1)}_{n-1}p_{(n-1)k}, \qquad n \geq 1.
\end{aligned}
$$

This, together with (2.6) and (3.3), shows that if $P_n(x) = p_{nk}(x)$, $n \geq 0$, then $\{P_n(x)\}$ satisfies (3.1) and (3.2) with $l = 0$. Hence,

$$
(3.5) \qquad P_n(x) = P^{(0)}_n(x), \qquad n \geq 0.
$$

The polynomials $\{P_n(x)\}$ are called the link polynomials of the blocks (1.1) defining $\{p_n(x)\}$.

Let

$$
\begin{aligned}
(3.6) \quad & W((P^{(l)}_n(x), P^{(l+1)}_{n-1}(x)) \\
&= \Delta_l(2, k-1)\begin{vmatrix} P^{(l)}_n(x), & P^{(l+1)}_{n-1}(x) \\ P^{(l)}_{n+1}(x), & P^{(l+1)}_n(x) \end{vmatrix}, \qquad n \geq 0,
\end{aligned}
$$

be the Casorati determinants of $\{P_n^{(l)}(x)\}$. Then

$$W((P_0^{(l)}(x), P_{-1}^{(l+1)}(x)) = \Delta_l(2, k-1)$$

and

$$W(P_n^{(l)}(x), P_{n-1}^{(l+1)}(x)) = \Delta_{n+l-1}(2, k-1)\prod_{j=1}^{n} a_{j+l}^{(0)}a_{j+l-1}^{(1)}\cdots a_{j+l-1}^{(k-1)}, \qquad n \geq 1. \tag{3.7}$$

Since $W(P_n^{(l)}(x), P_{n-1}^{(l+1)}(x))$ is not identically zero, $\{P_n^{(l)}(x)\}$ and $\{P_{n-1}^{(l+1)}(x)\}$ are linearly independent solutions of (3.1).

Let $\{Q_n(x)\}$ be a system of polynomials satisfying (3.1) for $n \geq 1$. Then

$$Q_n(x) = AP_n^{(l)}(x) + BP_{n-1}^{(l+1)}(x), \qquad n \geq 0, \tag{3.8}$$

where

$$A_0 = Q_0(x), \quad B = Q_1(x) - Q_0(x)P_1^{(l)}(x). \tag{3.9}$$

This follows from $\{P_n^{(l)}(x)\}$, $\{P_{n-1}^{(l+1)}(x)\}$ being a basis of solutions of (3.1).

For example, it is readily seen that $\{p_{nk}^{(lk)}(x)\}$ satisfies the recurrence relation (3.1) for $n \geq 1$, and a calculation based on (3.8) and (3.9) gives

$$p_{nk}^{(lk)}(x) = P_n^{(l)}(x) + a_l^{(0)}\Delta_l(2, k-1)\frac{\Delta_{l-1}(2, k-2)}{\Delta_{l-1}(2, k-1)}P_{n-1}^{(l+1)}(x), \tag{3.10}$$

which holds for $l, n \geq 0$. Observe that $p_{nk}(x) = p_{nk}^{(0)}(x) = P_n^{(0)}(x) = P_n(x)$. On the other hand, if $i = 1, 2, \dots, k-1$ and $Q_n(x) = p_{(n+1)k-i}^{(lk+j)}(x)$, then $\{Q_n(x)\}$ satisfies (3.1), with $l+1$ in the place of l, for $n \geq 1$. A calculation based on (2.15), (2.16), (2.17) and (3.9) then gives

$$\begin{aligned} p_{(n+1)k-i}^{(lk+i)}(x) = \Delta_l(i+1, k-1)P_n^{(l+1)}(x) + a_{l+1}^{(0)}\frac{\Delta_{l+1}(2, k-1)}{\Delta_l(2, k-1)} \\ \times [p_{k-2}^{(lk+1)}p_{k-i}^{(lk+i)} - p_{k-1}^{(lk+1)}p_{k-i-1}^{(lk+i)}]P_{n-1}^{(l+2)} \end{aligned} \tag{3.11}$$

for $n \geq 0$, and it is easily verified that

$$\begin{aligned} W(p_{k-2}^{(lk+1)}, p_{k-i-1}^{(lk+i)}) &= [p_{k-2}^{(lk+1)}p_{k-i}^{(lk+i)} - p_{k-1}^{(lk+1)}p_{k-i-1}^{(lk+i)}] \\ &= a_l^{(i)}\cdots a_l^{(k-1)}p_{i-2}^{(lk+1)} \\ &= a_l^{(i)}\cdots a_l^{(k-1)}\Delta_l(2, i-2). \end{aligned} \tag{3.12}$$

Thus we have established the explicit representation

$$\begin{aligned}(3.13)\qquad p_{(n+1)k-i}^{(lk+i)}(x) &= \Delta_l(i+1\,,\,k-1)P_n^{(l+1)}(x)\\ &\quad + a_{l+1}^{(0)}a_l^{(i)}\cdots a_l^{(k-1)}\Delta_l(2\,,\,i-2)\\ &\quad \times \frac{\Delta_{l+1}(2\,,\,k-1)}{\Delta_l(2\,,\,k-1)}P_{n-1}^{(l+2)}(x)\end{aligned}$$

which holds for $n \geq 0$, $l \geq 0$, $i = 1, 2, \ldots, k-1$. When $i = 1$ we have

$$(3.14)\qquad p_{(n+1)k-1}^{(lk+1)}(x) = \Delta_l(2\,,\,k-1)P_n^{(l+1)}(x)\,, \qquad n \geq 0\,,$$

and when $\Delta_n(2\,,\,k-1)$ is independent of n,

$$(3.15)\qquad p_{nk}^{(lk)}(x) = P_n^{(l)}(x) + a_l^{(0)}\Delta_{l-1}(2\,,\,k-2)P_{n-1}^{(l+1)}(x)\,,$$

$$(3.16)\qquad p_{(n+1)k-1}^{(lk+1)}(x) = \Delta_l(2\,,\,k-1)P_n^{(l+1)}(x)\,,$$

$$\begin{aligned}(3.17)\quad p_{(n+1)k-i}^{(lk+i)}(x) &= \Delta_l(i+1\,,\,k-1)P_n^{(l+1)}(x)\\ &\quad + a_{l+1}^{(0)}a_l^{(i)}\cdots a_l^{(k-1)}\Delta_l(2\,,\,i-2)P_{n-1}^{(l+2)}(x)\,,\\ &\qquad n \geq 0\,, i = 2, 3, \ldots, k-1.\end{aligned}$$

Let

$$(3.18)\qquad P^{(l)}(x) = \lim_{n\to\infty} \frac{P_{n-1}^{(l+1)}(x)}{P_n^{(l)}(x)}\,, \qquad x \in \mathbb{C} - \mathbb{R}.$$

The Stieltjes transform of the measure of orthogonality of $\{p_n^{(lk+i)}(x)\}$ is

$$(3.19)\qquad X_{i,l}(x) = \lim_{n\to\infty} \frac{p_{n-1}^{(lk+i+1)}(x)}{p_n^{(lk+i)}(x)}\,, \qquad x \in \mathbb{C} - \mathbb{R}\,,$$

$l \geq 0$, $i = 0, 1, 2, \ldots, k-1$. From (3.10), (3.13), and (3.3), we obtain the following formulae

$$(3.20)\qquad X_{0,0}(x) = \Delta_0(2\,,\,k-1)P^{(0)}(x)\,, \qquad i, l = 0\,,$$

$$\begin{aligned}(3.21)\quad &X_{0,l}(x)\\ &= \frac{\Delta_{l-1}(2\,,\,k-1)\Delta_l(2\,,\,k-1)P^{(l)}(x)}{\Delta_{l-1}(2\,,\,k-1) + a_l^{(0)}\Delta_l(2\,,\,k-1)\Delta_{l-1}(2\,,\,k-2)P^{(l)}(x)}\,,\\ &\qquad l > 0\,,\end{aligned}$$

and

$$X_{i,l}(x)=\frac{N_{i,l}}{D_{i,l}}, \qquad l\geq 0,\ i=1,2,\dots,k-1, \tag{3.22}$$

where

$$\begin{aligned}N_{i,l}=&\Delta_l(2,k-1)\Delta_l(i+2,k-1)\\&+a_{l+1}^{(0)}a_l^{(i+1)}\cdots a_l^{(k-1)}\Delta_{l+1}(2,k-1)\Delta_l(2,i-1)P^{(l+1)}(x)\end{aligned}$$

and

$$\begin{aligned}D_{i,l}=&\Delta_l(2,k-1)\Delta_l(i+1,k-1)\\&+a_{l+1}^{(0)}a_l^{(i)}\cdots a_l^{(k-1)}\Delta_{l+1}(2,k-1)\Delta_l(2,i-2)P^{(l+1)}(x),\end{aligned}$$

for $l\geq 0$, $i=1,2,\dots,k-1$. When $\Delta_n(2,k-1)$ is independent of n, the above relationships simplify to

$$X_{0,0}(x)=\Delta_0(2,k-1)P^{(0)}(x), \tag{3.23}$$

$$X_{0,l}(x)=\frac{\Delta_l(2,k-1)P^{(l)}(x)}{1+a_l^{(0)}\Delta_{l-1}(2,k-2)P^{(l)}(x)}, \qquad l>0,\ i=0, \tag{3.24}$$

and

$$\begin{aligned}&X_{i,l}(x)\\&=\frac{\Delta_l(i+2,k-1)+a_{l+1}^{(0)}a_l^{(i+1)}\cdots a_l^{(k-1)}\Delta_l(2,i-1)P^{(l+1)}(x)}{\Delta_l(i+1,k-1)+a_{l+1}^{(0)}a_l^{(i)}\cdots a_l^{(k-1)}\Delta_l(2,i-2)P^{(l+1)}(x)},\end{aligned} \tag{3.25}$$

$l\geq 0$, $i=1,2,\dots,k-1$.

REMARK 3.1. When $\Delta_n(2,k-1)$ is independent of n, (3.1) becomes

$$\begin{aligned}&[(x-b_{n+l}^0\Delta_{n+l}(2,k-1)-a_{n+l}^{(1)}\Delta_{n+l}(3,k-1)\\&\qquad\qquad -a_{n+l}^{(0)}\Delta_{n+l-1}(2,k-2)]P_n^{(l)}(x)\\&=P_{n+1}^{(l)}(x)+a_{n+l}^{(0)}a_{n+l-1}^{(1)}\cdots a_{n+l-1}^{(k-1)}P_{n-1}^{(l)}(x), \qquad n\geq 1,\end{aligned} \tag{3.26}$$

and (3.2) continues to hold.

4. Connection with polynomial mappings. Let $\{q_n(x)\}$ be a system of polynomials such that $q_0(x)=1$ and for every n, $q_n(x)$ has degree n and positive leading coefficient. In addition, assume that the polynomial set $\{q_n(x)\}$ is orthonormal with respect to a probability measure μ whose support is contained in $[-s,s]$, $0<s<+\infty$. Let

$T(x)$ be a polynomial of degree $k \geq 2$ with simple zeros such that $T(x) \geq s$ whenever $T'(x) = 0$. We say that $T(x)$ is a polynomial mapping for $\{q_n(x)\}$. Choose $W(x) = k^{-1}T'(x)$, and let $\{p_n(x)\}$ be the system of monic orthogonal polynomials obtained from $\{q_n(x)\}$ via the polynomial mapping $T(x)$ (with $W(x)$ as above) in the sense of Geronimo and Van Assche **[12]**. Assume $\{p_n(x)\}$ is given by (1.1) and (1.2). It follows from (2.3) of **[12]** that

$$p_{nk}(x) = c^{-n}\sqrt{\lambda_n}q_n(T(x)), \qquad n \geq 0, \tag{4.1}$$

where c is the leading coefficient of $T(x)$ and

$$\lambda_n = \frac{a_n^{(0)}}{a_0^{(0)}}\prod_{j=0}^{n-1} a_j^{(0)}a_j^{(1)}\cdots a_j^{(k-1)}, \qquad n \geq 0. \tag{4.2}$$

More explicitly, let $T(x)$ and $W(x)$ be as above, and assume that a system of polynomials $\{Q_n(x)\}$ is given by

$$\begin{gathered}(x - C_n)Q_n(x) = A_nQ_{n+1}(x) + B_nQ_{n-1}(x), \qquad n \geq 0, \\ Q_{-1}(x) = 0, \quad Q_0(x) = 1.\end{gathered} \tag{4.3}$$

Let $\{q_n(x)\}$ be the corresponding system of orthonormal polynomials; i.e.,

$$q_n(x) = \frac{Q_n(x)}{\sqrt{\Lambda_n}}, \qquad n \geq 0,$$

where

$$\Lambda_n = \int_{-s}^{s} Q_n^2(x)\, d\mu(x).$$

If $T(x) = c\widehat{T}(x)$ with $\widehat{T}(x)$ monic, then

$$\begin{aligned}&(\widehat{T}(x) - c^{-1}C_n)p_{nk}(x) \\ &\quad = p_{nk+k}(x) + c^{-2}A_{n-1}B_np_{(n-1)k}(x), \qquad n \geq 1, \\ &p_0(x) = 1, \qquad p_k(x) = \widehat{T}(x) - c^{-1}C_0,\end{aligned} \tag{4.4}$$

so that

$$p_{nk}(x) = c^{-n}A_0\cdots A_{n-1}Q_n(T(x)), \qquad n \geq 0. \tag{4.5}$$

We also say that $\{p_n(x)\}$ is obtained from $\{Q_n(x)\}$ via the polynomial mapping $T(x)$. Our next result gives a sufficient condition for $\Delta_n(2, k-1)$ to be independent of n.

Theorem 4.1. *Let $\{p_n(x)\}$, as in (1.1) and (1.2), be obtained from $\{Q_n(x)\}$, given by (4.3), via the polynomial mapping $T(x)$. Then $\Delta_n(2, k-1)$ must be independent of n.*

Proof. Let $T(x) = c\widehat{T}(x)$ and $\widehat{T}(x)$ monic. Then (4.4) and (4.5) hold, and from (3.1) with $1 = 0$ and (3.5) we obtain

$$\begin{aligned}
&[(x - b_n^{(0)})\Delta_n(2, k-1)\Delta_{n-1}(2, k-1) \\
&\quad - a_n^{(1)}\Delta_n(3, k-1)\Delta_{n-1}(2, k-1) \\
&\quad - a_n^{(0)}\Delta_n(2, k-1)\Delta_{n-1}(2, k-2) \\
&\qquad\qquad - \Delta_{n-1}(2, k-1)(\widehat{T}(x) - c^{-1}C_n)]p_{nk} \\
&= [\Delta_n(2, k-1)a_n^{(0)}a_{n-1}^{(1)}\cdots a_{n-1}^{(k-1)} \\
&\qquad - \Delta_{n-1}(2, k-1)c^{-2}A_{n-1}B_n]p_{(n-1)k}, \qquad n \geq 1.
\end{aligned}$$

Since the left-hand side is either 0 or a polynomial of degree at least nk, whereas the right-hand side has degree $nk - 1$ at the most, both sides must vanish. Thus,

$$\Delta_n(2, k-1) = \Delta_{n-1}(2, k-1) = \Delta_0(2, k-1),$$
$$a_n^{(0)}a_{n-1}^{(1)}\cdots a_{n-1}^{(k-1)} = c^{-2}A_{n-1}B_n, \qquad n \geq 1.$$

Remark 4.1. The preceding results also imply

$$\widehat{T}(x) = (x - b_0^{(0)})\Delta_0(2, k-1) - a_0^{(1)}\Delta_0(3, k-1) - c^{-1}C_0, \tag{4.6}$$

$$\begin{aligned}
C_n = C_0 + c(&a_n^{(0)}\Delta_{n-1}(2, k-2) \\
&+ a_n^{(1)}\Delta_n(3, k-1) - a_0^{(1)}\Delta_0(3, k-1)), \qquad n \geq 1,
\end{aligned} \tag{4.7}$$

and

$$a_n^{(0)} + a_n^{(1)} = a_0^{(1)}, \qquad n \geq 1. \tag{4.8}$$

Remark 4.2. We shall see in §5 that the condition on $\Delta_n(2, k-1)$ being independent of n is not sufficient for $\{p_n(x)\}$ to be obtained by means of a polynomial mapping. Assume, however, that (1.7) holds and that

$$\Delta_n(x) := a_n^{(0)}\Delta_{n-1}(2, k-2) + a_n^{(1)}\Delta_n(3, k-1) - a_0^{(1)}\Delta_0(3, k-1) \tag{4.9}$$

is independent of x (which implies that (4.8) holds). Let $0 < s < +\infty$ be such that the inverse image of $[-M, M]$ under $\Delta_0(1, k-1)$ is contained in $[-s, s]$, and choose c such that $c\Delta_0(1, k-1) \geq M$ at all points where $\Delta_0'(1, k-1) = 0$. Let

$$T(x) = c\Delta_0(1, k-1). \tag{4.10}$$

Since $\Delta_0(1, k-1) = p_k(x)$, $\Delta_0(1, k-1)$ must have real and simple zeros. Let $\{Q_n(x)\}$ be defined by

$$(x - C_n)Q_n(x) = Q_{n+1} + a_n^{(0)} a_{n-1}^{(1)} \cdots a_{n-1}^{(k-1)} Q_{n-1}(x), \qquad n \geq 1, \tag{4.11}$$

$$Q_0(x) = 1, \quad Q_1(x) = x,$$

where $C_0 = 0$, $C_n = c\Delta_n(x)$, $n \geq 1$. Then

$$p_{nk}(x) = c^{-n} Q_n(T(x)), \qquad n \geq 0, \tag{4.12}$$

and $T(x)$ is a polynomial mapping for $\{Q_n(x)\}$. Hence $\{p_n(x)\}$ can be obtained via a polynomial mapping.

5. Sieved orthogonal polynomials. Let $\{p_n(x)\}$ be given by

$$(x - b_n^{(j)})p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)} p_{nk+j-1}(x), \qquad n \geq 1, \tag{5.1}$$
$$p_{-1}(x) = 0, \qquad p_0(x) = 1,$$

with

$$b_n^{(j)} = 0, \quad 1 \leq j \leq k-1; \quad a_n^{(j)} = \frac{1}{4}, \quad 2 \leq j \leq k-1; \qquad n \geq 0. \tag{5.2}$$

Then $\{p_n^{(i)}(x)\}$, $i = 0, 1, 2, \ldots$, is called a system of sieved orthogonal polynomials. When $k > 2$, $\{p_n(x)\}$ is called a system of sieved orthogonal polynomials of the first kind, and $\{p_n^{(1)}(x)\}$ a system of the second kind. Curiously, because of historical reasons (see [2]) $\{p_n^{(1)}(x)\}$ is not the system of sieved polynomials of the second kind of the system $\{p_n(x)\}$. Instead, the system of sieved polynomials of the second kind of $\{p_n(x)\}$ is the system of polynomials $\{q_n^{(1)}(x)\}$ with $\{q_n(x)\}$ determined by

$$(x - \tilde{b}_n^{(j)})q_{nk+j}(x) = q_{nk+j+1}(x) + \tilde{a}_n^{(j)} q_{nk+j-1}(x), \qquad n \geq 0,\ 0 \leq j \leq k-1, \tag{5.3}$$

where

$$\tilde{a}_n^{(0)} = a_n^{(1)}, \quad \tilde{a}_n^{(1)} = a_n^{(0)}, \quad \tilde{a}_n^{(j)} = \frac{1}{4}, \qquad 2 \leq j \leq k-1, \qquad n \geq 0, \tag{5.4}$$

$$\tilde{b}_n^{(0)} = b_n^{(0)}, \quad \tilde{b}_n^{(j)} = 0, \qquad 1 \leq j \leq k-1, \ n \geq 0.$$

When $k = 2$, the above definition is applicable provided that we choose $\tilde{a}_1^{(0)} = 1/4$ instead of $\tilde{a}_1^{(0)} = a_0^{(1)}$ in (5.4).

If $\{p_n(x)\}$ is a system of sieved polynomials of the first kind, then $\Delta_n(2, k-1) = \widehat{U}_{k-1}(x)$, $\Delta_n(2, k-2) = \Delta_n(3, k-1) = \widehat{U}_{k-2}(x)$, and their monic link polynomials $\{P_n(x)\}$ satisfy

$$\begin{aligned}(5.5)\qquad & [(x - b_n^{(0)})\widehat{U}_{k-1}(x) - (a_n^{(0)} + a_n^{(1)})\widehat{U}_{k-2}(x)]P_n(x) \\ & \quad = P_{n+1}(x) + 4^{2-k}a_n^{(0)}a_{n-1}^{(1)}P_{n-1}(x), \qquad n \geq 1, \\ & P_0(x) = 1, \qquad P_1(x) = (x - b_0^{(0)})\widehat{U}_{k-1}(x) - a_0^{(1)}\widehat{U}_{k-2}(x),\end{aligned}$$

where $\{\widehat{U}_n(x)\}$ (see [18]) is the system of monic Chebyshev polynomials of the second kind: $\widehat{U}_{-1}(x) = 0$, $\widehat{U}_0(x) = 1$; $x\widehat{U}_n(x) = \widehat{U}_{n+1} + \frac{1}{4}\widehat{U}_{n-1}(x)$, $n \geq 0$. This follows from (3.26). Relation (5.5) can also be written in the form

$$\begin{aligned}(5.6)\qquad & 2^{1-k}[T_k(x) - b_n^{(0)}U_{k-1}(x) + (1 - 2(a_n^{(0)} + a_n^{(1)}))]P_n(x) \\ & \quad = P_{n+1}(x) + 4^{2-k}a_n^{(0)}a_{n-1}^{(1)}P_{n-1}(x), \qquad n \geq 1, \\ & P_0(x) = 1, \\ & P_1(x) = 2^{1-k}[T_k(x) - b_0^{(0)}U_{k-1}(x) + (1 - 2a_0^{(1)})U_{k-2}(x)],\end{aligned}$$

where $U_n(x) = 2^n\widehat{U}_n(x) = \sin(n+1)\theta/\sin\theta$, if $x = \cos\theta$, and $T_0(x) = 1$, $T_n(x) = \frac{1}{2}((U_n(x) - U_{n-2}(x))$, $n \geq 0$, are Chebyshev polynomials of the second and first kinds, respectively. It follows that if

$$(5.7)\qquad a_0^{(1)} = \tfrac{1}{2}; \quad b_n^{(0)} = 0, \quad a_{n+1}^{(0)} + a_{n+1}^{(1)} = \tfrac{1}{2}, \qquad n \geq 0,$$

in which case $\{p_n(x)\}$ is called a system of sieved random walk polynomials of the first kind (see [7], [9]), then

$$(5.8)\qquad p_{nk}(x) = P_n(x) = \frac{1}{2^{n(k-1)}}\widehat{Q}_n(T_k(x))$$

where $\{\widehat{Q}_n(x)\}$ is the system of orthogonal polynomials determined by

$$\begin{aligned}(5.9)\qquad & x\widehat{Q}_n(x) = \widehat{Q}_{n+1}(x) + 4a_n^{(0)}a_{n-1}^{(1)}\widehat{Q}_{n-1}(x), \qquad n \geq 0, \\ & \widehat{Q}_{-1}(x) = 0, \quad \widehat{Q}_0(x) = 1.\end{aligned}$$

In other words, $\{\widehat{Q}_n(x)\}$ is the system of monic polynomials of the system $\{Q_n(x)\}$ given by

$$\begin{aligned}(5.10)\qquad & xQ_n(x) = A_nQ_{n+1}(x) + B_nQ_{n-1}(x), \\ & Q_{-1}(x) = 0, \quad Q_0(x) = 1\end{aligned}$$

with $A_n = 2a_n^{(1)}$, $B_n = 2a_n^{(0)}$, $n \geq 1$, $A_0 = 1$, $B_0 = 0$. Relation (5.8), which can also be written

$$p_{nk}(x) = \frac{A_0 \cdots A_{n-1}}{2^{n(k-1)}} Q_n(T_k(x)), \tag{5.11}$$

means that $\{p_n(x)\}$ is obtained from $\{Q_n(x)\}$ (or $\{\widehat{Q}_n(x)\}$) via the polynomial mapping $T(x) = T_k(x)$. Since $A_n + B_n = 1$, $n \geq 0$, $\{Q_n(x)\}$ is a system of random walk polynomials ([7], [9]). The converse is a consequence of the following theorem.

THEOREM 5.1. *Let $\{p_n(x)\}$ be a system of sieved polynomials of the first kind, and assume that $\{p_n(x)\}$ is obtained from the system of orthogonal polynomials $\{Q_n(x)\}$,*

$$\begin{gathered}(x - C_n)Q_n(x) = A_n Q_{n+1}(x) + B_n Q_{n-1}(x), \qquad n \geq 0, \\ Q_{-1}(x) := 0, \quad Q_1(x) := 1,\end{gathered} \tag{5.12}$$

by means of the polynomial mapping $T(x)$. If $k > 2$, then

$$b_n^{(0)} = b_0^{(0)}, \quad a_{n+1}^{(0)} + a_{n+1}^{(1)} = a_0^{(1)}, \qquad n \geq 0, \tag{5.13}$$

and $Q_n(x) = R_n(x - C_0)$, where $\{R_n(x)\}$ is a system of symmetric polynomials.

Proof. Assume $\{p_n(x)\}$ is obtained from (5.12) by means of the polynomial mapping $T(x) = c\widehat{T}(x)$, with $\widehat{T}(x)$ a monic polynomial of degree k. It follows at once that

$$p_{nk}(x) = c^{-n} A_0 \cdots A_{n-1} Q_n(T(x)), \qquad n \geq 1; p_0(x) = 1, \tag{5.14}$$

so that

$$\begin{aligned}&(\widehat{T}(x) - c^{-1}C_n)p_{nk}(x) \\ &\quad = p_{nk+k}(x) + c^{-2}A_{n-1}B_n p_{(n-1)k}(x), \qquad n \geq 1, \\ &\qquad p_0(x) := 1, \quad p_k(x) = \widehat{T}(x) - c^{-1}C_0.\end{aligned} \tag{5.15}$$

Thus,

$$\widehat{T}(x) - c^{-1}C_0 = \widehat{T}_k(x) - b_0^{(0)}\widehat{U}_{k-1}(x) + \tfrac{1}{2}[1 - 2a_0^{(1)}]\widehat{U}_{k-2}(x) \tag{5.16}$$

and

$$\begin{aligned}\widehat{T}(x) - c^{-1}C_n = \widehat{T}_k(x) &+ \tfrac{1}{2}[1 - (a_n^{(0)} + a_n^{(1)})]\widehat{U}_{k-2}(x) \\ &- b_n^{(0)}\widehat{U}_{k-1}(x), \qquad n \geq 1.\end{aligned} \tag{5.17}$$

Therefore,

$$(5.18)\qquad c^{-1}(C_n - C_0) = [a_0^{(1)} - (a_n^{(0)} + a_n^{(1)})]\widehat{U}_{k-2}(x) + (b_0^{(0)} - b_n^{(0)})\widehat{U}_{k-1}(x), \qquad n \geq 1,$$

so that

$$(5.19)\qquad b_n^{(0)} = b_0^{(0)}, \quad a_{n+1}^{(0)} + a_{n+1}^{(1)} = a_0^{(1)}, \quad C_n = C_0, \qquad n \geq 0.$$

Hence (5.18) holds, and if $R_n(x) = Q_n(x+C_0)$, $n \geq 0$, then $\{R_n(x)\}$ is a system of symmetric orthogonal polynomials and $Q_n(x) = R_n(x - C_0)$. □

COROLLARY 5.1. *Assume the polynomial mapping of Theorem* 5.1 *is* $T(x) = c\widehat{T}_k(x)$, $c \geq 2^{k-1}$, *and that* $\{p_n(x)\}$ *is obtained from the system* $\{Q_n(x)\}$ *by means of the mapping* $T(x)$. *If* $k > 2$, *then* $\{p_n(x)\}$ *is a system of sieved random walk polynomials of the first kind. If, in addition,* $c = 2^{k-1}$, *then* $Q_n(x)$ *is a system of random walk polynomials.*

Proof. From (5.15),

$$\widehat{T}_k(x) - c^{-1}C_0 = \widehat{T}_k(x) - b_0^{(0)}\widehat{U}_{k-1}(x) + \tfrac{1}{2}(1 - 2a_0^{(1)})\widehat{U}_{k-2}(x).$$

It follows that $C_0 = b_0^{(0)} = 0$ and $a_0^{(1)} = \frac{1}{2}$. Also, from (5.17),

$$\widehat{T}_k(x) - c^{-1}C_n = \widehat{T}_k(x) - b_n^{(0)}U_{k-1}(x) + \tfrac{1}{2}[1 - 2(a_n^{(0)} + a_n^{(1)})],$$

so that $b_n^{(0)} = C_n = 0$ and $a_n^{(0)} + a_n^{(1)} = \frac{1}{2}$, $n \geq 1$. On the other hand, if $c = 2^{k-1}$,

$$(5.20)\qquad T_k(x)Q_n(T_k(x)) = A_nQ_{n+1}(T_k(x)) + B_nQ_{n-1}(T_k(x)), \qquad n \geq 1.$$

Also,

$$(5.21)\qquad T_k(x)Q_n'(T_k(x)) = 2a_n^{(1)}Q_{n+1}'(T_k(x)) + 2a_n^{(0)}Q_{n-1}'(T_k(x)), \qquad n \geq 1,$$

with

$$(5.22)\qquad Q_0'(T_k(x)) = 1; \quad Q_n'(T_k(x)) = \frac{2^{n(k-2)}}{a_0^{(1)}a_1^{(1)}\cdots a_{n-1}^{(1)}}p_{nk}(x), \qquad n \geq 1,$$

as follows from (5.10) and (5.11). Hence, from (5.20) and (5.22), $Q_n(x) = Q'_n(x)$, and then $A_n = 2a_n^{(1)}$, $B_n = 2a_n^{(0)}$, $n \geq 1$. Thus, $A_n + B_n = 1$, $n \geq 1$, and, since $A_0 = 2a_0^{(1)} = 1$, it follows that $\{Q_n(x)\}$ is as random walk polynomial system.

Theorem 5.1 and Corollary 5.1 generalize results in **[9]** from the case of symmetric polynomials to general polynomials which are not necessarily symmetric.

REMARK 5.1. The system $\{p_n(x)\}$ of sieved Pollaczek polynomials of the first kind (see **[8]**) has the recurrence coefficients

$$(5.23)\quad a_n^{(0)} = \frac{n}{4(n+a+\lambda)}, \quad a_n^{(1)} = \frac{n+2\lambda}{4(n+a+\lambda)}, \quad a_k^{(j)} = \frac{1}{4}, \quad 2 \leq j \leq k-1, \quad n \geq 0,$$
$$b_n^{(0)} = \frac{b}{n+a+\lambda}, \quad b_n^{(j)} = 0, \quad 1 \leq j \leq k-1, \quad n \geq 0.$$

It follows from Theorem 5.1 that if $k > 2$ and $a \neq 0$, it cannot be obtained from any system of orthogonal polynomials via a polynomial mapping. On the other hand, if $a = b = 0$, then $\{p_n(x)\}$ is a system of sieved random walk polynomials, namely, the sieved ultraspherical polynomials of the first kind of Al-Salam, Allaway and Askey **[2]**, and

$$(5.24)\qquad p_{nk}(x) = \frac{2(\lambda)_n}{(\lambda)_n 2^{nk}} Q_n(T_k(x)), \qquad n \geq 0,$$

where

$$(5.25)\quad xQ_n(x) = \frac{n+2\lambda}{2(n+\lambda)} Q_{n+1}(x) + \frac{n}{2(n+\lambda)} Q_{n-1}(x), \quad n \geq 1,$$
$$Q_0(x) = 1, \quad Q_1(x) = 2x.$$

This follows from (5.11). It is readily seen that

$$(5.26)\qquad Q_n(x) = \frac{n!}{(2\lambda)_n} C_n(x, \lambda), \qquad n \geq 0,$$

where (see **[18]**)

$$(5.27)\quad 2(n+\lambda)C_n(x, \lambda) = (n+1)C_{n+1}(x, \lambda) + (n+2\lambda-1)C_{n-1}(x, \lambda), \qquad n \geq 0,$$
$$C_{-1}(x, \lambda) = 0, \quad C_0(x, \lambda) = 1$$

is the system of ultraspherical polynomials. Thus,

$$(5.28)\qquad p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n} C_n(T_k(x), \lambda), \qquad n \geq 0.$$

We also observe that if under the remaining assumptions of the sieved ultraspherical polynomials of the first kind, i.e.,

$$(5.29)\qquad \begin{aligned} &a_n^{(0)} = \frac{n}{4(n+\lambda)}, \quad a_n^{(1)} = \frac{n+2\lambda}{4(n+\lambda)}, \qquad n \ge 1, \\ &a_n^{(j)} = \frac{1}{4}, \quad b_n^{(0)} = b_n^{(1)} = b_n^{(j)} = 0, \\ &\qquad\qquad\qquad 2 \le j \le k-1, \qquad n \ge 0, \end{aligned}$$

we change $a_0^{(1)}$ from $1/2$ to $\alpha/2$, $\alpha \neq 1$, then, if $k > 2$, $\{p_{nk}(x)\}$ cannot be obtained from any system of orthogonal polynomials by means of polynomial mappings (because $a_n^{(0)} + a_n^{(1)} = 1/2 \neq \alpha/2 = a_0^{(1)}$). However, it easily follows that

$$(5.30)\quad \begin{aligned} p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n}&[C_n(T_k(x), \lambda) \\ &+ 2\lambda(1-\alpha)U_{k-2}(x)C_{n-1}^{(1)}(T_k(x), \lambda) \\ &+ 2\lambda(1-\alpha)C_{n-1}^{(2)}(T_k(x), \lambda)], \\ &\qquad n \ge 0, \end{aligned}$$

or equivalently,

$$(5.31)\quad \begin{aligned} p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n}&[\alpha C_n(T_k(x), \lambda) \\ &+ 2\lambda(1-\alpha)xU_{k-1}(x)C_{k-1}^{(1)}(T_k(x), \lambda)], \\ &\qquad n \ge 1, \end{aligned}$$

where $\{C_n^{(i)}(x, \lambda)\}$ denotes the system of ith-associated polynomials of $\{C_n(x, \lambda)\}$. Note that if $k = 2$, (5.30) shows that $\{p_{nk}(x)\}$ originates via a polynomial mapping.

REMARK 5.2. Let $\{p_{nk}(x)\}$ be given by (5.1) and (5.2), and assume that $\{p_{nk}(x)\}$ is obtained from the system (5.12) by means of a polynomial mapping $T(x)$. It follows from the proof of Theorem 3.1 that if $k \ge 2$ then $b_n^{(0)} = b_0^{(0)}$, $n \ge 0$, i.e., $b_n^{(0)}$ is independent of n. The general (non-symmetric) sieved Pollaczek polynomials do not satisfy this condition (as $b \neq 0$). Hence, they cannot be obtained via polynomial mappings, even if $k = 2$. However, the symmetric sieved Pollaczek polynomials ($b = 0$ in (5.23)) can be obtained via a polynomial mapping when $k = 2$. In fact,

$$(5.32)\qquad p_{2n}(x) = \frac{n!}{4^n(\lambda)_n} P_n(T_2(x)), \qquad n \ge 0,$$

where $P_n(x) = P_n(x, \lambda, a, a)$, $n \geq 0$, is the system of the Pollaczek polynomials

$$(5.33)\quad \begin{gathered} 2[(n+\lambda+a)x+a]P_n(x) \\ = (n+1)P_{n+1}(x) + (n+2\lambda-1)P_{n-1}(x), \qquad n \geq 0, \\ p_{-1}(x) = 0, \quad P_0(x) = 1. \end{gathered}$$

Thus, Theorem 5.1 cannot be extended to the case $k = 2$.

REMARK 5.3. It is usually assumed that $a_0^{(1)} = 1/2$ for sieved polynomials of the first kind (perhaps for historical reasons, because this was indeed the case for the sieved ultra-spherical and random walk polynomials in **[2]**, **[7]**). Here we drop this assumption, and some interesting results will come about. For example, the sieved ultra-spherical polynomials of the first kind in **[2]** (i.e., $a_n^{(j)}$ given by (5.29) with $a_0^{(1)} = 1/2$ and $\lambda > 0$) are orthogonal with respect to an absolutely continuous measure whose support is $[-1, 1]$. However, if $a_0^{(1)}$ is changed to $\alpha/2$ where $\alpha = \frac{2\lambda k}{2\lambda(k-1)+1}$, and $\lambda > 3/2$, the absolutely continuous part of the orthogonality measure of the resulting polynomials $\{p_{nk}(x)\}$ still has $[-1, 1]$ as its support, but now the measure carries masses at the end points ± 1 when k is even. To see this, observe that, from (5.31),

$$p_{nk}(1) = \frac{n!}{2^{nk}(\lambda)_n}[\alpha C_n(1, \lambda) + 2\lambda(1-\alpha)kC_{n-1}^{(1)}(1, \lambda)], \qquad n \geq 1.$$

But

$$C_n(1, \lambda) = \frac{(2\lambda)_n}{n!}, \quad C_{n-1}^{(1)}(1, \lambda) = \frac{1}{2\lambda - 1}\left[\frac{(2\lambda)_n}{n!} - 1\right], \qquad n \geq 0,$$

as follows from (5.27). Hence

$$p_{nk}(x) = \frac{n!}{2^{nk}(\lambda)_n}\left[\frac{2\lambda k - \alpha(2\lambda(k-1)+1)}{2\lambda - 1}\frac{(2\lambda)_n}{n!} - \frac{2\lambda k(1-\alpha)}{2\lambda - 1}\right], \qquad n \geq 1,$$

and, if $\alpha = \frac{2\lambda k}{2\lambda(k-1)+1}$, then

$$p_{nk}(1) = \frac{n!}{2^{nk}(\lambda)_n} \cdot \frac{2\lambda k(\alpha - 1)}{2\lambda - 1}, \qquad n \geq 1.$$

Let μ denote the orthogonality measure of $\{p_n(x)\}$. The measure μ is compactly supported and its absolutely continuous part has support $[-1, 1]$. Furthermore,

$$\int_{-\infty}^{+\infty} p_n(x)p_m(x)\, d\mu(x) = \lambda_n\delta_{mn}, \qquad m, n \geq 0,$$

where

$$\lambda_0 = 1; \quad \lambda_{kn} = \frac{\alpha}{4^{kn}} \cdot \frac{(2\lambda)_n \cdot n!}{(\lambda)_n^2(\lambda+n)},$$
$$\lambda_{nk+j} = \frac{\alpha}{4^{nk+j}} \cdot \frac{(2\lambda)_{n+1} \cdot n!}{(\lambda)_{n+1}^2},$$
$$n \geq 1, \quad 1 \leq j \leq k-1.$$

It follows that

$$\frac{p_{nk}^2(1)}{\lambda_{kn}} = \frac{n!(\lambda+k)}{(2\lambda)_n\alpha}\left[\frac{2\lambda k(1-\alpha)}{2\lambda-1}\right]^2, \qquad n \geq 1,$$

and, since $\alpha \neq 1$, that

$$\frac{p_{nk}^2(1)}{\lambda_{kn}} \sim \left[\frac{2\lambda k(1-\alpha)}{2\lambda-1}\right]^2 \cdot \frac{\Gamma(2\lambda)}{\alpha} \cdot n^{2-2\lambda}, \qquad n \geq 1.$$

Since $\lambda > 3/2$, $\sum_{n=0}^{\infty} p_{nk}^2(1)/\lambda_{kn}$ converges. Moreover, it follows from (2.5) that

$$U_{k-1}(x)p_{nk+j}(x) = 2^{k-j}U_{j-1}(x)p_{nk+k}(x) + 2^{-j}\frac{n+2\lambda}{n+\lambda}U_{k-j-1}(x), \qquad 1 \leq j \leq k-1.$$

Thus

$$k\frac{p_{nk+j}(1)}{\sqrt{\lambda_{kn+j}}} = j\sqrt{\frac{n+1}{n+\lambda+1}} \cdot \frac{p_{(n+1)k}(1)}{\sqrt{\lambda_{(n+1)k}}} + (k-j)\sqrt{\frac{n+2\lambda}{n+\lambda}}\frac{p_{nk}(1)}{\sqrt{\lambda_{nk}}}$$

from which we deduce (using the inequality $(a+b)^2 \leq 2(a^2+b^2)$) that

$$\sum_{k=0}^{\infty}\frac{p_{nk+j}^2(1)}{\lambda_{nk+j}} < +\infty, \qquad j = 1, 2, \ldots, k-1.$$

Hence $\sum_{n=0}^{\infty} p_n^2(1)/\lambda_n < +\infty$, and from [**4**, p. 13], we conclude that μ has a mass at $x = 1$ and, thus, also at $x = -1$. We observe that if $k > 2$, this conclusion cannot be obtained from the theory of polynomial mappings as presented in [**12**].

REMARK 5.4. Under the circumstances above it can be shown that

$$\mu(\{-1\}) = \mu(\{1\}) = \frac{2\lambda-3}{2(2\lambda-1)} \tag{5.34}$$

when $k = 2$ (see [**10**]).

We finally give an example of how our procedure can be advantageous over other treatments of sieved orthogonal polynomials. To this purpose we shall consider an example of sieved orthogonal polynomials recently dealt with by Al-Salam and Ismail **[1]**: the sieved associated Pollaczek polynomials. Contrary to ours, their treatment is historical, and the polynomials are obtained from the associated q-Pollaczek polynomials (see **[3]**) by the same limit process as in **[8]**, **[13]**. Then, the limit process is used to establish generating functions for the polynomials, a very delicate matter, and the asymptotic behavior and the Stieltjes transform of the orthogonality measure are determined via Darboux's method (**[15]**, Chap. VIII). We follow a more direct approach.

We recall that the system of associated Pollaczek polynomials $\{R_n(x)\}$ is determined (see **[16]**) by the recurrence relations

(5.35) $$\begin{aligned} &2[(\lambda+n+a+c)x+b]R_n(x) \\ &\quad = (n+c+1)R_{n+1}(x)+(n+c+2\lambda-1)R_{n-1}(x), \\ &\qquad n\geq 0, \quad R_{-1}(x)=0, \quad R_0(x)=1. \end{aligned}$$

The notation $R_n(x) = P_n(x;\lambda, a, b, c)$ is also used. We observe that if $P_n(x;\lambda, a, b) = P_n(x;\lambda, a, b, 0)$ and $c = 1, 2, \ldots,$ then $\{P_n(x;\lambda, a, b, c)\}$ is the system of cth-associated polynomials of $\{P_n(x;\lambda, a, b)\}$. The latter system is simply called the system of Pollaczek polynomials.

If $\lambda > 0$ and $a, c \geq 0$, $\{R_n(x)\}$ is a system of orthogonal polynomials (other cases of orthogonality are possible). Let

(5.36) $$R(x, t) = \sum_{n=0}^{\infty} R_n(x)t^{n+c}.$$

By showing from the recurrence relation (5.35) that

(5.37) $$\frac{\partial R(x, t)}{\partial t} - \frac{2((a+\lambda)x-\lambda t+b)}{t^2-2xt+1}R(x, t) = \frac{t^{c-1}}{t^2-2xt+1}$$

and

(5.38) $$R(x, 0)=1, \quad c=0; \qquad R(x, 0)=0, \quad c>0,$$

it follows that

(5.39) $$\begin{aligned} R(x, t) &= c(1-\beta t)^A(1-\alpha t)^B \\ &\quad \times \int_0^t u^{c-1}(1-\beta u)^{-A-1}(1-\alpha u)^{-B-1}\,du, \\ &\qquad c>0, \end{aligned}$$

and

$$R(x, t) = (1-\beta t)^A (1-\alpha t)^B, \qquad c = 0, \tag{5.40}$$

where

$$\alpha = \alpha(x) = x + \sqrt{x^2-1}, \quad \beta = \beta(x) = x - \sqrt{x^2-1} \tag{5.41}$$

and

$$A = -\lambda + 2\frac{ax+b}{\alpha-\beta}, \quad B = -\lambda - 2\frac{ax+b}{\alpha-\beta}. \tag{5.42}$$

From (5.39) and (5.40), and observing that $R_n^{(1)}(x)$ is $R_n(x)$ with $c+1$ instead of c, it can be deduced (via Darboux's method, for example) that the Stieltjes transform of the orthogonality measure μ of $\{R_n(x)\}$:

$$R(x) = \lim_{n\to\infty} \frac{R_{n-1}^{(1)}(x)}{R_n(x)} = \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t-x}, \qquad x \in \mathbb{C} - \mathbb{R}, \tag{5.43}$$

is

$$R(x) = \beta \int_0^{1]} (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\, du, \qquad c = 0 \tag{5.44}$$

and

$$R(x) = \frac{c+1}{c} \frac{\int_0^{1]} u^c (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\, du}{\int_0^{1]} u^{c-1} (1-\beta^2 u)^{-A-1}(1-u)^{-B-1}\, du}, \qquad c > 0. \tag{5.45}$$

We observe that the integrals in (5.44) and (5.45) are Hadamard integrals. As as matter of fact

$$\begin{aligned} &\int_0^{1]} u^c (1-zu)^{-A-1}(1-u)^{-B-1}\, du \\ &\quad = \frac{\Gamma(c+1)\Gamma(-B)}{\Gamma(-B+c+1)}\, {}_2F_1\left(\left. \begin{matrix} A+1,\ c+1 \\ -B+c+1 \end{matrix} \right| z \right), \\ &\qquad\qquad |z| < 1, \ c > -1, \end{aligned} \tag{5.46}$$

and the integral makes sense as long as B is not an integer ≥ 0 (and not only when $\mathrm{Re}(B) < 0$). This was discussed in §1.

The branch $\sqrt{x^2-1}$ of the square root of x^2-1 in (5.41) is so chosen that $\sqrt{x^2-1} \sim x$ as $x \to \infty$.

Relation (5.44) can be obtained from (5.45) by taking

$$c\int_0^1 u^{c-1}\left(1-\frac{\beta}{\alpha}u\right)^{-A-1}(1-u)^{-B-1}\,du = 1 \tag{5.47}$$

when $c = 0$.

We begin by considering the system $\{q_n(x)\}$ determined by

$$\begin{aligned}(x-b_n^{(j)})q_{nk+j}(x) = q_{nk+j+1}(x)+a_n^{(j)}q_{nk+j-1}(x), \qquad n \ge 0,\\ 0 \le j \le k-1,\ q_{-1}(x) = 0, \quad q_0(x) = 1.\end{aligned} \tag{5.48}$$

We assume $k \ge 2$ and

$$a_n^{(0)} = \frac{n+2\lambda+c}{4(n+\lambda+a+c)}, \quad a_n^{(1)} = \frac{n+c}{4(n+\lambda+a+c)}, \qquad n \ge 0. \tag{5.49}$$

$$b_n^{(0)} = \frac{b}{n+\lambda+a+c}, \quad b_n^{(j)} = 0,\ 1 \le j \le k-1, \qquad n \ge 0. \tag{5.50}$$

and

$$a_n^{(j)} = \frac{1}{4}, \qquad n \ge 0,\ 2 \le j \le k-1. \tag{5.51}$$

Thus, the system $p_n(x) = q_n^{(1)}(x)$, $n \ge 0$, will be the system of sieved associated Pollaczek polynomials of the second kind. Clearly $\{p_n^{(r)}(x)\}$, their system of associated polynomials of order r, is the system of monic polynomials of the orthogonal polynomials $\{Q_n^{(\lambda,r)}(x)\}$ in **[1]**, for $0 \le r < k$.

Let $\{P_n(x)\}$ denote the link polynomials of $\{q_n(x)\}$. Then $\{p_n^{(r)}(x)\}$ can be represented in terms of the polynomials $\{P_n^{(1)}(x)\}$ and $\{P_n^{(2)}(x)\}$ via (3.15)–(3.17). Now, $\{P_n^{(1)}(x)\}$ satisfies

$$\begin{aligned}\frac{1}{2^{k-1}}\Bigg[T_k(x)+\frac{b}{n+\lambda+a+c+1}U_{k-1}(x)\\ +\frac{a}{n+\lambda+a+c+1}U_{k-2}(x)\Bigg]P_n^{(1)}(x)\\ = P_{n+1}^{(1)}(x)+4^{1-k}\frac{n+2\lambda+c+1}{n+\lambda+a+c+1}\cdot\frac{n+c}{n+\lambda+a+c}P_{n-1}^{(1)}(x),\\ n \ge 1\end{aligned} \tag{5.52}$$

and the initial conditions

$$(5.53)\quad \begin{aligned} P_0^{(1)}(x) &= 1\,, \\ P_1^{(1)}(x) &= \frac{1}{2^{k-1}}\left[T_k(x) + \frac{b}{\lambda+a+c+1}U_{k-1}(x) \right. \\ &\qquad \left. + \frac{c}{\lambda+a+c+1}U_{k-1}(x)\right]. \end{aligned}$$

If

$$(5.54)\qquad \widetilde{R}_n(x) = \frac{2^{nk}(\lambda+a+c+1)_n}{(c+1)_n}P_n^{(1)}(x)\,, \qquad n \geq 0\,,$$

then

$$(5.55)\quad \begin{aligned} 2[(n+\lambda+a+c+1)T_k(x) + bU_{k-1}(x) + aU_{k-2}(x)]\widetilde{R}_n(x) \\ = (n+c+1)\widetilde{R}_{n+1}(x) + (n+2\lambda+c+1)\widetilde{R}_{n-1}(x)\,, \\ n \geq 0\,, \end{aligned}$$

and

$$(5.56)\quad \begin{aligned} \widetilde{R}_0(x) &= 1\,, \\ \widetilde{R}_1(x) &= \frac{2}{c+1}[(\lambda+a+c+1)T_k(x) + bU_{k-1}(x) + cU_{k-2}(x)]. \end{aligned}$$

As in the case of the Pollaczek polynomials, it can be shown that

$$(5.57)\quad \begin{aligned} \sum_{n=0}^{\infty}\widetilde{R}_n(x)t^{n+c} &= c(1-\beta^k t)^{A-1}(1-\alpha^k t)^{B-1} \\ &\quad \cdot \int_0^t u^{c-1}(1-\beta^k u)^{-A}(1-\alpha^k u)^{-B}\,du \end{aligned}$$

where $\alpha = \alpha(x)$, $\beta = \beta(x)$ and

$$(5.58)\quad \begin{aligned} A &= -\lambda + 2\frac{2T_k(x) + bU_{k-1}(x) + aU_{k-2}(x)}{\beta^k - \alpha^k} \\ &= -\lambda + 2\frac{ax+b}{\alpha-\beta}\,, \\ B &= -\lambda - 2\frac{aT_k(x) + bU_{k-1}(x) + aU_{k-2}(x)}{\beta^k - \alpha^k} \\ &= -\lambda - 2\frac{ax+b}{\alpha-\beta}. \end{aligned}$$

We note that $\alpha^k(x) = \alpha(T_k(x))$, $\beta^k(x) = \beta(T_k(x))$. From (5.57) it can be deduced (via Darboux's method, for example, in the same manner as it is done for the polynomials $\{R_n(x)\}$), that

$$(5.59)\qquad \widetilde{R}(x) = \lim_{n\to\infty} \frac{\widetilde{R}^{(1)}_{n-1}(x)}{\widetilde{R}_n(x)} = \frac{c+1}{c}\beta^k \frac{\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du}{\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du}.$$

On the other hand,

$$(5.60)\qquad P_n^{(2)}(x) = \frac{(c+2)_n}{2^{nk}(\lambda+a+c+2)_n}\widetilde{R}_n^{(1)}(x).$$

Therefore,

$$(5.61)\qquad P^{(1)}(x) = \lim_{n\to\infty}\frac{P^{(2)}_{n-1}(x)}{P^{(1)}_n(x)} = 2^k\frac{(\lambda+a+c+1)}{c+1}R(x), \qquad x\in\mathbb{C}-\mathbb{R}.$$

Now, it follows from (3.25) that the continued fraction $X_r(x)$ of $\{p_n^{(r)}(x)\}$, i.e., of $\{q_n^{(r+1)}(x)\}$, and thus of $\{Q_n^{(\lambda,r)}(x)\}$, is, for $0\le r<k-1$,

$$(5.62)\qquad X_r(x) = 2\frac{2^{k-2}U_{k-r-2}(x)+a_1^{(0)}U_r(x)P^{(1)}(x)}{2^{k-1}U_{k-r-1}(x)+a_1^{(0)}U_{r-1}(x)P^{(1)}(x)} = 2A/B$$

where

$$A = cU_{k-r-2}(x)\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du + (2\lambda+c+1)U_r(x)\beta^k\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du$$

and

$$B = cU_{k-r-1}(x)\int_0^1 u^{c-1}(1-\beta^2u)^{-A}(1-u)^{B}\,du + (2\lambda+c+1)U_{r-1}(x)\beta^k\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du.$$

which, after using the identities

$$(5.63)\quad \begin{aligned}(2\lambda+c+1)&\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &=(-A)\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\\ &\quad+(-B)\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B-1}\,du\\ &=c\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &\quad-A(1-\beta^{2k})\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\end{aligned}$$

and

$$(5.64)\qquad U_j(x)=\frac{\alpha^{j+1}-\beta^{j+1}}{\alpha-\beta},$$

becomes

$$(5.65)\qquad X_r(x)=2\beta C/E$$

where

$$\begin{aligned}C=c&\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &+A\beta^{2k}(1-\alpha^{2r+2})\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\end{aligned}$$

and

$$\begin{aligned}E=c&\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du\\ &+A\beta^{2k}(1-\alpha^{2r})\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\end{aligned}$$

which is (3.5) of **[1]**. Observe that when $c=r=0$, we obtain (using (5.47)) that

$$(5.66)\, X_0(x)=2\left[\beta+(\beta-\alpha)\beta^{2k}A\int_0^1(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du\right]$$

which is (3.39) of **[8]**.

As for the case $r = k-1$, we need to calculate the continued fraction of $\{p_n^{(k-1)}(x)\}$, or the same, of $\{q_n^{(k)}(x)\}$. According to (3.24), this is

(5.67)
$$X_{k-1}(x) = \frac{2^{1-k}U_{k-1}(x)P^{(1)}(x)}{1 + 2^{2-k}a_1^{(0)}U_{k-2}(x)P^{(1)}(x)}.$$

A calculation as above readily gives

(5.68)
$$X_{k-1}(x) = \frac{2\beta(\lambda + a + c + 1)\int_0^1 u^c(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du}{D}$$

where

$$\begin{aligned} D = c&\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A}(1-u)^{-B}\,du \\ &+ A\beta^{2k}(1-\alpha^{2k-2})\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B}\,du. \end{aligned}$$

The above procedure can also be applied to the k-sieved associated Pollaczek polynomials of the first kind $P_n^{(\lambda,r)}(x) = P_n^{(\lambda,r)}(x; a, b, c)$, $k \geq 2$, $n \geq 0$, $r = 0, 1, 2, \ldots, k-1$. These are given by the recurrence relation

(5.69)
$$\begin{aligned} &2xP_n^{(\lambda,r)}(x) = P_{n+1}^{(\lambda,r)}(x) + P_{n-1}^{(\lambda,r)}(x),\ k|n+r, \qquad n \geq 0, \\ &2[(m + a + c + \lambda)x + b]P_{mk-r}^{(\lambda,r)}(x) \\ &\qquad = (m + c + 2\lambda)P_{mk-r+1}^{(\lambda,r)}(x) + mP_{mk-r-1}^{(\lambda,r)}(x),\ n + r = mk, \\ &\qquad\qquad m \geq 0, \end{aligned}$$

and the initial conditions

(5.70)
$$P_{-1}^{(\lambda,r)}(x) = 0, \quad P_0^{(\lambda,r)}(x) = 1.$$

For simplicity we will assume that b is a real number and $\lambda > 0$, $a, c \geq 0$, but other cases of orthogonality can be similarly handled.

It is readily verified that the system of monic polynomials of $\{p_n^{(\lambda,r)}(x)\}$ is the associated system $\{p_n^{(r)}(x)\}$ of order r of the orthogonal polynomial set $\{p_n(x)\}$ given by the blocks

(5.71)
$$(x - b_n^{(j)})p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)}p_{nk+j-1}(x)$$

for $n \geq 0$, $j = 0, 1, 2, \ldots, k-1$, and the initial conditions

(5.72)
$$p_{-1}(x) = 0, \quad p_0(x) = 1,$$

where

$$(5.73)\quad \begin{aligned} b_n^{(0)} &= \frac{-b}{n+a+c+\lambda}; \quad b_n^{(j)} = 0, \qquad j = 1, 2, \dots, k-1, \\ a_n^{(0)} &= \frac{n+c}{4(n+a+c+\lambda)}, \quad a_n^{(1)} = \frac{n+c+2\lambda}{4(n+a+c+\lambda)} \\ a_n^{(j)} &= \frac{1}{4}, \qquad j = 2, 3, \dots, k-1, n \geq 0. \end{aligned}$$

The link polynomials of $\{p_n(x)\}$ satisfy

$$(5.74)\quad \begin{aligned} &2^{1-k}\left[T_k(x) + \frac{a}{n+\lambda+a+c}U_{k-2}(x) + \frac{b}{n+\lambda+a+c}U_{k-1}(x)\right]P_n(x) \\ &= P_{n+1}(x) + 2^{-2k}\frac{n+c}{n+\lambda+a+c-1}\cdot\frac{n+c+2\lambda-1}{n+\lambda+a+c}P_{n-1}(x), \\ &\qquad n \geq 0, \end{aligned}$$

and the initial conditions

$$(5.75)\quad P_{-1}(x) = 0, \quad P_0(x) = 1.$$

If we let

$$(5.76)\quad Q_n(x) = \frac{2^{nk}(\lambda+a+c)_n}{(c+1)_n}P_n(x), \qquad n \geq 0,$$

then $Q_{-1}(x) = 0$, $Q_0(x) = 1$ and

$$(5.77)\quad \begin{aligned} &2[(n+\lambda+a+c)T_k(x) + aU_{k-2}(x) + bU_{k-1}(x)]Q_n(x) \\ &\quad = (n+c+1)Q_{n+1}(x) + (n+c+2\lambda-1)Q_{n-1}(x), \\ &\qquad n \geq 0. \end{aligned}$$

Also

$$(5.78)\quad Q_n^{(1)}(x) = \frac{2^{nk}(\lambda+a+c)_n}{(c+2)_n}P_n^{(1)}(x), \qquad n \geq 0,$$

and, as before, we obtain

$$(5.79)\quad \begin{aligned} &\lim_{n\to\infty}\frac{Q_{n-1}^{(1)}(x)}{Q_n(x)} \\ &= \frac{c+1}{c}\beta^k\frac{\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du}{\int_0^1 u^{c-1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du} \end{aligned}$$

for $x \in \mathbb{C} - \mathbb{R}$, where $\alpha = \alpha(x)$, $\beta = \beta(x)$ are given by (5.41) and $A = A(x)$, $B = B(x)$ by (5.42) or (5.58). Observe that $\alpha(T_k(x)) = \alpha^k(x)$, $\beta(T_k(x)) = \beta^k(x)$. Thus,

$$(5.80) \qquad P^{(0)}(x) = \lim_{n\to\infty} \frac{P_{n-1}^{(1)}(x)}{P_n(x)} = 2^k \frac{(\lambda + a + c)}{c+1} \lim_{n\to\infty} \frac{Q_{n-1}^{(1)}(x)}{Q_n(x)}$$

and therefore, if $x \in \mathbb{C} - \mathbb{R}$, then

(5.81)

$$P^{(0)}(x) = 2^k \frac{\lambda + a + c}{c} \beta^k \frac{\int_0^1 u^c (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}{\int_0^1 u^{c-1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}$$

where as before

$$c \int_0^1 u^{c-1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du = 1, \qquad c = 0.$$

Also,

(5.82)

$$\begin{aligned} P^{(1)}(x) &= \lim_{n\to\infty} \frac{P_{n-1}^{(2)}(x)}{P_n^{(1)}(x)} \\ &= 2^k \frac{\lambda + a + c + 1}{c+1} \beta^k \frac{\int_0^1 u^{c+1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}{\int_0^1 u^{c} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}. \end{aligned}$$

From (5.81) and (5.82) and from relations (4.23) and (3.25), we obtain for the continued fraction $X_r(x)$ of $\{P_n^{(\lambda, r)}(x)\}$ the following evaluation

$$\begin{aligned} (5.83) \quad X_0(x) &= \frac{1}{2^{k-1}} U_{k-1}(x) P^{(0)}(x) \\ &= 2 \frac{\lambda + a + c}{c} \beta^k \\ &\quad \times U_{k-1}(x) \frac{\int_0^1 u^c (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du}{\int_0^1 u^{c-1} (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du} \end{aligned}$$

which reduces to

$$(5.84) \quad X_0(x) = 2(\lambda + a) \beta^k U_{k-1}(x) \int_0^1 (1 - \beta^{2k} u)^{-A-1} (1-u)^{-B-1}\, du$$

when $c = 0$, and

$$X_r(x) = 2\frac{2^{k-2}U_{k-r-1}(x) + a_1^{(0)}U_{r-1}(x)P^{(1)}(x)}{2^{k-2}U_{k-r}(x) + a_1^{(0)}U_{r-2}(x)P^{(1)}(x)} = 2\frac{A+B}{C+D} \tag{5.85}$$

where

$$A = U_{k-r-1}(x)\int_0^1 u^c(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du,$$

$$C = U_{k-r}(x)\int_0^1 u^{c+1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du,$$

$$B = U_{r-1}(x)\beta^k\int_0^1 u^{c+1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du,$$

$$D = U_{r-2}(x)\beta^k\int_0^1 u^{c+1}(1-\beta^{2k}u)^{-A-1}(1-u)^{-B-1}\,du$$

for $c \geq 0$ and $r = 1, 2, \ldots, k-1$.

Acknowledgments. The authors thank G. Rodriguez of the Universidad de los Andes, Bogotá, for his help. The first author also thanks Colciencias, Bogotá, for partial travel support to attend the First U.S.A.–U.S.S.R. Conference on Approximation Theory, Tampa, Florida, March, 1990, where some of the results in this paper were presented.

References

[1] N. A. Al-Salam and M. E. H. Ismail, *On sieved orthogonal polynomials* VIII: *Sieved associated Pollaczek polynomials*, J. Approximation Theory, (1991), to appear.

[2] W. Al-Salam, W. R. Allaway and R. Askey, *Sieved ultraspherical polynomials*, Trans. Amer. Math. Soc., **234** (1984), 39–55.

[3] R. Askey and M. E. H. Ismail, *A generalization of ultraspherical polynomials*, in *Studies in Pure Mathematics* (P. Erdös, Ed.), Birkhäuser, Basel, 1983, 55–78.

[4] ——, *Recurrence relations, continued fractions and orthogonal polynomials*, Mem. Amer. Math. Soc., **300** (1984), 110 pp.

[5] H. Bremerman, *Distributions, Complex Variables and Fourier Transforms*, Addison-Wesley, Reading, Mass., 1965.

[6] J. A. Charris, C. P. Gomez and G. Rodriguez, *Two systems of orthogonal polynomials related to the Pollaczek polynomials*, to appear, Rev. Col. de Mat.

[7] J. A. Charris and M. E. H. Ismail, *On sieved orthogonal polynomials*, II: *Sieved random walk polynomials*, Canad. J. Math., **38** (1986), 397–415.

[8] ——, *On sieved orthogonal polynomials*, V: *Sieved Pollaczek polynomials*, SIAM J. Math. Anal., **18** (1987), 1177–1218.

[9] ——, *On sieved orthogonal polynomials*, VII: *Generalized polynomial mappings*, to appear, Trans. Amer. Math. Soc.

[10] J. A. Charris and G. Rodriguez-Blanco, *On systems of orthogonal polynomials with inner and end-point masses*, Rev. Col. de Mat., **24** (1990), 153–177.

[11] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[12] J. Geronimo and W. Van Assche, *Orthogonal polynomials on several intervals via a polynomial mapping*, Trans. Amer. Math. Soc., **308** (1986), 559–581.

[13] M. E. H. Ismail, *On sieved orthogonal polynomials*, I: *Symmetric Pollaczek analogues*, SIAM J. Math. Anal., **16** (1985), 89–111.

[14] S. Lang, *Real Analysis*, Addison-Wesley, Reading, Mass., 1984.

[15] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[16] F. Pollaczek, *Sur une famille de polynomes orthogonaux a quatre parametres*, C.R. Acad. Sci. Paris, **230** (1950), 2254–2256.

[17] ——, *Sur une generalization des polynomes de Jacobi*, Memor. Sci. Mathematiques, **131** (1956), Gauthier-Villars, Paris.

[18] E. D. Rainville, *Special Functions*, Macmillan, New York, 1974.

[19] J. Shohat and J. D. Tamarkin, *The Problem of Moments*, Math. Surveys, Vol. 1, Amer. Math. Soc., Providence, R.I., 1963.

[20] H. A. Slim, *On co-recursive orthogonal polynomials and their applications to potential scattering*, J. Math. Anal. Appl., **136** (1988), 1–19.

[21] J. C. Wheeler, *Modified moments and continued fraction coefficients for the diatomic linear chain*, J. Chem. Phys., **80** (1984), 472–476.

Received April 2, 1991 and in revised form August 8, 1991. Research partially supported by NSF grants DMS8814026, DMS8912423 and INT8803099.

J. Charris
The National University of Colombia
Bogota, Colombia

and

M.E.H. Ismail
University of South Florida
Tampa, FL 33620-5700

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

ASYMPTOTIC RADIAL SYMMETRY FOR SOLUTIONS OF $\Delta u + e^u = 0$ IN A PUNCTURED DISC

K. S. CHOU AND TOM Y. H. WAN

In this paper a representation formula for solutions of the equation

$$(*) \qquad \Delta u + 2Ke^u = 0, \quad K \text{ a constant},$$

in a punctured disc in terms of multi-valued meromorphic functions is found. As application it is deduced that a necessary and sufficient condition for a solution of $(*)$, $K > 0$, being asymptotic radially symmetric is

$$\int e^u < \infty.$$

1. Introduction. In [3], L. A. Caffarelli, B. Gidas, and J. Spruck proved that non-negative smooth solutions of the conformally invariant equation

$$(1) \qquad \Delta u + u^{(n+2)/(n-2)} = 0, \qquad u \geq 0,$$

in a punctured n-dimensional ball, $n \geq 3$, with an isolated singularity at the origin, are asymptotically radial. More precisely, if u is a solution of (1), then

$$u(x) = (1 + o(1))\psi(|x|) \quad \text{as } x \to 0,$$

for some radial singular solution $\psi(r)$.

Geometrically speaking, to solve equation (1) is to find locally a conformal metric on a conformally flat n-dimensional manifold with constant scalar curvature. Therefore, its two-dimensional analogue is

$$(2) \qquad \Delta u + e^u = 0.$$

In this paper, we shall establish a similar asymptotic radial symmetry result for a smooth solution u of (2) in the punctured disc, $D^* = D\backslash\{0\}$, $D = \{z \in \mathbb{C} | |z| < 1\}$, with an isolated singularity at the origin, under

$$(3) \qquad \int_{D^*} e^u < +\infty.$$

Unlike the higher dimensional case, as one will see, that the integrability condition (3) is necessary for u being asymptotically radial.

We point out that the isolated singularities or the behaviour at infinity of (2) in a punctured ball $B_1(0)\backslash\{0\} = \{x \in \mathbb{R}^3 : 0 < |x| < 1\}$ in 3-dimensions have been studied by M. Bidaut-Véron and L. Véron **[2]**.

2. Results. Our approach to this problem is based on a classical result of Liouville which gives a representation of solutions of equation (2) in a simply-connected domain by analytic functions. We extend this representation to a punctured disc, and then deduce the result from analytic function theory.

Let us first recall Liouville's theorem.

THEOREM 1 (*Liouville* **[6]**; *see also* **[1]**). *Let Ω be a simply-connected domain in $\mathbb{R}^2$. Then all real solutions of*

$$\Delta u + 2Ke^u = 0 \quad \text{in } \Omega, \qquad K \text{ a constant}, \tag{4}$$

are of the form

$$u = \log \frac{|f'|^2}{(1 + (K/4)|f|^2)^2}, \tag{5}$$

where $f(z)$ is a locally univalent meromorphic function in Ω.

COROLLARY 2. *All solutions of equation* (4) *in $\Omega = \mathbb{R}^2$ with $K > 0$ and*

$$\int_{\mathbb{R}^2} e^u < \infty$$

are of the form

$$u(x) = \log \frac{16\lambda^2}{(4 + \lambda^2 K|x - x_0|^2)^2}, \quad \lambda > 0, \ x_0 \in \mathbb{R}^2.$$

Proof. Let u and f be given in (5). Observe that Theorem 1 implies that $e^u|dz|^2 = f^* g_K$, where g_K denotes the standard metric on $\mathbb{S}^2$ with curvature K. By the integrability assumption f cannot have an essential singularity at infinity, for otherwise f would cover $\mathbb{S}^2$ (possibly except one point) infinitely many times near infinity, which is impossible. Therefore $\lim_{z\to\infty} f(z) = \infty$ or some $z_0 \in \mathbb{C}$. By compositing with an inversion, we may assume the former case holds. Then f maps $\mathbb{S}^2$ onto $\mathbb{S}^2$. Since $\mathbb{C}$ cannot cover $\mathbb{S}^2$ (notice that $f'(z) \neq 0$ for all $z \in \mathbb{C}$), f does not have poles in $\mathbb{C}$. This means $f: \mathbb{C} \to \mathbb{C}$ is a covering map and therefore it assumes the form $f(z) = \alpha z + \beta$ for some $\alpha \neq 0$ and β in $\mathbb{C}$. A substitution into (5) gives the desired conclusion. $\square$

Corollary 2 was previously proved by Chen and Li **[4]** by the method of moving planes. From (5), one can see that the integrability condition is also necessary for asymptotic radial symmetry. All nonradial solutions, which arise from transcendental functions, satisfy $\int e^u = \infty$.

Theorem 1 is, in general, not true for domains which are not simply-connected. For instance, the function $u = -\log 4r(1 + \frac{K}{4}r)^2$ is a solution of equation (4) in the punctured disc D^*, with an isolated singularity at the origin. Yet it is easy to see that this solution is given by a multi-valued analytic function $f(z) = z^{1/2}$ instead of a single-valued analytic function in the punctured disc via the formula (5).

We now give an extension of Liouville's theorem for the punctured disc.

THEOREM 3. *Real solutions of the equation* (4) *are of the form* (5), *with* f *a multi-valued locally univalent meromorphic function satisfying*:

1. *When* $K > 0$, $f(z) = g(z)z^\alpha$, $\alpha \in \mathbb{R}$, *or* $\varphi(\sqrt{z})$,
2. *when* $K = 0$, $f(z) = g(z)z^\alpha$ *or* $g(z) + c\log z$, $\alpha \in \mathbb{R}$, $c \in \mathbb{C}$; *and*
3. *when* $K < 0$, $f(z) = h(z)z^\beta$, $\beta \geq 0$.

Here g, φ, *and* h *are single-valued analytic functions in* D^*, D^*, *and* D *respectively,* $\varphi(z)\varphi(-z) = 1$, $h(0) \neq 0$, *and* $|h(D)| < 1$.

Proof. Consider the universal cover $\widetilde{D}^* = (0, 1] \times \mathbb{R}$ of the punctured disc. Let $\pi(r, \theta) = re^{i\theta}$ be the projection and let $\tilde{g} = dr^2 + \frac{1}{r^2}d\theta^2 = \pi^*|dx|^2$. It follows from Theorem 1 that there exists a local univalent meromorphic function $\tilde{h}(z)$ on $\widetilde{D}^*$ such that $e^{\tilde{u}}\tilde{g} = \tilde{h}^*g_K$, where $\tilde{u} = \pi^*u = u \circ \pi$ and now g_K denotes the standard metric on the two dimensional space form S_K with curvature K. Let $\tau\colon \widetilde{D}^* \to \widetilde{D}^*$ be the map $\tau(r, \theta) = (r, \theta + 2\pi)$. Then

$$\tau^*\tilde{h}^*g_K = \tau^*(e^{\tilde{u}}\tilde{g}) = e^{\tilde{u}}\tilde{g} = \tilde{h}^*g_K.$$

Therefore, $\tilde{h} \circ \tau \circ \tilde{h}^{-1}$ is a local isometry of S_K. By a result in differential geometry (Corollary 6.4, p. 256 in **[5]**), $\tilde{h} \circ \tau \circ \tilde{h}^{-1}$ can be extended uniquely to a global isometry of S_K. Locally

$$\tilde{h} \circ \tau = \rho \circ \tilde{h}, \qquad \rho \in \mathrm{Isom}(S_K).$$

Since $\widetilde{D}^*$ is simply connected, this holds globally. Moreover, ρ is analytic since $\tilde{h}$ and τ are analytic. Therefore, there exists a locally univalent multi-valued meromorphic function $h(z) = \tilde{h}(\pi^{-1}z)$ satisfying $h(ze^{2\pi i}) = \rho(h(z))$, $\rho \in \mathrm{Isom}(S_K)$, ρ analytic, in D^* such that

$$u = \log \frac{|h'|^2}{(1 + (K/4)|h|^2)^2}.$$

Here $h(ze^{2\pi i})$ denotes the value of h after a turn along the circle centered at the origin with radius $|z|$.

By a change of coordinates, we only need to prove the theorem for $K = 4$, $K = 0$, and $K = -4$, where now ρ is an analytic isometry of the standard unit sphere, the Eucidean plane, and the Poincaré disc respectively.

For $K = 4$, ρ is given by

$$\frac{w - a}{1 + \overline{a}w} = e^{i\theta} \frac{z - a}{1 + \overline{a}z}$$

and

$$\frac{w - a}{1 + \overline{a}w} = e^{i\theta} \frac{1 + \overline{a}z}{z - a}$$

for some $a \in \mathbb{C}$ and $\theta \in [0, 2\pi)$. In the first case, let

$$f(z) = \frac{h(z) - a}{1 + \overline{a}h(z)}.$$

Then f satisfies

$$f(ze^{2\pi i}) = e^{i\theta} f(z), \qquad \forall z \in D^*.$$

Consider the function

$$g(z) = f(z) z^{-\alpha}$$

on D^*, where $\alpha = \theta/2\pi$. We have

$$\begin{aligned} g(ze^{2\pi i}) &= f(ze^{2\pi i})(ze^{2\pi i})^{-\alpha} \\ &= f(z)e^{i\theta} z^{-\alpha} e^{-2\pi\alpha i} = g(z) \end{aligned}$$

for all $z \in D^*$. Hence $g(z)$ is a single-valued function and therefore analytic in D^*. So $f(z)$ takes the form $g(z)z^{\alpha}$. Using the fact that $w = (z - a)/(1 + \overline{a}z)$ is an isometry of the standard unit sphere,

$$u = \log \frac{|h'|^2}{(1 + |h|^2)^2} = \log \frac{|f'|^2}{(1 + |f|^2)^2},$$

which proves the first case.

In the second case, letting

$$f(z) = \frac{h(z) - a}{1 + \overline{a}h(z)},$$

we have $f(ze^{4\pi i}) = f(z)$. Hence there exists a single-valued analytic function φ in the punctured disc satisfying $f(z) = e^{i\theta/2}\varphi(\sqrt{z})$. The condition $f(ze^{2\pi i})f(z) = e^{i\theta}$ implies $\varphi(z)\varphi(-z) = 1$. The proof of the positive case is completed.

For $K = 0$, we notice that analytic isometries of the Euclidean plane are of the form $w = e^{i\theta}z + c$, which can be represented by $w - a = e^{i\theta}(z-a)$ or $w = z + c$. Similar argument as in the positive case gives us the desired result.

Finally, for $K = -4$, analytic isometries of the Poincaré disc are in one of the following forms:

$$\frac{w-a}{1-\bar{a}w} = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \quad \text{with } |a| < 1,$$
$$\frac{w-e^{i\theta_1}}{w-e^{i\theta_2}} = k\frac{z-e^{i\theta_1}}{z-e^{i\theta_2}}, \quad \text{with } k > 1,\ \theta_1 \neq \theta_2 \in \mathbb{R},$$
$$\frac{w-e^{i\theta}}{w+e^{i\theta}} = \frac{z-e^{i\theta}}{z+e^{i\theta}} + c, \quad \text{with } \theta \in \mathbb{R},\ c \in \mathbb{C}.$$

Using the same argument as above one can show that f assumes one of the following forms:

(i) $g(z)z^{\alpha}$,
(ii) $e^{i\theta_1}(e^{i\theta_2} - g(z)z^{i\alpha})/(e^{-i\theta_2} - g(z)z^{i\alpha})$, and
(iii) $e^{i\alpha}(1 + g(z) + \alpha\log z)/(1 - g(z) - \alpha\log z)$,

where g is analytic in D^*, and α, θ_1, θ_2, $\theta \in \mathbb{R}$. Observe that in (5) ($K = -4$) u becomes singular at $|f| = 1$. Hence, by the analyticity of f and the regularity of u, the image of f lies either inside or outside D. Replacing f by $1/f$ if $|f| > 1$, we may assume $f(D^*)$ is contained in D. This immediately implies that the expression in (i) can be rewritten as $h(z)z^{\beta}$ where $h(0) \neq 0$ and $\beta \geq 0$.

In the following let h stand for an analytic function in D with $h(0) \neq 0$. We shall show that in (ii) and (iii) $\alpha = 0$ and $g(z) = h(z)$, and consequently they are special cases of (i). To see this first observe that in case (ii) the image of D^* under the map $g(z)z^{i\alpha}$ lies in a half plane, which, modulo a rotation, may be taken to be the upper half plane. We have

$$0 < \arg(g(z)z^{i\alpha}) = \arg g(z) + \alpha\log|z| < \pi \pmod{2\pi}.$$

Applying the maximum principle to $\operatorname{Im} g(z)$ in the annulus $r_j < |z| < r_{j_0}$, $r_j = e^{-2j\pi/|\alpha|}$, $j > j_0$, j_0 large, we conclude that $\operatorname{Im} g(z) > 0$ for all z in a deleted neighborhood of 0. Hence 0 cannot be an essential singularity of g. Now we can write $g(z) = h(z)z^k$, $k \in \mathbb{Z}$. Then the inequality

$$0 < \arg(g(z)z^{i\alpha}) = \arg h(z) + \alpha \log|z| + k \arg z < \pi \pmod{2\pi}$$

implies $\alpha = k = 0$. Similarly one can show that in (iii) $\alpha = 0$ and $g(z) = h(z)$. This completes our proof of the theorem. □

Now we can deduce an asymptotic radial symmetry result for equation (4) from Theorem 3. First we need a lemma from complex analysis.

LEMMA 4. *Suppose that $g(z)$ is a holomorphic function in D^* which has an essential singularity at the origin. Then the multi-valued function $f(z) = z^\alpha g(z)$, $\alpha \in \mathbb{R}$, takes all values infinitely many times except at most one value.*

Proof. Consider the single-valued function $\phi(z) = z^{k-\alpha} f(z) = z^k g(z)$, where k is an integer such that $k > \alpha$. Since g has an essential singularity at the origin, so has ϕ. The sequence

$$\phi_n(z) = \phi\left(\frac{z}{2^n}\right)$$

is not a normal sequence on some annulus Γ: $r/4 < |z| < 2r$. In particular, the sequence is not a normal sequence on intersection Ω of Γ with any sector: $|\arg z - \arg z_0| < \varepsilon$, in the unit disc. Therefore the sequence

$$f_n(z) = f\left(\frac{z}{2^n}\right)$$

cannot be normal on Ω. Now, applying the Montel theorem [7], we see that for any $a \in \mathbb{C}$, except at most one point, there exist infinitely many n such that f_n takes the value a in Ω. This implies that f takes the value a infinitely many times in the sector. □

THEOREM 5. *Let u be a smooth real solution of the equation* (4) *with $K > 0$ in the punctured disc D^*. Then u is asymptotically radial, more precisely,*

$$u(z) = \alpha \log|z| + O(1) \quad \text{as } |z| \to 0,\ \alpha > -2,$$

if and only if

$$\int_{D^*} e^u < +\infty .$$

Proof. By Theorem 3, the metric $e^u|dz|^2$ is the pull-back of the spherical metric with curvature K via the holomorphic map f. Moreover f is a covering map on $D\backslash\{z<0\}$ since $f'\neq 0$ for all $z\in D^*$. If g takes the value ∞ infinitely many times, then so does f. This implies $e^u|dz|^2$ has infinite volume, i.e. $\int_{D^*} e^u = +\infty$. So we may assume g takes ∞ for finitely many times. Then g is holomorphic near the essential singularity and we can apply Lemma 4 (in case $f(z) = g(z)z^\alpha$) to conclude that f covers the image of f in the sphere infinitely many times. Thus $\int_{D^*} e^u = +\infty$. Therefore, the integrability condition implies that g at most has a pole at the origin. Simple calculation now establishes the asymptotic radial symmetry of the solution u. □

REMARK. Theorem 5 no longer holds for $K = 0$. In fact, it is straightforward to show that $\int e^u|dz|^2 < \infty$ for some deleted neighborhood of 0 if and only if $f(z) = h(z)z^\alpha$, $h(0)\neq 0$ and $\alpha > 0$. In particular, all radially symmetric solutions corresponding to $f(z) = h(z)z^k + c\log z$, $k\in\mathbb{Z}$, $c\neq 0$, satisfy $\int e^u|dz|^2 = \infty$ in any deleted neighborhood of 0.

On the other hand, Theorem 5 holds for $K<0$. In fact, all solutions are asymptotic radially symmetric and satisfy $\int e^u|dz|^2 < \infty$.

Acknowledgment. This work was partially supported by an Earmarked Grant for Research, Hong Kong. The authors are indebted to Professor Wei-yue Ding for many suggestions, especially those on the first draft of this paper. They also would like to thank Thomas K. K. Au for useful discussion.

REFERENCES

[1] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman Advanced Publishing Program, London, 1980.

[2] M. Bidaut-Véron and L. Véron, *Groupe conforme de S^2 et propriétés limites des solutions de $-\Delta u = \lambda e^u$*, C. R. Acad. Sci. Paris Série I, **308** (1989), 493–498.

[3] L. A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math., **42** (1989), 271–297.

[4] W. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J., **63** (1991), 615–623.

[5] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Interscience Publishers, John Wiley & Sons, 1963.

[6] J. Liouville, *Sur l'équation aux Dérivées Partielles* $\partial^2 \log \lambda / \partial u \partial v \pm 2\lambda a^2 = 0$, J. de Math., **18** (1), (1853), 71–72.

[7] P. Montel, *Leçons sur les Familles Normals de Fonctions Analytiques et leurs Applications*, Gauthier-Villars, Paris, 1927.

Received April 28, 1992 and in revised form July 6, 1992.

THE CHINESE UNIVERSITY OF HONG KONG
SHATIN, N. T., HONG KONG
E-mail address, K. S. Chou: b114733@cucsc.bitnet
E-mail address, T. Y. H. Wan: tomwan@cuhk.hk

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

KNOTS WITH ALGEBRAIC UNKNOTTING NUMBER ONE

MICAH E. FOGEL

Every knot, K, in S^3 has associated to it an equivalence class of matrices based on S-equivalence of Seifert matrices. When the knot is altered by changing a crossing, the S-equivalence class of the new knot is related to that of the original knot in a very specific way. This change in the Seifert matrices can be studied without regard to the underlying geometric situation, leading to a theory of *algebraic crossing changes*. Thus, the *algebraic unknotting number* may be defined as the smallest number of these algebraic crossing changes necessary to convert a Seifert matrix for the knot into a matrix for the unknot. A straightforward test of some well-known knot invariants will reveal that the algebraic unknotting number is one.

In [**4**], Murakami defined an operation on Seifert matrices that he called an *algebraic unknotting operation.* He showed that any geometric crossing change induced an algebraic unknotting operation on a suitably chosen Seifert matrix. Since any knot could be changed into any other knot by a sequence of crossing changes, any Seifert matrix could be transformed into any other Seifert matrix by a sequence of algebraic unknotting operations and S-equivalences. For knots K_1 and K_2 the *algebraic Gordian distance* from K_1 to K_2 is the minimum number of algebraic unknotting operations needed in such a sequence. The *algebraic unknotting number,* $u_a(K)$, is then the algebraic Gordian distance of K from the unknot, i.e. the minimum number of algebraic unknotting operations needed to reduce a Seifert matrix for K to a matrix S-equivalent to the zero matrix.

Since every crossing change induces an algebraic unknotting operation, there is the inequality $u_a(K) \leq u(K)$ where $u(K)$ is the regular geometric unknotting number of the knot. And in many cases $u_a(K)$ is the appropriate object of study rather than $u(K)$ because only the algebraic information contained in a Seifert matrix is used. Such is the case in Murasugi's result on signatures [**5**] and Nakanishi's theorem about minor indices [**6**]. Also, results depending only on the abelian invariants (notably Lickorish [**3**] as generalized by Cochran and Lickorish [**1**]) apply to $u_a(K)$ since all of the homology information about the cyclic covers is contained in the Seifert matrix (see, for instance, [**2**, §§8 and 9]).

In the case $u_a(K) = 1$, Murakami was able to prove that the Alexander module $H_1(C_K)$ (where C_K is the infinite cyclic cover of the knot exterior and t acts by covering translations) is a cyclic $Z[t, t^{-1}]$-module. In addition, there is a generator, g, of this module with $\beta(g, g) = \pm 1/\Delta$, where $\beta(\cdot, \cdot)$ is the Blanchfield pairing and Δ is the Alexander polynomial of the knot. This paper contains a proof of the converse, providing a complete algebraic characterization of knots with $u_a(K) = 1$:

THEOREM. *A knot K with Alexander polynomial Δ_K can be changed by a single crossing change into a knot K' with trivial Alexander polynomial if and only if the Alexander module is cyclic and has a generator g with $\beta(g, g) = \pm 1/\Delta$.*

In addition, the proof often allows direct calculation of the necessary crossing change. A somewhat unfortunate application of the theorem will be made to the knot 8_{10}.

1. Some special surgery curves. Crossing changes will be examined via surgery on a well controlled class of curves in S^3. All knots, curves, and disks will be tame, and oriented when convenient. The orientations chosen will be noted, but they are only used for calculation and will be irrelevant to the outcome. The notation z will be used to denote $t^{1/2} - t^{-1/2}$ and for a matrix M, the calligraphic letter $\mathscr{M}$ will be used to represent the skew-Hermitianized form $t^{1/2}M - t^{-1/2}M^T$ (t^{-1} being considered the conjugate of t).

A disk D in S^3 will be said to be *nice with respect to K* if D and K intersect in two points and $\operatorname{lk}(\partial D, K) = 0$. A simple closed curve γ in $S^3 - K$ is a *nice surgery curve for K* if it bounds a nice disk. Any knot and nice surgery curve pair can be isotoped to look like Figure 1. Clearly ± 1 surgery on a nice surgery curve yields a single crossing change in the knot, and any single crossing change can be effected by ± 1 surgery along a suitably chosen curve.

In Figure 1, a Seifert surface can be chosen for the knot so that the two strands of the knot cobound a band in the surface, and the surface does not meet the curve γ. Generators for the homology of this surface can be chosen so that one of them (to be called g_0) runs over this band from right to left, and the rest of the generators do not

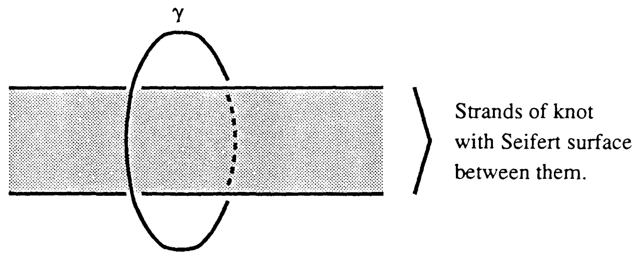

FIGURE 1

go over the band at all. A Seifert matrix for K has the form

$$M = \begin{pmatrix} & V & & * \\ & & & \vdots \\ & & & * \\ * & \cdots & * & x \end{pmatrix}, \qquad \mathscr{M} = \begin{pmatrix} & \mathscr{V} & & * \\ & & & \vdots \\ & & & * \\ * & \cdots & * & xz \end{pmatrix}$$

where $x = \operatorname{lk}(g_0, g_0^+)$, g_0^+ being the pushoff of g_0 from the Seifert surface in the (arbitrarily) chosen positive direction. V is the linking matrix for all the generators that don't go over the shown band, and their pushoffs.

If instead of the mundane curve shown in Figure 1, a nice surgery curve which has n full twists is used, the resulting knot has a Seifert surface which looks like Figure 2 (next page) (for -1 surgery). Note that n could be zero. This operation adds two generators to the homology of the Seifert surface, and the new Seifert matrix is

$$M' = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & M & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & -1 & 0 \\ \hline 0 & \cdots & 0 & 0 & n & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mp 1 \end{array}\right),$$

$$\mathscr{M}' = \left(\begin{array}{cccc|cc} & & & & 0 & 0 \\ & \mathscr{M} & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & -t^{1/2} & 0 \\ \hline 0 & \cdots & 0 & t^{-1/2} & nz & t^{-1/2} \\ 0 & \cdots & 0 & 0 & t^{1/2} & \mp z \end{array}\right).$$

It is then simple to calculate the Alexander polynomial of the new knot, which is given by $\Delta_{K'} = \det(\mathscr{M}')$, by expanding along the bottom

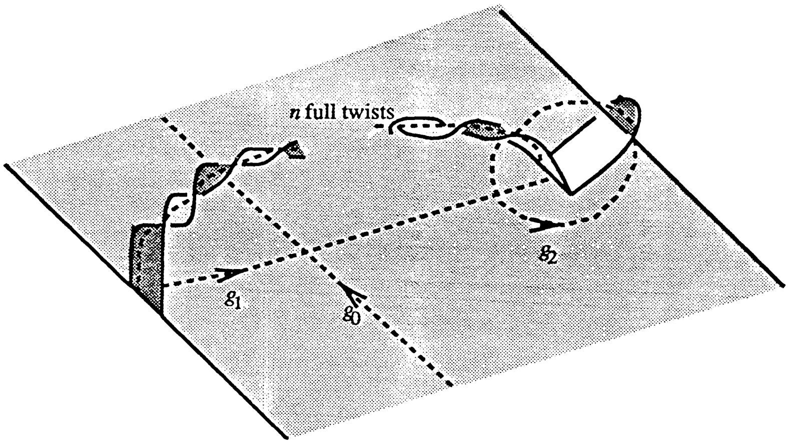

FIGURE 2

row of the matrix. The result is $(1 \mp nz^2)\det(\mathscr{M}) \mp z\det(\mathscr{V})$. Denote by L the knot obtained from K when $n = 0$ in the above, so that $\Delta_L = \det(\mathscr{M}) \mp z\det(\mathscr{V})$. Then noting that $\det(\mathscr{M}) = \Delta_K$, the result is

$$\Delta_{K'} = \Delta_L \mp nz^2\Delta_K .$$

Since n may be any integer, ± 1 surgery along a properly chosen curve can add *any* integral multiple of $z^2\Delta_K$ to Δ_L.

The key fact is that with proper choice of (nice) surgery curve, any *polynomial* multiple of Δ_K can be added to Δ_L, as long as the result satisfies the well known conditions $\Delta(1) = 1$ and $\Delta(t) = \Delta(t^{-1})$ for an Alexander polynomial. This is done by examining curves that wrap around the knot as in Figure 3. Each a_i is a nonzero integer, and if $a_i < 0$ then all the crossings in the magnified view of box i are reversed.

LEMMA 1. *The knot obtained from* ± 1 *surgery on the curve shown in Figure* 3 *has Alexander polynomial*

$$(1)\quad \Delta_{K'} = \Delta_L \mp \Delta_K \left[z^2 \left(n + \sum a_i \right) \right.$$
$$\left. - \sum \operatorname{sign}(a_i)(t^{|a_i|}(t-1) + t^{-|a_i|}(t^{-1}-1)) \right] .$$

Proof. Figure 4 shows box i after ± 1 surgery is performed along the curve. The figure shows part of a Seifert surface for the new knot K' along with a set of homology generators for the new handles. Note

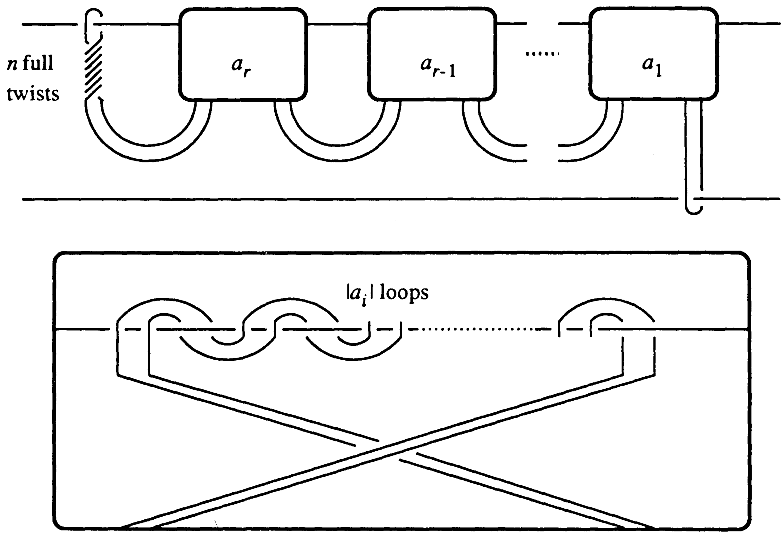

FIGURE 3

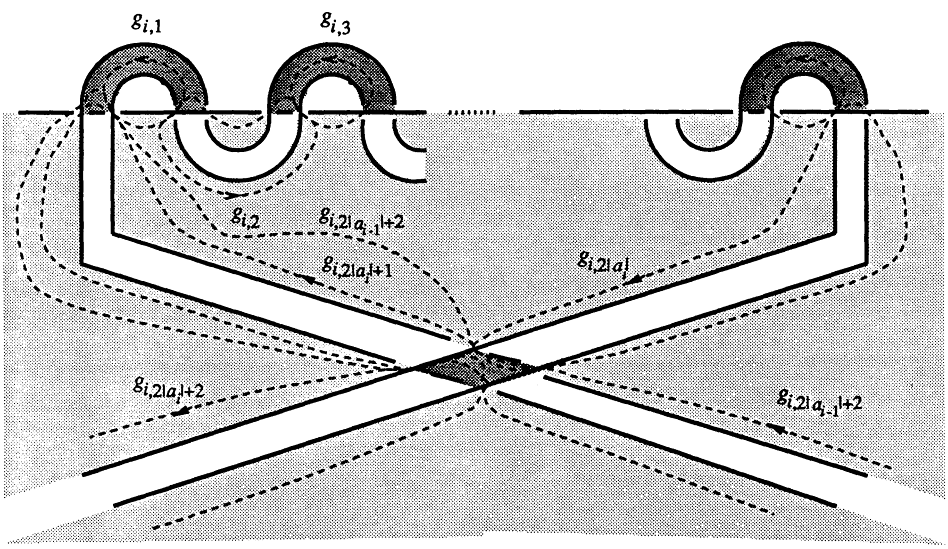

FIGURE 4

that the generators for one box do not interact with those of another box except that $\mathrm{lk}(g^+_{i,2|a_i|+2}, g_{i+1,1}) = -1$ or $\mathrm{lk}(g_{i,2|a_i|+2}, g^+_{i+1,1}) = 1$ depending on whether a_{i+1} is positive or negative.

Thus the Seifert matrix coming from this set of generators has the form

$$M' = \left(\begin{array}{cccc|ccc|cccc|cc}
 & & & & 0 & & & & & & & & \\
 & M & & & \vdots & & & & & & & & \\
 & & & & 0 & & & & & & & & \\
 & & & & c_1 & 0 \cdots & 0 & & & & & & \\ \hline
0 & \cdots & 0 & d_1 & \multicolumn{3}{|c|}{} & 0 & & & & & \\
 & & & 0 & \multicolumn{3}{|c|}{\text{Block for}} & \vdots & & & & & \\
 & & & \vdots & \multicolumn{3}{|c|}{\text{box } 1} & 0 & & & & & \\
 & & & 0 & \multicolumn{3}{|c|}{} & c_2 & & & & & \\ \cline{5-7}
 & & & & 0 & \cdots\ 0 & d_2 & & & & & & \\
 & & & & & & & \ddots & & & & & \\
 & & & & & & & & c_r & 0 \cdots & 0 & & \\ \cline{9-11}
 & & & & & & & d_r & \multicolumn{3}{|c|}{} & 0 & \\
 & & & & & & & 0 & \multicolumn{3}{|c|}{\text{Block for}} & \vdots & \\
 & & & & & & & \vdots & \multicolumn{3}{|c|}{\text{box } r} & 0 & \\
 & & & & & & & 0 & \multicolumn{3}{|c|}{} & -1 & 0 \\ \cline{9-13}
 & & & & & & & & 0 & \cdots\ 0 & 0 & \multicolumn{1}{|c}{n} & 0 \\
 & & & & & & & & & & 0 & \multicolumn{1}{|c}{-1} & \mp 1
\end{array}\right)$$

where $(c_i, d_i) = (-1, 0)$ if $a_i > 0$, and $(0, 1)$ if $a_i < 0$.
Each block is square of size $2|a_i| + 2$ and has the form

$$M_r = \begin{pmatrix}
1 & -1 & & & & & 0 & 0 \\
0 & 0 & -1 & & & & \vdots & \vdots \\
 & 0 & 1 & -1 & & & & \\
 & & 0 & \ddots & \ddots & & & \\
 & & & \ddots & 1 & -1 & \vdots & \vdots \\
 & & & & 0 & 0 & -1 & 0 \\
-1 & 0 & \cdots & & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & & \cdots & 0 & 1 & 0
\end{pmatrix}$$

for $a_r > 0$, while if $a_r < 0$ then this is replaced with the negative of its transpose. When this block is skew-Hermitianized in order to calculate the Alexander polynomial, it becomes

(2)

$$\mathcal{M}_r = \begin{pmatrix} z & -t^{1/2} & & & & & t^{-1/2} & 0 \\ t^{-1/2} & 0 & -t^{1/2} & & & & \vdots & \vdots \\ & t^{-1/2} & z & -t^{1/2} & & & & \\ & & t^{-1/2} & \ddots & \ddots & & & \\ & & & \ddots & z & -t^{1/2} & \vdots & \vdots \\ & & & & t^{-1/2} & 0 & -t^{1/2} & 0 \\ -t^{1/2} & 0 & \cdots & & \cdots & t^{-1/2} & 0 & -t^{-1/2} \\ 0 & 0 & \cdots & & \cdots & 0 & t^{1/2} & 0 \end{pmatrix}$$

or the negative of the transpose if $a_i < 0$.

Now the calculation of $\Delta_{K'}$ is routine. The determinant of $\mathcal{M}'$ can be expanded along the bottom row and last column. The result is

$$t^{1/2}\det\begin{pmatrix} \mathcal{M} & * & & & \\ * & \boxed{\mathcal{M}_1} & * & & \\ & * & \ddots & * & \\ & & * & \boxed{\mathcal{M}_r} & 0 \\ & & & * & t^{-1/2} \end{pmatrix} \pm z\det\begin{pmatrix} \mathcal{M} & * & & & \\ * & \boxed{\mathcal{M}_1} & * & & \\ & * & \ddots & * & \\ & & * & \boxed{\mathcal{M}_r} & * \\ & & & * & nz \end{pmatrix}$$

$$= (1 \mp nz^2)\det\begin{pmatrix} \mathcal{M} & * & & \\ * & \boxed{\mathcal{M}_1} & * & \\ & * & \ddots & * \\ & & * & \mathcal{M}_r \end{pmatrix} \mp z\det\begin{pmatrix} \mathcal{M} & * & & \\ * & \boxed{\mathcal{M}_1} & * & \\ & * & \ddots & * \\ & & * & \widetilde{\mathcal{M}_r} \end{pmatrix}$$

where $\widetilde{\mathcal{M}_r}$ means $\mathcal{M}_r$ with the last row and column removed.

The determinant in the first term can be computed easily by using the form of the blocks $\mathscr{M}_i$ given in formula (2). The determinant can be expanded repeatedly along the last row and column until all that is left is $\det(\mathscr{M}) = \Delta_K$.

The second term is a bit more complex. It can be expanded along the row and column corresponding to the top row and column of $\widetilde{\mathscr{M}_r}$. The result is

$$\det(\widetilde{\mathscr{M}_r})\cdot\det\begin{pmatrix} \mathscr{M} & * & & \\ * & \boxed{\mathscr{M}_1} & * & \\ & * & \ddots & * \\ & & * & \mathscr{M}_{r-1} \end{pmatrix}$$

$$+\det\begin{pmatrix} \mathscr{M} & * & & \\ * & \boxed{\mathscr{M}_1} & * & \\ & * & \ddots & * \\ & & * & \widetilde{\mathscr{M}_{r-1}} \end{pmatrix}\cdot\det\begin{pmatrix} \widetilde{\mathscr{M}_r}\text{ with first} \\ \text{row and column} \\ \text{removed} \end{pmatrix}.$$

The large determinant in the first term reduces to Δ_K as before. And the smaller matrix in the second term can be expanded along the first row and column repeatedly until it reduces to 1. So inductively, the result is (note that $\widetilde{\mathscr{M}} = \mathscr{V}$)

$$\Delta_{K'} = (1 \mp nz^2)\Delta_K \mp z(\Delta_K \textstyle\sum \det \widetilde{M_i} + \det \mathscr{V}) \tag{3}$$

It remains to calculate $\det(\widetilde{\mathscr{M}_i})$. This is done by taking the matrix $\mathscr{M}_i$ as given in (2) and deleting its last row and column. This leaves a matrix whose last row and column have two nonzero entries each. Expanding along the last row and column gives four terms, as follows (assume $a_i > 0$):

$$\det\begin{pmatrix} 0 & -t^{1/2} & & & & \\ t^{-1/2} & z & -t^{1/2} & & & \\ & t^{-1/2} & 0 & \ddots & & \\ & & \ddots & \ddots & -t^{1/2} & \\ & & & t^{-1/2} & z & -t^{1/2} \\ & & & & t^{-1/2} & 0 \end{pmatrix}$$

$$+\, t\det\begin{pmatrix} -t^{1/2} & & & & & \\ z & -t^{1/2} & & & & \\ t^{-1/2} & 0 & -t^{1/2} & & & \\ & \ddots & \ddots & \cdots & & \\ & & t^{-1/2} & z & -t^{1/2} & \\ & & & t^{-1/2} & 0 & -t^{1/2} \end{pmatrix}$$

$$+\, t^{-1}\det\begin{pmatrix} t^{-1/2} & 0 & -t^{1/2} & & & \\ & t^{-1/2} & z & \ddots & & \\ & & t^{-1/2} & \ddots & -t^{1/2} & \\ & & & \ddots & 0 & -t^{1/2} \\ & & & & t^{-1/2} & z \\ & & & & & t^{-1/2} \end{pmatrix}$$

$$+\det\begin{pmatrix} z & -t^{1/2} & & & & \\ t^{-1/2} & 0 & -t^{1/2} & & & \\ & t^{-1/2} & z & \ddots & & \\ & & \ddots & \ddots & -t^{1/2} & \\ & & & t^{-1/2} & 0 & -t^{1/2} \\ & & & & t^{1/2} & z \end{pmatrix}$$

whose values are 0, $-t^{a_i+1/2}$, $t^{-a_i-1/2}$, and $a_i z$ respectively. If $a_i < 0$ then since $\widetilde{\mathscr{M}_i}$ is of odd size, the determinant of the negative of its transpose is opposite in sign. Therefore, for any a_i, $\det(\widetilde{\mathscr{M}_i}) = a_i z - \operatorname{sign}(a_i)(t^{|a_i|+1/2} - t^{-|a_i|-1/2})$. Inserting this in (3) and noting that $\Delta_L = \Delta_K \mp z\det(\mathscr{Y})$ gives the desired result.

COROLLARY 2. *For knots, K, appearing as in Figure 3, with Alexander polynomial Δ_K, it is possible to make one crossing change to obtain* *a knot, K', with*

$$\Delta_{K'} = \Delta_L + \Delta_K b(t)$$

where $b(t)$ is any polynomial subject to the constraint that $\Delta_{K'}$ is a knot polynomial.

Proof. Since Δ_L is the Alexander polynomial for some knot, the requirements on $b(t)$ will be that $b(1) = 0$ and $b(t^{-1}) = b(t)$. But looking at formula (1) it is clear that n and a series of a_i may be chosen to give any b of this form. Therefore, there is a nice surgery curve on which ± 1 surgery—which changes exactly one crossing in the knot—gives a knot with the desired polynomial.

2. How nice are nice surgery curves? In this section some properties of nice surgery curves are developed to show that they are useful for more than just making large Seifert matrices.

LEMMA 3. *All of the different surgery curves, γ, given by different choices of n and series of a_i are homotopic in $S^3 - K$.*

Proof. Obvious, since homotopy in $S^3 - K$ allows γ to pass through itself.

Denote by C_K the infinite cyclic cover of the exterior of K. The Alexander module of K is $H_1(C_K)$ and is presented as a $Z[t, t^{-1}]$-module by the matrix $t^{1/2}\mathscr{M}$. Fix a strand of the knot and consider a small loop going around this strand. A lift of this loop represents t in the Alexander module (see Figure 5).

Any nice surgery curve γ can be isotoped to appear as in Figure 5. The can be seen by sliding the top half of Figure 1 around the knot to the left until it comes near the bottom half. Since the nice surgery curve bounds a nice disk, and both intersections of this disk with K

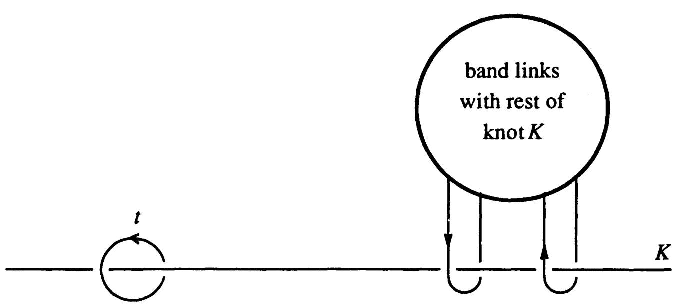

FIGURE 5

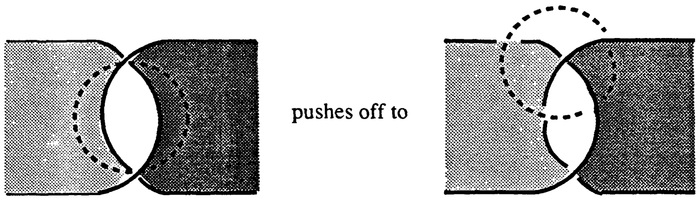

FIGURE 6

are clearly visible in Figure 5, the rest of the disk forms a band that meanders through the knot but never meets it. Let u be a simple closed curve that starts near the strand of K on the band, follows the band through the knot back to K, and then loops around the strand of K enough times so that $\text{lk}(u, K) = 0$. The original surgery curve is homotopic to the product curve $tut^{-1}u^{-1}$. This procedure works in reverse as well; any simple closed curve u with $\text{lk}(u, K) = 0$ gives rise to a nice surgery curve (in fact, a whole family of them) homotopic to $tut^{-1}u^{-1}$.

Both u and γ have linking number zero with the knot K, so both lift to C_K. Call these lifts $\tilde{u}$, $\tilde{\gamma}$. Homologically there is the relation $[\tilde{\gamma}] = (t-1)[\tilde{u}]$. Now note that $t-1$ is invertible mod Δ in $Z[t, t^{-1}]$. Thus for any homology class $\tilde{v}$ in the Alexander module, $(t-1)[\tilde{u}] = [\tilde{v}]$ can be solved for $[\tilde{u}]$. Choosing a representative of the class $[\tilde{u}]$ and projecting it down into S^3 yields a simple closed u with $\text{lk}(u, K) = 0$. Using the construction at the end of the previous paragraph completes the proof of

LEMMA 4. *Any class of curves in the Alexander module can be represented by the lift of a nice surgery curve.*

In light of Lemma 3 and the fact that homotopic curves have homotopic (hence homologous) lifts, it is possible to choose a nice surgery curve representing any element of the Alexander module, yet still have the full power of the previous section available to alter the knot polynomial.

LEMMA 5. *Any nice surgery curve can be taken to be the pushoff of curve on a Seifert surface of the type shown in Figure 6.*

Proof. The moves in Figure 7 show how this can be done.

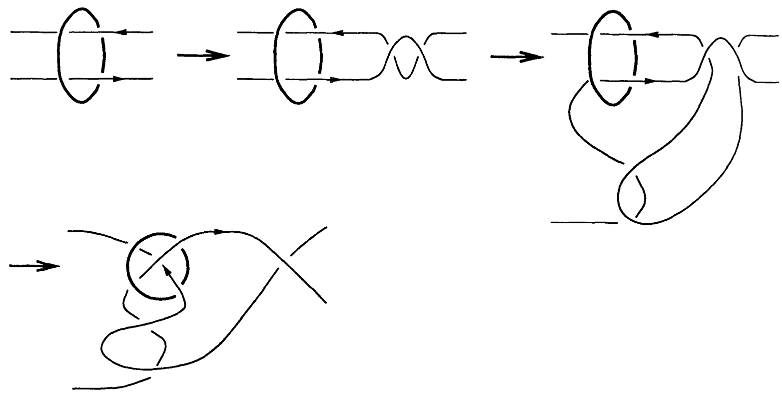

FIGURE 7

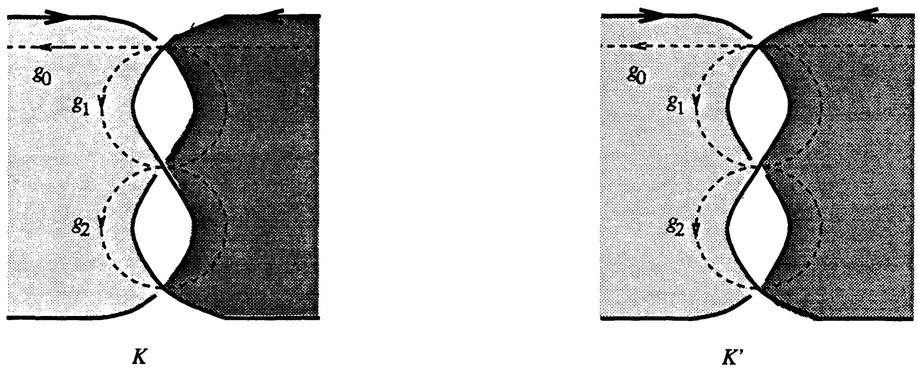

FIGURE 8

3. The main result.

THEOREM. *A knot K with polynomial Δ_K can be changed into K' with $\Delta_{K'} = 1$ by a single crossing change if and only if the Alexander module $H_1(C_K)$ is a cyclic $Z[t, t^{-1}]$-module of order Δ_K and has a generator g such that $\beta(g, g) = \pm 1/\Delta$.*

Proof. Assume that K can be changed into K' with $\Delta_{K'} = 1$ by a single crossing change and that K, K' are as shown in Figure 8, where pieces of Seifert surfaces are also shown. This situation can always be obtained by using Reidemeister moves of type II and III to get a local picture like Figure 8 near the crossing to be changed and then moving it to the top or bottom of a knot diagram, finally using the Seifert circle method to choose the Seifert surface. A simple calculation yields

(4)

$$M = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & V & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & x & -1 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 1 \end{array}\right), \quad M' = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & V & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & x & -1 & 0 \\ \hline 0 & \cdots & 0 & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right)$$

as the Seifert matrices with skew-Hermitianized forms

$$\mathscr{M} = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & \mathscr{V} & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & xz & -t^{1/2} & 0 \\ \hline 0 & \cdots & 0 & t^{-1/2} & 0 & t^{-1/2} \\ 0 & \cdots & 0 & 0 & -t^{1/2} & z \end{array}\right)$$

$$\mathscr{M}' = \left(\begin{array}{cccc|cc} & & & * & 0 & 0 \\ & \mathscr{V} & & \vdots & \vdots & \vdots \\ & & & * & 0 & 0 \\ * & \cdots & * & xz & -t^{1/2} & 0 \\ \hline 0 & \cdots & 0 & t^{-1/2} & z & t^{-1/2} \\ 0 & \cdots & 0 & 0 & -t^{1/2} & 0 \end{array}\right).$$

Taking the determinants of these gives

$$\Delta_{K'} = \det \begin{pmatrix} & & & * \\ & \mathscr{V} & & \vdots \\ & & & * \\ * & \cdots & * & xz \end{pmatrix}$$

so that this matrix has determinant one. Thus $\Delta_K = 1 + z\det(\mathscr{V})$. Note also that the Alexander dual to g_2^+ in the picture of K is a nice surgery curve, $+1$-surgery around which gives the knot K'.

Now $t^{1/2}\mathscr{M}$ is a presentation matrix for the Alexander module of K. Since the determinant of the upper left corner of the matrix is a power of t, which is a unit of $Z[t, t^{-1}]$, the first bunch of generators can all be expressed in terms of the generator corresponding to g_1, and the last column of $\mathscr{M}$ shows that this can in turn be expressed in terms of the generator corresponding to g_2. The presentation matrix has determinant Δ_K, proving the assertion that the Alexander module is cyclic of order Δ_K. It is generated by the lift g of the Alexander dual of g_2.

The Blanchfield pairing is given by $\beta(a, b) = z\bar{a}\mathscr{M}^{-1}b$ where a, b are vectors in terms of the spanning set used for calculation of $\mathscr{M}$ (see, for instance [**2**, §8]). Since $g = (0, \dots, 0, 1)$ in these coordinates, $\beta(g, g)$ is simply the bottom right entry in $z\mathscr{M}^{-1}$. Direct computation using the cofactor expansion of the inverse of a matrix gives $\beta(g, g) = z\det(\mathscr{Y})/\Delta_K$. Noting that the Blanchfield pairing takes values in $Q(t)/Z[t, t^{-1}]$ and that $\Delta_K = 1 + z\det(\mathscr{Y})$ gives the desired result $\beta(g, g) = -1/\Delta_K$.

Had a right crossing been made into a left crossing, the calculations would all be the same by switching all the crossings in Figure 8. Everything is the same except that Δ_K is now $1 - z\det(\mathscr{Y})$, which changes the end result to $\beta(g, g) = +1/\Delta_K$.

(The foregoing is essentially Murakami's proof. It should be noted that if $\beta(g, g) = \pm 1/\Delta_K$ then $a \mapsto \Delta_K\beta(g, a)$ is an epimorphism from $H_1(C_K)$ to $Z[t, t^{-1}]/\Delta_K Z[t, t^{-1}]$, and a simple argument based on the fact that $Q[t, t^{-1}]$ is a PID proves the kernel of this map to be zero. Hence, the Blanchfield pairing condition alone is enough to imply the generating condition.)

The converse is a matter of applying the lemmas in the correct order. Assume that the Alexander module for K is cyclic with generator g such that $\beta(g, g) = \pm 1/\Delta_K$. Use Lemma 4 to find a nice surgery curve γ whose lift to C_K is in the homology class g. Use Lemma 5 to arrange the knot and surgery curve to look like Figure 8; if $\beta(g, g) = +1/\Delta_K$ we first change all the crossings in Figure 8.

Once the knot is in this position, its Seifert matrix is given by formula (4). From the above calculation, $\beta(g, g) = z\det(\mathscr{Y})/\Delta_K$. Therefore $z\det(\mathscr{Y}) = \pm 1 + b(t)\Delta_K$ (recall that the Blanchfield pairing takes values in the quotient ring $Q(t)/Z[t, t^{-1}]$ so that values of the numerator are only determined up to adding a multiple of the denominator), where if the plus sign is chosen, the crossings are reversed in Figure 8. But since this is the case, a quick calculation yields $\Delta_{K'} = \Delta_K \pm z\det(\mathscr{Y})$, where again the plus sign is taken if the crossings in Figure 8 have been reversed. Substituting yields

$$\Delta_{K'} = 1 + b(t)\Delta_K.$$

Now since "1" is a knot polynomial, the condition on $b(t)$ to make $\Delta_{K'}$ a knot polynomial are exactly those necessary to apply Corollary 2. And because of Lemma 3, we may alter the surgery curve as necessary in Corollary 2 and still have a curve that will lift to g. Therefore surgery on this new nice surgery curve—which changes exactly one crossing—yields a knot with polynomial 1.

On occasion, the necessary surgery curve can be found explicitly. Suppose, for instance, that a Seifert surface is chosen and the corresponding Seifert matrix calculated. When this matrix is skew-Hermitianized, it becomes a presentation matrix for the Alexander module, with generators the lifts of the Alexander duals to generators of the homology of the Seifert surface.

If this matrix can be column-reduced to a matrix whose only nonzero entries are on the diagonal and any one row, and the diagonal elements in the other columns are $\pm t^n$, then a single generator for the Alexander module has been found—namely the generator g corresponding to the given row. If the matrix cannot be so reduced, then some basis change in the homology of the Seifert surface allows it to be reduced. However, discovering the necessary basis change may not be a simple problem. But if such a generator can be found then $\beta(g, g)$ can be easily calculated from the Seifert matrix. The problem then becomes whether or not a multiple of this generator can be found with $\beta(f(t)g, f(t)g) = \pm 1/\Delta$. This is a problem in Hermitian residues, which again may be difficult to solve.

Assuming both the difficulties in the previous paragraph can be overcome, and some multiple of the known generator has been found, it is now simple to extract the required surgery curve. For tg projects down to the loop whose lift is g, conjugated by the loop whose lift represents t. Thus, the projection of $f(t)g$ can be found in $S^3 - K$, and that will be the needed surgery curve.

4. An application to the knot 8_{10}. Figure 9 (next page) shows two pictures of the knot 8_{10}, the standard picture that appears in knot tables and one for which a Seifert surface is more obvious. The second picture is shown with generators for the homology of the surface. This knot has proven to be a stumbling block in the determination of the unknotting numbers of prime knots with small crossing number. This is because it can easily be unknotted with two crossing changes, yet all of the lower bounds (four-ball genus, minor index, half the signature) are one. The knot is thought to have unknotting number two.

8_{10}

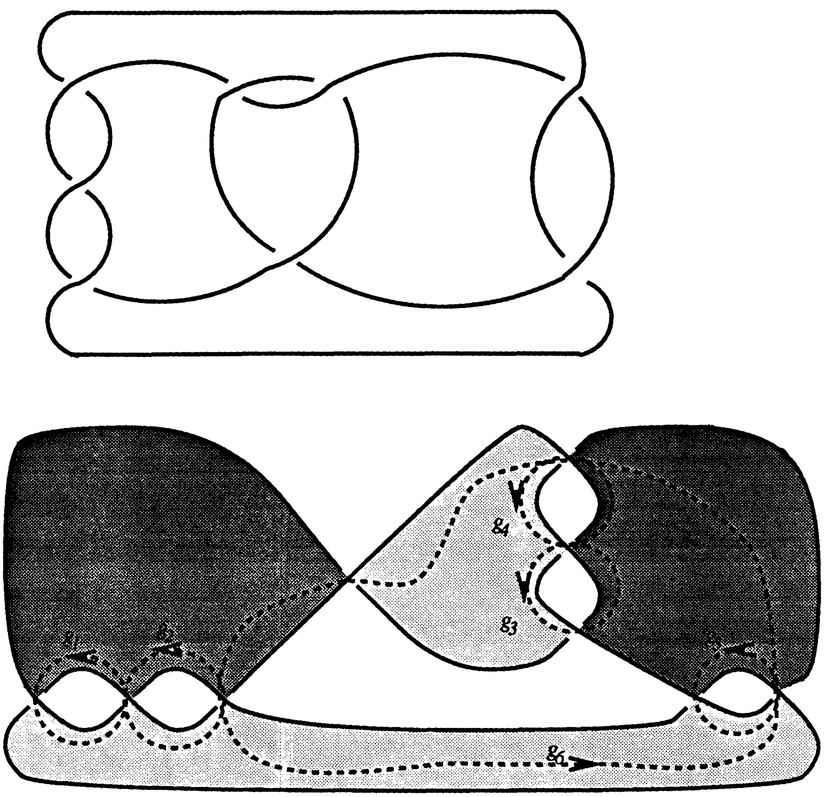

FIGURE 9

A Seifert matrix can be read from Figure 9 as

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

Thus a presentation matrix for the Alexander module is given by

$$\mathscr{M} = \begin{pmatrix} -z & -t^{-1/2} & 0 & 0 & 0 & 0 \\ t^{1/2} & -z & 0 & 0 & 0 & -t^{-1/2} \\ 0 & 0 & z & t^{-1/2} & 0 & 0 \\ 0 & 0 & -t^{1/2} & z & 0 & -t^{-1/2} \\ 0 & 0 & 0 & 0 & -z & t^{-1/2} \\ 0 & t^{1/2} & 0 & t^{1/2} & -t^{1/2} & -z \end{pmatrix}.$$

With a lot of tedium, this matrix can be column reduced to one which is upper triangular with ones on the main diagonal except for the first entry, which is $\Delta = t^3 - 3t^2 + 6t - 7 + 6t^{-1} - 3t^{-2} + t^{-3}$. So the homology of the infinite cyclic cover is a cyclic $Z[t, t^{-1}]$ module, generated by the first generator in the presentation. Call this generator g.

Now consider $\beta(f(t)g, f(t)g)$ for some $f(t)$. Using the definition $\beta(a, b) = z\bar{a}\mathscr{M}^{-1}b$ gives

$$\beta(f(t)g, f(t)g) = zf(t^{-1})f(t)\mathscr{M}_{1,1}/\Delta$$

where $\mathscr{M}_{1,1}$ is the cofactor of the $(1, 1)$-entry of $\mathscr{M}$. A little calculation yields $z\mathscr{M}_{1,1} = -(z^6 + 2z^4 + 2z^2)$. Since the Blanchfield pairing takes values in $Q(t)/Z[t, t^{-1}]$ and the denominator here is Δ, we can reduce $z\mathscr{M}_{1,1}$ modulo Δ to arrive at

$$\beta(f(t)g, f(t)g) = (z^4 + z^2 + 1)f(t^{-1})f(t)/\Delta.$$

Choosing $f(t) = t^3 - t^2 + 2t - 1$ and substituting into the above formula yields $\beta(f(t)g, f(t)g) = 1/\Delta$. Furthermore, $f(t)$ inverts modulo Δ (its inverse is $t^4 - 3t^3 + 4t^2 - 2t + 1$ up to multiplication by units). Therefore, $f(t)g$ also generates the homology of the infinite cyclic cover of 8_{10}, so that applying the theorem proves

COROLLARY 6. *The knot* 8_{10} *has algebraic unknotting number one.*

In this case, a further simplification exists in finding explicitly the crossing change, namely that the lifts of the Alexander duals to the homology generators of the Seifert surface actually form a Z-basis for the Alexander module, so that the surgery curve can actually be calculated in terms of the Alexander duals themselves. When this is done and the curve is suitably modified by the moves in Lemma 1 to trivialize the Alexander polynomial, the resulting curve can be simplified by Reidemeister moves to appear as in Figure 10 (next page). When the surgery is performed, the resulting knot can be reduced to the (at most) 14-crossing knot shown in Figure 11 (next page), which is $6^{**}1.(3, 2)\bar{1}.1.1.\bar{2}\,\bar{1}.2$ in the Conway notation. The Alexander polynomial of this knot can indeed be calculated to be trivial.

In The Introduction this was cited as an unfortunate result. This is because this corollary shows that abelian methods, or any other methods dealing with the Seifert matrix, cannot be used to show that the unknotting number of 8_{10} is not one. More delicated procedures must be found.

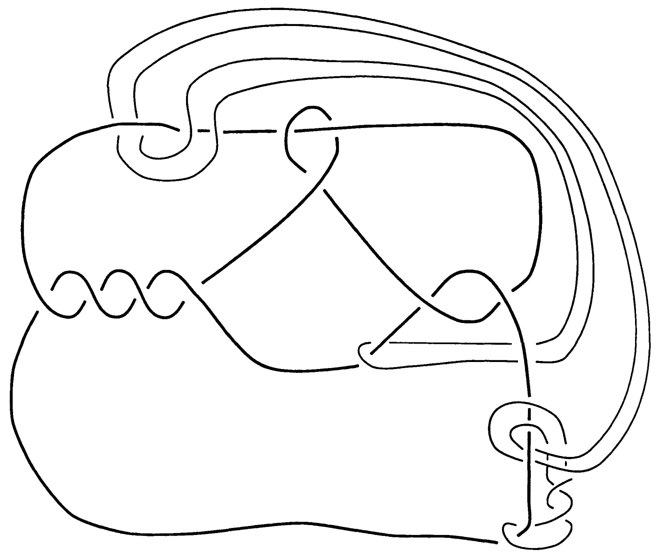

FIGURE 10

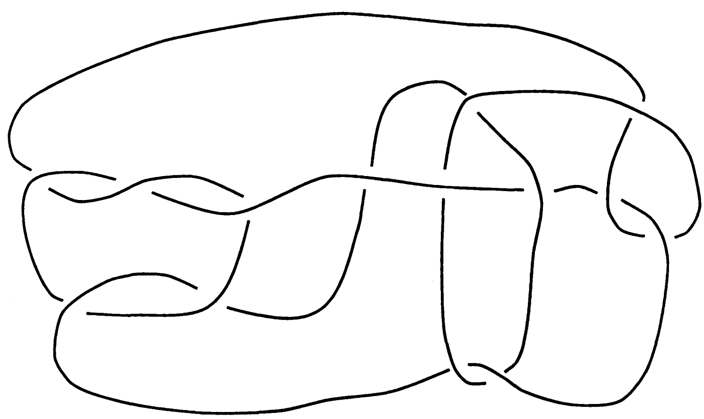

FIGURE 11

REFERENCES

[1] T. D. Cochran and W. B. R. Lickorish, *Unknotting information from 4-manifolds*, Trans. Amer. Math. Soc., **297** (1986), 125–142.

[2] C. McA. Gordon, *Some Aspects of Classical Knot Theory*, Lecture Notes in Math., vol. 685, Springer-Verlag, Berlin and New York, 1978, pp. 1–60.

[3] W. B. R. Lickorish, *The unknotting number of a classical knot*, Contemp. Math., **44** (1985), 117–121.

[4] H. Murakami, *Algebraic unknotting operation*, Q&A In General Topology, **8** (1990), 283–292.

[5] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc., **117** (1965), 387–422.

[6] Y. Nakanishi, *A note on unknotting number*, Math. Sem. Notes, Kobe Univ., **9** (1981), 99–108.

Received December 14, 1991 and in revised form March 3, 1993.

UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

THE STRUCTURE OF CLOSED NONPOSITIVELY CURVED EUCLIDEAN CONE 3-MANIFOLDS

KERRY N. JONES

A structure theorem is proven for closed Euclidean 3-dimensional cone manifolds with all cone angles greater than 2π and cone locus a link (no vertices) which allows one to deduce precisely when such a manifold is homotopically atoroidal, and to construct its characteristic submanifold (torus decomposition) when it is not. A by-product of this structure theorem is the result that any Seifert-fibered submanifold of such a manifold admits a fibration with fibers parallel to the cone locus. This structure theorem is applied to several examples arising as branched covers over universal links.

0. Introduction. Much of the recent progress in 3-manifold topology has to do with the link between topology and geometry in 3-manifolds. There has been a great deal of work in the last decade on homogeneous Riemannian metrics on 3-manifolds, spurred on by the tantalizing prospect of the Thurston Geometrization Conjecture. At the same time, there has been a renewed interest in branched covers, as a result of the notion of a *universal link*, a link in S^3 which has the property that all closed, orientable 3-manifolds are obtained as branched covers over S^3, branched over this fixed link (see, for example, **[HLM]**). It had, of course, long been known that all such 3-manifolds were representable as branched covers over the 3-sphere, but in the older construction, it was a very simple kind of branched cover (namely a 3-fold cover) over a possibly very complicated link in the 3-sphere. One advantage of the newer branched cover construction is that many geometric structures on the fixed link in S^3 lift to the branched covers and thus, to all 3-manifolds. So, it seems likely that by moving the complication from the link to the branched covering map itself we may gain some real insight into the geometry of 3-manifolds.

One particular kind of geometric structure which has this lifting property is that of a *cone manifold* structure (see, for example, **[A-R]**, **[Ho]** and **[Jo1]**). The purpose of this paper is to give a structure theorem for 3-manifolds possessing a certain type of cone manifold structure, namely, a Euclidean cone manifold structure without vertices and with cone angles greater than 2π. These are the "nonpositively curved" cone manifolds referred to in the title. It will become clear subsequently why we refer to these as nonpositively curved. This kind of cone manifold structure is possessed, for example, by all branched covers over the figure-eight knot with branching indices greater than 2 and all branched covers over the Borromean rings with branching indices greater than 1 (both the figure-eight knot and the Borromean rings are universal).

More specifically, we will prove

THEOREM 2.1. *Let M be a closed, orientable 3-dimensional Euclidean cone manifold with no vertices and all cone angles $> 2\pi$. Then there is a canonical compact 2-complex C in M such that*

(1) *the components of the complement of C (denoted by $M_1, \ldots, M_n$) are each the interior of a compact Seifert-fibered manifold (possibly with boundary)*

(2) *each M_i may be given a convex Euclidean cone metric*

(3) *M is atoroidal if and only if each M_i is an open solid torus.*

Note that here (and consistently throughout this paper) *atoroidal* means homotopically atoroidal, i.e., admitting no nonperipheral π_1-injectively *immersed* tori.

We will also deduce some corollaries of this structure theorem, including results related to the Jaco-Shalen/Johannson torus decomposition of these manifolds, restricting the kinds of geometric structures that can be present in these manifolds. We will also be able to reproduce (only for manifolds of this type) Casson, Jungreis and Gabai's recent result (see **[Ga]**) that manifolds with π_1-injectively immersed tori but no incompressible tori must be Seifert-fibered.

We will then apply this theorem to several illustrative examples. The manifolds to which this theorem applies are known to be irreducible and in fact to have universal cover $\mathbb{R}^3$, so finding the tori in these manifolds is the key to understanding how they fit into the Thurston Geometrization Program.

1. Cone manifolds. We will begin by making a few brief definitions and state some preliminary results. More details may be found in **[Jo2]**.

DEFINITION. A *Euclidean cone manifold* is a metric space obtained as the quotient space of a disjoint union of a collection of geodesic n-simplices in $\mathbb{E}^n$ by an isometric pairing of codimension-one faces in such a combinatorial fashion that the underlying topological space is a manifold.

Such a space possesses a flat Riemannian metric on the union of the top-dimensional cells and the codimension-1 cells. On each codimension-2 cell, the structure is completely described by an angle, which is the sum of the dihedral angles around all of the codimension-2 simplicial faces which are identified to give the cell. The *cone locus* of a cone manifold is the closure of all the codimension-2 cells for which this angle is not 2π (the Riemannian metric may be extended smoothly over all cells whose angle is 2π). For the purposes of this paper, we are interested in the 3-dimensional case in which the singular locus is a link (which must have constant cone angle on each component) and we make this blanket assumption throughout the remainder of the paper.

One particularly useful feature of the cone manifold structure is its close relationship with the notion of a branched cover. Recall that a branched covering map is a continuous map of pairs $\rho : (\hat{M}, \hat{L}) \to (M, L)$ where $\hat{M}, M$ are n-manifolds, and $\hat{L}, L$ are $(n-2)$-complexes, which restricts to a covering map both on $\hat{L}$ and on the complement of $\hat{L}$ (we will make the stipulation that $\hat{L}$ be saturated with respect to ρ for technical convenience). The important result is that if M is a cone manifold with the cone locus contained in L, then $\hat{M}$ is a cone manifold with the cone locus contained in $\hat{L}$. In particular, cone metrics may be lifted to true covers as well as branched covers (a covering map is clearly a branched covering map with any downstairs branch set whatever). Branched covering maps of degree d, branched over a fixed branch set L are in one-to-one correspondence with conjugacy classes of transitive representations of $\pi_1(M-L)$ into S_d (that is, representations whose image acts transitively on the set $\{0, 1, \dots, d-1\}$). We also note that the cone angles in the lifted cone manifold structure are the downstairs cone angles multiplied by the branching indices of the branched covering (we will need this in our examples).

Geodesics in a Euclidean cone manifold are of three different types: straight lines joining points on the cone locus which join in such a way as to have an angle of at least π measured in either direction, straight lines disjoint from the cone locus, and straight lines contained in the cone locus. One consequence of the nature of geodesics in Euclidean cone manifolds is that when a geodesic encounters a point of cone angle less than 2π, that geodesic may not be extended beyond that point, since no possible direction of an extension will have the required angle measure. Conversely, however, when a geodesic encounters a cone point with angle greater than 2π there are an infinite number of distinct ways to continue.

As mentioned earlier, there is a very strong analogy between cone angle and curvature, as one might expect by considering, for example, the Gauss-Bonnet theorem. More specifically, cone angles greater than 2π act like negative curvature and cone angles less than 2π act like positive curvature. To be precise, we have the following

PROPOSITION 1.1. *Let M be a Euclidean cone 3-manifold with cone locus a link. If all the cone angles of M are less than 2π, M admits a smooth Riemannian metric of nonnegative sectional curvature. If all the cone angles of M are greater than 2π, M admits a smooth Riemannian metric of nonpositive sectional curvature.*

Proof. One constructs a metric of bounded sectional curvature which is flat outside of a tubular neighborhood of the cone locus. See **[Jo1]**, Theorems 2.1 and 2.2. Similar techniques are used in **[G-Th]** with hyperbolic cone manifolds. □

One of the most useful aspects of this smoothing technique is that it gives us immediately that the universal cover of a Euclidean cone manifold with singular locus a link and all cone angles greater than 2π is $\mathbb{R}^3$ (apply the Cartan-Hadamard theorem to the smooth metric). In particular, such a manifold is irreducible.

By being a bit more careful with the smoothing, we can also deduce the following theorem, which is an analogue (and consequence) of a minimal surface result in Riemannian geometry due to Schoen and Yau **[S-Y]**.

PROPOSITION 1.2. *Let M be a compact Euclidean cone 3-manifold with cone locus a link and all cone angles greater than 2π. Then, any π_1-injective map of a torus into M is homotopic to a totally geodesic torus (in the cone metric) which contains some component of the cone locus.*

Proof. See **[Jo2**, Lemma 3.1**]** for the details. Essentially, one shows that one can take a sufficiently tight smoothing to which one applies the Schoen and Yau minimality result and obtains a totally geodesic torus in the smooth metric which is homotopic to a totally geodesic torus in the cone metric. This torus can be translated in a normal direction and remains totally geodesic until it hits some component of the cone locus, which it must in fact contain. □

This result will be the key to the proof of part (3) of Theorem 2.1.

2. Structure theorem.

THEOREM 2.1. *Let M be a closed, orientable 3-dimensional Euclidean cone manifold with no vertices and all cone angles $> 2\pi$. Then there is a canonical compact 2-complex C in M such that*

(1) *the components of the complement of C (denoted by $M_1, \ldots, M_n$) are each the interior of a compact Seifert-fibered manifold (possibly with boundary)*
(2) *each M_i may be given a convex Euclidean cone metric*
(3) *M is atoroidal if and only if each M_i is an open solid torus.*

Proof. We will construct this decomposition by working in $\hat{M}$, the universal cover of M. We will mimic, in some sense, the usual Dirichlet domain construction of differential geometry.

Begin with disjoint metrically regular tubular neighborhoods of the cone locus in $\hat{M}$. Expand the radius of these tubular neighborhoods equivariantly. When two of the neighborhoods touch, continue expanding in such a way as to maintain the product structure of each neighborhood. That is, after the first point at which two of these bump into each other, each neighborhood will be a round tubular neighborhood with a flat side cut off by a plane parallel to the core geodesics of both of the intersecting neighborhoods (see Fig. 2.1 on next page). These boundary "ribbons" intersect (nontransversely) in parallelograms (generically—they *coincide* if the core geodesics of the intersecting neighborhoods are parallel) and, as the neighborhoods continue to expand, the ribbons widen until they bump into another ribbon (or possibly the round part of another neighborhood if a tangency of the round parts occurs exactly at a "corner" of the cross section). Note that at all times the cross section of each neighborhood is convex. Note also that this expansion cannot continue indefinitely (all cross sections must eventually be compact polygons) since a regular neighborhood of the cross section is imbedded under the projection to M, which has finite volume.

When the expansion of these convex product neighborhoods has been carried as far as it will go, the union of all the boundaries form

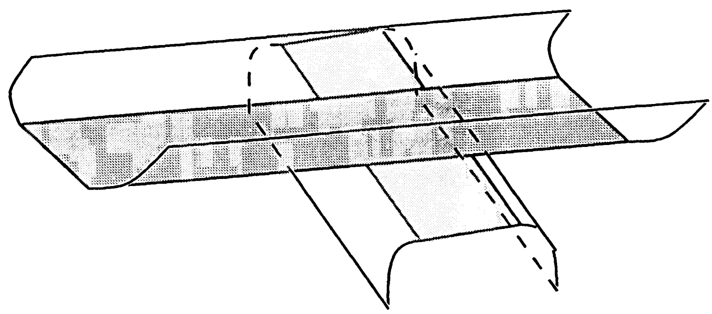

FIGURE 2.1

an invariant (under the actions of the deck transformations on $\hat{M}$) 2-complex C_1 whose complement is a collection of open parallellepipeds with convex base (and a singular core geodesic) and a collection of open Euclidean solid polyhedra. We note that each of these Euclidean polyhedra (the components that do *not* contain a cone geodesic) has compact faces, since each face is the portion of a ribbon *between* two of the nontransverse intersections with other ribbons. We need to eliminate these Euclidean polyhedra. First, however, we will note the following lemma, which will be useful subsequently.

LEMMA 2.2. *Let α be a cone geodesic in a Euclidean cone manifold M satisfying the hypotheses of Theorem* 2.1. *Let $\hat{\alpha}$ be a component of the preimage of α in $\hat{M}$ and let Γ_α be the deck transformation on $\hat{M}$ with minimum translation distance which leaves $\hat{\alpha}$ invariant* (*i.e., the deck transformation that "rolls up" $\hat{\alpha}$ into α*). *Then, Γ_α rotates a tubular neighborhood of $\hat{\alpha}$ by an angle rationally related to the cone angle at α.*

Proof of Lemma. Since the deck transformations act by isometries and the preceding construction is geometrically canonical, any deck transformation that leaves a cone geodesic invariant must leave the component of the complement of C_1 containing that cone geodesic invariant also. In particular, the isometry must take polygonal cross sections to polygonal cross sections and so must act locally as a translation composed with a rotation rationally related to the cone angle at the center point (other symmetries of the polygon are ruled out by orientability). □

Now, we will eliminate the Euclidean polyhedra in the complement of C_1 (at the cost of convexity of the complement) by cutting each of these Euclidean regions up by considering the shortest path from an interior point to the boundary. The set of points that admit shortest paths to two or more faces (including those whose unique shortest path is to the intersection of two faces) is an invariant 2-complex which decomposes the polyhedron into contractible bounded polyhedra. We now alter C_1 by removing the faces which are part of the boundary of one of these Euclidean polyhedra and adding in the 2-complex which subdivides each polyhedron to yield a 2-complex C_2. The complement of C_2 consists entirely of polyhedra which retract to a cone geodesic. They are convex parallelepipeds with non-convex "warts" attached to them along the faces which were between the intersections with the other ribbons. C_2 is still invariant under the action of the deck transformations on $\hat{M}$ and, since each component of the complement has exactly one cone geodesic in it, has the property that the components of the complement are left invariant only by a deck transformation that has an invariant cone geodesic. In particular, using Lemma 2.2, we see that the complementary regions project to open solid tori in M which may be canonically Seifert-fibered by the projections of lines parallel to the singular core geodesic (actually the Seifert-fibration is canonical only on the complement of C_1, but it may be extended to the complement of C_2 in an obvious, but non-canonical, fashion—this will cause us no difficulties, as we will only need the fibration to be canonical near faces which are in both C_1 and C_2).

Next, we will define a new invariant 2-complex C_3 by removing all the interiors of all the noncompact faces from C_2. These are all infinite strips which bisect an infinite strip cobounded by two parallel cone geodesics. We note that this can be done without disturbing the Seifert-fibration on the complement, since the Seifert-fibrations on the two sides of all of the removed faces agree. If this face removal leaves any isolated geodesics in C_3, remove them also. Note that these may be additional singular fibers for the complement of C_3—it is no longer true that all singular fibers of the fibration are cone geodesics. Singular fibers of order 2 can also be introduced which bisect a type-II face (see definition below) if that face is glued to an image of itself under a deck transformation.

Now, let C be the projection of C_3 to M. We claim that C has the desired properties.

Let us now proceed to verify the conclusions of the Theorem: (1) is clear from the construction. (2) follows from the following construction: let $\hat{\alpha}$ be a cone geodesic in $\hat{M}$. Let $N(\hat{\alpha})$ be the convex

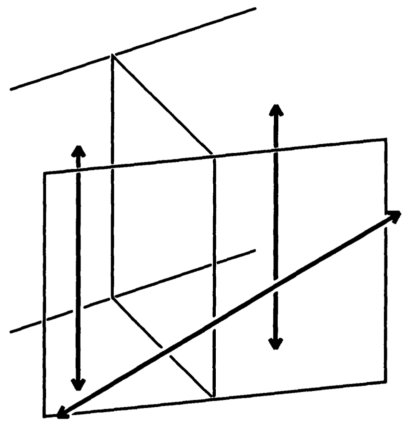

FIGURE 2.2

parallelepiped obtained by expanding a tubular neighborhood of $\hat{\alpha}$ until it hits either another cone geodesic *or* the perpendicular bisector of the strip cobounded by $\hat{\alpha}$ and some parallel cone geodesic $\hat{\beta}$. We will refer to the former faces as "type-I faces" and the latter as "type-II faces." $N(\hat{\alpha})$ has compact cross section since a regular neighborhood of a polygon similar to the cross section, but shrunk by a factor of two is imbedded under projection to M. Now, consider the collection of $N(\hat{\beta})$ for all $\hat{\beta}$ parallel to $\hat{\alpha}$ (here parallel means "cobounding a totally geodesic flat strip"). These may be glued along the type-II faces to give a new parallelepiped $P(\hat{\alpha})$ which is still convex since any type-I face which is adjacent to a type-II face corresponds to a cone geodesic $\hat{\gamma}_1$ which is not parallel to the core geodesic and thus causes a type-I face adjacent to the corresponding type-II face in the adjacent parallelepiped (adjacent across the type-II face) making an angle of π with the first type-I face (see Fig. 2.2) unless another geodesic $\hat{\gamma}_2$ cuts it off exactly at the vertex, causing an angle less than π.

Now, it need not be the case that $N(\hat{\alpha})$ projects to an open solid torus in M, or that $P(\hat{\alpha})$ projects to a Seifert-fibered subset of M, but it is true that $P(\hat{\alpha})$ is homeomorphic to a component $M_{\hat{\alpha}}$ of the complement of C_3 whose stabilizer Γ leaves $P(\hat{\alpha})$ invariant (it is generated by deck transformations that either "roll up" or permute the cone geodesics of $M_{\hat{\alpha}}$ which are also the cone geodesics of $P(\hat{\alpha})$). and thus, $M_{\hat{\alpha}}$ projects to a Seifert-fibered subspace of M that is homeomorphic to $P(\hat{\alpha})/\Gamma$ which is the interior of a compact convex Euclidean cone manifold.

(3) is somewhat more difficult to verify: we will define an associated convex cone 2-manifold (similar to the technique used in **[Jo2]**) which

has the property that M is atoroidal if and only if the 2-manifold has no closed geodesics. (3) will follow from this. First, we will define the *associated 2-orbifold* for M and subsequently define the *associated 2-manifold* for M.

For each cone geodesic α in M, take a copy of the cross section of $N(\hat{\alpha})$, then take a quotient of this cross section under the rotation guaranteed by Lemma 2.2 and denote this quotient by $O(\alpha)$. $O(\alpha)$ is a convex "cone orbifold"—an orbifold in which the cone angles at singular points are not necessarily $2\pi/n$ where n is the order of the isotropy group. Thus, in a cone orbifold, one needs to record the cone angle at a singularity separately from the order of the local isotropy group. Now, some boundary edges of the collection of cone orbifolds will correspond to type-II faces of the $N(\hat{\alpha})$ and some will correspond to type-I faces (note that the rotation of which $O(\alpha)$ is the quotient preserves face type). Take the collection of $O(\alpha)$ for all cone geodesics α in M and glue corresponding type-II faces together—this will perhaps introduce new orbifold singularities at vertices of the $O(\alpha)$ and perhaps at the midpoints of edges (these must have isotropy order 2). Note that we must orient the cone locus to fix a normal direction for the $O(\alpha)$ in order to insure that the gluing is well-defined. The components of this new cone orbifold (which we will denote by $O(M)$) are the base orbifolds for the Seifert fibrations on the various M_i.

Now, we are ready to define an *associated 2-manifold* for M, which we will denote by $\hat{O}(M)$ (note that this is slightly different from the definition in **[Jo2]**—the 2-manifold in **[Jo2]** is the union of the cross sections of the $P(\hat{\alpha})$ which is the universal cover of the 2-manifold we will define here). We use the fact that all orbifolds (with two families of exceptions) have a finite cover which is a manifold and take $\hat{O}(M)$ to be the union of the minimal-degree manifold covers for each component of $O(M)$. This is perhaps not uniquely defined, but we really only need some compact manifold cover, so our definition will be sufficient for our purposes here. We need only show that none of the components of $O(M)$ are "bad" orbifolds (in Thurston's terminology, see **[Sc]**, **[Th]**). The bad orbifolds, however, all have underlying space S^2 and a simple Gauss-Bonnet argument shows that S^2 can admit a Euclidean cone metric only when there are at least 3 cone points with cone angles less than 2π. But the only cone points on $O(M)$ that have cone angle less than 2π are points that have nontrivial isotropy groups, and thus the orbifold structure must have at least three singularities. But all of the bad orbifolds have fewer than three singularities.

To see that $\hat{O}(M)$ has the property claimed, we use Proposition 1.2 to see that any injectively immersed torus is homotopic to a totally geodesic torus containing some cone geodesic α and thus corresponds to a closed geodesic in any component of the 2-manifold which contains a cross section of $N(\hat{\alpha})$. To see this, lift the torus to a totally geodesic plane in $\hat{M}$ which contains a geodesic $\hat{\alpha}$ and observe that this plane stays entirely in $P(\hat{\alpha})$ and thus meets any cross section of $P(\hat{\alpha})$ in a geodesic which projects to a closed geodesic in $\hat{O}(M)$. Furthermore, any closed geodesic in $\hat{O}(M)$ corresponds to a totally geodesic (and hence π_1-injective) immersed torus in M. Thus, M is atoroidal if and only if there are no closed geodesics in the associated 2-manifold. It should be noted that, in general, a torus corresponds to several distinct geodesics in $\hat{O}(M)$ which form an equivariant family with respect to the orbifold covering projection to $O(M)$.

It only remains to show that the associated 2-manifold of M contains no closed geodesics if and only if each component of the complement of C is a solid torus. Since each component of $\hat{O}(M)$ is a Euclidean cone manifold with all cone angles greater than 2π, there will be closed geodesics in each free homotopy class of loops in $\hat{O}(M)$. Thus, M is atoroidal if and only if each component of $\hat{O}(M)$ is simply connected. Since the 2-sphere does not admit a Euclidean cone metric with all cone angles greater than 2π, no component of $\hat{O}(M)$ can be a 2-sphere. Thus, the only obstruction to the existence of tori in M is the possibility that each component of $\hat{O}(M)$ is a disk. But, the only orbifolds that are covered by a manifold disk are disks with a single orbifold singularity and all of the Seifert-fibered spaces corresponding to these bases are solid tori (again, see [Sc]). □

Actually, somewhat more can be said than the preceding theorem. For each one of the M_i which is not an open solid torus, we observe that we can find a collection of disjoint 2-sided embedded tori (one for each end of M_i) which are parallel to C and saturated with respect to the Seifert fibration on M_i (since each end of the interior of an orientable Seifert-fibered manifold with boundary is a product of a torus with an open interval). Each of these tori must in fact be incompressible, since this torus fibers over a boundary curve of the associated 2-manifold to M_i. This boundary curve is homotopically nontrivial and hence homotopic to a geodesic in the 2-manifold which is covered by a totally geodesic torus (hence π_1-injective) in M.

Thus, if there is more than one M_i, the manifold must be Haken unless all M_i are solid tori, in which case the manifold is atoroidal. In particular, if M admits an injectively immersed torus, there must be some M_i that is not a solid torus, and if M admits no incompressible tori, there must be only one M_i. Thus, we recover the result (only for manifolds of this form) that a manifold that admits an injectively immersed torus but not an incompressible torus must be Seifert-fibered (see **[Ga]**).

Furthermore, these tori form a collection T containing the canonical collection of tori in the Jaco-Shalen/Johannson torus decomposition (see **[J-S]** and **[Jh]**). To see this, we observe that each torus in T cuts off a "collar" from its associated M_i. The components of the complement of T thus fall into one of three categories:

(1) a manifold homeomorphic to a non-solid torus component of the complement of C

(2) a manifold consisting of a union of solid torus components of the complement of C, together with one or more collars and components of C

(3) a manifold consisting of collars and components of C.

We observe that each of these components must be Seifert-fibered or atoroidal: a component in the first category is clearly Seifert-fibered. For a component, N, in the second or third category, we observe that each collar may be extended metrically (away from the component in question) until the torus boundary is totally geodesic in the cone metric. This cannot necessarily be accomplished in M, since the geodesic homotopic to the boundary curve in the associated 2-manifold need not be simple (also, the surface covering the geodesic might be a one-sided Klein bottle instead of a torus), but it can certainly be done metrically by working (for example) in the cover of M corresponding to the fundamental group of the particular torus in question. This metric extension is homeomorphic to N. Repeat this procedure for all collars of N. We now have a Euclidean cone manifold with totally geodesic boundary (note that it may have cone locus on the boundary) which we may double to obtain a closed Euclidean cone manifold (call it N') which either has no cone locus (possible only if N was in the third category) and is hence a Euclidean manifold and thus Seifert-fibered or has nonempty cone locus and satisfies the hypotheses of Theorem 2.1. Note now that in N', all π_1-injective tori may be homotoped to the doubling tori and, thus, all tori are peripheral in each half (using standard free product with amalgamation results).

Note that we are *not* asserting that the atoroidal pieces obtained in this way are not Seifert-fibered also—there are some spaces that are both atoroidal and Seifert-fibered (the I-bundles over the torus and Klein bottle).

Finally, we observe that there is a restriction on the kinds of geometries that the Seifert-fibered pieces can possess—the base orbifold *must* be negatively curved (since there *are* cone points on the associated 2-manifold it must have negative Euler characteristic). So, a maximal proper Seifert-fibered submanifold of a manifold of this type must have $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{E}^3$ geometry (for the components that consist of collars only and have empty cone locus) and, if the whole manifold is Seifert-fibered, it must have $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{SL_2\mathbb{R}}$ geometry (again, see **[Sc]** for the relevant definitions—for a different proof of a slightly weaker result, see **[Jo1**, Chapter 5]).

We collect these results in the following

COROLLARY 2.2. *If M is a Euclidean cone manifold satisfying the hypotheses of Theorem 2.1, then*

(1) *if M admits a π_1-injective torus but no incompressible torus, M must be Seifert-fibered*
(2) *the collection of boundary-parallel tori in each non-solid torus component of M_i forms a collection of tori containing the Jaco-Shalen/Johannson characteristic tori*
(3) *if M is Seifert-fibered, it must have $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{SL_2\mathbb{R}}$ geometry*
(4) *a maximal proper Seifert-fibered submanifold of M must have $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$ geometry.*

3. Examples. The easiest way to get examples of cone manifolds of this type is to consider *sufficiently branched* covers over Euclidean orbifolds, that is, branched covers over a topological space which admits a Euclidean orbifold structure in which the downstairs branching locus is equal to the singular locus of the orbifold and the branching indices over each component are greater than or equal to the order of the isotropy group of that component in the orbifold fundamental group of the base. Two particularly accessible orbifolds to use in this context are the figure-eight knot and the 6^2_2 link (see the link tables in **[Ro]**) since both of these have had their lattice of branched covers calculated up to degree 10 (**[He]**, **[Jo3]**). These links are of interest since they are both non-torus rational links and hence universal **[HLM]**.

First, we note that much of the actual calculation of the 2-complex

C is unnecessary if all we are interested in is, say, the homeomorphism types of the various components of the complement of C. In this case, we really need only calculate the associated 2-manifolds of M corresponding to the various parallel classes of cone geodesics and look at how the parallelepipeds over them fit together. This can be done quite conveniently in the case of sufficiently branched covers over orbifolds by simply examining the monodromy of the branched cover.

First, the figure-eight knot (a more detailed development of whose geometry may be found in **[Jo2]**): S^3 admits a Euclidean orbifold structure with cone angle $2\pi/3$ along the figure-eight knot. Therefore, any branched cover over S^3, branched over the figure-eight knot with all branching indices greater than 2 admits a Euclidean cone manifold structure satisfying the hypotheses of Theorem 2.1. Let us fix some notation by letting K denote the figure-eight knot and $\varphi\colon \pi_1(S^3 - K) \to S_d$ be a homomorphism with transitive image in S_d (that is, whose image acts transitively on $\{0, 1, \dots, d-1\}$). Then, φ is the monodromy of a degree d cover of $S^3 - K$ and thus a degree d branched cover of S^3, branched over K. We will use the presentation

$$\langle a, b, c, d : d^{-1}b^{-1}c, b^{-1}aba^{-1}c, a^{-1}d^{-1}c\rangle$$

for $\pi_1(S^3 - K)$ and note that the group is generated by a and c so that we need only specify φ on these generators. Then, a component of the cone locus corresponds to a cycle in $\varphi(a)$ of length 4 or greater. For each such cycle of length q, we have a parallelepiped with base a $2q$-gon which is the universal cover of a product neighborhood of the component of the cone locus. It is possible that two or more of these cycles represent the same component of the cone locus if φ of the longitude of the knot $(ba^{-1}c^{-1}ad)$ takes one cycle to another. Let us label the vertices of each polygon in the order of each cycle of $\varphi(a)$ by the labels $0, 1, \dots, d-1$ alternating with $0', 1', \dots, (d-1)'$. We may ascertain which vertices of the polygonal cross-section correspond to type-I faces and which correspond to type-II faces by the following calculation: writing permutation actions on the right, and denoting the set of fixed points of a permutation σ by $\operatorname{fix}(\sigma)$ we define

$$\begin{aligned} F = {} & \operatorname{fix}(\varphi(d^3))\varphi(a^{-1}cab^{-1}) \cap \operatorname{fix}(\varphi(d^3))\varphi(b^{-1}) \\ & \cap \operatorname{fix}(\varphi(a^3))\varphi(b^{-1}) \cap \operatorname{fix}(\varphi(a^3))\varphi(cab^{-1}). \end{aligned}$$

Then, we set

$$G = \{j \mid \operatorname{orbit}(\langle \varphi(ba^{-1}c^{-1}ad)\rangle, j) \subset F\}.$$

Then, a vertex with a label i is a type-II face if and only if $i \in G$ and it is glued to the vertex with label $(i\varphi(bd^{-1}))'$. From this information, we can compute the associated 2-manifold.

For example, if we set

$$\varphi(a) = (0\,2\,1)(3\,4\,7)(5\,6\,9\,8)$$

and

$$\varphi(c) = (0\,5\,2\,4\,9\,7)(6\,8)$$

(which is branched cover number 43 in **[He]**), we find that there is one component of cone locus (cone angle $= 8\pi/3$) whose associated 2-orbifold is a disk with two orbifold singularities, of orders 2 and 3. The 2-fold singularity comes from the fact that the monodromy of the longitude in this cover rotates the disk normal to the cone locus through an angle of $4\pi/3$, yielding a quotient orbifold with four vertices in the boundary, each having angle $2\pi/3$. The 3-fold singularity comes from the fact that two adjacent faces of this orbifold correspond to type-II faces which are glued to each other, yielding the orbifold asserted above. Thus, the torus decomposition of this space consists of an atoroidal Euclidean piece (which is in fact a twisted I-bundle over the Klein bottle) and the Seifert-fibered space which fibers over the disk with two exceptional fibers, of orders 2 and 3 (the trefoil knot complement).

Using another of Hempel's examples (number 37), we set

$$\varphi(a) = (0\,2\,1)(3\,7\,5\,8\,4\,9\,6)$$

and

$$\varphi(c) = (0\,3\,5\,6)(2\,4\,8\,7)$$

and calculate that here there is also one component of cone locus (this time with cone angle $14\pi/3$) whose associated 2-manifold is a disk (there are no type-II faces) with only one cone point and thus we have an atoroidal manifold (which is in fact computed to be hyperbolic by Jeff Weeks' computer program *snappea*).

At this point, a remark is in order about how the definitions for F and G were obtained: this computation is done in detail in **[Jo2]** and consists of examining the flat planes extending out from the cone locus in the direction of a potentially parallel component of cone locus and checking which components of the branching locus are intersected transversely along the way—for the two components to be truly parallel (and thus separated by a type-II face) it must be the case that all components of the branching locus encountered must *not* be in the cone locus (in this case, they must be in the 3-fold branching locus). The definitions for F and G are merely codifications of these intersection conditions in terms of the monodromy of the branched cover.

The 6_2^2 link is somewhat more complicated than the figure-eight knot because it is a 2-component link. In fact, the Euclidean orbifold structure has different cone angles on the two link components even though there is an involution of S^3 that takes one component to the other. The Euclidean orbifold structure has cone angle $2\pi/3$ on one component and π on the other.

We will use the presentation

$$\langle a, b, c, d, e : ab^{-1}d^{-1}ba^{-1}b^{-1}, de^{-1}a^{-1}, ca^{-1}e^{-1}, bece^{-1}\rangle$$

for the fundamental group of the 6_2^2 link complement and note that it is generated by a and b (which are meridians of the two components) and thus we need only specify φ on these two elements. We will use the orbifold structure in which a has cone angle $2\pi/3$ and b has cone angle π.

There are two distinct types of associated 2-manifolds here, the ones corresponding to components of the cone locus that cover the a component and the b component, respectively. For the former, as before, we set

$$\begin{aligned} F' = {} & \operatorname{fix}(\varphi(b^2)) \cap \operatorname{fix}(\varphi(c^2)) \\ & \cap \operatorname{fix}(\varphi(a^3))\varphi(ec^{-1}) \cap \operatorname{fix}(\varphi(a^3))\varphi(b^{-1}a^{-1}ec^{-1}) \\ & \cap \operatorname{fix}(\varphi(d^2))\varphi(a^{-1}ec^{-1}) \cap \operatorname{fix}(\varphi(d^2))\varphi(ec^{-1}) \end{aligned}$$

and let

$$G' = \{j \mid \operatorname{orbit}(\langle\varphi(ce^{-1}a^{-1}b)\rangle, j) \subset F'\}$$

and compute that the type-II faces run between the vertices labelled i (where $i \in G'$) and $(i\varphi(ce^{-1}d^{-1}))'$.

For the components of the cone locus that cover the b component of the 6_2^2 link, we set

$$\begin{aligned} F'' = {} & \operatorname{fix}(\varphi(a^3)) \cap \operatorname{fix}(\varphi(c^2)) \\ & \cap \operatorname{fix}(\varphi(b^2))\varphi(a^{-1}) \cap \operatorname{fix}(\varphi(a^3))\varphi(e^{-1}) \end{aligned}$$

and let

$$G'' = \{j \mid \operatorname{orbit}(\langle\varphi(ea^{-1}ba^{-1})\rangle, j) \subset F''\}$$

and we set

$$\begin{aligned} F''' = {} & \operatorname{fix}(\varphi(d^2)) \cap \operatorname{fix}(\varphi(a^3))\varphi(ae^{-1}) \\ & \cap \operatorname{fix}(\varphi(c^2))\varphi(ae^{-1}) \cap \operatorname{fix}(\varphi(a^3))\varphi(c^{-1}ae^{-1}) \end{aligned}$$

and let

$$G''' = \{j \mid \text{orbit}(\langle \varphi(ea^{-1}ba^{-1})\rangle, j) \subset F'''\}.$$

For covers over the 6_2^2 link, we have the type-II faces running between vertices labelled i and $i\varphi(ea)$ where $i \in G''$ and also between i' and $(i\varphi(d))'$ where $i \in G'''$. We note also that crossing a type-II face around a cone geodesic that covers b reverses the orientation of the geodesic.

We will again apply this procedure to two examples. For the first (10.56 in **[Jo3]**), we set

$$\varphi(a) = (0\,1\,2)(3\,4\,5)(6\,7\,8\,9), \qquad \varphi(b) = (0\,3)(1\,4)(2\,6\,7\,5\,8\,9).$$

We find that there are two components of cone locus, one of which (covering b) has associated 2-orbifold a Möbius band with one order-2 singularity and the other of which (covering a) has associated 2-manifold a disk with one cone point. Thus, we have a manifold whose torus decomposition consists of a Seifert-fibered space over the Möbius band with one singular fiber of order 2 and an atoroidal manifold with one cusp.

For our second example, we set

$$\varphi(a) = (0\,1\,2)(3\,4\,5)(6\,7\,8\,9), \qquad \varphi(b) = (0\,1\,3\,4)(2\,6\,5\,8)(7\,9).$$

This is example 10.49 in **[Jo3]**.

Here, we again have two components of cone locus (both of the 4-cycles in $\varphi(b)$ are on the same component of cone locus) and we find that there are no type-II faces, so that we have an atoroidal manifold (which is in fact hyperbolic—again courtesy of *snappea*).

References

[A-R] I. R. Aitchison and J. H. Rubinstein, *An introduction to polyhedral metrics of non-positive curvature on 3-manifolds*, LMS Lecture Note Series, **151** (1990), 127–162.

[G-Th] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math., **89** (1987), 1–12.

[Ga] David Gabai, *Convergence groups are Fuchsian groups*, preprint.

[HLM] H. M. Hilden, M. T. Lozano and J. M. Montesinos, *On knots that are universal*, Topology, **24** (1985), 499–504.

[He] John Hempel, *The lattice of branched covers over the figure eight knot*, Topology Appl., **34** (1990), 609–642.

[Ho] Craig Hodgson, *Geometric structures on 3-dimensional orbifolds: notes on Thurston's proof*, preprint.

[J-S] William H. Jaco and Peter B. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc., **21** (1979).

[Jh] Klaus I. Johannson, *Homotopy equivalences of 3-manifolds with boundary*, Lect. Notes in Math., vol. 761, Springer-Verlag, Berlin and New York, 1979.

[Jo1] Kerry N. Jones, *Cone manifolds in 3-dimensional topology: applications to branched covers*, thesis, Rice University, 1990.

[Jo2] ——, *Injectively immersed tori in branched covers over the figure eight knot*, Proc. Edinburgh Math. Soc., **36** (1993), 231–259.

[Jo3] ——, *Branched covers of the* 6_2^2 *link up to degree* 10, computer file available from the author.

[Ro] Dale Rolfsen, *Knots and Links*, Publish or Perish, Berkeley, 1976.

[S-Y] R. Schoen and Shing-Tung Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Ann. of Math., **110** (1979), 127–142.

[Sc] Peter Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc., **15** (1983), 401–487.

[Th] William Thurston, *The geometry and topology of three-manifolds*, Princeton University Course Notes, 1979.

Received April 27, 1992 and in revised form November 25, 1992.

THE UNIVERSITY OF TEXAS
AUSTIN, TX 78712

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

ON THE FROBENIUS MORPHISM OF FLAG SCHEMES

MASAHARU KANEDA

Dedicated to Professor C. W. Curtis on the occasion of his 65th birthday

We give a new proof to V. B. Mehta and A. Ramanthan's theorem that the Schubert subschemes in a flag scheme are all simultaneously compatibly split, using the representation theory of infinitesimal algebraic groups. In particular, the present proof dispenses with the Bott-Samelson schemes.

Let K be a perfect field of positive characteristic p. If A is a K-algebra and $r \in \mathbf{Z}$, one defines a new K-algebra $A^{(r)}$ by the ring homomorphism $K \to A$ such that $\xi \mapsto \xi^{p^{-r}}$. Given a K-scheme $\mathfrak{X}$ we will denote by $\mathfrak{X}^{(r)}$ the K-scheme having the same underlying topological space as that of $\mathfrak{X}$ but with the structure sheaf $\mathscr{O}_{\mathfrak{X}} \otimes_K K^{(-r)}$, which we regard as a sheaf of K-algebras by the usual multiplication of K on $K^{(-r)}$ from the right. If $\mathscr{F}$ is an $\mathscr{O}_{\mathfrak{X}}$-module, we set $\mathscr{F}^{(r)} = \mathscr{F} \otimes_K K^{(-r)}$; it comes equipped with the structure of an $\mathscr{O}_{\mathfrak{X}^{(r)}}$-module. If $r > 0$, the morphism $F_{\mathfrak{X}}^r \colon X \to X^{(r)}$ that is the identity on the underlying topological spaces and such that $a \otimes \xi \mapsto a^{p^r}\xi$ for each $a \in \Gamma(\mathfrak{V}, \mathscr{O}_{\mathfrak{X}})$ and $\xi \in K^{(-r)}$ with $\mathfrak{V}$ open in $\mathfrak{X}$ is called the rth Frobenius morphism of $\mathfrak{X}$.

If K is algebraically closed, Hartshorne **[HASV]**, (III.6.4) showed that on the projective spaces over K, the direct image of any invertible sheaf under the Frobenius morphism splits into a direct sum of invertible sheaves; this was crucial for B. Haastert **[Haas]** to prove the $\mathscr{D}$-affinity of the projective spaces. We will compute in §1 which invertible sheaf enters as a direct summand.

More generally, we say after V. B. Mehta and A. Ramanathan **[MR]** that $\mathfrak{X}$ is Frobenius split iff the structural morphism $F_{\mathfrak{X}}^{\natural} \colon \mathscr{O}_{\mathfrak{X}^{(1)}} \to F_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}$ admits a left inverse, called a Frobenius splitting, so that $\mathscr{O}_{\mathfrak{X}^{(1)}}$ is a direct summand of $F_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}$. If σ is a Frobenius splitting of $\mathfrak{X}$ and if $\mathfrak{Y}$ is a closed subscheme of $\mathfrak{X}$ defined by an ideal sheaf $\mathscr{I}$, we say σ splits $\mathfrak{Y}$ iff $\sigma(F_{\mathfrak{X}*}\mathscr{I}) \subseteq \mathscr{I}^{(1)}$, in which case $\mathfrak{Y}$ will also be Frobenius split, said to be compatibly split in $\mathfrak{X}$.

Mehta and Ramanathan showed that the flag schemes are Frobenius split with all the Schubert subschemes compatibly split. Their result has various applications, e.g., to their simple proof of Kempf's (resp. Demazure's) vanishing theorem of the higher cohomology of dominant (resp. ample) invertible sheaves on the flag schemes (resp. the Schubert schemes).

In §3 we will rederive a part of their theorem that the flag schemes are Frobenius split, using the representation theory of infinitesimal algebraic groups. Along the same line one can find a particularly nice splitting of each flag scheme that splits all its Schubert subschemes; that we will do in §4.

We will let $K\mathbf{Alg}$ (resp. $\mathbf{Mod}_{\mathfrak{X}}$) denote the category of K-algebras (resp. $\mathscr{O}_{\mathfrak{X}}$-modules). Also $\mathbf{Sch}_K$ (resp. $\mathbf{Grp}_K$) is the category of K-schemes (resp. K-group schemes). If $\mathfrak{G}$ is a K-group, $\mathfrak{G}\,\mathbf{Mod}$ will denote the category of $\mathfrak{G}$-modules.

The §4 is largely due to the referee, who kindly communicated a sketch of the arguments. We have also revised the proof in (3.2) of the surjectivity of a nonzero G_rB-homomorphism from $St_r \otimes_K St_r$ into $\widehat{Z}_r(2(p^r-1)\rho)$. Formerly the argument was borrowed from Jantzen's book [J], (II.11.13).

The author is grateful to the referee for generously sharing his/her ideas with him. Thanks are also due to Akiyama S. for a helpful suggestion to (1.3).

1. Projective spaces. In this section we assume K is algebraically closed and consider the case $\mathfrak{X} = \mathbf{P}^N$ the projective N-space over K.

(1.1) As $\mathbf{P}^N$ is defined over $\mathbf{F}_p$, $(\mathbf{P}^N)^{(1)} \simeq \mathbf{P}^N$. We will denote by F the composite of $F_{\mathbf{P}^N}$ with the isomorphism.

The invertible $\mathscr{O}_{\mathfrak{X}}$-modules are parametrized by $\mathbf{Z}$: if $\mathscr{O}(1)$ is Serre's twisting sheaf, we let $\mathscr{O}(n) = \mathscr{O}(1)^{\otimes_n}$ (resp. $\mathscr{O}(-n)^{-1}$) if $n \geq 0$ (resp. $n < 0$).

By [HASV], (III.6.4) for any $n \in \mathbf{Z}$ there are $n_i \in \mathbf{Z}$ such that

$$F_*\mathscr{O}(n) \simeq \coprod_{i=0}^{p^N-1} \mathscr{O}(n_i) \quad \text{in } \mathbf{Mod}_{\mathfrak{X}}.$$

We will compute the n_i in this section.

(1.2) If $n = n' + pn''$ with $n' \in [0, p-1]$ and $n'' \in \mathbf{Z}$, then

$$(1) \qquad F_*\mathscr{O}(n) \simeq F_*(\mathscr{O}(n') \otimes_{\mathscr{O}_{\mathfrak{X}}} F^*\mathscr{O}(n'')) \simeq F_*\mathscr{O}(n') \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}(n'')$$

by the projection formula; hence we have only to compute $F_*\mathscr{O}(n)$,

$n \in [0, p-1]$. Fix such n. Then (cf. **[Haas]**, p. 400)

$$\text{(2)} \qquad \exists \theta_i \in \mathbf{N} \text{ with } \sum_{i \geq 0} \theta_i = p^N \colon F_* \mathscr{O}(n) = \coprod_{i \geq 0} (\mathscr{O}(-i)^{\oplus \theta_i}).$$

Let S_m be the mth homogeneous part of the polynomial algebra in $N+1$ indeterminates over K. Then for each $j \in \mathbf{N}$ we have as K-linear spaces

$$\begin{aligned}
\text{(3)}\ S_{n+jp} &\simeq \Gamma(\mathfrak{X}, \mathscr{O}(n+jp)) \simeq \mathbf{Mod}_{\mathfrak{X}}(\mathscr{O}_{\mathfrak{X}}, \mathscr{O}(n+jp)) \\
&\simeq \mathbf{Mod}_{\mathfrak{X}}(F^* \mathscr{O}_{\mathfrak{X}}, \mathscr{O}(n+jp)) \\
&\simeq \mathbf{Mod}_{\mathfrak{X}}(\mathscr{O}_{\mathfrak{X}}, F_* \mathscr{O}(n+jp)) \\
&\simeq \mathbf{Mod}_{\mathfrak{X}}\left(\mathscr{O}_{\mathfrak{X}}, \coprod_i \mathscr{O}(j-i)^{\oplus \theta_i}\right) \quad \text{by the projection formula} \\
&\simeq \coprod_i S_{j-i}^{\oplus \theta_i},
\end{aligned}$$

hence

$$\text{(4)} \qquad \binom{n+jp+N}{N} = \sum_i \theta_i \binom{j-i+N}{N}.$$

In order to compute the θ_i, we will agree that for each $t \in \mathbf{Z}$ and $m \in \mathbf{N}$

$$\text{(5)} \quad \binom{t}{m} = \frac{1}{m!} \frac{d^m}{dx^m}\bigg|_{x=1} x^t = \begin{cases} 1 & \text{if } m = 0, \\ \dfrac{t(t-1)\cdots(t-m+1)}{m!} & \text{if } m \geq 1. \end{cases}$$

(1.3) Lemma. (i) *For each* $r \in \mathbf{N}$

$$\theta_r = \sum_{i=0}^{r} (-1)^i \binom{N+1}{i} \binom{n+(r-i)p+N}{N}.$$

(ii) *If* $r \geq N+1$ *or* $n+N \geq (N+1-r)p$, *then* $\theta_r = 0$.

(iii) $\theta_N = \begin{cases} \binom{p-n-1}{N} & \textit{if } p-n-1 \geq N, \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (i) We will argue by induction on r. If $r = 0$, take $j = 0$ in (1.2) (4) to verify the assertion. If $r \geq 1$, take $j = r$ in (1.2)(4) to get

$$(1)\quad \theta_r = \binom{n+rp+N}{N} - \sum_{i=0}^{r-1} \theta_i \binom{r-i+N}{N}$$
$$= \binom{n+rp+N}{N}$$
$$- \sum_{s=0}^{r-1} \binom{n+sp+N}{N} \sum_{k=0}^{r-1-s} (-1)^k \binom{N+1}{k} \binom{r-s-k+N}{N}$$

by the induction hypothesis; hence one has only to show

$$(2)\quad -\sum_{k=0}^{t-1} (-1)^k \binom{N+1}{k} \binom{t-k+N}{N} = (-1)^t \binom{N+1}{t} \quad \forall t \in [1, r].$$

Assume first $t-1 \le N$. Then the left-hand side of (2) is

$$(3)\qquad -\sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} \binom{t-k+N}{N} + (-1)^t \binom{N+1}{t};$$

hence it will be enough to show

$$(4)\qquad \sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} \binom{t-k+N}{N} = 0 \quad \forall t \ge 1.$$

But the left-hand side is

$$(5)\qquad \sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} \frac{1}{N!} \frac{d^N}{dx^N}\bigg|_{x=1} x^{t-k+N}$$
$$= \frac{1}{N!} \frac{d^N}{dx^N}\bigg|_{x=1} \{x^{t-1}(x-1)^{N+1}\} = 0,$$

using the Leibniz rule.

If $t-1 > N$, then the left-hand side of (2) is

$$(6)\qquad -\sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} \binom{t-k+N}{N} = 0 \quad \text{by (4)}$$

while the right-hand side of (2) is 0 as $t \ge N+2$. Hence (i) holds.

(ii) If $r \ge N+1$, then

$$(7)\qquad \theta_r = \sum_{i=0}^{N+1} (-1)^i \binom{N+1}{i} \binom{n+(r-i)p+N}{N} \quad \text{by (i)}$$
$$= \frac{1}{N!} \frac{d^N}{dx^N}\bigg|_{x=1} \{x^{n+(r-N+1)p+N}(x-1)^{p(N+1)}\} = 0,$$

using the Leibniz rule again. Likewise the rest.

(1.4) We summarize the foregoing computations in

PROPOSITION. *If* $n \in [0, p-1]$ *and* $n' \in \mathbf{N}$, *then*

$$F_*\mathscr{O}(n+pn') \simeq \coprod_{i=0}^{N} \mathscr{O}(n'-i)^{\oplus \theta_i} \quad \text{in } \mathbf{Mod}_{\mathfrak{X}}$$

with $\theta_i = \sum_{j=0}^{i}(-1)^j\binom{N+1}{j}\binom{n+(i-j)p+N}{N}$ *as in* (1.3).

2. Preliminaries. In this section we recall some standard facts of the Frobenius splittings and of the representation theory of algebraic groups. We will also introduce the notations in (2.5) to be used in §3 and §4.

(2.1) Let $\mathfrak{X}$ be a K-scheme. If $\mathfrak{V}$ is open in $\mathfrak{X}$, one can identify $\mathscr{O}_{\mathfrak{X}^{(1)}}(\mathfrak{V}) = \Gamma(\mathfrak{V}, \mathscr{O}_{\mathfrak{X}^{(1)}})$ with $\mathscr{O}_{\mathfrak{X}}(\mathfrak{V})^{(1)}$. Then the structure morphism $F_{\mathfrak{X}}^{\mathfrak{f}}(\mathfrak{V})\colon \mathscr{O}_{\mathfrak{X}^{(1)}}(\mathfrak{V}) \to (F_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}})(\mathfrak{V})$ is just the p^rth power map. Hence a Frobenius split K-scheme is reduced **[R]**, Remark 1.3(i).

(2.2) LEMMA (cf. **[R]**, Corollary 1.11 and **[MR]**, Lemma 1). *Let* $\mathfrak{X}$ *be a* K*-scheme Frobenius split by* $\sigma \in \mathbf{Mod}_{\mathfrak{X}^{(1)}}(F_{\mathfrak{X}*}\mathscr{O}_{\mathfrak{X}}, \mathscr{O}_{\mathfrak{X}^{(1)}})$.

(i) *If* $\mathfrak{X}_1$ *and* $\mathfrak{X}_2$ *are closed subschemes of* $\mathfrak{X}$ *both split by* σ, *then so is* $\mathfrak{X}_1 \cap \mathfrak{X}_2$.

(ii) *Let* $\mathfrak{Y}$ *be a closed subscheme of* $\mathfrak{X}$ *split by* σ. *If the underlying space* $|\mathfrak{Y}|$ *of* $\mathfrak{Y}$ *is Noetherian, then each irreducible component of* $\mathfrak{Y}$ *given the reduced closed structure is also split by* σ.

Proof. (i) If $\mathscr{I}_i$ is the ideal sheaf of $\mathfrak{X}_i$, the ideal sheaf of $\mathfrak{X}_1 \cap \mathfrak{X}_2$ is $\mathscr{I}_1 + \mathscr{I}_2$. Then

$$\sigma(F_{\mathfrak{X}*}(\mathscr{I}_1 + \mathscr{I}_2)) \subseteq \mathscr{I}_1^{(1)} + \mathscr{I}_2^{(1)} = (\mathscr{I}_1 + \mathscr{I}_2)^{(1)}, \tag{1}$$

and hence $\mathfrak{X}_1 \cap \mathfrak{X}_2$ is split by σ.

(ii) If $|\mathfrak{Y}| = |\mathfrak{Y}_1| \cup \cdots \cup |\mathfrak{Y}_r|$ is a decomposition into the irreducible components of $\mathfrak{Y}$, each of which is given the reduced closed structure, put $\mathfrak{V} = |\mathfrak{X}| \backslash (|\mathfrak{Y}_2| \cup \cdots \cup |\mathfrak{Y}_r|)$. Then $|\mathfrak{Y}_1 \cap \mathfrak{V}| = |\mathfrak{Y} \cap \mathfrak{V}|$. As both $\mathfrak{Y}_1 \cap \mathfrak{V}$ and $\mathfrak{Y} \cap \mathfrak{V}$ are reduced, $\mathfrak{Y}_1 \cap \mathfrak{V} = \mathfrak{Y} \cap \mathfrak{V}$; hence

$$\mathfrak{Y}_1 \cap \mathfrak{V} \text{ is split by } \sigma|_{\mathfrak{V}^{(1)}} \text{ in } \mathfrak{V}. \tag{2}$$

Let $\mathscr{P}$ be the ideal sheaf of $\mathfrak{Y}_1$ in $\mathfrak{X}$. To see that $\sigma(F_{\mathfrak{X}*}\mathscr{P}) \subseteq \mathscr{P}^{(1)}$, the problem being local we may assume $\mathfrak{X} = \mathfrak{Sp}_K A$ for some K-algebra A. Then $\mathscr{P} = \tilde{\mathfrak{p}}$ and $\sigma(F_{\mathfrak{X}*}\mathscr{P}) = \widetilde{\mathfrak{I}}$ for some ideals $\mathfrak{p}$ and

$\mathfrak{I}$ of A with $\mathfrak{p} \subseteq \mathfrak{I}$. As $\mathfrak{Y}_1$ is reduced and irreducible, $\mathfrak{p}$ is prime. By (2) there is $f \in A \backslash \mathfrak{p}$ such that $\mathfrak{p}_f = \mathfrak{I}_f$ in A_f; hence $\mathfrak{I} = \mathfrak{p}$, as desired.

(2.3) Let $\mathfrak{G}$ be an affine algebraic K-group scheme, $\mathfrak{H}$ a subgroup scheme of $\mathfrak{G}$, and $\pi: \mathfrak{G} \to \mathfrak{G}/\mathfrak{H}$ the quotient morphism. $\mathfrak{G}/\mathfrak{H}$ is a K-scheme (cf. **[J]**, (I.5.6)(8)), and π is open and affine (cf. **[J]**, (I.5.7)(3), (1)).

If M is an $\mathfrak{H}$-module and if $\mathfrak{V}$ is open in $\mathfrak{G}/\mathfrak{H}$, we set

$$
\begin{aligned}
(1) \qquad & \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, M)^{\mathfrak{H}} \\
&= \{f \in \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, M) \mid \\
&\qquad f(A)(xh) = h^{-1} f(A)(x) \ \forall x \in (\pi^{-1}\mathfrak{V})(A), \\
&\qquad\qquad h \in \mathfrak{H}(A),\ A \in K\mathbf{Alg}\}.
\end{aligned}
$$

One defines an $\mathscr{O}_{\mathfrak{G}/\mathfrak{H}}$-module $\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M)$ by

$$
(2) \qquad \mathfrak{V} \mapsto \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, M)^{\mathfrak{H}}.
$$

The correspondence $M \mapsto \mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M)$ defines an exact functor from $\mathfrak{H}$ **Mod** into the category of quasicoherent $\mathscr{O}_{\mathfrak{G}/\mathfrak{h}}$-modules. $\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M)$ carries also a structure of $\mathfrak{G}$-linearization.

If we let $\mathfrak{H}$ operate on the coordinate algebra $K[\mathfrak{G}]$ of $\mathfrak{G}$ (resp. M) by the right regular action (resp. as given), and take the $\mathfrak{H}$-fixed point set of $M \otimes_K K[\mathfrak{G}]$, we get a left exact functor

$$
(3) \qquad \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}}: \mathfrak{H}\ \mathbf{Mod} \to \mathfrak{G}\ \mathbf{Mod} \quad \text{via} \quad M \mapsto (M \otimes_K K[\mathfrak{G}])^{\mathfrak{H}},
$$

where the $\mathfrak{G}$-module structure on $(M \otimes_K K[\mathfrak{G}])^{\mathfrak{H}}$ is given by the left regular action on $K[\mathfrak{G}]$. Then

$$
(4) \qquad \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}}(M) \simeq \Gamma(\mathfrak{G}/\mathfrak{H}, \mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M)) \quad \text{in } \mathfrak{G}\ \mathbf{Mod}.
$$

The functor $\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}}$ is right adjoint to the forgetful functor $\mathfrak{G}$ **Mod** $\to$ $\mathfrak{H}$ **Mod**: If V is a $\mathfrak{G}$-module, one has a K-linear isomorphism

$$
(5) \qquad \mathfrak{G}\ \mathbf{Mod}(V, \mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}}(M)) \to \mathfrak{H}\ \mathbf{Mod}(V, M) \quad \text{via } f \mapsto e_M \circ f
$$

with an inverse given by $g \mapsto \hat{g}$ such that

$$
\begin{aligned}
(6) \qquad & \hat{g}(v)(A)(x) = (g \otimes_K A)(x^{-1}(v \otimes 1)), \\
&\qquad\qquad v \in V,\ x \in \mathfrak{G}(A),\ A \in K\mathbf{Alg},
\end{aligned}
$$

where $e_M = M \otimes_K \varepsilon_{\mathfrak{G}} \in \mathfrak{H}\ \mathbf{Mod}(\mathrm{ind}_{\mathfrak{H}}^{\mathfrak{G}}(M), M)$ such that $\sum m_i \otimes a_i \mapsto \sum \varepsilon_{\mathfrak{G}}(a_i) m_i$ with $\varepsilon_{\mathfrak{G}}$ the counit of the Hopf algebra $K[\mathfrak{G}]$, or e_M is the evaluation at the neutral element of $\mathfrak{G}(K)$ under the identification

(4). The isomorphism (5) is called a Frobenius reciprocity. One has also the tensor identity (cf. **[J]**, (I.3.6)) in $\mathfrak{G}$ **Mod**:

(7) $$V \otimes_K \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(M) \xrightarrow{\sim} \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(V \otimes_K M)$$

such that the image of $v \otimes f$ sends $x \in \mathfrak{G}(A)$ into $(x^{-1}(v \otimes 1)) \otimes_A f(x)$, $A \in K\mathbf{Alg}$.

(2.4) Let $\mathfrak{K}$ be a subgroup of $\mathfrak{H}$ and $q\colon \mathfrak{G}/\mathfrak{K} \to \mathfrak{G}/\mathfrak{H}$ the natural morphism. One has (cf. **[J]**, (I.5.19)(5))

(1) $$\mathscr{L}_{\mathfrak{G}/\mathfrak{H}} \circ \operatorname{ind}_{\mathfrak{K}}^{\mathfrak{H}} \simeq q_* \mathscr{L}_{\mathfrak{G}/\mathfrak{K}} \quad \text{on } \mathfrak{K}\ \mathbf{Mod}$$

such that if $\mathfrak{V}$ is an affine open of $\mathfrak{G}/\mathfrak{H}$ and $M \in \mathfrak{K}$ **Mod**, the following commutative diagram results:

(2) $$\begin{tikzcd} K[\pi^{-1}\mathfrak{V}] \otimes_K \operatorname{ind}_{\mathfrak{K}}^{\mathfrak{H}}(M) \arrow[r] & K[\pi^{-1}\mathfrak{V}] \otimes_K M \\ \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, \operatorname{ind}_{\mathfrak{K}}^{\mathfrak{H}}(M))^{\mathfrak{H}} \arrow[r, "\sim"] \arrow[u, hook] & \mathbf{Sch}_K(\pi^{-1}\mathfrak{V}, M)^{\mathfrak{K}} \arrow[u, hook] \end{tikzcd}$$

where π is the quotient morphism $\mathfrak{G} \to \mathfrak{G}/\mathfrak{H}$ and the top horizontal map is given by $a \otimes f \mapsto a \otimes e_M(f)$.

Taking the global sections of (1) yields the transitivity of inductions:

(3) $$\operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}} \circ \operatorname{ind}_{\mathfrak{K}}^{\mathfrak{H}} \simeq \operatorname{ind}_{\mathfrak{K}}^{\mathfrak{G}}.$$

If $L \in \mathfrak{K}$ **Mod**, the transitivity of inductions makes the following diagram commute:

(4) $$\begin{tikzcd} \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(\operatorname{ind}_{\mathfrak{K}}^{\mathfrak{H}}(L)) \arrow[r, "\sim"] \arrow[d, "e_{\operatorname{ind}_{\mathfrak{K}}^{\mathfrak{H}}(L)}"'] & \operatorname{ind}_{\mathfrak{K}}^{\mathfrak{G}}(L) \arrow[d, "e_L"] \\ \operatorname{ind}_{\mathfrak{K}}^{\mathfrak{H}}(L) \arrow[r, "e_L"'] & L. \end{tikzcd}$$

(2.5) We now fix the notations to be used throughout the rest of the paper. G will denote a semisimple simply connected K-group with a maximal torus T, both split over $\mathbf{Z}$, and R the root system of G relative to T with a positive system R^+. We choose a Borel subgroup B of G containing T such that the roots of the unipotent radical U of B are $-R^+$, and set $\mathfrak{X} = G/B$.

Let $W = N_G(T)/T$ be the Weyl group of G. If α is a simple root, let s_α be the reflexion in W associated to α, and let $l\colon W \to \mathbf{N}$ be the length function on W with respect to $\{s_\alpha \mid \alpha \text{ simple}\}$. If $w_0 \in W$ with $w_0 R^+ = -R^+$, set $U^+ = w_0 U w_0^{-1}$. Then $\{wU^+B\}_{w \in W}$ provides an open covering of G.

As $B = T \ltimes U$, $\mathbf{Grp}_K(B, GL_1) \simeq \mathbf{Grp}_K(T, GL_1)$, which we will denote by X. X has the structure of an abelian group, called the weight lattice, such that $(\lambda + \mu)(t) = \lambda(t)\mu(t)$, $t \in T$, $\lambda, \mu \in X$. Define a partial order on X such that $\lambda \leq \mu$ iff $\mu - \lambda \in \sum_{\alpha \in R^+} \mathbf{N}\alpha$. Let X^+ be the set of dominant weights, and put $\rho = \frac{1}{2}\sum_{\alpha\in R^+} \alpha \in X^+$.

If M is a T-module, one can write $M = \coprod_{\lambda \in X} M_\lambda$ with $M_\lambda = \{m \in M \mid t(m \otimes 1) = m \otimes \lambda(t) \ \ \forall t \in T(A),\ A \in K\mathbf{Alg}\}$. We say λ is a weight of M iff $M_\lambda \neq 0$.

By abuse of notation we let $\lambda \in X$ also denote the 1-dimensional B-module defined by λ. One has (cf. [J], (II.2.6))

(1) $$\operatorname{ind}_B^G(\lambda) \neq 0 \quad \text{iff} \quad \lambda \in X^+,$$

in which case (cf. [J], (II.2.2))

(2) $\operatorname{ind}_B^G(\lambda)$ has the highest weight λ with $\dim \operatorname{ind}_B^G(\lambda)_\lambda = 1$.

If $G_r = \ker F_G^r$, $F_{\mathfrak{X}}^r \colon \mathfrak{X} \to \mathfrak{X}^{(r)}$ factors through the natural morphism $q \colon \mathfrak{X} \to G/G_rB$ to induce an isomorphism $F \colon G/G_rB \to \mathfrak{X}^{(r)}$ in $\mathbf{Sch}_K$ (cf. [J], (I.9.5)) so that the diagram

(3) $$\begin{tikzcd} \mathfrak{X} \arrow[r, "F_{\mathfrak{X}}^r"] \arrow[d, "q"'] & \mathfrak{X}^{(r)} \\ G/G_rB \arrow[ur, "\sim"', "F"] & \end{tikzcd}$$

commutes. If $B_r = \ker F_B^r$ and $U_r^+ = \ker F_{U^+}^r$, the multiplication induces an isomorphism of K-schemes $U_r^+ \times B_r \to G_r$ (cf. [J], (II.3.2)).

For simplicity we set

(4) $$\widehat{Z}_r = \operatorname{ind}_B^{G_rB} \colon B\ \mathbf{Mod} \to G_rB\ \mathbf{Mod}.$$

As $G_rB/B \simeq U_r^+$ is affine,

(5) $\widehat{Z}_r$ is exact (cf. [J], (I.5.13)).

If $\lambda \in X$, then (cf. [J], (II.9.2))

(6) $\widehat{Z}_r(\lambda)$ has highest weight λ with $\dim \widehat{Z}_r(\lambda)_\lambda = 1$

and

(7) $$\widehat{Z}_r(\lambda)^* \simeq \widehat{Z}_r(-\lambda + 2(p^r - 1)\rho) \quad \text{in } G_rB\ \mathbf{Mod}.$$

Also (cf. [J], (II.9.5))

(8) $\operatorname{soc}_{G_rB} \widehat{Z}_r(\lambda)$ is simple of highest weight λ.

In particular (cf. [J], (II.3.18)),

(9) $\widehat{Z}_r((p^r-1)\rho)$ is simple and admits a structure of G-module,

called the rth Steinberg module and denoted St_r.

One has by (2.4)(1)

$$q_*\mathscr{L}_{G/B}(M) \simeq \mathscr{L}_{G/G_rB}(\widehat{Z}_r(M)) \quad \forall M \in B\ \mathbf{Mod}. \tag{10}$$

As B is defined over $\mathbf{F}_p$, $B^{(r)} \simeq B$ in $\mathbf{Grp}_K$ (cf. [J], (I.9.5)); hence one can make M into a G_rB-module, denoted $M^{[r]}$, through the quotient morphism $G_rB \to G_rB/G_r$ composed with the isomorphism $G_rB/G_r \xrightarrow{\sim} B^{(r)}$ induced by F_G^r. One has

$$F_*\mathscr{L}_{G/G_rB}(M^{[r]}) \simeq \mathscr{L}_{G/B}(M)^{(r)} \quad \text{in} \mathbf{Mod}_{\mathfrak{X}^{(r)}}. \tag{11}$$

That is given in each $(wU^+B/B)^{(r)}$, $w \in W$, by the following commutative diagram:

$$\begin{array}{ccc}
\Gamma((wU^+B/B)^{(r)}, F_*\mathscr{L}_{G/G_rB}(M^{[r]})) & \xleftarrow{\sim} & \Gamma((wU^+B/B)^{(r)}, \mathscr{L}_{G/B}(M)^{(r)}) \\
\wr\| & & \wr\| \\
\mathbf{Sch}_K(wU^+, M^{[r]})^{U_r^+} & & \mathbf{Sch}_K(wU^+, M) \otimes_K K^{(-r)} \\
\wr\| & & \wr\| \\
K[wU^+]^{U_r^+} \otimes_K M^{[r]} & \longleftarrow & M \otimes_K (K[wU^+] \otimes_K K^{(-r)})
\end{array} \tag{12}$$

with the bottom horizontal map given by $m \otimes a \otimes \xi \mapsto a^{p^r}\xi \otimes m$.

(2.6) We examine next the inverse image $q^*\mathscr{L}_{G/G_rB}(V)$, $V \in G_rB\ \mathbf{Mod}$. As the quotient morphism $G \to G/G_rB$ is not locally trivial, the argument of [J], (I.5.17)(1) does not apply as it is. One could consult [CPS], (3.1.2) and (2.7), but we prefer to write down an explicit proof of the following fact:

PROPOSITION. *Let $s \in \mathbf{N}$, $r \in \mathbf{Z}^+$, and let $q_s\colon G/G_sB \to G/G_{r+s}B$ be the natural morphism. If V is a $G_{r+s}B$-module, the imbedding*

$$\mathscr{L}_{G/G_{r+s}B}(V) \to q_{s*}\mathscr{L}_{G/G_sB}(V)$$

induces an isomorphism

$$q_s^*\mathscr{L}_{G/G_{r+s}B}(V) \to \mathscr{L}_{G/G_sB}(V)$$

that makes, in each $wU^+x_s = wU^+B/G_sB$, $w \in W$, *the following diagram commutative*:

$$\begin{array}{ccc} \Gamma(wU^+x_s, q_s^*\mathscr{L}_{G/G_{r+s}B}(V)) & \xrightarrow{\sim} & \Gamma(wU^+x_s, q_s^*\mathscr{L}_{G/G_sB}(V)) \\ \downarrow & & \downarrow \\ K[wU^+/U_s^+]\otimes_{K[wU^+/U_{r+s}^+]}(K[wU^+]\otimes_K V) & \longrightarrow & K[wU^+]\otimes_K V \end{array}$$

where the bottom horizontal map is given by $b\otimes c\otimes v \mapsto bc\otimes v$.

Proof. By taking the direct limit we may assume $\dim V < \infty$. Let $\pi_s: G \to G/G_sB$ and $\pi_s': G \to G/G_{r+s}B$ be the quotient morphisms so that $q_s \circ \pi_s = \pi_s'$. Define $\psi: q_s^*\mathscr{L}_{G/G_{r+s}B}(V) \to \mathscr{L}_{G/G_sB}(V)$ to be the adjoint of the imbedding $\mathscr{L}_{G/G_{r+s}B}(V) \to q_{s*}\mathscr{L}_{G/G_sB}(V)$.

Assume first $s = 0$. As $\{wU^+x_0\}_{w\in W}$ is an open covering of $\mathfrak{X}$, to see that ψ is invertible, we have only to check it in each wU^+x_0, $w \in W$, then only in U^+x_0 by the W-equivariance; hence it is enough to show that the map

$$\begin{aligned}(1)\qquad & \Gamma(U^+x_0, \mathscr{O}_{\mathfrak{X}})\otimes_{\Gamma(U^+x_r, \mathscr{O}_{G/G_rB})}\mathbf{Sch}_K(U^+B, V)^{G_rB} \\ & \to \mathbf{Sch}_K(U^+B, V)^B\end{aligned}$$

is invertible. But the left-hand side is isomorphic to

$$\begin{aligned}(2)\qquad & K[U^+]\otimes_{K[U^+/U_r^+]}\mathbf{Sch}_K(U^+, V)^{U_r^+} \\ & \simeq K[U^+]\otimes_{K[U^+]^{(r)}}\mathrm{ind}_{U_r^+}^{U^+}(V) \\ & \simeq K[U^+]\otimes_{K[U^+]^{(r)}}(V\otimes_K K[U^+])^{U_r^+}\end{aligned}$$

while the right-hand side is isomorphic to $V\otimes_K K[U^+]$. Hence we are reduced to showing that the map

$$\begin{aligned}(3)\qquad & K[U^+]\otimes_{K[U^+]^{(r)}}(V\otimes_K K[U^+]) \to V\otimes_K K[U^+] \\ & \text{via } a\otimes m\otimes b \mapsto m\otimes ab\end{aligned}$$

induces an isomorphism upon restriction to

$$K[U^+]\otimes_{K[U^+]^{(r)}}(V\otimes_K K[U^+])^{U_r^+}.$$

We will argue by induction on $\dim V$. Note that

$$(4)\qquad V^{U_r^+} \neq 0 \quad \text{if } V \neq 0$$

as U_r^+ is unipotent. In particular, if $\dim V = 1$, $V = V^{U_r^+}$; hence

$$(5)\qquad (V\otimes_K K[U^+])^{U_r^+} \simeq V\otimes_K(K[U^+]^{U_r^+}) \simeq V\otimes_K K[U^+]^{(r)},$$

and the assertion follows.

Assume next $\dim V > 1$. As $U^+/U_r^+ \simeq U^{(r)}$ is affine, $\mathrm{ind}_{U_r^+}^{U^+}$ is exact. Also $K[U^+]$ is free of rank $p^{r|R^+|}$ over $K[U^+]^{(r)}$. Hence we get a commutative diagram of column exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K[U^+] \otimes_{K[U^+]^{(r)}} \mathrm{ind}_{U_r^+}^{U^+}(V^{U_r^+}) \arrow[r] \arrow[d] & V^{U_r^+} \otimes_K K[U^+] \arrow[d] \\
K[U^+] \otimes_{K[U^+]^{(r)}} \mathrm{ind}_{U_r^+}^{U^+}(V) \arrow[r] \arrow[d] & V \otimes_K K[U^+] \arrow[d] \\
K[U^+] \otimes_{K[U^+]^{(r)}} \mathrm{ind}_{U_r^+}^{U^+}(V/V^{U_r^+}) \arrow[r] \arrow[d] & (V/V^{U_r^+}) \otimes_K K[U^+] \arrow[d] \\
0 & 0.
\end{tikzcd} \tag{6}$$

By the induction hypothesis, the top and the bottom horizontal maps are isomorphic; therefore so is the middle, as claimed.

If $s > 0$, we have by the above a commutative diagram

$$\begin{tikzcd}
\pi_s^* q_s^* \mathscr{L}_{G/G_{r+s}B}(V) \arrow[r, "\pi_s^*\psi"] \arrow[d, equal] & \pi_s^* \mathscr{L}_{G/G_sB}(V) \arrow[d, "\wr"] \\
\pi_s'^* \mathscr{L}_{G/G_{r+s}B}(V) \arrow[r, "\sim"'] & \mathscr{L}_{G/B}(V).
\end{tikzcd} \tag{7}$$

Hence at each $x \in \mathfrak{X}$ we have an isomorphism

$$\begin{aligned}
(8)\quad (\pi_s^*\psi)_x = \mathscr{O}_{\mathfrak{X},x} \otimes_{\mathscr{O}_{G/G_sB,\pi_s(x)}} \psi_{\pi_s(x)} :\ & \\
\mathscr{O}_{\mathfrak{X},x} \otimes_{\mathscr{O}_{G/G_sB,\pi_s(x)}} \mathscr{O}_{G/G_sB,\pi_s(x)} \otimes_{\mathscr{O}_{G/G_{r+s}B,\pi_s'(x)}} & \mathscr{L}_{G/G_{r+s}B}(V)_{\pi_s'(x)} \\
\to \mathscr{O}_{\mathfrak{X},x} \otimes_{\mathscr{O}_{G/G_sB,\pi_s(x)}} & \mathscr{L}_{G/G_sB}(V)_{\pi_s(x)}.
\end{aligned}$$

But $\mathscr{O}_{\mathfrak{X},x}$ is free of rank $p^{s|R^+|}$ over $\mathscr{O}_{\mathfrak{X}^{(s)},x}$; hence $\mathscr{O}_{\mathfrak{X},x}$ is faithfully flat over $\mathscr{O}_{G/G_sB,\pi_s(x)}$, so $\psi_{\pi_s(x)}$ is already isomorphic, from which we conclude that $\psi\colon q_s^*\mathscr{L}_{G/G_{r+s}B}(V) \to \mathscr{L}_{G/G_sB}(V)$ is an isomorphism.

(2.7) One can likewise show

Proposition. *Let* $(\mathfrak{G}, \mathfrak{H}) = (G, B)$ *or* (G, G_rB). *Let* $M, M' \in \mathfrak{H}\,\mathbf{Mod}$.

(i) *The natural morphism*

$$\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M) \otimes_{\mathscr{O}_{\mathfrak{G}/\mathfrak{H}}} \mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M') \to \mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M \otimes_K M')$$

is invertible.

(ii) *If* M *is finite dimensional,* $\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M)$ *is locally free of rank* $\dim M$, *and the natural morphism* $\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M^*) \to \mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M)^\vee$ *is invertible.*

(2.8) **Lemma.** *Let* $(\mathfrak{G}, \mathfrak{H}) = (G, B)$ *or* (G, G_rB). *If* L, M, *and* N *are* $\mathfrak{H}$*-modules with* L *and* M *finite dimensional, put*

$$\begin{aligned} M_1 &= \mathbf{Mod}_{\mathfrak{G}/\mathfrak{H}}(\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(L), \mathfrak{L}_{\mathfrak{G}/\mathfrak{H}}(M)), \\ M_2 &= \mathbf{Mod}_{\mathfrak{G}/\mathfrak{H}}(\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(M), \mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(N)), \\ M_3 &= \mathbf{Mod}_{\mathfrak{G}/\mathfrak{H}}(\mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(L), \mathscr{L}_{\mathfrak{G}/\mathfrak{H}}(N)). \end{aligned}$$

Then one has a commutative diagram of K*-linear spaces*

$$\begin{tikzcd} M_1 \otimes_K M_2 \arrow[r, "c"] & M_3 \\ \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(L^* \otimes_K M) \otimes_K \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(M^* \otimes_K N) \arrow[u, "\wr"] \arrow[r, "\mu"'] & \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(L^* \otimes_K N), \arrow[u, "\wr"'] \end{tikzcd}$$

where c *is the composition, the vertical isomorphisms are the natural ones, and* $\mu \in \mathfrak{G}\,\mathbf{Mod}$, *such that the diagram*

$$\begin{tikzcd} \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(L^* \otimes_K M) \otimes_K \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(M^* \otimes_K N) \arrow[r, "\mu"] \arrow[d, "e_{L^* \otimes_K M} \otimes_K e_{M^* \otimes_K N}"] & \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(L^* \otimes_K N) \arrow[d, "e_{L^* \otimes_K N}"] \\ L^* \otimes_K M \otimes_K M^* \otimes_K N \arrow[r, "L^* \otimes_K \nu \otimes_K N"'] & L^* \otimes_K N \end{tikzcd}$$

commutes if ν *is the natural map.*

Proof. Let $\psi_1 \in \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(L^* \otimes_K M)$, $\psi_2 \in \operatorname{ind}_{\mathfrak{H}}^{\mathfrak{G}}(M^* \otimes_K N)$, $\psi_3 = \mu(\psi_1 \otimes_K \psi_2)$, and $\tilde{\psi}_1 \in M_1$, $\tilde{\psi}_2 \in M_2$, $\tilde{\psi}_3 \in M_3$ corresponding to ψ_1, ψ_2, ψ_3, respectively. We must show

$$\tilde{\psi}_2 \circ \tilde{\psi}_1 = \tilde{\psi}_3. \tag{1}$$

One has

$$\psi_3 = (L^* \otimes_K \nu \otimes_K N) \circ (\psi_1 \otimes_K \psi_2) \quad \text{in } \mathbf{Sch}_K(\mathfrak{G}, L^* \otimes_K N). \tag{2}$$

If $\mathfrak{V}$ is an affine open of $\mathfrak{X}$, one can write

$$\mathrm{res}^{\mathfrak{G}}_{\pi^{-1}\mathfrak{V}}(\psi_1) = \sum_i a_i \otimes f_i \otimes m_i \quad \text{in } K[\pi^{-1}\mathfrak{V}] \otimes_K L^* \otimes_K M,$$

and

$$\mathrm{res}^{\mathfrak{G}}_{\pi^{-1}\mathfrak{V}}(\psi_2) = \sum_j b_j \otimes g_j \otimes n_j \quad \text{in } K[\pi^{-1}\mathfrak{V}] \otimes_K M^* \otimes_K N.$$

Then by (2)

$$\mathrm{res}^{\mathfrak{G}}_{\pi^{-1}\mathfrak{V}}(\psi_3) = \sum_{i,j} a_i b_j \otimes g_j(m_i) f_i \otimes n_j \qquad \text{in } K[\pi^{-1}\mathfrak{V}] \otimes_K L^* \otimes_K N. \tag{3}$$

If $v = \sum_k c_k \otimes l_k \in (K[\pi^{-1}\mathfrak{V}] \otimes_K L)^{\mathfrak{H}} \simeq \Gamma(\mathfrak{V}, \mathscr{L}(L))$, then

$$\tilde{\psi}_3(\mathfrak{V})(v) = \sum_{i,j,k} a_i b_j c_k \otimes g_j(m_i) f_i(l_k) n_j \quad \text{by (3)} \tag{4}$$

while

$$\begin{aligned} ((\tilde{\psi}_2 \circ \tilde{\psi}_1)(\mathfrak{V}))(v) &= \tilde{\psi}_2(\mathfrak{V}) \left(\sum_{i,k} a_i c_k \otimes f_i(l_k) m_i \right) \\ &= \sum_{i,j,k} a_i c_k b_j \otimes f_i(l_k) g_j(m_i) n_j; \end{aligned} \tag{5}$$

hence $\tilde{\psi}_2 \circ \tilde{\psi}_1 = \tilde{\psi}_3$ in $\mathfrak{V}$, as desired.

(2.9) Let $M \in B$ **Mod**, and denote the isomorphism $\mathscr{L}_{G/G_rB}(\widehat{Z}_r(M)) \to q_*\mathscr{L}_{G/B}(M)$ (resp. $q^*\mathscr{L}_{G/G_rB}(\widehat{Z}_r(M)) \to \mathscr{L}_{G/B}(\widehat{Z}_r(M))$) of (2.4) (resp. (2.6)) by θ_1 (resp. θ_2). One readily verifies

LEMMA. *If $\tilde{a}\colon q^*q_*\mathscr{L}_{G/B}(M) \to \mathscr{L}_{G/B}(M)$ is the adjunction, then*

$$\tilde{a} \circ q^*\theta_1 = \mathscr{L}_{G/B}(e_M) \circ \theta_2.$$

(2.10) Let M' be another B-module. If $\dim M < \infty$, one gets from (2.9) a commutative diagram of K-linear spaces

(1)

$$\begin{tikzcd}
\mathbf{Mod}_{\mathfrak{X}}(q^*q_*\mathscr{L}_{\mathfrak{X}}(M), \mathscr{L}_{\mathfrak{X}}(M')) \arrow[r, "\sim"] & \mathbf{Mod}_{\mathfrak{X}}(q^*\mathscr{L}_{G/G_rB}(\widehat{Z}_r(M)), \mathscr{L}_{\mathfrak{X}}(M')) \\
\mathbf{Mod}_{\mathfrak{X}}(\mathscr{L}_{\mathfrak{X}}(M), \mathscr{L}_{\mathfrak{X}}(M')) \arrow[r] \arrow[u, "\mathbf{Mod}_{\mathfrak{X}}(\tilde{a}, \mathscr{L}_{\mathfrak{X}}(M'))"] \arrow[d, dash, "\wr"'] & \mathbf{Mod}_{\mathfrak{X}}(\mathscr{L}_{\mathfrak{X}}(\widehat{Z}_r(M)), \mathscr{L}_{\mathfrak{X}}(M')) \arrow[u, "\wr"', "\mathbf{Mod}_{\mathfrak{X}}(\theta_2, \mathscr{L}_{\mathfrak{X}}(M'))"] \arrow[d, dash, "\wr"'] \\
\mathrm{ind}_B^G(M^* \otimes_K M') \arrow[r, "\mathrm{ind}_B^G(e_M^* \otimes_K M')"'] & \mathrm{ind}_B^G(\widehat{Z}_r(M)^* \otimes_K M'),
\end{tikzcd}$$

where the top (resp. middle) horizontal map is $\mathbf{Mod}_{\mathfrak{X}}(q^*\theta_1, \mathscr{L}_{\mathfrak{X}}(M'))$ (resp. $\mathbf{Mod}_{\mathfrak{X}}(\mathscr{L}_{\mathfrak{X}}(e_M), \mathscr{L}_{\mathfrak{X}}(M')))$ and $e_M^* \in B\ \mathbf{Mod}(M^*, \widehat{Z}_r(M)^*)$ is the dual of e_M.

On the other hand, let $L \in G_rB\ \mathbf{Mod}$ with $\dim L < \infty$, $\tau_1 \in G_rB\ \mathbf{Mod}(L^* \otimes_K \widehat{Z}_r(M'), \widehat{Z}_r(L^* \otimes_K M'))$ the tensor identity (2.3)(7), and $\tau_2 \in G\ \mathbf{Mod}(\mathrm{ind}_{G_rB}^G(\widehat{Z}_r(L^* \otimes_K \widehat{Z}_r(M'))), \mathrm{ind}_B^G(L^* \otimes_K M'))$ the transitivity of inductions (2.4)(3). If θ_1' (resp. θ_2^L) is θ_1 (resp. θ_2) with M (resp. $\widehat{Z}_r(M)$) replaced by M' (resp. L), one has a commutative diagram of K-linear spaces

(2)
$$\begin{tikzcd}
\mathbf{Mod}_{G/G_rB}(\mathscr{L}_{G/G_rB}(L), q_*\mathscr{L}_{\mathfrak{X}}(M')) \arrow[r, "\sim"] & \mathbf{Mod}_{\mathfrak{X}}(q^*\mathscr{L}_{G/G_rB}(L), \mathscr{L}_{\mathfrak{X}}(M')) \\
\mathbf{Mod}_{G/G_rB}(\mathscr{L}_{G/G_rB}(L), \mathscr{L}_{G/G_rB}(\widehat{Z}_r(M'))) \arrow[u, "\mathbf{Mod}_{G/G_rB}(\mathscr{L}_{G/G_rB}(L), \theta_1')"', "\wr"] \arrow[d, dash, "\wr"'] & \mathbf{Mod}_{\mathfrak{X}}(\mathscr{L}_{\mathfrak{X}}(L), \mathscr{L}_{\mathfrak{X}}(M')) \arrow[u, "\wr", "\mathbf{Mod}_{\mathfrak{X}}(\theta_2^L, \mathscr{L}_{\mathfrak{X}}(M'))"'] \arrow[d, dash, "\wr"'] \\
\mathrm{ind}_{G_rB}^G(L^* \otimes_K \widehat{Z}_r(M')) \arrow[r, "\sim"] & \mathrm{ind}_B^G(L^* \otimes_K M'),
\end{tikzcd}$$

where the top (resp. bottom) isomorphism is an adjunction (resp. $\tau_2 \circ \mathrm{ind}_{G_rB}^G(\tau_1)$).

Then putting together (1) and (2) yields

(2.11) Lemma. *If $M, M' \in B\ \mathbf{Mod}$ with $\dim M < \infty$, one has a commutative diagram of K-linear spaces*

$$\begin{tikzcd}
\mathbf{Mod}_{\mathfrak{X}}(\mathscr{L}_{\mathfrak{X}}(M), \mathscr{L}_{\mathfrak{X}}(M')) \arrow[dd] & \mathrm{ind}_B^G(M^* \otimes_K M') \arrow[l, "\sim"'] \arrow[d, "\mathrm{ind}_B^G(e_M^* \otimes_K M')"] \\
& \mathrm{ind}_B^G(\widehat{Z}_r(M)^* \otimes_K M') \\
\mathbf{Mod}_{G/G_rB}(q_*\mathscr{L}_{\mathfrak{X}}(M), q_*\mathscr{L}_{\mathfrak{X}}(M')) & \mathrm{ind}_{G_rB}^G(\widehat{Z}_r(M)^* \otimes_K \widehat{Z}_r(M')), \arrow[l, "\sim"'] \arrow[u, "\wr", "\tau_2 \circ \mathrm{ind}_{G_rB}^G(\tau_1)"']
\end{tikzcd}$$

where the left vertical map is given by $f \mapsto q_*f$,

$$\tau_1 \in G_rB\ \mathbf{Mod}(\widehat{Z}_r(M)^* \otimes_K \widehat{Z}_r(M'), \widehat{Z}_r(\widehat{Z}_r(M)^* \otimes_K M'))$$

is the tensor identity, and

$$\tau_2 \in G\ \mathbf{Mod}(\mathrm{ind}_{G_rB}^G(\widehat{Z}_r(\widehat{Z}_r(M)^* \otimes_K M')), \mathrm{ind}_B^G(\widehat{Z}_r(M)^* \otimes_K M'))$$

is the transitivity of inductions.

3. Flag schemes.

(3.1) As $F\colon G/G_rB \to \mathfrak{X}^{(r)}$ is invertible, to see that $\mathfrak{X} = G/B$ is Frobenius split, one has only to show that

$$F_*^{-1}((F_{\mathfrak{X}}^r)^{\mathfrak{f}}) \in \mathbf{Mod}_{G/G_rB}(\mathscr{O}_{G/G_rB}, q_*\mathscr{O}_{\mathfrak{X}})$$

admits a left inverse.

One has $\mathrm{soc}_{G_rB}\,\widehat{Z}_r(K) = K$ by (2.5)(8); hence one has the inclusion $i \in G_rB\ \mathbf{Mod}(K, \widehat{Z}_r(K))$. As $\mathscr{L}_{G/G_rB}$ is exact, $\mathscr{L}_{G/G_rB}(i)$ induces monic $\mathscr{O}_{G/G_rB} \to q_*\mathscr{O}_{\mathfrak{X}}$. On the other hand,

$$\mathbf{Mod}_{G/G_rB}(\mathscr{O}_{G/G_rB}, q_*\mathscr{O}_{\mathfrak{X}}) \simeq \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \simeq K. \tag{1}$$

Hence we may assume

$$F_*^{-1}((F_{\mathfrak{X}}^r)^{\mathfrak{f}}) = \mathscr{L}_{G/G_rB}(i). \tag{2}$$

(3.2) THEOREM. *The imbedding $\mathscr{L}_{G/G_B}(i)$ splits to yield*

$$q_*\mathscr{O}_{\mathfrak{X}} \simeq \mathscr{O}_{G/G_rB} \oplus \mathscr{L}_{G/G_rB}(\widehat{Z}_r(K)/K) \quad in \mathbf{Mod}_{G/G_rB}.$$

Proof. Put $i^\vee = \mathbf{Mod}_{G/G_rB}(\mathscr{L}_{G/G_rB}(i), \mathscr{O}_{G/G_rB})$. Our objective is to show that $i^\vee$ is surjective. If $i^* \in G_rB\ \mathbf{Mod}(\widehat{Z}_r(K)^*, K)$ is the dual of i, one has a commutative diagram of K-linear spaces

$$\begin{array}{ccc}
\mathbf{Mod}_{G/G_rB}(\mathscr{L}_{G/G_rB}(\widehat{Z}_r(K)), \mathscr{O}_{G/G_rB}) & \xrightarrow{i^\vee} & \mathbf{Mod}_{G/G_rB}(\mathscr{L}_{G/G_rB}(K), \mathscr{O}_{G/G_rB}) \\
\wr\| & & \wr\| \\
\mathrm{ind}_{G_rB}^G(\widehat{Z}_r(K)^*) & \longrightarrow & \mathrm{ind}_{G_rB}^G(K) \\
{\scriptstyle e_{\widehat{Z}_r(K)^*}}\downarrow & & \downarrow{\scriptstyle e_K} \\
\widehat{Z}_r(K)^* & \xrightarrow[i^*]{} & K,
\end{array} \tag{1}$$

where the middle horizontal map is $\mathrm{ind}_{G_rB}^G(i^*)$.

As e_K is invertible and as i^* is surjective, one has only to show

$$e_{\widehat{Z}_r(K)^*} \text{ is surjective.} \tag{2}$$

Recall the tensor identity (2.3)(7)

$$St_r \otimes_K St_r \simeq \widehat{Z}_r((p^r-1)\rho \otimes_K St_r) \quad \text{in } G_rB\ \mathbf{Mod}. \tag{3}$$

As $\widehat{Z}_r$ is exact, a surjective

$$(p^r-1)\rho \otimes_K e_{(p^r-1)\rho} \in B\ \mathbf{Mod}((p^r-1)\rho \otimes_K St_r, 2(p^r-1)\rho) \tag{4}$$

induces a surjective

$$\begin{aligned}(5)\qquad & \widehat{Z}_r((p^r-1)\rho \otimes_K e_{(p^r-1)\rho}) \\ & \qquad \in G_rB\ \mathbf{Mod}(St_r \otimes_K St_r, \widehat{Z}_r(2(p^r-1)\rho)).\end{aligned}$$

But $\widehat{Z}_r(2(p^r-1)\rho) \simeq \widehat{Z}_r(K)^*$ by (2.5)(7); hence one gets a surjective

$$(6)\qquad \phi \in G_rB\ \mathbf{Mod}(St_r \otimes_K St_r, \widehat{Z}_r(K)^*).$$

Then ϕ induces $\hat{\phi} \in G\ \mathbf{Mod}(St_r \otimes_K St_r, \mathrm{ind}_{G_rB}^G(\widehat{Z}_r(K)^*))$ by the Frobenius reciprocity such that

$$(7)\qquad e_{\widehat{Z}_r(K)^*} \circ \hat{\phi} = \phi;$$

hence $e_{\widehat{Z}_r(K)^*}$ must be surjective, as desired.

(3.3) Remarks. (i) Unlike the case of the projective spaces, $q_*\mathscr{O}_{G/B}$ does not in general split into a direct sum of invertible sheaves **[Haas]**, (4.5.5).

(ii) If $s \in \mathbf{N}$, one can make as in (2.5)(11) a G_rB-module M into a $G_{r+s}B$-module, denoted also by $M^{[s]}$. Then in $G_{r+s}B$ **Mod**

$$(1)\qquad \mathrm{ind}_{G_sB}^{G_{r+s}B}(K) \simeq \mathrm{ind}_{G_sB/G_s}^{G_{r+s}B/G_s}(K) \simeq (\mathrm{ind}_B^{G_rB}(K))^{[s]}.$$

If $q_s\colon G/G_sB \to G/G_{r+s}B$ is the natural morphism, one has commutative diagram in $\mathbf{Sch}_K$

$$(2)\qquad \begin{array}{ccccc} G/B & \xrightarrow{\sim} & (G/B)^{(s)} & \xleftarrow{\sim} & G/G_sB \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q^{(s)}} & & \downarrow{\scriptstyle q_s} \\ G/G_rB & \xrightarrow[\sim]{} & (G/G_rB)^{(s)} & \xleftarrow[\sim]{} & G/G_{r+s}B; \end{array}$$

hence the natural morphism $\mathscr{O}_{G/G_{r+s}B} \to q_{s*}\mathscr{O}_{G/G_sB}$ splits to yield

$$(3)\qquad q_{s*}\mathscr{O}_{G/G_sB} \simeq \mathscr{O}_{G/G_{r+s}B} \oplus \mathscr{L}_{G/G_{r+s}B}((\widehat{Z}_r(K)/K)^{[s]})$$

in $\mathbf{Mod}_{G/G_{r+s}B}$.

(iii) The cup product $St_r \otimes_K St_r \to \mathrm{ind}_B^G(2(p^r-1)\rho)$ in G **Mod**, induced by the multiplication $(p^r-1)\rho \otimes_K (p^r-1)\rho \to 2(p^r-1)\rho$, turns out to be surjective (cf. **[J]**, (II.14.20)). On the other hand, one has K-linear isomorphisms

$$\begin{aligned}
&G\,\mathbf{Mod}(St_r \otimes_K St_r,\ \mathrm{ind}_{G_rB}^{G}(\widehat{Z}_r(K)^*)) \\
&\quad \simeq G\,\mathbf{Mod}(St_r \otimes_K St_r,\ \mathrm{ind}_{G_rB}^{G}(\widehat{Z}_r(2(p^r-1)\rho))) \\
&\quad \simeq G\,\mathbf{Mod}(St_r \otimes_K St_r,\ \mathrm{ind}_{B}^{G}(2(p^r-1)\rho)) \\
&\qquad\qquad\qquad\qquad \text{by the transitivity of inductions} \\
&\quad \simeq B\,\mathbf{Mod}(St_r \otimes_K St_r,\ 2(p^r-1)\rho) \\
&\qquad\qquad\qquad\qquad \text{by the Frobenius reciprocity} \\
&\quad \simeq K.
\end{aligned}$$

It follows that $\hat{\phi}$ is surjective, hence every morphism $q_*\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{G/G_rB}$, and, a fortiori, every Frobenius splitting of G/B, is provided by $St_r \otimes_K St_r$.

4. Schubert schemes.

(4.1) Let $\tilde{\phi}$ be the K-linear map

$$St_r \otimes_K St_r \to \mathbf{Mod}_{G/G_rB}(q_*\mathscr{O}_{\mathfrak{X}},\ \mathscr{O}_{G/G_rB})$$

induced by $\hat{\phi}$ of (3.2). One has K-linear isomorphisms

$$\begin{aligned}
(1)\quad &G_rB\,\mathbf{Mod}(St_r \otimes_K St_r,\ K) \\
&\simeq G_rB\,\mathbf{Mod}(St_r,\ St_r) \quad \text{as } St_r \text{ is self-dual} \\
&\simeq B\,\mathbf{Mod}(St_r,\ (p^r-1)\rho) \quad \text{by the Frobenius reciprocity} \\
&\simeq K \quad \text{as } (p^r-1) \text{ is the highest weight of } St_r \\
&\simeq G\,\mathbf{Mod}(St_r,\ St_r) \simeq G\,\mathbf{Mod}(St_r \otimes_K St_r^*,\ K) = K\,\mathrm{Tr},
\end{aligned}$$

where Tr is the trace map of the K-linear endomorphisms of St_r. Hence we may assume in (3.2)

$$(2)\qquad i^* \circ \phi = \mathrm{Tr}.$$

In particular, if $v_- \in St_{r,-(p^r-1)\rho}\backslash 0$ and $v_+ \in St_{r,(p^r-1)\rho}\backslash 0$, then

$$(3)\qquad \mathrm{Tr}(v_- \otimes v_+) \neq 0$$

as one can regard v_+ as the dual of v_-. Hence one can take the splitting of $\mathscr{L}_{G/G_rB}(i)$ to be

$$(4)\qquad \sigma = \tilde{\phi}(v_- \otimes v_+).$$

We will show

(4.2) THEOREM. *Let* $w \in W$. *If* $\mathscr{I}_w$ *is the ideal sheaf of the Schubert scheme* $\mathfrak{X}(w) = \overline{U^+wB/B}$ *in* $\mathfrak{X}$, *then*

$$\sigma(q_*\mathscr{I}_w) \subseteq F_*^{-1}(\mathscr{I}_w^{(r)}).$$

Hence $F_*\sigma$ *splits all the Schubert subschemes of* $\mathfrak{X}$.

(4.3) Let $\mathfrak{Y}$ be a closed subscheme of $\mathfrak{X}$ with the underlying topological space $|\mathfrak{X}|\backslash|U^+B/B|$. If α is a simple root, the Schubert scheme $\mathfrak{X}(s_\alpha)$ is an irreducible component of $\mathfrak{Y}$. In $w \in W$ with $l(w) \geq 2$, there are simple roots α_1 and α_2 such that $l(s_{\alpha_1} w s_{\alpha_2}) = l(w) - 2$. Then

(1) $\qquad s_{\alpha_1} w \neq w s_{\alpha_2}$ with $l(s_{\alpha_1} w) = l(w s_{\alpha_2}) = l(w) - 1$.

It follows that

(2) $\quad \mathfrak{X}(w)$ is an irreducible component of $\mathfrak{X}(s_{\alpha_1} w) \cap \mathfrak{X}(w s_{\alpha_2})$.

Hence in order to get (4.2), it will suffice by (2.2) to show that

(3) $\qquad F_*\sigma$ splits $\mathfrak{Y}$.

(4.4) Let $j \in \mathrm{ind}_B^G(\rho)_\rho$ be such that $j = 1$ in U^+ regarded as an element of $\mathbf{Sch}_K(G, \rho)_\rho^B$ [J], (II.2.6), and let

$$\tilde{j} \in \mathbf{Mod}_{\mathfrak{X}}(\mathscr{L}_{\mathfrak{X}}(-\rho), \mathscr{O}_{\mathfrak{X}}) \simeq \Gamma(\mathfrak{X}, \mathscr{L}_{\mathfrak{X}}(\rho))$$

corresponding to j. If $w \in W$, one has a commutative diagram

(1)
$$\begin{array}{ccc}
\Gamma(wU^+B/B, \mathscr{L}_{\mathfrak{X}}(-\rho)) & \xrightarrow{\tilde{j}(wU^+B/B)} & \Gamma(wU^+B/B, \mathscr{O}_{\mathfrak{X}}) \\
\wr\| & & \wr\| \\
\mathbf{Sch}_K(wU^+, -\rho) & & \mathbf{Sch}_K(wU^+, K) \\
\wr\| & & \wr\| \\
K[wU^+] & \xrightarrow[j|_{wU^+}]{} & K[wU^+].
\end{array}$$

If $j|_{wU^+} = 0$, then j would vanish in wU^+B that is open in G, hence in the whole of G, contradicting the choice of j. It follows that

(2) $\qquad \tilde{j}$ is monic.

(4.5) LEMMA. $\operatorname{Supp}(\operatorname{coker}\tilde{j}) = |\mathfrak{X}|\backslash|U^+B/B|$.

Proof. As $\tilde{j}$ is invertible in U^+B/B,

$$\operatorname{Supp}(\operatorname{coker}\tilde{j}) \subseteq |\mathfrak{X}|\backslash|U^+B/B|. \tag{1}$$

On the other hand, if α is a simple root, one finds $j = 0$ in $U^+s_\alpha B$ (cf. [J], (II.2.6)); hence

$$\operatorname{Supp}(\operatorname{coker}\tilde{j}) \supseteq |U^+s_\alpha B/B|. \tag{2}$$

But $\operatorname{Supp}(\operatorname{coker}\tilde{j})$ is closed in $\mathfrak{X}$ as $\mathscr{L}_\mathfrak{X}(-\rho)$ is quasicoherent. Hence

$$\begin{aligned} \operatorname{Supp}(\operatorname{coker}\tilde{j}) &\supseteq \bigcup_{\alpha \text{ simple}} |\mathfrak{X}(s_\alpha)| \\ &= \bigcup_{w\in W\backslash 1} |U^+wB/B| = |\mathfrak{X}|\backslash|U^+B/B|, \end{aligned} \tag{3}$$

and the assertion follows.

(4.6) We take $\mathfrak{Y}$ to be the closed subscheme of $\mathfrak{X}$ defined by the ideal sheaf $\operatorname{im}\tilde{j}$. One has a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{L}_\mathfrak{X}(-\rho)^{(r)} \arrow[r, "\tilde{j}^{(r)}"] \arrow[d] & \mathscr{O}_\mathfrak{X}^{(r)} \arrow[r] \arrow[d, "(F_\mathfrak{X}^r)^\natural"] & (\operatorname{coker}\tilde{j})^{(r)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F^r_{\mathfrak{X}*}\mathscr{L}_\mathfrak{X}(-\rho) \arrow[r, "F^r_{\mathfrak{X}*}\tilde{j}"'] & \mathscr{O}_\mathfrak{X} \arrow[r] & F^r_{\mathfrak{X}*}(\operatorname{coker}\tilde{j}) \arrow[r] & 0, \end{tikzcd} \tag{1}$$

where the left vertical morphism is given by $f \mapsto f^{p^r}j^{p^r-1}$. If $\tilde{j}_r = F_*^{-1}(\tilde{j}^{(r)})$, hitting F_*^{-1} on (1) yields by (2.5)(11) a commutative diagram of short exact sequences

(2)

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{L}_{G/G_rB}(-p^r\rho) \arrow[r, "\tilde{j}_r"] \arrow[d] & \mathscr{O}_{G/G_rB} \arrow[r] \arrow[d, "\tilde{i}"] & \operatorname{coker}\tilde{j}_r \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & q_*\mathscr{L}_\mathfrak{X}(-\rho) \arrow[r, "q_*\tilde{j}"'] & q_*\mathscr{O}_\mathfrak{X} \arrow[r] & q_*(\operatorname{coker}\tilde{j}) \arrow[r] & 0, \end{tikzcd}$$

with $j_r = \tilde{j}_r(G/G_rB) \in \operatorname{ind}^G_{G_rB}(p^r\rho)_{p^r\rho}\backslash 0$.

Our objective is to show

(4.7) **Proposition.** $\sigma(\operatorname{im}(q_*\tilde{j})) \subseteq \operatorname{im}\tilde{j}_r$.

Proof. Put

$$
\begin{aligned}
(1)\qquad M_1 &= \mathbf{Mod}_{G/G_rB}(q_*\mathscr{O}_{\mathfrak{X}}, \mathscr{O}_{G/G_rB}),\\
M_2 &= \mathbf{Mod}_{G/G_rB}(q_*\mathscr{L}_{\mathfrak{X}}(-\rho), q_*\mathscr{O}_{\mathfrak{X}}),\\
M_3 &= \mathbf{Mod}_{G/G_rB}(q_*\mathscr{L}_{\mathfrak{X}}(-\rho), \mathscr{L}_{G/G_rB}(-p^r\rho)),\\
M_4 &= \mathbf{Mod}_{G/G_rB}(\mathscr{L}_{G/G_rB}(-p^r\rho), \mathscr{O}_{G/G_rB}),\\
I_1 &= \operatorname{ind}_{G_rB}^G(\widehat{Z}_r(K)^*),\\
I_2 &= \operatorname{ind}_{G_rB}^G(\widehat{Z}_r(K) \otimes_K \widehat{Z}_r(-\rho)^*).
\end{aligned}
$$

One has in G **Mod**

$$
\begin{aligned}
(2)\qquad \operatorname{ind}_{G_rB}^G(\widehat{Z}_r(-\rho)^*) &\simeq \operatorname{ind}_{G_rB}^G(\widehat{Z}_r((2p^r-1)\rho)) \quad \text{by (2.5)(7)}\\
&\simeq \operatorname{ind}_B^G((2p^r-1)\rho) \quad \text{by the transitivity of inductions}
\end{aligned}
$$

and

$$
\begin{aligned}
(3)\qquad & \operatorname{ind}_{G_rB}^G(\widehat{Z}_r(-\rho)^* \otimes_K -p^r\rho)\\
&\simeq \operatorname{ind}_{G_rB}^G(\widehat{Z}_r((2p^r-1)\rho) \otimes_K -p^r\rho)\\
&\simeq \operatorname{ind}_{G_rB}^G(\widehat{Z}_r((p^r-1)\rho)) \quad \text{by the tensor identity}\\
&\simeq St_r \quad \text{by the tensor identity again.}
\end{aligned}
$$

Hence one gets by (2.8) and (2.11) a commutative diagram of K-linear spaces

(4)

$$
\begin{tikzcd}
St_r \otimes_K St_r \otimes_K \mathbf{Mod}_{\mathfrak{X}}(\mathscr{L}_{\mathfrak{X}}(-\rho), \mathscr{O}_{\mathfrak{X}}) \arrow[d, "\tilde{\phi}\otimes_K q_*"'] & St_r \otimes_K St_r \otimes_K \operatorname{ind}_B^G(\rho) \arrow[l, "\sim"'] \arrow[d, "\hat{\phi}\otimes_K \nu_1"] \\
M_1 \otimes_K M_2 \arrow[d, "c_1"'] & I_1 \otimes_K I_2 \arrow[l, "\sim"'] \arrow[d, "\nu_2"] \\
\mathbf{Mod}_{G/G_rB}(q_*\mathscr{L}_{\mathfrak{X}}(-\rho), \mathscr{O}_{G/G_rB}) & \operatorname{ind}_B^G((2p^r-1)\rho) \arrow[l, "\sim"'] \\
M_3 \otimes_K M_4 \arrow[u, "c_2"] & St_r \otimes_K \operatorname{ind}_{G_rB}^G(p^r\rho), \arrow[l, "\sim"'] \arrow[u, "\nu_3"']
\end{tikzcd}
$$

where c_1 and c_2 are compositions, and ν_1, ν_2, ν_3 are some nonzero G-homomorphisms.

If $\tilde{v}_- \in M_3$ is v_- under the isomorphism (3), then (4.7) will follow from

$$c_1 \circ (\tilde{\phi} \otimes_K q_*)(v_- \otimes_K v_+ \otimes_K \tilde{j}) = c_2(\tilde{v}_- \otimes_K \tilde{j}_r), \tag{5}$$

which translates through (4) into

$$\nu_2 \circ (\hat{\phi} \otimes_K \nu_1)(v_- \otimes_K v_+ \otimes_K j) = \nu_3(v_- \otimes_K j_r). \tag{6}$$

We actually need (6) to hold only up to $K^\times$.

Consider an imbedding $\hat{e}_{p^r\rho} \in G\,\mathbf{Mod}(\mathrm{ind}_{G_rB}^G(p^r\rho), \mathrm{ind}_B^G(p^r\rho))$ and put $j_r' = \hat{e}_{p^r\rho}(j_r)$. One has K-linear isomorphisms

$$\begin{aligned}
(7)\qquad & G\,\mathbf{Mod}(St_r \otimes_K St_r \otimes_K \mathrm{ind}_B^G(\rho), \mathrm{ind}_B^G((2p^r-1)\rho)) \\
&\simeq B\,\mathbf{Mod}(St_r \otimes_K St_r \otimes_K \mathrm{ind}_B^G(\rho), (2p^r-1)\rho) \\
&\qquad\qquad \text{by the Frobenius reciprocity} \\
&\simeq K \quad \text{by (2.5)(2)} \\
&\simeq G\,\mathbf{Mod}(St_r \otimes_K \mathrm{ind}_{G_rB}^G(p^r\rho), \mathrm{ind}_B^G((2p^r-1)\rho)) \\
&\simeq G\,\mathbf{Mod}(St_r \otimes_K \mathrm{ind}_B^G(p^r\rho), \mathrm{ind}_B^G((2p^r-1)\rho)).
\end{aligned}$$

Hence if $\mu_1 \in G\,\mathbf{Mod}(St_r \otimes_K St_r \otimes_K \mathrm{ind}_B^G(\rho), \mathrm{ind}_B^G((2p^r-1)\rho))$ and $\mu_2 \in G\,\mathbf{Mod}(St_r \otimes_K \mathrm{ind}_B^G(p^r\rho), \mathrm{ind}_B^G((2p^r-1)\rho))$ are the cup products, we are reduced to showing

$$\mu_1(v_- \otimes_K v_+ \otimes_K j) = \mu_2(v_- \otimes_K j_r') \quad \text{up to } K^\times. \tag{8}$$

But we have another cup product

$$\mu_3 \in G\,\mathbf{Mod}(St_r \otimes_K \mathrm{ind}_B^G(\rho), \mathrm{ind}_B^G(p^r\rho))$$

such that

$$\mu_1 = \mu_2 \circ (St_r \otimes_K \mu_3). \tag{9}$$

By the weight consideration we must have

$$\mu_3(v_+ \otimes j) = j_r' \quad \text{up to } K^\times; \tag{10}$$

hence (8) follows, as desired.

(4.8) Finally, if P is a parabolic subgroup of G containing B, let $\overline{\pi}: G/B \to G/P$ be the natural morphism. As G/B and G/P are both projective over K, $\overline{\pi}$ is projective. Also $\overline{\pi}_*\mathscr{O}_{G/B} = \mathscr{O}_{G/P}$, as $\overline{\pi}$ is locally trivial with $\mathscr{O}_{P/B}(P/B) = K$. Hence one gets from **[MR]**, Proposition 4,

Corollary. *$\overline{\pi}_*(F_*\sigma)$ splits all the Schubert subshcemes of G/P.*

References

[CPS] E. Cline, B. Parshall and L. Scott, *A Mackey imprimitivity theory for algebraic groups*, Math. Z., **182** (1983), 447–471.

[Haas] B. Haastert, *Über Differentialoperatoren und $\mathscr{D}$-moduln in positiver Charakteristik*, Manuscripta Math., **58** (1987), 385–415.

[HASV] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Springer LNM, **156** (1970).

[J] J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, Orlando, 1987.

[MR] V. B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. Math., **122** (1985), 27–40.

[R] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, IHES Pub. Math., **65** (1987), 61–90.

Received January 27, 1992 and in revised form April 19, 1993.

Department of Mathematics
Faculty of Science
Nigata University
Nigata 950-21 Japan
E-mail address: F00500@sinet.ad.jp

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

MÖBIUS-INVARIANT HILBERT SPACES IN POLYDISCS

H. TURGAY KAPTANOĞLU

We define the Dirichlet space $\mathscr{D}$ on the unit polydisc $\mathbb{U}^n$ of $\mathbb{C}^n$. $\mathscr{D}$ is a semi-Hilbert space of of holomorphic functions, contains the holomorphic polynomials densely, is invariant under compositions with the biholomorphic automorphisms of $\mathbb{U}^n$, and its semi-norm is preserved under such compositions. We show that $\mathscr{D}$ is unique with these properties. We also prove $\mathscr{D}$ is unique if we assume that the semi-norm of a function in $\mathscr{D}$ composed with an automorphism is only equivalent in the metric sense to the semi-norm of the original function. Members of a subclass of $\mathscr{D}$ given by a norm can be written as potentials of $\mathscr{L}^2$-functions on the n-torus $\mathbb{T}^n$. We prove that the functions in this subclass satisfy strong-type inequalities and have tangential limits almost everywhere on $\partial\mathbb{U}^n$. We also make capacitory estimates on the size of the exceptional sets on $\partial\mathbb{U}^n$.

1. Introduction. *Möbius-invariant spaces.* Let $\mathbb{U}$ be the open *unit disc* in $\mathbb{C}$ and $\mathbb{T}$ be the *unit circle* bounding it. The open *unit polydisc* $\mathbb{U}^n$ and the *torus* $\mathbb{T}^n$ in $\mathbb{C}^n$ are the cartesian products of n unit discs and n unit circles, respectively. $\mathbb{T}^n$ is the distinguished boundary of $\mathbb{U}^n$ and forms only a small part of the topological boundary $\partial\mathbb{U}^n$ of $\mathbb{U}^n$. We denote by $\mathscr{M}$ the group of all *biholomorphic automorphisms* of $\mathbb{U}^n$ (the *Möbius group*). The subgroup of *linear* automorphisms in $\mathscr{M}$ is denoted by $\mathscr{U}$. The space of holomorphic functions with domain $\mathbb{U}^n$ will be called $\mathscr{H}(\mathbb{U}^n)$ and will carry the topology of *uniform convergence on compact subsets* of $\mathbb{U}^n$.

A *semi-inner product* on a complex vector space $\mathscr{H}$ is a sesquilinear functional on $\mathscr{H} \times \mathscr{H}$ with all the properties of an inner product except that it is possible to have $\langle\langle a, a\rangle\rangle = 0$ when $a \neq 0$. $\|a\| = \sqrt{\langle\langle a, a\rangle\rangle}$ is the associated *semi-norm*. We assume $\langle\langle \cdot , \cdot\rangle\rangle$ is not identically zero.

DEFINITION 1.1. $\mathscr{H}$ is called a *Hilbert space* of holomorphic functions on $\mathbb{U}^n$ if

(i) $\mathscr{H}$ is a linear subspace of $\mathscr{H}(\mathbb{U}^n)$,
(ii) the semi-inner product $\langle\langle \cdot , \cdot\rangle\rangle$ of $\mathscr{H}$ is complete,
(iii) $\mathscr{H}$ contains all (holomorphic) polynomials,
(iv) polynomials are dense in $\mathscr{H}$ in the topology of the semi-norm $\|\cdot\|$ of $\mathscr{H}$.

A space $\mathscr{H}$ of functions on $\mathbb{U}^n$ is *$\mathscr{M}$-invariant* if $f \circ \Psi \in \mathscr{H}$ whenever $f \in \mathscr{H}$ and $\Psi \in \mathscr{M}$. An $\mathscr{M}$-invariant Hilbert space $\mathscr{H}$ of holomorphic functions on $\mathbb{U}^n$ will be called an *$\mathscr{M}$-space* for brevity. *$\mathscr{U}$-invariance* and *$\mathscr{U}$-space* have similar definitions.

$\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of nonnegative integers, positive integers, integers, and real numbers, respectively. A *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a point in $\mathbb{N}^n$. $\sum_\alpha$ indicates a summation with α running over all the points in $\mathbb{N}^n$, and $\sum'_{\alpha \in \mathbb{I}}$ is a summation where we consider only those α in the index set $\mathbb{I}$ with all positive components. Let also $D_j = \partial/\partial z_j$ and $\bar{D}_j = \partial/\partial \bar{z}_j$. The following abbreviated notations will be used:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$
$$\alpha! = \alpha_1! \cdots \alpha_n!, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

The *Dirichlet space* $\mathscr{D}(\mathbb{U}^n)$ is the class of $f(z) = \sum_\alpha f_\alpha z^\alpha \in \mathscr{H}(\mathbb{U}^n)$ with

$$\|f\|_{\mathscr{D}}^2 = \sum_\alpha \alpha_1 \cdots \alpha_n |f_\alpha|^2 = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \alpha_1 \cdots \alpha_n |f_\alpha|^2 < \infty. \tag{1.1}$$

Equivalently, $\mathscr{D}(\mathbb{U}^n)$ is the class of those $f \in \mathscr{H}(\mathbb{U}^n)$ with

$$\|f\|_{\mathscr{D}}^2 = \int_{\mathbb{U}^n} |D_1 \cdots D_n f|^2 \, d\mu_n < \infty,$$

where μ_n is the Lebesgue maesure on $\mathbb{U}^n$ normalized so that $\mu_n(\mathbb{U}^n) = 1$. The semi-norm $\|\cdot\|_{\mathscr{D}}$ is obtained from the semi-inner product

$$\langle\langle f, g \rangle\rangle_{\mathscr{D}} = \sum_\alpha \alpha_1 \cdots \alpha_n f_\alpha \bar{g}_\alpha = \int_{\mathbb{U}^n} (D_1 \cdots D_n f) \overline{(D_1 \cdots D_n g)} \, d\mu_n.$$

Main results. In this work, we first prove *two* theorems which show that the Dirichlet space is unique among $\mathscr{M}$-spaces that have certain properties.

THEOREM A. *Let $\mathscr{H}$ be an $\mathscr{M}$-space and suppose that*

$$\|f\| = \|f \circ \Psi\| \qquad (f \in \mathscr{H}, \ \Psi \in \mathscr{M}). \tag{1.2}$$

Then

$$\|f\| = C \, \|f\|_{\mathscr{D}} \qquad (f \in \mathscr{H}),$$

where $C = \|z_1 \cdots z_n\|^2$. Thus $\mathscr{H}$ is $\mathscr{D}(\mathbb{U}^n)$.

Note that the assumption on the semi-norm is equivalent to $\langle\langle f \circ \Psi, g \circ \Psi\rangle\rangle = \langle\langle f, g\rangle\rangle$, and the conclusion implies that $\langle\langle f, g\rangle\rangle = C\langle\langle f, g\rangle\rangle_{\mathscr{D}}$, for all $f, g \in \mathscr{H}$ and $\Psi \in \mathscr{M}$.

For the second theorem, we need a strengthening of condition (ii) of Definition 1.1. To derive it, we write the Taylor series expansion at 0 of an $f \in \mathscr{H}$ by seperating the higher and lower dimensional terms. For example, when $n = 2$, using (z, w) for (z_1, z_2) and (k, l) for (α_1, α_2), we write

$$f(z, w) = f_{00} + \sum_{k=1}^{\infty} f_{k0} z^k + \sum_{l=1}^{\infty} f_{0l} w^l + \sum_{k,l=1}^{\infty} f_{kl} z^k w^l.$$

Since we assume conditions (iii) and (iv) of Definition 1.1, we can define a *norm* on $\mathscr{H}$ by

$$|||f|||^2 = |f_{00}|^2 + \sum_{k=1}^{\infty} |f_{k0}|^2 \|z^k\|_1^2 + \sum_{l=1}^{\infty} |f_{0l}|^2 \|w^l\|_1^2 + \sum_{k,l=1}^{\infty} |f_{kl}|^2 \|z^k w^l\|^2,$$

where $\|\cdot\|$ is still the semi-norm of $\mathscr{H} \subset \mathscr{H}(\mathbb{U}^n)$ and $\|\cdot\|_1$ denotes the semi-norm of a similar $\mathscr{H}' \subset \mathscr{H}(\mathbb{U})$. We already know (see **[1]**) that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_{\mathscr{D}}$ in $\mathbb{U}$. Since our proof of Theorem B is by induction on the dimension of the polydisc, the above definition of $|||\cdot|||$ makes sense. Now we make the alternate assumption

(ii)′ $\mathscr{H}$ is complete in the norm $|||\cdot|||$.

A similar condition was assumed in **[1]**, whereas **[4]** assumes (ii).

THEOREM B. *Let $\mathscr{H}$ be an $\mathscr{M}$-space in the sense modified by* (ii)′ *and assume that there is a positive constant $\delta < 1$ such that*

$$\delta\|f\| \le \|f \circ \Psi\| \le \frac{1}{\delta}\|f\| \qquad (f \in \mathscr{H},\ \Psi \in \mathscr{M}). \tag{1.3}$$

Then there exists positive constants K_1 and K_2 such that

$$K_1\|f\|_{\mathscr{D}} \le \|f\| \le K_2\|f\|_{\mathscr{D}} \qquad (f \in \mathscr{H}).$$

Thus $\mathscr{D}(\mathbb{U}^n)$ is unique again.

The proofs of these theorems will be presented in §2.

Next, we consider a subspace of the Dirichlet space, one that is defined by a genuine norm similar to $|||\cdot|||$. This space is not $\mathscr{M}$-invariant any more, but the stronger conditions on it allow us to prove that it has *tangential* limits as we approach $\partial\mathbb{U}^n$. In fact, tangential limits exist for a wider class of functions which are potentials of certain functions in $\mathscr{L}^2(\mathbb{T}^n)$. The precise definitions and theorems are stated in §3. Theorems C and D at the end of that section are the major results in this direction.

In earlier work, Arazy and Fisher [1] proved, under slightly different hypotheses, the analogs of Theorem A and Theorem B in $\mathbb{U}$. Zhu [5] found the equivalent of Theorem A for the unit ball in $\mathbb{C}^n$ when $n \geq 2$. Nagel, Rudin and Shapiro [3] obtained the unit-disc versions of Theorems C and D.

After the submission of the manuscript, we were informed by a referee that in the preprint *Invariant Hilbert Spaces of Analytic Functions on Bounded Symmetric Domains* by J. Arazy and S. D. Fisher, results analogous to Theorems A and B were established for all irreducible bounded symmetric domains.

NOTATION. λ_n is the Lebesgue measure on $\mathbb{T}^n$ both normalized to have mass 1; i.e., it is the Haar measure on the compact abelian group $\mathbb{T}^n$. If $p \in [1, \infty)$, its *conjugate* is $q = p/(p-1)$. The $\mathscr{L}^p$- and ℓ^p-spaces will have their usual meaning. z_j will usually be an element of $\mathbb{U}$ and ζ_j of $\mathbb{T}$. Apart from the usual big $\mathscr{O}$ notation, we will use $u \sim v$ to mean both $u = \mathscr{O}(v)$ and $v = \mathscr{O}(u)$, and $u \approx v$ to mean u/v has a finite positive limit.

The *Poisson integral* of an $f \in \mathscr{L}^1(\mathbb{T}^n)$ is

$$P[f](z) = \int_{\mathbb{T}^n} f(\zeta) \prod_{j=1}^{n} \frac{1-|z_j|^2}{|1-z_j\bar{\zeta}_j|^2}\, d\lambda_n(\zeta) \qquad (z \in \mathbb{U}^n),$$

and its *Cauchy integral* is

$$C[f](z) = \int_{\mathbb{T}^n} f(\zeta) \prod_{j=1}^{n} \frac{1}{1-z_j\bar{\zeta}_j}\, d\lambda_n(\zeta) \qquad (z \in \mathbb{U}^n),$$

where the products are called the *Poisson kernel* $P(z, \zeta)$ and the *Cauchy kernel* $C(z, \zeta)$ for $\mathbb{U}^n$, respectively. These transforms have the following invariance properties: If $f \in \mathscr{L}^1(\lambda_n)$, $\Psi \in \mathscr{M}$, and $U \in \mathscr{U}$, then

$$P[f \circ \Psi] = P[f] \circ \Psi \qquad \text{and} \qquad C[f \circ U] = C[f] \circ U.$$

The automorphisms of $\mathbb{U}^n$ for $n \geq 2$ are generated by the following three subgroups: *rotations* in each variable separately

$$R_\theta(z) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n),$$

Möbius transformations in each variable separately

$$\Phi_w(z) = \left(\phi_{w_1}(z_1), \ldots, \phi_{w_n}(z_n)\right),$$

and the *coordinate permutations*. Here $\theta \in [-\pi, \pi]^n$ and $w \in \mathbb{U}^n$ are fixed, Möbius transformations are in the form

$$\phi_w(z) = \frac{w - z}{1 - \bar{w}z} \qquad (w \in \mathbb{U},\ z \in \overline{\mathbb{U}}), \tag{1.4}$$

and the coordinate permutations are nothing but the $n!$ members of the symmetric group $\mathcal{S}_n$ on n objects. Thus an arbitrary $\Psi \in \mathcal{M}$ can be written in the form

$$\Psi(z) = \left(e^{i\theta_1}\phi_{w_1}(z_{\sigma(1)}), \ldots, e^{i\theta_n}\phi_{w_n}(z_{\sigma(n)})\right),$$

for some $w \in \mathbb{U}^n$ and $\theta \in [-\pi, \pi]^n$, and $\sigma \in \mathcal{S}_n$. $\mathcal{U}$ is generated by $\sigma \in \mathcal{S}_n$ and the rotations R_θ. Each Möbius transformation Φ_w is an *involution* (its inverse is itself) exchanging 0 and w. $\mathcal{M}$ acts *transitively* on $\mathbb{U}^n$: if $a, b \in \mathbb{U}^n$, then $\Phi_b \circ \Phi_a \in \mathcal{M}$ moves a to b (and b to a). Finally, $\mathcal{M}^*$ denotes the component of the identity in $\mathcal{M}$; i.e., $\mathcal{M}^*$ is $\mathcal{M}$ without the action of $\mathcal{S}_n$.

2. Uniqueness of the Dirichlet space. We start by showing that $\mathcal{D}(\mathbb{U}^n)$ has all the properties of a Hilbert space in the sense of Definition 1.1. Clearly the polynomials are in $\mathcal{D}(\mathbb{U}^n)$ and $\|z^\alpha\|^2 = \alpha_1 \cdots \alpha_n$ for all $\alpha \in \mathbb{N}^n$. A quick look at (1.1) shows that the polynomials are dense in $\mathcal{D}(\mathbb{U}^n)$ with respect to $\|\cdot\|_{\mathcal{D}}$. Again from (1.1), identifying g by $\{g_\alpha\}$, we see that $\mathcal{D}(\mathbb{U}^n)$ is a weighted ℓ^2-space, hence every Cauchy sequence $\{f_j\}$ in $\mathcal{D}(\mathbb{U}^n)$ converges in $\|\cdot\|_{\mathcal{D}}$ to some $f \in \mathcal{D}(\mathbb{U}^n)$ represented by $\{f_\alpha\}$ for $\alpha \in \mathbb{Z}_+^n$. To show that f is holomorphic, let $f_m(z) = \sum_{k=1}^m \sum'_{|\alpha|=k} f_\alpha z^\alpha$ and pick $\varepsilon > 0$. For any $0 < r < 1$ and positive integers $m > l > n$,

$$\begin{aligned}
\sup_{z \in r\overline{\mathbb{U}}^n} \left|(f_m - f_l)(z)\right| &= \sup_{z \in r\overline{\mathbb{U}}^n} \left| \sum_{k=l+1}^m \sum_{|\alpha|=k}{}' f_\alpha z^\alpha \right| \le \sum_{k=l+1}^m \sum_{|\alpha|=k}{}' |f_\alpha|\, r^{|\alpha|} \\
&\le \left(\sum_{k=l+1}^m \sum_{|\alpha|=k}{}' |f_\alpha|^2 \right)^{1/2} \left(\sum_{k=l+1}^m \sum_{|\alpha|=k}{}' r^{2|\alpha|} \right)^{1/2} \\
&< \left(\sum_{k=l+1}^m \sum_{|\alpha|=k} \alpha_1 \cdots \alpha_n |f_\alpha|^2 \right)^{1/2} \left(\sum_{k=l+1}^m k^n r^{2k} \right)^{1/2}.
\end{aligned}$$

The first factor is less than ε when l and m are large enough because $f \in \mathcal{D}(\mathbb{U}^n)$, and the second factor is bounded as $l, m \to \infty$. Hence

$f(z) = \sum_\alpha' f_\alpha z^\alpha$ is uniformly convergent on compact subsets of $\mathbb{U}^n$, and this proves $f \in \mathscr{H}(\mathbb{U}^n)$. Note that we need not know f_α if $\alpha \in \mathbb{N}^n \setminus \mathbb{Z}_+^n$. These coefficients of f can be taken arbitrarily as long as f remains holomorphic.

LEMMA 2.1. *$\mathscr{M}$ is generated by $\mathscr{S}_n$, rotations $R_\omega(z) = (e^{i\omega} z_1, z_2, \ldots, z_n)$ with $\omega \in [-\pi, \pi]$, and Möbius transformations of the form $\Phi_t(z) = (\phi_t(z_1), z_2, \ldots, z_n)$ with $0 \le t < 1$.*

PROPOSITION 2.2. *$\mathscr{D}(\mathbb{U}^n)$ is $\mathscr{M}$-invariant.*

Proof. The integral form of the Dirichlet semi-norm uses the measure μ_n which is invariant under rotations and permutations. Thus $\mathscr{D}(\mathbb{U}^n)$ is $\mathscr{U}$-invariant. To prove invariance under Möbius transformations, in view of Lemma 2.1, it suffices to consider

$$w = \Phi_r(z) = \left(\frac{r - z_1}{1 - r z_1}, z_2, \ldots, z_n\right).$$

Then $D_1^z(f \circ \Phi_r) = (D_1^w f) dw_1/dz_1$ and

$$\begin{aligned} |D_1^z D_2 \cdots D_n(f \circ \Phi_r)|^2 &= |D_1^w D_2 \cdots D_n f|^2 \frac{(r^2-1)^2}{|1 - r z_1|^4} \\ &= |D_1^w D_2 \cdots D_n f|^2 J_{\Re}\Phi_r(z), \end{aligned}$$

since $dw_1/dz_1 = (r^2 - 1)/(1 - r z_1)^2$, where $J_{\Re}\Phi_r$ is the real Jacobian of Φ_r. Therefore

$$\begin{aligned} \|f \circ \Phi_r\|_{\mathscr{D}}^2 &= \int_{\mathbb{U}^n} \left|D_1^z D_2 \cdots D_n(f \circ \Phi_r)(z)\right|^2 d\mu_n(z) \\ &= \int_{\mathbb{U}^n} \left|D_1^w D_2 \cdots D_n f(w)\right|^2 \frac{(r^2-1)^2}{|1 - r z_1|^4} \frac{1}{J_{\Re}\Phi_r(z)} d\mu_n(w) \\ &= \int_{\mathbb{U}^n} \left|D_1^w D_2 \cdots D_n f(w)\right| d\mu_n(w) = \|f\|_{\mathscr{D}}^2. \end{aligned}$$

□

Note that when $n \ge 2$, $\mathscr{D}(\mathbb{U}^n)$ does not put any conditions on the infinitely many power series coefficients of f, those with at least one $\alpha_j = 0$, i.e., those in $\mathbb{N}^n \setminus \mathbb{Z}_+^n$. Thus if each term in the Taylor expansion of some $f \in \mathscr{H}(\mathbb{U}^n)$ depends on fewer than n variables, then $\|f\|_{\mathscr{D}} = 0$ and $f \in \mathscr{D}$. The Dirichlet space can also be thought of as a quotient space of holomorphic functions satisfying (1.1) where the functions whose Taylor series differ by terms depending on at most $n-1$ variables are identified. Trivially any holomorphic function f of fewer than n variables or any constant f has $\|f\|_{\mathscr{D}} = 0$ and is in $\mathscr{D}(\mathbb{U}^n)$. For comparison, when $n = 1$, only constants (a one-dimensional subspace) have zero Dirichlet semi-norm.

We can define a *modified* Dirichlet space $\tilde{\mathscr{D}}(\mathbb{U}^n)$ similar to $\mathscr{D}(\mathbb{U}^n)$ by considering a norm instead of a semi-norm. This requires some control on *all* the power series coefficients of $f \in \mathscr{H}(\mathbb{U}^n)$. For simplicity let's look at the case $n = 2$. With notation as before, let

$$\begin{aligned}\|f\|_{\tilde{\mathscr{D}}}^2 &= |f_{00}|^2 + \sum_{k=1}^{\infty} k|f_{k0}|^2 + \sum_{l=1}^{\infty} l|f_{0l}|^2 + \sum_{k,l=1}^{\infty} kl|f_{kl}|^2 \\ &= \left|f(0,0)\right|^2 + \int_{\mathbb{U}} \left|D_1 f(z,0)\right|^2 d\mu_1(z) \\ &\quad + \int_{\mathbb{U}} \left|D_2 f(0,w)\right|^2 d\mu_1(w) \\ &\quad + \int_{\mathbb{U}^2} \left|D_1 D_2 f(z,w)\right|^2 d\mu_2(z,w).\end{aligned}$$

This norm is $\mathscr{U}$-invariant, but not $\mathscr{M}$-invariant; in fact, none of its first three terms is preserved under compositions with Φ_r.

Proof of Theorem A. First, $\langle\!\langle z^\alpha, z^\beta \rangle\!\rangle = 0$ if $\alpha \neq \beta$. To see this, assume, without loss of generality, $\alpha_1 \neq \beta_1$. Let ω be an irrational multiple of π and consider the rotation $R_\omega(z) = (e^{i\omega} z_1, z_2, \dots, z_n)$. By the $\mathscr{M}$-invariance of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$,

$$\begin{aligned}\langle\!\langle z^\alpha, z^\beta \rangle\!\rangle &= \langle\!\langle z^\alpha \circ R_\omega, z^\beta \circ R_\omega \rangle\!\rangle \\ &= \langle\!\langle e^{i\alpha_1\omega} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}, e^{i\beta_1\omega} z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \rangle\!\rangle \\ &= e^{i(\alpha_1-\beta_1)\omega} \langle\!\langle z^\alpha, z^\beta \rangle\!\rangle,\end{aligned}$$

and the desired orthogonality result follows.

Put $C_\alpha = \langle\!\langle z^\alpha, z^\alpha \rangle\!\rangle$. Note that C_α is defined only for $\alpha \in \mathbb{N}^n$. If β is another multi-index and $\beta = \sigma(\alpha)$ for some $\sigma \in \mathscr{S}_n$, then by the $\mathscr{M}$-invariance of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ again, $C_\alpha = C_\beta$.

Now let $0 < r_j < 1$, $\Psi = (\phi_{r_1}, \dots, \phi_{r_n})$, and consider $f(z) = \prod_{j=1}^n (1 - r_j z_j) \in \mathscr{H}$. Then since z^α is orthogonal to z^β for $\alpha \neq \beta$, we get

$$\langle\!\langle f, f\rangle\!\rangle = \left\|\prod_{j=1}^{n}(1-r_jz_j)\right\|^2 = \left\|1-\sum_{j=1}^{n} r_jz_j+\cdots+(-1)^m\prod_{j=1}^{n} r_jz_j\right\|^2$$
$$= C_{(0,\dots,0)} + \left(\sum_{j=1}^{n} r_j^2\right)C_{(1,0,\dots,0)}$$
$$+\left(\sum_{l>j=1}^{n} r_j^2r_l^2\right)C_{(1,1,0,\dots,0)}+\cdots+\left(\prod_{j=1}^{n} r_j^2\right)C_{(1,\dots,1)}.$$

On the other hand,

$$(f\circ\Psi)(z) = \prod_{j=1}^{n}\left(1-r_j\frac{r_j-z_j}{1-r_jz_j}\right) = \prod_{j=1}^{n}\frac{1-r_j^2}{1-r_jz_j}$$
$$= \prod_{j=1}^{n}\left[(1-r_j^2)\sum_{\alpha_j=0}^{\infty} r_j^{\alpha_j}z_j^{\alpha_j}\right] = \left(\prod_{j=1}^{n}(1-r_j^2)\right)\left(\sum_{\alpha} r^\alpha z^\alpha\right);$$

now the density of the polynomials and the axioms of a Hilbert space imply

$$\langle\!\langle f\circ\Psi, f\circ\Psi\rangle\!\rangle = \left(\prod_{j=1}^{n}(1-r_j^2)^2\right)\sum_{\alpha} r^{2\alpha}C_\alpha.$$

Putting $x_j = r_j^2$ and using the $\mathscr{M}$-invariance of the semi-norm gives

$$\text{(2.1)}\quad C_{(0,\dots,0)} + \left(\sum_{j=1}^{n} x_j\right)C_{(1,0,\dots,0)} + \left(\sum_{l>j=1}^{n} x_jx_l\right)C_{(1,1,0,\dots,0)}$$
$$+\cdots+\left(\prod_{j=1}^{n} x_j\right)C_{(1,\dots,1)} = \left(\prod_{j=1}^{n}(1-x_j)^2\right)\sum_{\alpha} x^\alpha C_\alpha.$$

The constant terms $(C_{(0,\dots,0)})$ cancel, and if we set the coefficients of x_1, x_1x_2, $\dots$, and $x_1x_2\cdots x_n$ on either side equal to each other, we obtain, respectively,

$$C_{(1,0,\dots,0)} = C_{(1,0,\dots,0)} - 2C_{(0,\dots,0)},$$
$$C_{(1,1,0,\dots,0)} = C_{(1,1,0,\dots,0)} - 2^2C_{(1,0,\dots,0)},$$
$$\vdots$$
$$C_{(1,\dots,1)} = C_{(1,\dots,1)} - 2^nC_{(1,\dots,1,0)}.$$

These imply $C_{(0,\dots,0)} = 0$, $C_{(1,0,\dots,0)} = 0$, $\dots$, $C_{(1,\dots,1,0)} = 0$. After the elimination of the terms that are zero, (2.1) simplifies to

$$\sum_{\alpha}{}' x^{\alpha} C_{\alpha} = \frac{x_1 \cdots x_n \, C_{(1,\dots,1)}}{(1-x_1)^2 \cdots (1-x_n)^2}$$
$$= C_{(1,\dots,1)} \left(\sum_{\alpha_1=1}^{\infty} \alpha_1 x_1^{\alpha_1} \right) \cdots \left(\sum_{\alpha_n=1}^{\infty} \alpha_n x_n^{\alpha_n} \right);$$

and this implies

$$C_{\alpha} = \alpha_1 \cdots \alpha_n \, C_{(1,\dots,1)} \qquad (\alpha \in \mathbb{Z}_+^n).$$

Thus the norm of a monomial of fewer than n variables is zero. Since the polynomials are dense in $\mathscr{H}$, the same result is true for any $f \in \mathscr{H}$ whose Taylor expansion consists of monomials depending on fewer than n variables. But $C_{(1,\dots,1)} \neq 0$, because otherwise, since the polynomials are dense in $\mathscr{H}$, $\langle\langle \cdot , \cdot \rangle\rangle$ would be identically zero contrary to hypothesis. Then renaming $C_{(1,\dots,1)} = C$ completes the proof. □

Proof of Theorem B. We will only show how the two-variable case is obtained from the one-variable case. This then can be adapted to prove by induction the case for arbitrary $\mathbb{U}^n$. Unless explicitly stated, subscripted C's will denote positive constants that are independent of any parameters.

Step 1. We begin by introducing two other semi-inner products on $\mathscr{H}$. For $f, g \in \mathscr{H}$, let

$$[f, g] = \int_{\mathbb{T}^2} \langle\langle f \circ R_{\theta}, \; g \circ R_{\theta} \rangle\rangle \, d\lambda_2(\theta)$$

and

$$\langle f, g \rangle = m \langle\langle f \circ \Phi, \; g \circ \Phi \rangle\rangle,$$

where m is an *invariant mean* (see [2]) on the abelian subgroup

$$\mathscr{N} = \{ \Phi = (\phi_s, \phi_t) : 0 \le s, t < 1 \}$$

of $\mathscr{M}$. To actually make $\mathscr{N}$ abelian, in this proof we change our definition of a Möbius transformation so that $\phi_w(z)$ is the negative of what is given in (1.4). The earlier definition was adopted to make ϕ_w an involution, since it simplified calculations involving ϕ_w^{-1}. The required boundedness condition for the existence of this nonunique mean is furnished by (1.3). Rotations and $\mathscr{N}$, along with $\mathscr{S}_2$, suffice to generate $\mathscr{M}$, by Lemma 2.1.

Now $[\cdot\,,\,\cdot]$ is rotation-invariant

$$[f\,,\,g] = [f\circ R_\theta\,,\,g\circ R_\theta] \qquad (\theta\in[-\pi\,,\,\pi]^n)\,, \tag{2.2}$$

and $\langle\cdot\,,\,\cdot\rangle$ is $\mathscr{N}$-invariant

$$\langle f\,,\,g\rangle = \langle f\circ\Phi\,,\,g\circ\Phi\rangle \qquad (\Phi\in\mathscr{N}). \tag{2.3}$$

Moreover, (1.3) implies

$$\delta^2\|f\|^2 \le [f\,,\,f] \le \frac{1}{\delta^2}\|f\|^2 \qquad (f\in\mathscr{H})\,, \tag{2.4}$$

and

$$\delta^2\|f\|^2 \le \langle f\,,\,f\rangle \le \frac{1}{\delta^2}\|f\|^2 \qquad (f\in\mathscr{H})\,; \tag{2.5}$$

and combining these two, we further obtain

$$\delta^4[f\,,\,f] \le \langle f\,,\,f\rangle \le \frac{1}{\delta^4}[f\,,\,f] \qquad (f\in\mathscr{H}). \tag{2.6}$$

(2.4) and (2.5) show that the semi-norms associated to $[\cdot\,,\,\cdot]$ and $\langle\cdot\,,\,\cdot\rangle$ are both equivalent to $\|\cdot\|$.

As in the proof of Theorem A, the rotation-invariance of $[\cdot\,,\,\cdot]$ gives the orthogonality condition

$$[z^{k_1}w^{l_1}\,,\,z^{k_2}w^{l_2}] = 0 \qquad ((k_1\,,\,l_1)\ne(k_2\,,\,l_2))\,, \tag{2.7}$$

which leads to

$$[f\,,\,f] = \sum_{k\,,\,l=0}^{\infty} |f_{kl}|^2\,[z^kw^l\,,\,z^kw^l] \qquad (f\in\mathscr{H}).$$

Therefore, to prove the theorem, it suffices to show that

$$K_1\,kl \le [z^kw^l\,,\,z^kw^l] \le K_2\,kl \qquad ((k\,,\,l)\in\mathbb{N}^2)$$

or, equivalently,

$$K_3\,kl \le \langle z^kw^l\,,\,z^kw^l\rangle \le K_4\,kl \qquad ((k\,,\,l)\in\mathbb{N}^2)$$

for some positive constants K_1, K_2, K_3, and K_4. Clearly $K_2\ge K_1$ and $K_4\ge K_3$.

Step 2. Claim:

$$\langle z^kw^l\,,\,z^kw^l\rangle \ne 0 \qquad \text{if } k\ge 1 \text{ and } l\ge 1. \tag{2.8}$$

Suppose it is zero for some (N, M); then $[z^N w^M, z^N w^M] = 0$ also. Then for $0 \le s, t < 1$, if we use (2.3), (2.6) and power series expansion

$$
\begin{aligned}
0 &= \langle z^N w^M, z^N w^M \rangle \\
&= \left\langle \left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M, \left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M \right\rangle \\
&\ge \delta^8 \left[\left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M, \left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M \right] \\
&= \left[\sum_{k=0}^{\infty} c_{kN}(s) z^k \sum_{l=0}^{\infty} c'_{lM}(t) w^l, \sum_{k=0}^{\infty} c_{kN}(s) z^k \sum_{l=0}^{\infty} c'_{lM}(t) w^l \right] \\
&= \sum_{k,l=0}^{\infty} |c_{kN}(s)|^2 |c'_{lM}(t)|^2 [z^k w^l, z^k w^l].
\end{aligned}
$$

A tedious computation shows that the coefficients $c_{kN}(s) \neq 0$ for any k, N, and s as given above; the same is obviously true for $c'_{lM}(t)$. Thus $[z^k w^l, z^k w^l] = 0$. This means that every element in $\mathscr{H}$ has zero norm and contradicts our basic assumption that $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is not identically zero. Hence the claim is proved.

The one-variable result can be stated as

$$
(2.9) \qquad C_1 k \le \langle z^k, z^k \rangle_1 \le C_2 k \qquad (k \in \mathbb{N}).
$$

It is a consequence of condition (ii)′ of Definition 1.1 and of (2.8) that the subspace of $\mathscr{H}$ consisting of functions whose Taylor series expansion at 0 depend only on z is closed. Then (2.9) implies that, for *fixed* $M \in \mathbb{N}$,

$$
(2.10) \qquad C_3 k \le \langle z^k w^M, z^k w^M \rangle \le C_4 k \qquad (k \in \mathbb{N}),
$$

and we have a similar equation when the power of z is held constant. Of course, the constants C_3 and C_4 are different for different M. It is our aim to find their explicit dependence on M. If we had only finitely many M, we could pick C_3 and C_4 independently of M and the proof would be over. In the sequel, whenever we have only finitely many n or M, we will use this fact without further reference.

Step 3 (*upper bound*). Let $M \in \mathbb{N}$ be *fixed* and $k, j \in \mathbb{N}$. Put

$$
\alpha_{kM}^{jM} = \langle z^k w^M, z^j w^M \rangle, \qquad \beta_{kM} = \langle z^k w^M, z^k w^M \rangle,
$$
$$
b_{kM} = [z^k w^M, z^k w^M].
$$

By (2.3), $\alpha_{kM}^{jM} = \langle \phi_s^k w^M, \phi_s^j w^M \rangle$ for any $s \in [0, 1)$. Differentiate both sides of this equality with respect to s and set $s = 0$. Then take $k = j+1$ and add the resulting expressions from $j = 0$ to $j = N \geq 1$; and finally divide both sides by $N + 1$. The result is

$$\frac{\beta_{N+1,M}}{N+1} = \frac{2S_{NM}}{N(N+1)} - \frac{\alpha_{N+2,M}^{NM}}{N}, \tag{2.11}$$

where $S_{NM} = \frac{1}{2}\beta_{0M} + \sum_{k=1}^{N} \beta_{kM}$. In particular, $\alpha_{N+2,M}^{NM}$ is *real*. Now using (2.3) and (2.5), and letting $s^2 = \frac{N}{N+1}$, we obtain

$$\begin{aligned} \frac{1}{\delta^4}\beta_{1M} &= \frac{1}{\delta^4}\langle \phi_s w^M, \phi_s w^M \rangle \geq [\phi_s w^M, \phi_s w^M] \\ &= s^2 b_{0M} + (1-s^2)^2 \sum_{k=0}^{\infty} s^{2k} b_{k+1,M} \\ &\geq \frac{\delta^4}{(N+1)^2} \sum_{k=0}^{N} \left(\frac{N}{N+1}\right)^k \beta_{k+1,M} \geq \frac{\delta^4}{(N+1)^2 e} \sum_{k=0}^{N} \beta_{k+1,M}, \end{aligned}$$

which implies

$$S_{NM} \leq \left(\frac{e\beta_{1M}}{\delta^8} + \frac{\beta_{0M}}{2}\right) N^2.$$

Using (2.10) twice on the right side gives

$$S_{NM} \leq C_5 M N^2. \tag{2.12}$$

It is this inequality and its pair (2.15) below that allow us to pass from one variable to several variables.

As a special case, when $M = 0$, we get $S_{N0} = 0$ for all $N \geq 1$. Symmetric nature of the calculation shows also $S_{0N} = 0$ for all $N \geq 1$. It follows that

$$\beta_{N0} = \langle z^N, z^N \rangle = 0 \quad \text{and} \quad \beta_{0N} = \langle w^N, w^N \rangle = 0 \qquad (N \in \mathbb{Z}_+). \tag{2.13}$$

The inequality

$$\delta^8(\beta_{NM} + \beta_{N+2,M}) \leq \langle (z^N + z^{N+2}) w^M, (z^N + z^{N+2}) w^M \rangle$$

is a direct consequence of (2.6) and (2.7). Using this, after some routine calculation, (2.11) can be written as

$$\begin{aligned} 2\frac{\beta_{N+1,M}}{N+1} &\leq \frac{4S_{NM}}{N(N+1)} + \frac{2\beta_{N+2,M}}{N(N+2)} + (1-\delta^8)\left(\frac{\beta_{NM}}{N} + \frac{\beta_{N+2,M}}{N+2}\right) \\ &\leq 6C_5 M + (1-\delta^8)\left(\frac{\beta_{NM}}{N} + \frac{\beta_{N+2,M}}{N+2}\right), \end{aligned}$$

which is equivalent to $2\gamma_{N+1,M} \le (1-\delta^8)(\gamma_{NM}+\gamma_{N+2,M})$ if we let

$$\gamma_{NM} = \frac{\beta_{NM}}{N} - \frac{3C_5 M}{\delta^8}.$$

A result in [1] shows $\gamma_{N+1,M} \le |\gamma_{NM}|$ for positive N. Then using (2.8), we get

$$\frac{\beta_{NM}}{N} - \frac{3C_5 M}{\delta^8} \le \left|\frac{\beta_{1M}}{1} - \frac{3C_5 M}{\delta^8}\right| \le C_6 M.$$

Multiplying both sides by M, we conclude

$$\beta_{NM} \le \left(C_6 + \frac{3C_5}{\delta^8}\right) NM = K_4 NM,$$

which holds for $N \ge 1$ and $M \ge 0$, and for $M \ge 1$ and $N \ge 0$, by the symmetry of the computation.

Step 4 (*lower bound*). If we combine the result of Step 1 with (2.6), we also get

$$b_{kM} \le \frac{K_4}{\delta^4} kM. \tag{2.14}$$

We use (2.6), (2.7), (2.3), (2.14), and take $s^2 = \frac{N}{N+1}$ to calculate

$$\begin{aligned}
\delta^8 s^2 b_{1M} &\le \delta^8 [(1+sz)w^M, (1+sz)w^M] \\
&= \delta^4 \langle (1+s\phi_s)w^M, (1+s\phi_s)w^M \rangle \\
&\le (1-s^2)^2 \left[\frac{w^M}{1-sz}, \frac{w^M}{1-sz}\right] = \left(\frac{1}{N+1}\right)^2 \sum_{k=0}^{\infty} s^{2k} b_{kM} \\
&\le \frac{1}{(N+1)^2} \left(\sum_{k=0}^{mN} b_{kM} + \frac{K_2}{\delta^4} M \sum_{k=mN+1}^{\infty} k \left(\frac{N}{N+1}\right)^k \right),
\end{aligned}$$

where m will be determined shortly. After approximating the second sum by an integral, we have

$$\delta^8 \left(\frac{N}{N+1}\right) b_{1M} \le \frac{1}{(N+1)^2} \sum_{k=0}^{mN} b_{kM} + \frac{K_2}{\delta^4} M(m+1)e^{-m}.$$

Because of (2.10), M in the last term can be replaced with $C_8 b_{1M}$. Now choose m so large that $K_2 \delta^{-4}(m+1)e^{-m} \le \delta^8/3$. Using (2.10) once again and some simplification yields

$$\sum_{k=0}^{N} b_{kM} \ge C_9 M N^2, \tag{2.15}$$

which is the reverse inequality for (2.12). Combining (2.15) with (2.11), we obtain

$$\delta^4 C_9 M \le \frac{\beta_{N+1,M}}{N+1} + \left(\frac{\beta_{NM}}{N}\right)^{1/2}\left(\frac{\beta_{N+2,M}}{N+2}\right)^{1/2}. \tag{2.16}$$

Now let

$$\phi_s^N(z) = \left(\frac{z-s}{1-sz}\right)^N = \sum_{k=0}^{\infty} c_{kN}(s) z^k$$

and consider

$$\begin{aligned} b_{NM} &\ge \delta^4 \beta_{NM} = \delta^4 \langle \phi_s^N w^M, \phi_s^N w^M\rangle \ge \delta^8 [\phi_s^N w^M, \phi_s^N w^M] \\ &= \delta^8 \sum_{k=1}^{\infty} \left|c_{kN}(s)\right|^2 b_{kM} \ge \delta^8 \left|c_{N+1,N}(s)\right|^2 b_{N+1,M} \end{aligned}.$$

A calculation in [1] shows that $\left|c_{N+1,N}(s)\right|^2 \ge 1/2$ for all $s \in [0, 1)$ and $N \ge 1$. Thus there is a constant C_{10} such that

$$\beta_{N+1,M} \le C_{10} \beta_{NM}. \tag{2.17}$$

Now (2.16) and (2.17) together imply

$$\beta_{NM} \ge \left(\frac{\delta^4 C_9}{4C_{10}}\right) NM = K_3\, NM,$$

for $N \ge 1$ and $M \ge 0$, and for $M \ge 1$ and $N \ge 0$.

Step 5. The only term we have not yet accounted for is $\beta_{00} = \langle 1, 1\rangle$. Since it represents a one-dimensional subspace of $\mathscr{H}$, we now know $\mathscr{H} = \mathscr{D}(\mathbb{U}^2)$. To complete the proof, we will also show $\beta_{00} = b_{00} = 0$. To obtain a contradiction, suppose $b_{00} = [1, 1] \ne 0$. Let $f \in \mathscr{H}$, Φ be a Möbius transformation, and denote the power series coefficients of f and $f \circ \Phi$ by f_{kl} and f'_{kl}, respectively. Because of (2.13)

$$\|f\|^2 \ge \delta^2 [f, f] = \delta^2 \left(b_{00} |f(0, 0)|^2 + \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl}\right)$$

and

$$[f\circ\Phi, f\circ\Phi] = b_{00}\,|f(\Phi(0, 0))|^2 + \sum_{k,l=1}^{\infty} |f'_{kl}|^2 b_{kl}.$$

We have

$$\begin{aligned} \sum_{k,l=1}^{\infty} |f'_{kl}|^2 b_{kl} &\ge K_1 \sum_{k,l=1}^{\infty} |f_{kl}|^2 kl = K_1 \|f\circ\Phi\|_{\mathscr{D}}^2 = K_1 \|f\|_{\mathscr{D}}^2 \\ &= K_1 \sum_{k,l=1}^{\infty} |f_{kl}|^2 kl \ge \frac{K_1}{K_2} \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl}. \end{aligned}$$

Hence

$$\begin{aligned} b_{00}|f(0,0)|^2 + \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} &= [f, f] \geq \delta^8 [f\circ\Phi, f\circ\Phi] \\ &= \delta^8\left(b_{00}\,|f(\Phi(0,0))|^2 + \sum_{k,l=1}^{\infty} |f'_{kl}|^2 b_{kl}\right) \\ &= \delta^8\left(b_{00}\,|f(\Phi(0,0))|^2 + \frac{K_1}{K_2}\sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl}\right), \end{aligned}$$

from which we obtain

$$b_{00}|f(0,0)|^2 + \left(1-\delta^8\frac{K_1}{K_2}\right)\sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} \geq \delta^8 b_{00}\,|f(\Phi(0,0))|^2 .$$

The left hand side of this equation is finite since it is equivalent to $\|f\|^2$. Since $\Phi(0,0)$ can be any point in $\mathbb{U}^2$, it follows that every element of $\mathscr{H}$, *i.e.*, of $\mathscr{D}(\mathbb{U}^2)$, is bounded. But a Dirichlet space contains unbounded elements. In $\mathbb{U}$, this is seen most easily by the Area Theorem; in $\mathbb{U}^2$, we take an unbounded function depending only on one variable. Therefore $b_{00} = 0$ and we are done. □

COROLLARY 2.3. *Theorem* B *is true even if* (1.3) *holds only for* $\Psi \in \mathscr{M}^*$. *Theorem* A *is true even if* (1.2) *holds only for* $\Psi \in \mathscr{M}^*$.

Proof. The proof of Theorem B uses coordinate permutations nowhere. Theorem A is a consequence of Theorem B. □

3. Boundary behavior. *Dirichlet-type spaces.* This section requires some new notions that were studied in $\mathbb{U}$ in **[3]** and **[4]**. For each $\delta = (\delta_1, \dots, \delta_n)$ with each $0 \leq \delta_j \leq 1$, we define the *Dirichlet-type spaces* $\mathscr{D}_\delta(\mathbb{U}^n)$ to consist of those $f(z) = \sum_\alpha f_\alpha z^\alpha \in \mathscr{H}(\mathbb{U}^n)$ that satisfy

$$\|f\|_{\mathscr{D}_\delta} = \sum_\alpha \alpha_1^{2\delta_1}\cdots\alpha_n^{2\delta_n}|f_\alpha|^2 < \infty.$$

This definition makes sense even if some $\delta_j = 0$ if we interpret $0^0 = 1$. In fact, if all the $\delta_j = 0$, then $\mathscr{D}_\delta(\mathbb{U}^n) = \mathscr{H}^2(\mathbb{U}^n)$. The space corresponding to $\delta_1 = \cdots = \delta_n = 1$ consists of functions f with $D_1\cdots D_n f \in \mathscr{H}^2(\mathbb{U}^n)$. When $\delta_1 = \cdots = \delta_n = 1/2$, we have the Dirichlet space. For $n = 1$, all Dirichlet-type spaces are contained in $\mathscr{H}^2(\mathbb{U})$, but this is not true if $n > 1$.

Some subclasses of $\mathscr{D}_\delta(\mathbb{U}^n)$ have certain integral representations: If $F \in \mathscr{L}^2(\mathbb{T}^n)$, for $0 \le \delta_j < 1$ and $z \in \mathbb{U}^n$, set

$$f(z) = \int_{\mathbb{T}^n} F(\zeta) \prod_{j=1}^{n} \frac{1}{(1 - z_j\bar{\zeta}_j)^{1-\delta_j}} \, d\lambda_n(\zeta). \tag{3.1}$$

The product is the Cauchy kernel each of whose factors is raised to a fractional power. Omitting j, each factor can be expanded as

$$(1 - z\bar{\zeta})^{\delta - 1} = \sum_{\alpha=0}^{\infty} b_\alpha \, z^\alpha \bar{\zeta}^\alpha$$

where

$$b_\alpha = \frac{\Gamma(1 - \delta + \alpha)}{\Gamma(1 - \delta)\,\Gamma(1 + \alpha)} \sim \frac{1}{\alpha^\delta}.$$

In particular $b_0 = 1$. Let c_α be the $(\alpha_1, \ldots, \alpha_n)$*th Fourier coefficient* of F; i.e.,

$$c_\alpha = \int_{\mathbb{T}^n} \bar{\zeta}^\alpha \, F(\zeta) \, d\lambda_n(\zeta).$$

Setting $f_\alpha = b_{\alpha_1} \cdots b_{\alpha_n} c_\alpha$, we get

$$\begin{aligned} f(z) &= \int_{\mathbb{T}^n} \prod_{j=1}^{n} \sum_{\alpha_j=0}^{\infty} b_{\alpha_j} z_j^{\alpha_j} \bar{\zeta}_j^{\alpha_j} F(\zeta) \, d\lambda_n(\zeta) \\ &= \sum_\alpha b_{\alpha_1} \cdots b_{\alpha_n} z^\alpha \int_{\mathbb{T}^n} \bar{\zeta}^\alpha \, F(\zeta) \, d\lambda_n(\zeta) \\ &= \sum_\alpha b_{\alpha_1} \cdots b_{\alpha_n} z^\alpha \, c_\alpha = \sum_\alpha f_\alpha z^\alpha. \end{aligned}$$

Now

$$\|f\|_{\mathscr{D}_\delta} = \sum_\alpha \alpha_1^{2\delta_1} \cdots \alpha_n^{2\delta_n} |b_{\alpha_1}|^2 \cdots |b_{\alpha_n}|^2 |c_\alpha|^2 \sim \sum_\alpha |c_\alpha|^2 < \infty.$$

Hence $f \in \mathscr{D}_\delta(\mathbb{U}^n)$; i.e., any f given by (3.1) is in a Dirichlet-type space.

But not all $f \in \mathscr{D}_\delta(\mathbb{U}^n)$ have integral representations as in (3.1), because a Dirichlet-type space does not control all the power series coefficients of its members. However, we can define a space $\tilde{\mathscr{D}}_\delta(\mathbb{U}^n)$ similar to $\tilde{\mathscr{D}}(\mathbb{U}^n)$ in which an integral representation is possible. Let's concentrate on the case $n = 2$ again for simplicity. With obvious notation, $f \in \tilde{\mathscr{D}}_\delta(\mathbb{U}^2)$ if and only if

$$\|f\|^2_{\tilde{\mathscr{D}}_\delta} = |f_{00}|^2 + \sum_{k=1}^{\infty} k^{2\delta_1}|f_{k0}|^2 + \sum_{l=1}^{\infty} l^{2\delta_2}|f_{0l}|^2 + \sum_{k,l=1}^{\infty} k^{2\delta_1} l^{2\delta_2} |f_{kl}|^2 < \infty. \tag{3.2}$$

Given $f(z) = \sum_{kl} f_{kl} z^k w^l \in \tilde{\mathscr{D}}_\delta(\mathbb{U}^2)$, let $c_{kl} = f_{kl}/b_k b_l$ if $(k, l) \in \mathbb{N}^2$ (recall that $b_0 = 1$), and let $c_{kl} = 0$ otherwise. Then, using (3.2),

$$\sum_{(k,l)\in\mathbb{Z}^2} |c_{kl}|^2 = |c_{00}|^2 + \sum_{k=1}^{\infty} |c_{k0}|^2 + \sum_{l=1}^{\infty} |c_{0l}|^2 + \sum_{k,l=1}^{\infty} |c_{kl}|^2 \sim \|f\|^2_{\tilde{\mathscr{D}}_\delta} < \infty.$$

Thus there is an $F \in \mathscr{L}^2(\mathbb{T}^2)$ such that $\hat{F}(k, l) = c_{kl}$. Therefore

$$\begin{aligned} f(z) &= \sum_{k,l=0}^{\infty} f_{kl} z^k w^l = \sum_{k,l=0}^{\infty} c_{kl}\, b_k b_l\, z^k w^l \\ &= \sum_{k,l=0}^{\infty} b_k b_l\, z^k w^l \int_{\mathbb{T}^2} \bar{\zeta}_1^k \bar{\zeta}_2^l F(\zeta)\, d\lambda_2(\zeta) \\ &= \int_{\mathbb{T}^2} \frac{F(\zeta)\, d\lambda_2(\zeta)}{(1-\bar{\zeta}_1 z)^{1-\delta_1}(1-\bar{\zeta}_2 w)^{1-\delta_2}}. \end{aligned}$$

Clearly F is not unique. In fact, c_{kl} can be defined arbitrarily for $(k, l) \notin \mathbb{N}^2$ as long as we retain $\sum_{(k,l)\in\mathbb{Z}^2} |c_{kl}|^2 < \infty$.

Kernels and potentials. From now on, we will also use $e^{i\theta_j}$ for $\zeta_j \in \mathbb{T}$, $e^{i\varphi_j}$ for $\eta_j \in \mathbb{T}$, and $r_j e^{i\theta_j} = r_j \zeta_j$ for $z_j \in \mathbb{U}$. The point $(1, \dots, 1) \in \mathbb{T}^n$ corresponding to $\theta_1 = \cdots = \theta_n = 0$ will act like the origin in $\mathbb{R}^n$. Now the Poisson kernel takes the more familiar form

$$P_r(\theta) = \prod_{j=1}^{n} \frac{1 - r_j^2}{1 - 2r_j \cos\theta_j + r_j^2},$$

and it is considered as a function of θ indexed by r. So the $\mathscr{L}^p$-norm of a Poisson integral will be obtained by an integration on the θ-variable and will still depend on r.

A *kernel* K is a nonnegative $\mathscr{L}^1$-function on $\mathbb{T}^n$ which is even and decreasing in each $|\theta_j|$ when the other variables are kept fixed. We will also have $K(1, \dots, 1) = \infty$ and normalize as $\|K\|_1 = 1$. A *potential* is the convolution of an $\mathscr{L}^p$-function F on $\mathbb{T}^n$ with a kernel. Thus (3.1) defines $f(z) \in \tilde{\mathscr{D}}_\delta(\mathbb{U}^n)$ as a potential. The Poisson integral is simply the convolution with the Poisson kernel.

Let's define the *Bessel kernels* on the torus. For $0 < \delta_j \leq 1$, let

$$G_\delta(\zeta) = \prod_{j=1}^{n} g_{\delta_j}(\theta_j) = \prod_{j=1}^{n}\left(1 + \frac{1}{2}\sum_{\alpha_j \neq 0} |\alpha_j|^{-\delta_j}\zeta_j^{\alpha_j}\right)$$
$$= \prod_{j=1}^{n}\left(1 + \sum_{\alpha_j=1}^{\infty} \alpha_j^{-\delta_j}\cos(\alpha_j\theta_j)\right),$$

where each g_{δ_j} is a Bessel kernel on the unit circle. $g_\delta(0) = \infty$, g_δ is a decreasing function of $|\omega|$ for $\omega \neq 0$, $g_\delta(\omega) > 0$, and $g_\delta(-\omega) = g_\delta(\omega)$. Each $g_{\delta_j} \in \mathscr{L}^1(\mathbb{T})$, so $G_\delta \in \mathscr{L}^1(\mathbb{T}^n)$, and $\|G_\delta\|_1 = \prod_{j=1}^{n} \|g_{\delta_j}\|_1 = 1$. When $0 < \delta < 1$,

$$g_\delta(\omega) \approx \left|\sin\frac{\omega}{2}\right|^{\delta-1} \quad \text{as } \omega \to 0.$$

Also $g_1(\omega) = 1 - \log|2\sin(\omega/2)|$. $P_r[g_\delta] = P_r * g_\delta$ is the harmonic extension of g_δ to $\mathbb{U}$. As $r \to 1$, it satisfies the following:

$$\|P_r * g_\delta\|_q \sim (1-r)^{\delta - 1/p} \qquad (\delta p < 1), \tag{3.3}$$

$$\|P_r * g_\delta\|_q \sim \left(\log\frac{1}{1-r}\right)^{1/q} \qquad (\delta p = 1,\ p > 1), \tag{3.4}$$

$$\|P_r * g_1\|_\infty \sim \log\frac{1}{1-r} \qquad (\delta = 1,\ p = 1).$$

$P_r[G_\delta]$ possesses these properties in each variable seperately.

On the unit circle, for $0 < \delta < 1$, the *modified* Bessel kernels are

$$\tilde{g}_\delta(\omega) = (1 - e^{i\omega})^{\delta-1} \quad \text{and} \quad \tilde{g}_1(\omega) = \log\frac{1}{1-e^{i\omega}}.$$

On the torus, let $\tilde{G}_\delta(\zeta) = \prod_{j=1}^{n} \tilde{g}_{\delta_j}(\theta_j)$. These functions are not positive, so they are not properly kernels, but they are dominated by the Bessel kernels: There are constants $C_\delta > 0$ such that $|\tilde{G}_\delta| \leq C_\delta G_\delta$. If each δ_j is less than 1,

$$P_r[\tilde{G}_\delta](\theta) = \prod_{j=1}^{n} P_{r_j}[\tilde{g}_\delta](\theta_j) = \prod_{j=1}^{n}(1 - z_j)^{\delta_j - 1},$$

with a logarithmic term if some $\delta_k = 1$. For $F \in \mathscr{L}^p(\mathbb{T}^n)$, the map that takes F to $G_\delta * F$ is one-to-one, and the Cauchy integral of $\tilde{G}_\delta * F$ is the same as its Poisson integral:

$$P_r[\tilde{G}_\delta * F](\theta) = (P_r * \tilde{G}_\delta * F)(\theta)$$
$$= \int_{\mathbb{T}^n} F(\zeta)\prod_{j=1}^{n}\frac{1}{(1 - z_j\bar{\zeta}_j)^{1-\delta_j}}\, d\lambda_n(\zeta) = f(z).$$

Thus we have obtained the integral in (3.1). From now on, F and f will always be related as in this equation. $\delta_j = 1$ does not give rise to a Cauchy-type integral; so we will not pay any attention to this case any more.

Tangential limits. Define the *tangential approach regions* to the unit circle:

$$(3.5)\qquad \begin{aligned} A_{\gamma,c}(\varphi) &= \left\{ re^{i\theta} : 1 - r > c\left|\sin\frac{\theta-\varphi}{2}\right|^{\gamma} \right\}, \\ E_{\gamma,c}(\varphi) &= \left\{ re^{i\theta} : 1 - r > \exp\left(-c\left|\sin\frac{\theta-\varphi}{2}\right|^{-\gamma}\right) \right\}. \end{aligned}$$

$A_{\gamma,c}(\varphi)$ has *(polynomial) order of contact* γ, and $E_{\gamma,c}(\varphi)$ makes *exponential contact*, with $\mathbb{T}$. A function f defined in $\mathbb{U}$ has $A_\gamma(E_\gamma)$-*limit* L at $e^{i\varphi}$ if $f(z) \to L$ as $z \to e^{i\varphi}$ within $A_{\gamma,c}(E_{\gamma,c})$ for every $c > 0$. In [3], it was shown that Poisson integrals of the modified Bessel potentials have A_γ-limits a.e. on $\mathbb{T}$ if $\delta p < 1$ for $\gamma = \frac{1}{1-\delta p}$, and E_γ-limits a.e. on $\mathbb{T}$ if $\delta p = 1$ for $\gamma = q - 1$.

Let $Q = Q(\eta, s)$ be the cube centered at $\eta \in \mathbb{T}^n$ with sides $s = (s_1, \ldots, s_n)$, where each s_j has the same order as $\max\{s_j : 1 \le j \le n\} \to 0$. Its volume is $\lambda_n(Q) = s_1 \cdots s_n$. If $F \in \mathscr{L}^p(\mathbb{T}^n)$, its *Hardy-Littlewood maximal function* is

$$(M_pF)(\eta) = \sup_{0<s_1,\ldots,s_n\le 1}\left(\frac{1}{\lambda_n(Q)}\int_Q |f|^p\,d\lambda_n\right)^{1/p}.$$

M_1 is of weak type $(1, 1)$; and since $M_pF = \left(M_1|F|^p\right)^{1/p}$, M_p is of weak type (p, p). Thus there are C_p such that

$$\lambda_n\left(\{M_pF > t\}\right) \le \frac{C_p}{t^p}\|F\|_p^p \qquad (F \in \mathscr{L}^p(\mathbb{T}^n),\ t \in (0, \infty)).$$

The proofs of the following assertions are similar to the proofs given in [3] for $n = 1$ and will be omitted. Some of them are valid in more general situations. The first result is obtained using the straightforward inequality

$$\int_{\mathbb{T}^n} |F|\,G_\delta\,d\lambda_n \le (M_1F)(1, \ldots, 1)\int_{\mathbb{T}^n} G_\delta\,d\lambda_n = (M_1F)(1, \ldots, 1),$$

which holds for any $F \in \mathscr{L}^1(\mathbb{T}^n)$, and whose proof is also in [3].

THEOREM 3.1. *There is a* $C_p < \infty$ *such that for* $F \in \mathscr{L}^p(\mathbb{T}^n)$ *and* $\zeta, \eta \in \mathbb{T}^n$,

$$\left|(G_\delta * F)(\zeta)\right| \le C_p\left[(M_pF)(\eta)\left(\prod_{j=1}^n |\zeta_j - \eta_j|^{1/p}\right)\|G_\delta\|_q + (M_1F)(\eta)\right].$$

Convolution of two kernels is a kernel; so Theorem 3.1 holds with $P_r * G_\delta = P_r[G_\delta]$ in place of G_δ, which has the desired properties $\|P_r * G_\delta\|_1 = \|P_r\|_1 \, \|G_\delta\|_1 = 1$ and $\|P_r * G_\delta\|_q = \prod_{j=1}^n \|P_{r_j} * g_{\delta_j}\|_q$. The Hölder inequality gives $M_1 F \leq M_p F$. In addition, $|\zeta_j - \eta_j|$ can be replaced by $|\theta_j - \varphi_j|$, or even by $|\sin((\theta_j - \varphi_j)/2)|$, since they are all of the same order as $\zeta \to \eta$. Lastly, we can put $\tilde{G}_\delta$ in place of G_δ on the left side of the inequality since the latter dominates the former. Hence Theorem 3.1 yields

THEOREM 3.2. *If $F \in \mathscr{L}^p(\mathbb{T}^n)$, then, using $z = re^{i\theta}$, for all $e^{i\theta}$, $e^{i\varphi} \in \mathbb{T}^n$,*

$$(3.6) \qquad |f(z)| \leq C_p \, (M_p F)(\varphi) \left[1 + \prod_{j=1}^n \left| \sin \frac{\theta_j - \varphi_j}{2} \right|^{1/p} \|P_{r_j} * g_{\delta_j}\|_q \right].$$

For given φ, any bound on the product on the right side gives a bound on $f(z)$. This leads us to the tangential approach regions to $\mathbb{T}^n$. So fix an $\eta \in \mathbb{T}^n$. As $z \to \zeta$, all $r_j \to 1$; and because of (3.3) and (3.4), $\|P_r * G_\delta\|_q \sim \prod_{j=1}^n b_j$, where $b_j = (1 - r_j)^{\delta_j - 1/p}$ or $b_j = (\log(1/(1 - r_j)))^{1/q}$ depending on whether $\delta_j p < 1$ or $\delta_j p = 1$, respectively. In other words, an approach region should be determined by

$$\prod_{j=1}^n b_j \left| \sin \frac{\theta_j - \varphi_j}{2} \right|^{1/p} < c.$$

So define $B_{\gamma,c}(\eta)$ by

$$(3.7) \qquad B_{\gamma,c}(\eta) = B_{\gamma,c}(\varphi) = \left\{ z \in \mathbb{U}^n : \prod_{j=1}^n b_j^{-1/\gamma_j} \left| \sin \frac{\theta_j - \varphi_j}{2} \right| < c \right\}.$$

Each of the factors in the above product is related to one of the regions in (3.5). In particular, points in a cartesian product of one-dimensional approach regions such as $B_{\gamma_1,c_1}(\varphi_1) \times \cdots \times B_{\gamma_n,c_n}(\varphi_n)$, where each $B_{\gamma_j,c_j}(\varphi_j)$ is either $A_{\gamma_j,c_j}(\varphi_j)$ or $E_{\gamma_j,c_j}(\varphi_j)$, satisfy the criterion for being in $B_{\gamma,c}(\eta)$. Hence an approach region can make exponential contact with $\mathbb{T}^n$ in one (complex) direction and polynomial contact in another. For $\eta \in \mathbb{T}^n$, the *maximal functions* associated to these approach regions are defined as

$$(M_{G_\delta,\gamma,c} f)(\eta) = \sup\{ |f(z)| : z \in B_{\gamma,c}(\eta) \}.$$

A function f defined in $\mathbb{U}^n$ is said to have *B_γ-limit* L at $\eta \in \mathbb{T}^n$ if $f(z) \to L$ as $z \to \eta$ within $B_{\gamma,c}(\eta)$ for every $c > 0$.

THEOREM 3.3. *If* $F \in \mathscr{L}^p(\mathbb{T}^n)$ *and* $t \in (0, \infty)$, *there is a* $C = C(p, c)$ *such that*

$$\lambda_n\big(\{M_{G_\delta, \gamma, c} f > t\}\big) < \frac{C}{t^p} \|F\|_p^p ,$$

where

$$\gamma_j = \frac{1}{1 - \delta_j p} \quad \text{if } \delta_j p < 1 \quad \text{and} \quad \gamma_j = q - 1 \quad \text{if } \delta_j p = 1. \tag{3.8}$$

This follows from the weak-type-(p, p) estimate for M_p. The weak-type estimate gives rise to a convergence theorem via classical arguments. This is the content of part (i) of Theorem C. The case $p = 1$ of part (ii) of that theorem also follows from Theorem 3.3. Henceforth, p, δ, and γ will always be related by (3.8).

Capacities. For $E \subset \mathbb{T}^n$, let $T(G_\delta, p, E)$ be the set of all nonnegative $F \in \mathscr{L}^p(\mathbb{T}^n)$ such that $(G_\delta * F)(\zeta) \geq 1$ for all $\zeta \in E$. The *p-capacity* of E is

$$\Sigma_{G_\delta}(E) = \inf\{\, \|F\|_p^p : F \in T(G_\delta, p, E)\,\}.$$

$\Sigma_{G_\delta}(E) = 0$ implies $\lambda_n(E) = 0$. The functions $G_\delta * F$ are defined Σ_{G_δ}-almost everywhere. If $F \in T(P_r * G_\delta, p, E)$, then $\Sigma_{P_r}(E) \leq \Sigma_{P_r * G_\delta}(E)$. For $\eta \in \mathbb{T}^n$ and *fixed* $\rho > 1$, let

$$\Gamma(\eta) = \{\, z \in \mathbb{U}^n : |z_j - \eta_j| < \rho\big(1 - |z_j|\big),\ 1 \leq j \leq n\,\},$$

and set $S(E) = \mathbb{U}^n \setminus \bigcup_{\eta \notin E} \Gamma(\eta)$. $\Gamma(\eta)$ is the cartesian product of n sets each of which is asymptotic, as $z_j \to \eta_j$, to an angle-shaped approach region in $\mathbb{U}$ with vertex at η_j. For $\eta \in \mathbb{T}^n$, the *nontangential maximal function* is

$$(Nf)(\eta) = \sup\{\, |f(z)| : z \in \Gamma(\eta)\}.$$

For $W \subset \mathbb{U}^n$, $J_c^\gamma(W)$ is the set of $\eta \in \mathbb{T}^n$ for which W intersects $B_{\gamma, c}(\eta)$.

LEMMA 3.4. *There exists a constant* $b = b(n) > 0$ *such that if* $F \geq 0$ *on* $\mathbb{T}^n$, $F \geq 1$ *on* $E \in \mathbb{T}^n$, *and* $z \in S(E)$, *then* $P_r[F](z) > b$.

We will use this lemma with $G_\delta * F$ in place of F. It leads to the following lower estimate for capacities.

PROPOSITION 3.5. *If $F \in \mathscr{L}^1(\mathbb{T}^n)$ and $0 < t < \infty$, then $\{M_{G_\delta,\gamma,c}f > t\}$ is contained in $J_c^\gamma(S(\{Nf > t\}))$. Thus, there is a $C = C(p, c)$ such that if $t \in (0, \infty)$ and $F \in \mathscr{L}^1(\mathbb{T}^n)$, we have*

$$\lambda_n(\{M_{G_\delta,\gamma,c}f > t\}) \leq C\,\Sigma_{G_\delta}(\{Nf > t\}).$$

THEOREM 3.6. *For $1 < p < \infty$, there is a constant $C_p < \infty$ such that if $F \in \mathscr{L}^p(\mathbb{T}^n)$ and $F \geq 0$, then*

$$\int_0^\infty \Sigma_{G_\delta}(\{G_\delta * F > t\})\,d(t^n) \leq C_p \|F\|_p^p.$$

Combining Lemma 3.4, Proposition 3.5, and Theorem 3.6 with the fact that G_δ dominates $\tilde{G}_\delta$ and that $N(G_\delta * F) = G_\delta * NF$, we obtain the strong-type estimates in part (ii) of Theorem C.

THEOREM C. *Let $1 \leq p < \infty$, $F \in \mathscr{L}^p(\mathbb{T}^n)$, $0 < \delta_j < 1$, define f as in* (3.1)*, pick γ_j as in* (3.8)*, and for $\zeta \in \mathbb{T}^n$, construct $B_{\gamma,c}(\zeta)$ as in* (3.7).

(i) *The B_γ-limit of f exists a.e. $[\lambda_n]$ on $\mathbb{T}^n$*
(ii) *There are positive constants C_p such that*

$$\|M_{G_\delta,\gamma,c}f\|_p \leq C_p\|F\|_p \qquad (1 < p < \infty),$$
$$\lambda_n(\{M_{g_\delta,\gamma,c}f > t\}) \leq \frac{C_1}{t}\|F\|_1 \qquad (p = 1,\ 0 < t < \infty).$$

If $\zeta \in \partial\mathbb{U}^n \setminus \mathbb{T}^n$, then only one component of ζ, say the nth, has $|\zeta_n| = 1$. Then the first $n-1$ factors in the product in (3.6) are bounded as $z \to \zeta$. So in this case, it suffices to apply the one-variable result in the nth variable. The approach regions are restricted only in the nth component as in (3.5), and $(z_1, \ldots, z_{n-1})$ can approach $(\zeta_1, \ldots, \zeta_{n-1}) \in \partial\mathbb{U}^n$ in any manner whatsoever. Theorem C remains valid except that in part (ii), we would use one-dimensional norm and Lebesgue measure.

When $p = 2$ and all the $\delta_j = 1/2$, this theorem takes care of $\tilde{\mathscr{D}}(\mathbb{U}^n)$, but cannot deal with $\mathscr{D}(\mathbb{U}^n)$. Thus the functions in the modified Dirichlet space have tangential ($B_{(1,\ldots,1)}$-) limits at almost every boundary point of the unit polydisc. When $n = 1$, since $\mathscr{H}^2(\mathbb{U})$ includes all Dirichlet-type spaces, elements of $\mathscr{D}(\mathbb{U})$ have nontangential limits a.e. on $\mathbb{T}$.

Now we will look at the size of the exceptional sets. From Lemma 3.4, part (ii) of Theorem C, and the first part of Proposition 3.5, we obtain

LEMMA 3.7. *If $1 < p < \infty$, then for some $C = C(p, c)$*

$$\Sigma_{P_r}(J_c^\gamma(S(E))) \leq C\,\Sigma_{P_r * G_\delta}(E).$$

Hence for $F \in \mathscr{L}^p(\mathbb{T}^n)$,

$$\Sigma_{P_r}\big(\{M_{G_\delta,\gamma,c}f > t\}\big) \leq C\,\Sigma_{P_r * G_\delta}\big(\{Nf > t\}\big).$$

THEOREM 3.8. *If $1 < p < \infty$ and $F \in \mathscr{L}^p(\mathbb{T}^n)$, then*

$$\int_0^\infty \Sigma_{P_r}\big(\{M_{G_\delta,\gamma,c}f > t\}\big)\, d(t^n) \leq C\,\|F\|_p^p,$$

and thus

$$\Sigma_{P_r}\big(\{M_{G_\delta,\gamma,c}f > t\}\big) \leq C\frac{\|F\|_p^p}{t^p}.$$

This theorem is an analog of Theorem C in the language of capacities and proved similarly.

THEOREM D. *Let $1 < p < \infty$, $F \in \mathscr{L}^p(\mathbb{T}^n)$, and f be as in* (3.1).

(i) *There is a set $E_1 \subset \mathbb{T}^n$ with $\Sigma_{P_r * G_\delta}(E_1) = 0$ such that the nontangential limit of f exists at every point of $\mathbb{T}^n \backslash E_1$.*

(ii) *There is a set $E_2 \subset \mathbb{T}^n$ with $\Sigma_{P_r}(E_2) = 0$ such that the B_γ-limit of f exists at every point of $\mathbb{T}^n \backslash E_2$.*

This result is a consequence of the basic properties of capacities and Theorem 3.10. For points on $\partial\mathbb{U}^n \backslash \mathbb{T}^n$, the one-variable result can again be used to reach a similar conclusion. Hence if $p = 2$ and all the $\delta_j = 1/2$, the points on $\partial\mathbb{U}^n$ where the modified Dirichlet space does not have nontangential limits have zero capacity in some sense.

REFERENCES

[1] J. Arazy and S. D. Fisher, *The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces*, Illinois J. Math., **29** (1985), 449–462.

[2] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, I, 2nd ed., Grundlehren Math. Wiss., vol. 115, Springer, Berlin, 1979.

[3] A. Nagel, W. Rudin and J. H. Shapiro, *Tangential boundary behavior of functions in Dirichlet-type spaces*, Ann. of Math., **116** (1982), 331–360.

[4] W. Rudin, *The variation of holomorphic functions on tangential boundary curves*, Michigan Math. J., **32** (1985), 41–45.

[5] K. Zhu, *Möbius-invariant Hilbert spaces of holomorphic functions in the unit ball of* $\mathbb{C}^n$, Trans. Amer. Math. Soc., **323** (1991), 823–842.

Received May 11, 1992. Part of this work represents a part of the author's Ph.D. thesis completed under the supervision of Walter Rudin.The author wishes to thank Walter Rudin and Frank Forelli for several useful discussions. Supported under TBAG-ÇG2 by TÜB.ITAK.

MATHEMATICS DEPARTMENT
UNIVERSITY OF WISCONSIN
MADISON, WI 53706

Current address: Mathematics Department
Middle East Technical University
Ankara 06531 Turkey
E-mail address: mathtk@vm.cc.metu.edu.tr

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

THE COHOMOLOGY RING OF THE SPACES OF LOOPS ON LIE GROUPS AND HOMOGENEOUS SPACES

KATSUHIKO KURIBAYASHI

Let X be a simply connected space whose $\bmod p$ cohomology is isomorphic to that of a compact, simply connected, simple Lie group as an algebra over the Steenrod algebra. We determine the algebra structure of the $\bmod p$ cohomology of ΩX algebraically. Moreover we give a method to determine the algebra structure of the $\bmod p$ cohomology of the space of loops on a homogeneous space.

0. Introduction. Let G be a compact simply connected Lie group and ΩX the space of loops on a space X. In **[4]**, R. Bott has given a method to obtain generators of the Pontryagin ring $H_*(\Omega G)$ and has determined its Hopf algebra structure explicitly for $G = \mathrm{SU}(m)$, $\mathrm{Spin}(m)$ and G_2. By applying this method, T. Watanabe **[23]** has determined the Hopf algebra structure of $H_*(\Omega F_4)$. A. Kono and K. Kozima **[8]** have determined the Hopf algebra structure over the Steenrod algebra $\mathscr{A}(2)$ of $H_*(\Omega G; \mathbb{Z}/2)$ for $G = F_4, E_6, E_7$ and E_8, without using Bott's method. In order to determine the algebra structure, they have made use of the Eilenberg-Moore spectral sequence **[16]** which converges to $H^*(G; \mathbb{Z}/2)$ and whose E_2-term is isomorphic to $\mathrm{Ext}^{**}_{H_*(\Omega G;\mathbb{Z}/2)}(\mathbb{Z}/2, \mathbb{Z}/2)$. Moreover a homotopy fiber of $\Omega x_4\colon \Omega BG \to \Omega K(\mathbb{Z}, 4)$ has been used to examine the coalgebra structure, where $x_4\colon BG \to K(\mathbb{Z}, 4)$ is a map representing the generator of $H^4(BG)$. The consideration of the dual of those results (**[4]**, **[8]**, **[23]**) enables us to determine the Hopf algebra structure of the $\bmod p$ cohomology of ΩG for the Lie groups G. On the other hand, we can decide the coalgebra structure of $H^*(\Omega G; \mathbb{Z}/p)$ algebraically from the algebra $H^*(G; \mathbb{Z}/p)$ over the Steenrod algebra $\mathscr{A}(p)$. The following result is due to R. M. Kane **[5]**.

THEOREM 0.1. *Suppose that X is a simply connected H-space and* (0.1): *there exists a compact, simply connected, simple Lie group G such that $H^*(X; \mathbb{Z}/p) \cong H^*(G; \mathbb{Z}/p)$ as an algebra over the $\bmod p$ Steenrod algebra $\mathscr{A}(p)$.* (*We do not require the existence of any map between X and G which induces the isomorphism.*)

Then $H^*(\Omega X; \mathbb{Z}/p) \cong H^*(\Omega G; \mathbb{Z}/p)$ as a coalgebra.

This result motivates the conjecture that $H^*(\Omega X; \mathbb{Z}/p)$ is isomorphic, as an algebra, to $H^*(\Omega G; \mathbb{Z}/p)$ under the condition in Theorem 0.1. In this paper, we will show

THEOREM 0.2. *If X is a simply connected space and satisfies* (0.1), *then $H^*(\Omega X; \mathbb{Z}/p) \cong H^*(\Omega G; \mathbb{Z}/p)$ as an algebra.*

(*Note X is merely a simply connected space. We do not assume that it space is an H-space.*)

Theorem 0.2 is obtained as a consequence of algebraic calculation of the algebras $H^*(\Omega G; \mathbb{Z}/p)$. In particular, when $H_*(G)$ is p-torsion free, the algebra structure of $H^*(\Omega G; \mathbb{Z}/p)$ is determined by virtue of Proposition 1.6, which asserts that algebraic calculation of $H^*(\Omega X; \mathbb{Z}/p)$ is possible when $H^*(X; \mathbb{Z}/p)$ is an exterior algebra. In order to calculate the algebra $H^*(\Omega G; \mathbb{Z}/p)$, we make use of the Steenrod operations in the Eilenberg-Moore spectral sequence ([**15**], [**20**]) and [**10**, Theorem 2.3], which is an answer to extension problems in spectral sequences.

In the latter half of this paper, we examine the algebra structure of the cohomology rings of spaces of loops on homogeneous spaces. In [**19**], L. Smith has shown the following.

THEOREM ([**19**; Theorem P2]). *Let G be a compact simply connected Lie group, U a closed connected subgroup of G and $i\colon U \hookrightarrow G$ the inclusion map. Consider $H^*(U; \mathbb{Z}/p)$ as an $H^*(G; \mathbb{Z}/p)$ module via the map $i^*\colon H^*(G; \mathbb{Z}/p) \to H^*(U; \mathbb{Z}/p)$. Then if $H^*(G; \mathbb{Z}/p)$ is an exterior algebra on odd dimensional generators, there is a filtration $\{F^{-n}H^*(\Omega(G/U); \mathbb{Z}/p); n \geq 0\}$ such that $E_0^{**}(H^*(\Omega(G/U); \mathbb{Z}/p)) \cong \mathrm{Tor}^{**}_{H^*(G;\mathbb{Z}/p)}(\mathbb{Z}/p, H^*(U; \mathbb{Z}/p))$ as a Hopf algebra.*

From this theorem and [**10**; Theorem 2.4], we will obtain a proposition (Proposition 1.10) on the algebra structure of $H^*(\Omega(G/U); \mathbb{Z}/p)$. By applying our proposition, the mod p cohomology rings of

$$\Omega(\mathrm{SU}(m+n)/\mathrm{SU}(n)), \qquad \Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(n)),$$
$$\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(m) \times \mathrm{Sp}(n))$$

can be computed. But if G is not simply connected or $H^*(G; \mathbb{Z}/p)$ is not an exterior algebra, it is not easy to calculate the cohomology ring of $\Omega(G/U)$ in general. In order to determine the algebra structure of

$$H^*(\Omega(U(m+n)/U(m) \times U(n)); \mathbb{Z}/p),$$
$$H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p), \quad H^*(\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)); \mathbb{Z}/2),$$

we cannot apply Proposition 1.10 because $U(m+n)$ and $\mathrm{SO}(m+n)$ are not simply connected and $H^*(E_8; \mathbb{Z}/2)$ is not an exterior algebra. In the concrete, we will attempt to compute the mod p cohomology rings of

$$\Omega(\mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n)), \qquad \Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n))$$

and the mod 2 cohomology ring of

$$\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)).$$

This paper is organized as follows. In §1, we state our results. In §2, we prove them by using results of **[1]**, **[2]**, **[3]**, **[7]**, **[14]** and **[22]**.

1. Results. In this paper, we may denote p^f by $p[f]$ for any prime number p. $\mathbb{K}_p$ means a field of characteristic p. In this section, for algebras A and B, $A \cong B$ means that A is isomorphic to B as an algebra.

Let G be an exceptional Lie group. When $H^*(G)$ has p-torsion, the algebra structure of the mod p cohomology of the space of loops on the exceptional Lie group G is determined by considering the Eilenberg-Moore spectral sequence converging to $H^*(\Omega G; \mathbb{Z}/p)$.

THEOREM 1.1.

(1)
$$H^*(\Omega G_2; \mathbb{Z}/2) \cong \mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^4) \otimes \Gamma[w_{10}, y_8],$$
$$\deg s^{-1}x_3 = 2, \quad \deg y_8 = 8, \quad \deg w_{10} = 10.$$

(2)
$$H^*(\Omega F_4; \mathbb{Z}/2) \cong \mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^4) \otimes \Gamma[w_{10}, y_8, s^{-1}x_{15}, s^{-1}x_{23}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg y_8 = 8, \quad \deg w_{10} = 10.$$

(3)
$$H^*(\Omega E_6; \mathbb{Z}/2) \cong \mathbb{Z}/2[s^{-1}x_8]/(s^{-1}x_8^{16}) \otimes \{\otimes_{f\geq 1}\mathbb{Z}/2[e_f]/(e_f^8)\} \otimes \Gamma[w_{10}, s^{-1}x_{15}, s^{-1}x_{23}]$$
$$\deg s^{-1}x_i = i-1, \quad \deg w_{10} = 10, \quad \deg e_f = 2^{f+2}.$$

(4)
$$H^*(\Omega E_7; \mathbb{Z}/2) \cong \mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^{16}) \otimes \Gamma[w_{10}, w_{18}, w_{34}, y_{32}, s^{-1}x_{15}, s^{-1}x_{23}, s^{-1}x_{27}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg w_i = i, \quad \deg y_{32} = 32.$$

(5) $H^*(\Omega E_8;\mathbb{Z}/2)$
$$\cong \mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^{16})\otimes \mathbb{Z}/2[s^{-1}x_{15}]/(s^{-1}x_{15}^4)$$
$$\otimes\,\Gamma[w_{46}, w_{38}, w_{34}, w_{58}, y_{32}, y_{56}, s^{-1}x_{23}, s^{-1}x_{27}],$$
$$\deg s^{-1}x_i = i-1,\quad \deg w_i = i,\quad \deg y_i = i.$$

THEOREM 1.2.

(1) $$H^*(\Omega F_4;\mathbb{Z}/3)\cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^9)$$
$$\otimes\,\Gamma[w_{22}, y_{18}, s^{-1}x_{11}, s^{-1}x_{15}],$$
$$\deg s^{-1}x_i = i-1,\quad \deg y_{18}=18,\quad \deg w_{22}=22.$$

(2) $H^*(\Omega E_6;\mathbb{Z}/3)\cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^9)$
$$\otimes\,\Gamma[w_{22}, y_{18}, s^{-1}x_9, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{17}],$$
$$\deg s^{-1}x_i = i-1,\quad \deg y_{18}=18,\quad \deg w_{22}=22.$$

(3) $H^*(\Omega E_7;\mathbb{Z}/3)\cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^{27})\otimes\{\otimes_{f\geq 1}\mathbb{Z}/3[e_f]/(e_f^9)\}$
$$\otimes\,\Gamma[w_{22}, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{27}, s^{-1}x_{35}],$$
$$\deg s^{-1}x_i = i-1,\quad \deg w_{22}=22,\quad \deg e_f = 6\cdot 3^f.$$

(4) $H^*(\Omega E_3;\mathbb{Z}/3)$
$$\cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^{27})$$
$$\otimes\,\Gamma[w_{22}, w_{58}, y_{54}, s^{-1}x_{15}, s^{-1}x_{27}, s^{-1}x_{35}, s^{-1}x_{39}, s^{-1}x_{47}],$$
$$\deg s^{-1}x_i = i-1,\quad \deg y_{54}=54,\quad \deg w_i = i.$$

THEOREM 1.3.
$H^*(\Omega E_8;\mathbb{Z}/5)$
$$\cong \mathbb{Z}/5[s^{-1}x_3]/(s^{-1}x_3^{25})$$
$$\otimes\,\Gamma[w_{58}, y_{50}, s^{-1}x_{15}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{35}, s^{-1}x_{39}, s^{-1}x_{47}],$$
$$\deg s^{-1}x_i = i-1,\quad \deg y_{50}=50,\quad \deg w_{58}=58.$$

Before we state the algebra structure of the mod p cohomology of the space of loops on G whose integral cohomology has no p-torsion, let us define some notation.

NOTATION 1.4. Let k be a non-negative integer, p a prime number and $\wp^i$, Sq^i the Steenrod operations. Put $P(k, m)=\wp^{p^{k-1}\cdot m}\cdots\wp^m$ where $k>0$, $\wp^i = Sq^{2i}$ if $p=2$, and $P(0, m)=\mathrm{id}$.

The following lemma will be needed to study the Steenrod operations in the Eilenberg-Moore spectral sequence.

LEMMA 1.5. *Let H^* be a Hopf algebra over $\mathscr{A}(p)$. Suppose that H^* is isomorphic, as an algebra, to an exterior algebra on odd dimensional generators. Then we can choose generators x_i which satisfy the following properties.*

(1.1) $H^* \cong \Lambda(x_1, \dots, x_S)$, *where* $\deg x_i = 2m(i) + 1$.

$P(k, m(i))x_i = \varepsilon x_j$ *for any* $k \geq 0$ *and* i, *where* $\varepsilon = 0$ *or* 1, $\deg x_j = 2m(i)p^k + 1$ *and* $x_j = 0$ *if* $(QH^*)^{2m(i)p^k+1} = 0$.

Also, for any i *and* j $(i \neq j)$, *if* $P(k, m(i))x_i = P(k', m(i))x_j$, *then* $P(k, m(i))x_i = P(k', m(j))x_j = 0$.

In Proposition 1.6, we treat a space X which satisfies the following:

(A) X is a simply connected space and

$$H^*(X; \mathbb{K}_p) \cong \Lambda(x_{2\cdot m(1)+1}, \dots, x_{2\cdot m(s)+1}),$$

where $\deg x_{2m(i)+1} = 2m(i) + 1$ and $m(1) \leq \cdots \leq m(s)$.

(B) When $\mathbb{K}_p = \mathbb{Z}/p$, $H^*(X; \mathbb{Z}/p)$ has a Hopf algebra structure over $\mathscr{A}(p)$. Moreover if we choose generators $x_{2m(i)+1}$ satisfying (1.1), then one of the conditions (1.2) or (1.3) is satisfied for any $i \in J$, where $J = \{i | x_{2\cdot m(i)+1} \neq P(k, m(j))x_{2\cdot m(j)+1}$ for any $k > 0$ and $j\}$.

(1.2): $m(j) \cdot p[f] \neq m(i)$ for any $j \in J$ and $f \geq 1$.

(1.3): If there exist $j \in J$ and $f \geq 1$ such that $m(j) \cdot p[f] = m(i)$, then $f \leq k(j)$, where $k(j) = \min\{k | P(k, m(j))x_{2\cdot m(j)=1} = 0\}$. If $m(i) \cdot p[k(i) + f] = m(j) \cdot p[t]$ for some $j \in J$, $t < k(j)$ and $f \geq 1$, then $k(i) \geq k(j)$.

Let $\{F^n\}_{n\leq 0}$ be the decreasing filtration of $\Gamma = H^*(\Omega X; \mathbb{Z}/p)$ which is obtained from the Eilenberg-Moore spectral sequence converging to Γ. Roughly speaking, the condition (1.2) or (1.3) is sufficient for deciding whether, for any algebra generator x of Γ belonging to F^n, x^p and the algebra generators of Γ belonging to F^{n+1} are independent.

PROPOSITION 1.6. (1) *If* $p = 0$ *and* X *satisfies the condition* (A), *then*

$$H^*(\Omega X; \mathbb{K}_0) \cong \mathbb{K}_0[s^{-1}x_{2\cdot m(1)+1}, \dots, s^{-1}x_{2\cdot m(s)+1}],$$

where $\deg s^{-1}x_{2\cdot m(i)+1} = 2 \cdot m(i)$.

(2) *Suppose that* $\mathbb{K}_p$ *is a perfect field whose characteristic is non-zero,* X *satisfies the condition* (A), *and that* $m(1) \cdot p > m(s)$. *Then*

$$H^*(\Omega X; \mathbb{K}_p) \cong \Gamma[s^{-1}x_{2\cdot m(1)+1}, \dots, s^{-1}x_{2\cdot m(s)+1}],$$

where $\deg s^{-1}x_{2\cdot m(i)+1} = 2 \cdot m(i)$.

(3) *If* X *satisfies the conditions* (A) *and* (B), *then*

$$H^*(\Omega X; \mathbb{Z}/p) \cong \bigotimes_{i\in J} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2\cdot m(i)+1})]/(\gamma_{p[f]}(s^{-1}x_{2\cdot m(i)+1})^{p[k(i)]}) \right\},$$

where $\deg \gamma_{p[f]}(s^{-1}x_{2\cdot m(i)+1}) = 2\cdot m(i)\cdot p[f]$, *and* $\gamma_1(s^{-1}x_{2\cdot m(i)+1}) = s^{-1}x_{2\cdot m(i)+1}$. *Throughout Proposition* 1.6, $s^{-1}x_t$ *transgresses to* x_t.

By making use of Proposition 1.6, we can determine the algebra structure of the mod p cohomology of ΩG, where G is a compact, simply connected, simple Lie group whose integral cohomology has no p-torsion.

In Proposition 1.6, if X is a simply connected Lie group G whose type is $(2n(1)+1, \ldots, 2n(t)+1)$, then $S^{2n(1)+1}\times\cdots\times S^{2n(t)+1}$ is mod 0 equivalent to G. Therefore, Proposition 1.6 (1) holds clearly in this case. Since $S^3\times S^5\times\cdots\times S^{2n-1}\simeq_p \mathrm{SU}(n)$ (mod p-equivalence) if $p\geq n$, and $S^3\times S^7\times\cdots\times S^{4n-1}\simeq_p \mathrm{Sp}(n)$ if $p\geq 2n$, Proposition 1.6 (2) holds clearly in the cases where $X=\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$.

REMARK. In the assumption of Theorem 0.2, if the condition "G is simple" is omitted, then we cannot deduce the assertion of Theorem 0.2 by applying Proposition 1.6. In fact, the condition (1.3) does not hold in general for cohomology of semi-simple Lie groups. For example, let us consider the mod 3 cohomology ring

$$H^*(\mathrm{SU}(2)\times \mathrm{Spin}(20); \mathbb{Z}/3) \cong \Lambda(x_3)\otimes\Lambda(e_3, e_7, e_{11}, \ldots, e_{23}, e_{27}, e_{31}, e_{35})\otimes\Lambda(y_{19}).$$

If we take notice of the elements x_3 and y_{19}, then it follows that condition (1.3) is not satisfied because $m(j)=1$ and $m(i)=9$, that is $1\cdot 3^2=9$ and $f=2>1=k(j)$. This means that we cannot determine, by using our method, the mod 3 cohomology ring of the space of loops on a simply connected space X whose mod 3 cohomology is isomorphic to $H^*(\mathrm{SU}(2)\times\mathrm{Spin}(20);\mathbb{Z}/3)$.

Applying Proposition 1.6 (3), we have

THEOREM 1.7.

(1) $H^*(\Omega G_2; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{11}] & \text{if } p=3 \text{ or } p>5, \\ \bigotimes_{f\geq 0}\mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{25}) & \text{if } p=5. \end{cases}$$

(2) $H^*(\Omega F_4; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{23}] & \textit{if } p > 11, \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} & \\ \qquad \otimes\Gamma[s^{-1}x_{11}(5), s^{-1}x_{15}(7), s^{-1}x_{23}(11)] & \\ \qquad\qquad \textit{if } p = 5, 7 \textit{ or } 11. & \end{cases}$$

(3) $H^*(\Omega E_6; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_9, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{17}, s^{-1}x_{23}] & \\ \qquad \textit{if } p > 11, & \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} & \\ \otimes\Gamma[s^{-1}x_9, s^{-1}x_{11}(5), s^{-1}x_{15}(7), s^{-1}x_{17}, s^{-1}x_{23}(11)] & \\ \qquad \textit{if } p = 5, 7 \textit{ or } 11. & \end{cases}$$

(4)

$H^*(\Omega E_7; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{19}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{35}] & \\ \qquad \textit{if } p > 17, & \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} & \\ \otimes\Gamma[s^{-1}x_{11}(5), s^{-1}x_{15}(7), s^{-1}x_{19}, s^{-1}x_{23}(11), s^{-1}x_{27}(13), & \\ \qquad s^{-1}x_{35}(17)] & \\ \qquad \textit{if } p = 5, 7, 11, 13 \textit{ or } 17. & \end{cases}$$

(5) $H^*(\Omega E_8; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{15}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{35}, s^{-1}x_{39}, & \\ \qquad s^{-1}x_{47}, s^{-1}x_{59}] \quad \textit{if } p > 29, & \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} & \\ \otimes\Gamma[s^{-1}x_{15}(7), s^{-1}x_{23}(11), s^{-1}x_{27}(13), & \\ \quad s^{-1}x_{35}(17), s^{-1}x_{39}(19), s^{-1}x_{47}(23), s^{-1}x_{59}(29)] & \\ \qquad \textit{if } p = 5, 7, 11, 13, 17, 19, 23 \textit{ and } 29. & \end{cases}$$

Throughout Theorem 1.7, $\deg s^{-1}x_i = i-1$, $\deg s^{-1}x_i(q) = i-1$, *and* $s^{-1}x_i(q)$ *is removed from the divided polynomial algebra if* $p = q$. *Moreover* $s^{-1}x_i$ $(s^{-1}x_i(q))$ *transgresses to* x_i, *which is a suitable free algebra generator of* $H^*(G; \mathbb{Z}/p)$.

Before we state results about the cohomology rings of spaces of loops on classical groups, let us define the following

NOTATION 1.8. Let T be a set consisting of some natural numbers. Put $M(T, p) = \{n \in T | n \neq mp^f$ for any $m \in T$ and $f \geq 1\}$ and $t(m, k) = \min\{t | 2mp^t + 1 > k\}$ for $m \in M(T, p)$.

THEOREM 1.9.

(1)

$$H^*(\Omega \operatorname{Spin}(2n+1); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,4n-1)]}) \right\}$$

where $T = \{1, 3, \ldots, 2n-1\}$ *and* $p \neq 2$.

(2)

$$H^*(\Omega \operatorname{Spin}(2n); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,4n-5)]}) \right\} \otimes \Gamma[s^{-1}e'_{2n-1}]$$

where $T = \{1, 3, \ldots, 2n-3\}$ *and* $p \neq 2$.

(3)

$$H^*(\Omega \operatorname{SU}(n); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,2n-1)]}) \right\}$$

where $T = \{1, 2, \ldots, n-1\}$.

(4)

$$H^*(\Omega \operatorname{Sp}(n); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,4n-1)]}) \right\}$$

where $T = \{1, 3, \ldots, 2n-1\}$ *and* $p \neq 2$.

(5) $$H^*(\Omega \operatorname{Sp}(n); \mathbb{Z}/2) \cong \Gamma[s^{-1}x_3, s^{-1}x_7, \ldots, s^{-1}x_{4n-1}].$$

Throughout Theorem 1.9, *the free algebra generator* $s^{-1}e_i$ (*resp.* $s^{-1}e'_i$, $s^{-1}x_i$ *and* $s^{-1}x'_i$) *transgresses to an appropriate free algebra generator*

e_i *(resp.* e'_i, x_i *and* x'_i) *of* $H^*(G; \mathbb{Z}/p)$ *(see the proof of Theorem* 1.9 *in* §2).

Let G be a simply connected Lie group whose $\bmod p$ cohomology is an exterior algebra on odd dimensional generators, U a closed connected subgroup of G, and $i\colon U \hookrightarrow G$ the inclusion map. By [**13**; 7.20 Theorem (Samelson-Leray)], we see that the sub-Hopf algebra $H^*(G; \mathbb{Z}/p)\backslash\backslash i^*$ ($=$ sub-ker i^*; see [**18**; Notation, p. 312]) of $H^*(G; \mathbb{Z}/p)$ is an exterior algebra on odd dimensional generators. Moreover, from the method of construction of $H^*(G; \mathbb{Z}/p)\backslash\backslash i^*$ (see [**18**; Proposition 1.4]), we see that $H^*(G; \mathbb{Z}/p)\backslash\backslash i^*$ is a sub-Hopf algebra of $H^*(G; \mathbb{Z}/p)$ over $\mathscr{A}(p)$. Under the above conditions and notations, the following proposition holds.

PROPOSITION 1.10. *Suppose that the condition* (1.2) *or* (1.3) *is satisfied in the algebra*

$$H^*(G; \mathbb{Z}/p)\backslash\backslash i^* \cong \Lambda(x_{2m(1)+1}, \ldots, x_{2m(s)+1}),$$

where $x_{2m(i)+1}$ *are algebra generators satisfying* (1.1), *and that*

$$Q(H^*(U; \mathbb{Z}/p)//i^*)^{2m(i)\cdot p[k(i)+f]} = 0$$

for any $i \in J$ *and* $f \geq 0$. *Then*

$$\begin{aligned} &H^*(\Omega(G/U); \mathbb{Z}/p) \\ &\quad \cong \bigotimes_{i\in J} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2m(i)+1})]/(\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^{p[k(i)]}) \right\} \\ &\qquad \otimes H^*(U; \mathbb{Z}/p)//i^* \end{aligned}$$

as an algebra.

Applying Proposition 1.10, we have the following:

THEOREM 1.11.

(1)

$$\begin{aligned} &H^*(\Omega(\mathrm{SU}(m+n)/\mathrm{SU}(n)); \mathbb{Z}/p) \\ &\quad \cong \bigotimes_{s\in M(T,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-1)]}) \right\} \end{aligned}$$

where $T = \{n, n+1, \ldots, m+n-1\}$.

(2)

$$H^*(\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(n));\mathbb{Z}/p) \cong \bigotimes_{s\in M(T,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,4m+4n-1)]}) \right\}$$

where $T=\{2n+1, 2n+3, \dots, 2m+2n-1\}$ *and* $p\neq 2$.

(3)

$$H^*(\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(n));\mathbb{Z}/2) \cong \Gamma[s^{-1}x_{4n+3}, s^{-1}x_{4n+7}, \dots, s^{-1}x_{4m+4n-1}].$$

THEOREM 1.12.

$$H^*(\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(m)\times\mathrm{Sp}(n));\mathbb{Z}/p) \cong \Gamma[s^{-1}x_{4m+3}, s^{-1}x_{4m+7}, \dots, s^{-1}x_{4m+4n-1}] \otimes \Lambda(x'_3, x'_7, \dots, x'_{4n-1})$$

where $\deg s^{-1}x_j = j-1$, $\deg x'_i = i$ *and* $m\geq n$.

The following theorems are obtained by computing in the concrete.

THEOREM 1.13.

$$H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n));\mathbb{Z}/p) \cong \begin{cases} \bigotimes_{s\in M(T_1,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-3)]}) \right\} & \text{if } n \text{ is odd, } m \text{ is even,} \\ \bigotimes_{s\in M(T_2,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-5)]}) \right\} \\ \otimes\Gamma[s^{-1}e'_{m+n-1}] \quad \text{if } n \text{ and } m \text{ are odd,} \\ \bigotimes_{s\in M(T_3,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-3)]}) \right\} \\ \otimes\Lambda(x_{n-1}) \quad \text{if } n \text{ is even, } m \text{ is odd,} \\ \bigotimes_{s\in M(T_4,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-5)]}) \right\} \\ \otimes\Gamma[s^{-1}e'_{m+n-1}]\otimes\Lambda(x_{n-1}) \quad \text{if } n \text{ and } m \text{ are even,} \end{cases}$$

where $p\neq 2$, $T_1=\{n, n+2, \dots, m+n-2\}$, $T_2=\{n, n+2, \dots, m+n-3\}$, $T_3=\{n-1, n+1, \dots, m+n-2\}$ *and* $T_4=\{n-1, n+1, \dots, m+n-3\}$.

THEOREM 1.14. *When* $n \geq 2$,

$$
\begin{aligned}
&H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/2) \\
&\quad \cong \left\{ \bigotimes_{j \in L,\, s_j > 1} \Gamma[w_j] \right\} \\
&\quad \otimes \Lambda \otimes \left\{ \bigotimes_{m(j) \in M(T,2)} \mathbb{Z}/2[s^{-1}x_j]/(s^{-1}x_j^{2^{t(m(j),\, m+n-1)}}) \right\} \\
&\quad \otimes \left\{ \bigotimes_{j \in L'} \Gamma[w_j] \otimes \Lambda(s^{-1}x_j) \right\}
\end{aligned}
$$

where

$$
L = \{j \mid j = 2m(j)+1,\ n \leq j \leq m+n-1\}, \qquad T = \{m(j) \mid j \in L\},
$$
$$
L' = \{j \mid j = 2m(j),\ n \leq j < \min(2n, m+n)\},
$$
$$
\Lambda = \begin{cases} \displaystyle\bigotimes_{j \in L,\, s_j = 1} \Gamma[y_{4m(j)}] \quad \text{if } m+n-2 \not\equiv 0 \bmod 4 \text{ or } n > m, \\ \displaystyle\bigotimes_{j \in L,\, s_j = 1} \Gamma[y_{4m(j)}] \\ \displaystyle\otimes \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/2[\gamma_{2[f]}(y_{m+n-2})]/(\gamma_{2[f]}(y_{m+n-2})^4) \right\} \\ m(j) \neq (m+n-2)/4,\ (m+n-2)/2 \\ \qquad\qquad\qquad \text{if } m+n-2 \equiv \bmod 4 \text{ and } n \leq m, \end{cases}
$$

$j \cdot 2[s_j - 1] < m+n \leq j \cdot 2[s_j]$, $\deg w_j = j \cdot 2[s_j] - 2$, $\deg s^{-1}x_i = i-1$, $\deg y_t = t$.

THEOREM 1.15.

$$
\begin{aligned}
&H^*(\Omega(\mathrm{U}(m+n)/\mathrm{U}(m) \times \mathrm{U}(n)); \mathbb{Z}/p) \\
&\quad \cong \Gamma[\tau\rho_1, \tau\rho_2, \ldots, \tau\rho_n] \otimes \Lambda(s^{-1}c_1, s^{-1}c_2, \ldots, s^{-1}c_n),
\end{aligned}
$$

where $\deg \tau\rho_i = 2m + 2i - 2$, $\deg s^{-1}c_i = 2i - 1$, $m \geq n$.

THEOREM 1.16.

$$
\begin{aligned}
&H^*(\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)); \mathbb{Z}/2) \\
&\quad \cong \Lambda(e'_7, e'_{11}, e'_{13}, u_5, u_9, u_{17}, u_{29}) \\
&\quad\quad \otimes \Gamma[w_{38}, w_{34}, w_{46}, w_{58}, v_{22}, v_{26}, v_{28}]
\end{aligned}
$$

where $\deg e'_i = i$, $\deg u_j = j$, $\deg w_l = l$, $\deg v_m = m$.

Furthermore, $j^*(e_i) = e'_i$ *if* $i = 7, 11$ *or* 13, *and* $j^*(e_i) = 0$ *if* $i = 3, 5, 9, 15$, *or* 17, *where* j *is the inclusion map in the fibration*

$$\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)) \overset{j}{\hookrightarrow} \mathrm{SU}(9)/\mathbb{Z}/3 \to E_8$$

and

$$e_i \in H^*(\mathrm{SU}(9)/\mathbb{Z}/3; \mathbb{Z}/2) \cong \Lambda(e_3, e_5, \dots, e_{17}).$$

2. Proofs. In this section, we will prove all the results stated in §1.

Proof of Theorems 1.1, 1.2 *and* 1.3. Let (G, p) be one of pairs of the exceptional Lie group and the prime number in Theorem 1.1, 1.2 or 1.3. For an appropriate algebra generator x_i of $H^*(G; \mathbb{Z}/p)$, choose a continuous map f to the Eilenberg-MacLane space $K(\mathbb{Z}/p, \deg x_i)$ from G representing the generator x_i. We can compare the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ converging to $H^*(\Omega G; \mathbb{Z}/p)$ with the spectral sequence converging to $H^*(K(\mathbb{Z}/p, \deg x_i - 1); \mathbb{Z}/p)$ by using the morphism of spectral sequences which is induced by the map $f\colon G \to K(\mathbb{Z}/p, \deg x_i)$. By applying [**18**; Lemma 3.9], all differentials d_r are determined. This enables us to obtain the explicit form of E_∞^{**}. We have Theorems 1.1, 1.2 and 1.3 by virtue of [**10**; Theorem 2.4]. (Cf. the proof of Lemma 2.2.)

In order to prove Lemma 1.5, we will prepare a lemma.

NOTATION. Put $U = \{u | u \not\equiv 0 \bmod p\}$. For any $u \in U$, let $i(u)$ be the least integer i which satisfies $(QH^*)^{2up^i+1} \neq 0$.

LEMMA 2.1. *For any* $u \in U$, *put* $m = up^{i(u)}$. *Under the assumptions of Lemma* 1.5, *for any* l, *we can choose a basis* $\{x_1, \dots, x_v\}$ *for* $\bigotimes_{0\le t\le l+1}(QH^*)^{2mp^t+1}$ *so as to satisfy the following conditions.*

(i) $x_1, \dots, x_v$ *are primitive.*

(ii) *If* $\deg P(k, m(i))x_i \le 2mp^{l+1} + 1$ $(\deg x_i = 2m(i) + 1)$, *then* $P(k, m(i))x_i = \varepsilon x_j$, *where* $\varepsilon = 1$ *or* 0, $\deg x_j = 2m(i)p^k + 1$ *and* $x_j = 0$ *if* $(QH^*)^{2m(i)p^k+1} = 0$.

(iii) *For any* i *and* j $(i \neq j)$, *if*

$$\deg P(k, m(i))x_i = \deg P(k', m(j))x_j \le 2mp^{l+1} + 1$$

and

$$P(k, m(i))x_i = P(k', m(j))x_j,$$

then

$$P(k, m(i))x_i = P(k', m(j))x_j = 0.$$

Proof. All basis elements x_j can be replaced by primitive elements modulo decomposables, by the Samelson-Leray theorem and associativity of homology. Let us prove this lemma by induction on dimensions. Suppose that Lemma 2.1 holds up to an integer l, that is, we can choose a basis

$$M = \{P(t_i, mp^{l(i)})x_i\}_{i \in J, 0 \le t_i \le s(i)}$$

for $\bigoplus_{0 \le t \le l+1}(QH^*)^{2mp^t+1}$ so that x_i is primitive, where $\deg x_i = 2mp^{l(i)} + 1$ and $s(i)$ is the lesser of $l + 1 - l(i)$ and the integer t satisfying

$$P(t+1, mp^{l(i)})x_i = 0 \quad \text{and} \quad P(t, mp^{l(i)})x_i \neq 0.$$

We can see that basis elements of $(QH^*)^{2mp^{l+1}+1}$ can be uniquely expressed as $P(l+1-l(j), mp^{l(j)})x_j$. Let S be a subset

$$\{\wp^{mp^{l+1}} \cdot P(l+1-l(j), mp^{l(j)})x_j\}$$

of $(QH^*)^{2mp^{l+2}+1}$ which is obtained from the basis

$$\{P(l+1-l(j), mp^{l(j)})x_j\}.$$

Choose a maximal subset S' consisting of linearly independent elements of S. The subset S' is written as

$$\{\wp^{mp^{l+1}} \cdot P(l+1-l(j_i), mp^{l(j_i)})x_{j_i}\}_{1 \le i \le N}.$$

If there exists an integer $j \in J - \{j_1, \dots, j_N\}$ such that

$$\wp^{mp^{l+1}} \cdot P(l+1-l(j), mp^{l(j)})x_j \neq 0,$$

then, from the maximality of S', we have that

$$\begin{aligned} &\wp^{mp^{l+1}} \cdot P(l+1-l(j), mp^{l(j)})x_j \\ &\quad = \sum_{1 \le i \le N} (-\lambda_i)\wp^{mp^{l+1}} \cdot P(l+1-l(j_i) \cdot mp^{l(j_i)})x_{j_i}, \end{aligned}$$

where the coefficients λ_i are not all zero. Choose an element x_{j_t} of maximal degree from the elements x_{j_i} $(1 \le i \le N)$ such that $\lambda_i \neq 0$. Put $y_{j_t} = x_{j_t} + \sum_{0 \le i \le N, i \neq t} \lambda'_i P(l(j_t) - l(j_i), mp^{l(j_i)})x_{j_i}$, where $x_{j_0} = x_j$, $\lambda_0 = 1$ and $\lambda'_i = \lambda_i/\lambda_t$. By replacing x_{j_t} with y_{j_t} and $P(k, mp^{l(j_t)})x_{j_t}$ with $P(k, mp^{l(j_t)})y_{j_t}$ for all $k \le l+1-l(j_t)$, we see that $\wp^{mp^{l+1}} \cdot P(l+1-l(j_t), mp^{l(j_t)})y_{j_t} = 0$. The subset of H^* obtained from M by this replacement is a basis for H^* and satisfies the conditions (i), (ii) and (iii) up to the integer l.

If the argument started from the unique expression of the base of $(QH^*)^{2mp^{l+1}+1}$ continues infinitely, then we obtain infinitely many bases $P(l+1-l(j), mp^{l(j)})x_j$ of $(QH^*)^{2mp^{l+1}+1}$ such that $\wp^{mp^{l+1}} \cdot P(l+1-l(j), mp^{l(j)})x_j = 0$, which is a contradiction. Finally, by repeating the argument, we can obtain a basis $\{P(l+1-l(j), mp^{l(j)})x_j\}$ for $(QH^*)^{2mp^{l+1}+1}$ such that all non-zero elements

$$\wp^{mp^{l+1}} \cdot P(l+1-l(j), mp^{l(j)})x_j$$

are linearly independent. From such elements, we can obtain a basis for $\bigoplus_{0\le t\le l+2}(QH^*)^{2mp^t+1}$ which satisfies the conditions (i), (ii) and (iii). (Note that all basis elements x_j for QH^* are primitive.) Similarly, we can choose a basis for $\bigoplus_{0\le t\le 1}(QH^*)^{2mp^t+1}$ so as to satisfy (i), (ii) and (iii). This completes the proof of Lemma 2.1. □

Proof of Lemma 1.5. The vector spaces

$$P(k, up^{i(u)+r}) \cdot (QH^*)^{2up^{\iota(u)+r}+1} \quad \text{and} \quad \bigoplus_{0\le t}(QH^*)^{2u'p^{\iota(u')+t}+1}$$

do not intersect for any k, r and u, $u' \in U$ $(u \neq u')$. Therefore Lemma 1.5 follows from Lemma 2.1. □

Proof of Proposition 1.6 (1) *and* (2). Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence (with $\mathbb{K}_p$-coefficients) of the path-loop fibration $\Omega X \hookrightarrow PX \to X$. Put $\Gamma = H^*(X; \mathbb{K}_p)$.

(1) In the case where $p = 0$, since $\Gamma \cong \Lambda(x_{2m(1)+1}, \dots, x_{2m(s)+1})$, we see that $E_2 \cong \mathrm{Tor}_\Gamma^{**}(\mathbb{K}_0, \mathbb{K}_0) \cong \mathbb{K}_0[s^{-1}x_{2m(1)+1}, \dots, s^{-1}x_{2m(s)+1}]$. Since the total degree of each algebra generator in E_2^{**} is even, this spectral sequence collapses at the E_2-term. Hence, by [**12**; Example 11 (page 25)], we have (1).

(2) By the same argument as in the proof of (1), we can conclude that $E_0^{**} \cong E_\infty^{**} \cong E_2^{**} \cong \Gamma[s^{-1}x_{2m(1)+1}, \dots, s^{-1}x_{2m(s)+1}]$. Therefore, a subset $S = \{\gamma_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f\ge 0, 1\le i\le s}$ of $H^*(\Omega X; \mathbb{K}_p)$ is a p-simple system of generators. In order to apply [**10**; Theorem 2.4], we must verify that

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \notin N(S) \tag{2.1}$$

and

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^P \notin N^p(S) \quad \text{for any } i \ (1 \le i \le s) \tag{2.2}$$

(see [**10**; Notation 2.2]). If there exists some integer i such that $\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \in N(S)$, then we have an equation

$$\sum_j \lambda_j \alpha_j^p + w = \gamma_{p[f]}(s^{-1}x_{2m(i)+1}), \tag{2.3}$$

where $\alpha_j \in S$ and w represented by S does not have a term

$$\lambda \gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \qquad (\lambda \neq 0).$$

Comparing the degree of the elements in the equation, we see that $p^{f'+1} \cdot 2m(j) = p^f \cdot \deg s^{-1}x_{2m(i)+1} = p^f \cdot 2m(i)$ when the filtration degree of α_i is $p[f']$. Suppose that $m(j) < m(i)$. Then $f'+1 > f$ and so $p < p^{f'+1-f} = \frac{m(i)}{m(j)} \le \frac{m(s)}{m(1)}$. But this contradicts the assumption $\frac{m(s)}{m(1)} < p$. For a similar reason, the case $m(j) > m(i)$ does not occur. Hence we have that $m(j) = m(i)$. Thus each α_j in the equation (2.3) is written as $\gamma_{p[f-1]}(s^{-1}x_{2m(t_j)+1})$, where $m(t_j) = m(i)$. The element α_j^p is in a smaller filter than the filter including $\gamma_{p[f]}(s^{-1}x_{2m(i)+1})$. From the equation (2.3), we have that

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) - w = \sum_j \lambda_j \alpha_j^p. \tag{2.4}$$

Let l be the least of the filtration degrees of the terms in the left-hand side of (2.4). Consider the equation (2.4) in $E_0^{l,*}$. The right-hand side of (2.4) is zero and the left-hand side is non-zero. Finally, we obtain (2.1). In a similar manner, we have (2.2). From the above argument, we see that $h(\gamma_{p[f]}(s^{-1}x_{2m(i)+1})) = p$ for any i (see [**10**; Theorem 2.4]). Hence we have (2) by applying [**10**; Theorem 2.4]. □

In order to prove Proposition 1.6 (3) by using [**10**; Theorem 2.4], we must obtain a good p-simple system of generators for $H^*(\Omega X; \mathbb{Z}/p)$. First, applying the same argument as in the proof of (2), we can conclude that $E_0^{**} \cong E_\infty^{**} \cong E_2^{**} \cong \Gamma[s^{-1}x_{2m(1)+1}, \ldots, s^{-1}x_{2m(s)+1}]$, where $\{E_r, d_r\}$ is the Eilenberg-Moore spectral sequence (with $\mathbb{Z}/p$-coefficients) of the path loop fibration $\Omega X \hookrightarrow PX \to X$. Therefore, we can choose a subset $S = \{\gamma_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f \ge 0, 1 \le i \le s}$ of $H^*(\Omega X; \mathbb{Z}/p)$ as a p-simple system of generators for $H^*(\Omega X; \mathbb{Z}/p)$. The following lemma guarantees that we can choose a good p-simple system of generators.

LEMMA 2.2. *A p-simple system of generators*

$$\widetilde{S} = \{\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f \ge 0, 1 \le i \le s}$$

for $H^*(\Omega X\,;\mathbb{Z}/p)$ *which satisfies the following conditions* (2.5), (2.6) *and* (2.7) *can be organized from the system* S.

(2.5) $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^r+1})^p = \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^{r+1}+1})$ *for any* $i\in J$ *and* $0\le r\le k(i)-2$. (*About the integer* $k(i)$ *and the set* J *of integers, see the remarks following Lemma* 1.5.)

(2.6) $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \notin N(\widetilde{S})$.

(2.7) $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^{k(i)-1}+1})^p \notin N^p(\widetilde{S})$ *for any* $i\in J$.

Proof of Proposition 1.6 (3). Let AG be a subset

$$\{\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})\}_{i\in J\,,\,f\ge 0}$$

of $\widetilde{S}$. By Lemma 2.2, we see that the conditions of [**10**; Theorem 2.4] are satisfied and that $h(\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})) = p[k(i)]$. Thus we have (3) by virtue of [**10**; Theorem 2.4]. □

Lemma 2.2 can be proved by virtue of the following lemma.

LEMMA 2.3. *In the module* $F^{-p[f]}H^*(\Omega X\,;\mathbb{Z}/p)$, *if* $k(i)=1$, *then* $\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^p = w_0$ *and if* $k(i)>1$, *then* $\gamma_{p[f]}(s^{-1}x_{2m(i)p^t+1})^p = \gamma_{p[f]}(s^{-1}x_{2m(i)p^{t+1}+1}) + w_t$ *for any* $0\le t\le k(i)-2$, *where* $w_n\in F^{-p[f]+1}H^*(\Omega X\,;\mathbb{Z}/p)$. (*See Figure* 1.)

Proof. By [**15**], we know that the module E_r^{**} is an $\mathscr{A}(p)$-module and that the isomorphisms $E_0^{**}\cong E_\infty^{**}$ and $E_\infty^{**}\cong E_2^{**}$ are morphisms of $\mathscr{A}(p)$-modules, where $\mathscr{A}(p)$ is the Steenrod algebra. Let us consider $\wp_{EM}^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1})$ in E_∞^{**} for any $i\in J$. By identifying the $\mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p\,,\mathbb{Z}/p)$ which is obtained from the Koszul resolution and that which is obtained from the bar resolution, we can regard $\gamma_{p[f]}(s^{-1}x_{2m(i)+1})$ as

$$\underbrace{[x_{2m(i)+1}|x_{2m(i)+1}|\cdots|x_{2m(i)+1}]}_{p[f]}.$$

(See [**19**; Proposition 1.1] and [**11**; Proposition 1.2].) Therefore

$$\begin{aligned}
\wp_{EM}^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) &= \wp_{EM}^{m(i)\cdot p[f]}[x_{2m(i)+1}|\cdots|x_{2m(i)+1}]\\
&= [\wp^{m(i)}x_{2m(i)+1}|\cdots|\wp^{m(i)}x_{2m(i)+1}]\\
&\quad + \sum[\wp^{l(1)}x_{2m(i)+1}|\cdots|\wp^{l(n(f))}x_{2m(i)+1}],\\
&\qquad\qquad l(1)+\cdots+l(n(f)) = m(i)\cdot p[f],\\
&\qquad\qquad (l(1),\ldots,l(n(f)))\ne(m(i),\ldots,m(i)).
\end{aligned}$$

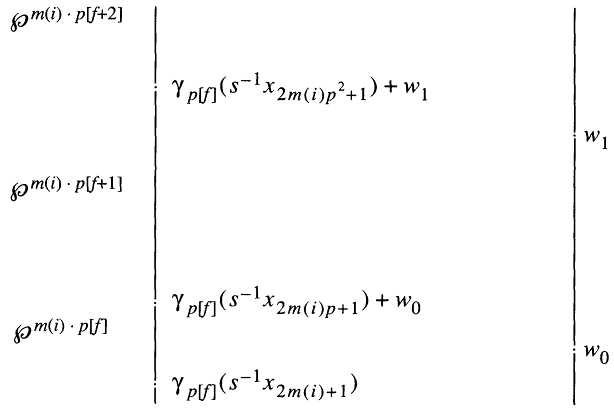

FIGURE 1

In the above last expression, the second summations are zero from the instability axiom of the Steenrod operation. From (1.1),

$$\wp^{m(i)}x_{2m(i)+1} = \begin{cases} x_{2m(i)p+1} & \text{if } k(i) > 1 \\ 0 & \text{if } k(i) = 1 \end{cases} \quad \text{in } H^*(X\,;\mathbb{Z}/p).$$

Note that there is an integer j such that $m(i)p = m(j)$ if $k(i) > 1$. Hence we obtain that

$$\wp_{EM}^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) = \begin{cases} \gamma_{p[f]}(s^{-1}x_{2m(i)p+1}) & \text{if } k(i) > 1, \\ 0 & \text{if } k(i) = 1, \end{cases}$$

in $E_\infty^{-p[f],*}$.

Therefore, we see that

$$\begin{aligned} \gamma_{p[f]}(s^{-1}x_{2m(i)+1})^p &= \wp^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \\ &= \begin{cases} \gamma_{p[f]}(s^{-1}x_{2m(i)p+1}) & \text{if } k(i) > 1, \\ 0 & \text{if } k(i) = 1, \end{cases} \end{aligned}$$

in $E_0^{-p[f],*}$, where $\wp^i$ is the ordinary Steenrod operation. This fact allows us to conclude that

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^p = \begin{cases} \gamma_{p[f]}(s^{-1}x_{2m(i)p+1}) + w_0 & \text{if } k(i) > 1 \\ w_0 & \text{if } k(i) = 1, \end{cases}$$

in $F^{-p[f]}H^*(\Omega X\,;\mathbb{Z}/p)$, where $w_0 \in F^{-p[f]+1}H^*(\Omega X\,;\mathbb{Z}/p)$. Using the same argument as above, it follows that the latter half of Lemma 2.3 holds.

Proof of Lemma 2.2. Put

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^r+1}) = \gamma_{p[f]}(s^{-1}x_{2m(i)p^r+1}) + w_0^{p[r-1]} + w_1^{p[r-2]} + \cdots + w_{r-1}$$

for $1 \le r \le k(i)-1$, and put $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) = \gamma_{p[f]}(s^{-1}x_{2m(i)+1})$. From Lemma 2.3, it follows that (2.5) holds.

Let $\widetilde{S}$ be the subset of $H^*(\Omega X; \mathbb{Z}/p)$ which is obtained from S by replacing $\gamma_{p[f]}(s^{-1}x_{2m(i)p^r+1})$ with $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^r+1})$ in S for any $i \in J$. Let us prove (2.6). If $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \in N(\widetilde{S})$ for some $i \in J$, then we have following:

$$(2.8) \qquad \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) = \sum_j \mu_j \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})^p + w$$

in $H^*(\Omega X; \mathbb{Z}/p)$, where $\mu_j \ne 0$ and w represented by $\widetilde{S}$ does not have a term $\lambda\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})$ $(\lambda \ne 0)$.

First let us consider the case where i satisfies (1.2). Choose an integer j in the right-hand side of the equality (2.8) such that $j \in J$. By comparing the degrees of the elements in the equality (2.8), we have that $2m(i) \cdot n(f) = p \cdot 2m(j) \cdot p[f(j)]$. From (1.2), we can conclude that $m(i) = m(j)$. Hence

$$(2.9) \qquad \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) \in F^{-p[f-1]}H^*(\Omega X; \mathbb{Z}p).$$

Choose an integer j so that $j \notin J$. Then there exist some integers $t \in J$ and $n \in \mathbb{N}$ such that $x_{2m(j)+1} = P(n, m(t))x_{2m(t)+1}$. Since $\gamma_{p[f(j)]}(s^{-1}x_{2m(j)+1}) = \gamma_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n]}$, from (2.8), we see that $2m(t) \cdot [f(j)] \cdot p[n+1] = 2m(i) \cdot p[f]$. From the condition (1.2), we have that $m(i) = m(t)$ and $f > f-n-1 = f(j)$. Thus we can conclude that

$$\begin{aligned}(2.10) \quad \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) &= P(n, m(t))\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1}) \\ &= P(n, m(t))\tilde{\gamma}_{p[f-n-1]}(s^{-1}x_{2m(t)+1}) \\ &\in F^{-p[f-n-1]}H^*(\Omega X; \mathbb{Z}/p).\end{aligned}$$

From (2.9) and (2.10), we see that the equality (2.8) causes a contradiction to the module structure of E_0^{**}. Thus we have (2.6).

Next let us consider the case that i satisfies the condition (1.3). Assume that there exists an element $\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})$ which satisfies $f(j) > f$ in (2.8). Applying the same argument as above, we see that there exist integers $t \in J$ and $n \in \mathbb{N}$ such that $\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) = \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n]}$. If $n+1 < k(t)$, then

$$\text{fil-deg}\, \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n+1]} = -p[f(j)].$$

Therefore, by using the usual argument of the filtration, we see that (2.8) causes a contradiction. Hence $n+1 \ge k(t)$. From the argument of the total degree in (2.8), we obtain that $p[f] \cdot 2m(i) = p[f(j)] \cdot p[n+1] \cdot 2m(t)$ and so $p[f] \cdot m(i) = p[f(j)+n+1] \cdot m(t)$. But this equality contradicts the condition(1.3) because $f(j) > f$ and $n+1 \geq k(t)$. Hence we conclude that $f(j) \leq f$ for any j in (2.8). Suppose that $f(j) = f$ for some j. From the condition (1.3) and the fact that $m(i) = p[n+1] \cdot m(t)$, where i and $t \in J$, it follows that

$$\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n+1]} \in F^{-p[f]+1}H^*(\Omega X; \mathbb{Z}/p)$$

if $n+1 = k(t)$ and that

$$\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n+1]} = \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(t)p^{n+1}+1})$$

if $n+1 < k(t)$. From (2.8), we have an equality:

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) = \sum \lambda_u \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(u)+1}) + w \quad \text{in } E_0^{-p[f],*},$$

where $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \neq \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(u)+1})$ and $\lambda_u \neq 0$. But this equality contradicts the fact that $\widetilde{S}$ is a p-simple system of generators for E_0^{**}. Finally, $f(j) < f$ for any j, which is a contradiction. We have (2.6).

Let us verify (2.7). If there exists an integer i such that

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p[k(i)-1]+1})^p \in N^p(\widetilde{S}),$$

then we have the following:

$$(2.11) \quad \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p[k(i)-1]+1})^p = \sum_j \lambda_j \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) + w'$$

in $H^*(\Omega X; \mathbb{Z}/p)$, where w' expressed by $\widetilde{S}$ does not include terms $\lambda\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})$ $(\lambda \neq 0)$.

Suppose that there exists an integer j in (2.11) such that $j \in J$. By applying the same argument as the proof of (2.6), we see that the equality (2.11) causes a contradiction. Hence it follows that $j \notin J$ for any j in (2.11). For any j, there exist integers $t_j \in J$ and n_j such that

$$\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t_j)+1})^{p[n_j]} = \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}).$$

From (2.11), we have the following equality:

$$(2.12) \quad \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})^{p[k(i)]} = \sum_j \lambda_j \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t_j)+1})^{p[n_j]} + w'.$$

We can suppose that the element

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p[k(i)-1]+1})^p \quad (= \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})^{p[k(i)]})$$

has the least degree of elements $\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})^{p[k(j)]}$ which belong to $N^p(\widetilde{S})$. Hence

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \neq 0 \quad \text{and} \quad \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t_j)+1}) \neq 0$$

in $\bigoplus_{u<p[f+k(i)]\cdot 2m(i)}(QH^*(\Omega X; \mathbb{Z}/p))^u$ (cf. the proof of [**10**; Proposition 2.5]). When the condition (1.2) is satisfied, it follows that the equality (2.3) induces a contradiction (compare the degrees of the elements in the left-hand side of (2.12) with those of the right-hand side). If the condition (1.3) is satisfied, then $k(i) \geq k(t_j)$ for any j. Therefore, by regarding $H^*(\Omega X; \mathbb{Z}/p)$ as an algebra which is a tensor product of monogenic algebras, we see that the equality (2.12) induces an equality contradicting the algebra structure of $H^*(\Omega X; \mathbb{Z}/p)$ (cf. the proof of [**10**; Proposition 2.5] and [**13**; 7.11 Theorem (Borel)]). Finally, we have (2.7). □

Proof of Theorem 1.7. By using the result in [**14**] concerning the Steenrod operation in $H^*(G; \mathbb{Z}/p)$ and Proposition 1.6 (3), we can have this theorem. □

Proof of Theorem 1.9. (1) As is known,

$$H^*(\mathrm{Spin}(2n+1); \ \mathbb{Z}/p) \cong \Lambda(e_3, e_7, \dots, e_{4n-1})$$

and

$$\wp^k e_{2m(i)+1} = \binom{m(i)}{k} e_{2m(i)+2k(p-1)+1} \tag{2.13}$$

if there exists the algebra generator $e_{2m(i)+2k(p-1)+1}$, and $\wp^k e_{2m(i)+1} = 0$ if indecomposable elements do not exist on the degree $2m(i) + 2k(p-1)+1$. Therefore the set $\{m(i) | i \in J\}$ is equal to $M(T, p)$ and the number $k(i)$ is equal to $t(m, n)$. By virtue of Proposition 1.6 (3), we have (1).

(2) Since $\mathrm{Spin}(2n-1) \times S^{2n-1} \simeq_p \mathrm{Spin}(2n)$, it follows that

$$\Omega\,\mathrm{Spin}(2n-1) \times \Omega S^{2n-1} \simeq_p \Omega\,\mathrm{Spin}(2n).$$

Hence we obtain (2) from (1). (In this case, since the condition (1.3) is satisfied, (2) can be proved by applying Proposition 1.6 (3) without using (1).)

(3) and (4). If $p \neq 2$, then (2.13) holds in

$$H^*(\mathrm{SU}(n); \ \mathbb{Z}/p) \cong \Lambda(e_3, e_5, \dots, e_{2n-1})$$

and

$$H^*(\mathrm{Sp}(n);\ \mathbb{Z}/p) \cong \Lambda(e_3, e_7, \dots, e_{4n-1}).$$

If $p = 2$, then $Sq^{2j}e_{2i-1} = \binom{i-1}{j}e_{2i+2j-1}$ in $H^*(\mathrm{SU}(n);\mathbb{Z}/2)$, where $e_{2t-1} = 0$ if $t > n$. By applying Proposition 1.6 (3), we can obtain (3) and (4).

(5) By considering the degrees of the subalgebra generators of $H^*(\Omega X;\mathbb{Z}/2)$, we see that $P(k, m)x_{2m+1} = 0$ for any m and $k > 0$. Therefore, $J = \{1, 2, \dots, n\}$, $m(i) = 2i-1$ and $k(i) = 1$ in Proposition 1.6 (3). We have (5) by Proposition 1.6 (3). □

The method used to prove Theorems 1.1, 1.2, 1.3, 1.7, 1.9 and [**10**; Theorem 2.4] is indeed algebraic, that is, properties of G as Lie groups are not used. Therefore we can have Theorem 0.2.

Proposition 1.10 can be deduced from the results of [**19**].

Proof of Proposition 1.10. By virtue of [**19**; Theorem P2], it follows that

$$\begin{aligned}(2.14)\quad E_0^{**}(H^*(\Omega(G/U);\mathbb{Z}/p)) &\cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p, H^*(U;\mathbb{Z}/p)) \\ &\cong H^*(U;\mathbb{Z}/p)//i^* \otimes \mathrm{Tor}^{**}_{\Gamma\backslash\backslash i^*}(\mathbb{Z}/p, \mathbb{Z}/p),\end{aligned}$$

as Hopf algebras (see [**19**; Proposition 1.5]), where $\Gamma = H^*(G;\mathbb{Z}/p)$,

$$\begin{aligned}&[H^*(U;\mathbb{Z}/p)//i^* \otimes \mathrm{Tor}^{**}_{\Gamma\backslash\backslash i^*}(\mathbb{Z}/p, \mathbb{Z}/p)]^{s,t} \\ &\quad = \bigoplus_{m+n=t} [(H^*(U;\mathbb{Z}/p)//i^*)^m \otimes \mathrm{Tor}^{s,n}_{\Gamma\backslash\backslash i^*}(\mathbb{Z}/p, \mathbb{Z}/p)].\end{aligned}$$

Moreover, from the proof of [**19**; Theorem P2], we see that the filtration $\{F^{-n}H^*(\Omega(G/U);\mathbb{Z}/p)\}$ is given from the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of a fibration $\Omega(G/U) \hookrightarrow U \xrightarrow{i} G$, and that the isomorphism (2.14) is as follows:

$$E_0^{**}(H^*(\Omega(G/U);\mathbb{Z}/p)) \cong E_\infty^{**} \cong E_2 \cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p, H^*(U;\mathbb{Z}/p)).$$

Therefore, we can conclude that

(2.15) the isomorphism (2.14) is a morphism of $\mathscr{A}(p)$-modules.

Since $H^*(U;\mathbb{Z}/p)//i^*$ is a Hopf algebra, by the Hopf-Borel theorem ([**13**; 7.11 Theorem]), it follows that $H^*(U;\mathbb{Z}/p)//i^*$ is isomorphic to

$$\Lambda(y_1, \dots, y_t) \otimes \mathbb{Z}/p[u_1, \dots, u_m]/(u_1^{p[f(1)]}, \dots, u_m^{p[f(m)]})$$

as an algebra, where y_i and u_i are appropriate algebra generators.

Since $\Gamma\backslash\backslash i^* = H^*(G;\mathbb{Z}/p)\backslash\backslash i^* \cong \Lambda(x_{2m(1)+1}, \dots, x_{2m(s)+1})$, we obtain that

$$\mathrm{Tor}^{**}_{\Gamma\backslash\backslash i^*}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \Gamma[s^{-1}x_{2m(1)+1}, \dots, s^{-1}x_{2m(s)+1}].$$

Let us express the element in E_0^{**} and its representative element with the same notation. Let S be a subset

$$\{y_i\}_{1\leq i\leq t} \cup \{u_j\}_{1\leq j\leq m} \cup \{\gamma_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f\geq 0,\, 1\leq i\leq s}$$

of $H^*(\Omega(G/U)\,;\mathbb{Z}/p)$, where $\deg \gamma_{p[f]}(s^{-1}x_{2m(i)+1}) = p[f]\cdot 2m(i)$. Then S is a p-simple system of generators for $H^*(\Omega(G/U)\,;\mathbb{Z}/p)$. From (2.15), by using the same argument as the proof of Proposition 1.6 (3), we can have this proposition. □

Proof of Theorem 1.11. Let i: $\mathrm{SU}(n) \to \mathrm{SU}(m+n)$ be the inclusion map. We know that $i^*(x_{2i-1}) = x_{2i-1}$ if $1 \leq i \leq n$ and that $i^*(x_{2i-1}) = 0$ if $n < i \leq m+n$, where i^* is the morphism of algebras from $H^*(\mathrm{SU}(m+n)\,;\mathbb{Z}/p) \cong \Lambda(x_3\,, x_5\,, \dots\,, x_{2m+2n-1})$ into $H^*(\mathrm{SU}(n)\,;\mathbb{Z}/p) \cong \Lambda(x_3\,, x_5\,, \dots\,, x_{2n-1})$, and the x_i are appropriate generators of each algebra. Hence we can conclude that

$$H^*(\mathrm{SU}(n)\,;\mathbb{Z}/p)//i^* = 0$$

and that

$$H^*(\mathrm{SU}(m+n)\,;\mathbb{Z}/p)\backslash\backslash i^* = \Lambda(x_{2n+1}\,, x_{2n+3}\,, \dots\,, x_{2m+2n-1})\,.$$

By applying Proposition 1.10, we can obtain Theorem 1.11 (1). Similarly, we have Theorem 1.11 (2) and (3). □

Proof of Theorem 1.12. Let i: $\mathrm{Sp}(m)\times\mathrm{Sp}(n) \to \mathrm{Sp}(m+n)$ be the inclusion map and Bi: $B\,\mathrm{Sp}(m)\times B\,\mathrm{Sp}(n) \to B\,\mathrm{Sp}(m+n)$ the map which is induced from i. We know that $Bi^*(q_i) = \sum_{j+k=i} q_j'\cdot q_k''$, where Bi^* is the morphism of algebras from

$$H^*(B\,\mathrm{Sp}(m+n)\,;\mathbb{Z}/p) \cong \mathbb{Z}/p[q_1\,, q_2\,, \dots\,, q_{m+n}]$$

into

$$H^*(B\,\mathrm{Sp}(m)\times B\,\mathrm{Sp}(n)\,;\mathbb{Z}/p) \cong \mathbb{Z}/p[q_1'\,, q_2'\,, \dots\,, q_m'\,, q_1''\,, \dots\,, q_n''],$$

q_i, q_i', and q_i'' are appropriate algebra generators of each algebra, and $\deg q_i = \deg q_i' = \deg q_i'' = 4i$. Therefore, we see that $i^*(x_{4i-1}) = x_{4i-1}' + x_{4i-1}''$ if $1\leq i\leq n$, $i^*(x_{4i-1}) = x_{4i-1}'$ if $n+1\leq i\leq m$, and $i^*(x_{4i-1}) = 0$ if $m+1\leq i$, where i^* is the morphism of algebras from i^*: $H^*(\mathrm{Sp}(m_n)\,;\mathbb{Z}/p) \cong \Lambda(x_3\,, x_7\,, \dots\,, x_{4m+4n-1})$ into

$$\begin{aligned} &H^*(\mathrm{Sp}(m)\times\mathrm{Sp}(n)\,;\mathbb{Z}/p) \\ &\qquad \cong \Lambda(x_3'\,, x_7'\,, \dots\,, x_{4m-1}'\,, x_3''\,, x_7''\,, \dots\,, x_{4n-1}''),\end{aligned}$$

and the algebra generators x_i, x_i' and x_i'' transgress to q_i q_i' and q_i'' respectively. Hence we can conclude that

$$H^*(\mathrm{Sp}(m)\times \mathrm{Sp}(n)\,;\,\mathbb{Z}/p)//i^* \quad \text{and} \quad H^*(\mathrm{Sp}(m+n)\,;\,\mathbb{Z}/p)\backslash\backslash i^*$$

are isomorphic to

$$\Lambda(x_3', x_7', \dots, x_{4n-1}') \quad \text{and} \quad \Lambda(x_{4m+3}, x_{4m+7}, \dots, x_{4m+4n-1})$$

respectively. We have Theorem 1.12 by virtue of Proposition 1.10. □

Proof of Theorem 1.13. Let p be an odd prime. As is known,

$$H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n)\,;\,\mathbb{Z}/p)$$
$$\cong \begin{cases} \Lambda(e_{2n+1}, e_{2n+5}, \dots, e_{2m+2n-3}) & \text{if } n \text{ is odd and } m \text{ is even},\\ \Lambda(e_{2n+1}, e_{2n+5}, \dots, e_{2m+2n-5}, e'_{m+n-1}) & \text{if } n \text{ and } m \text{ are odd},\\ \Lambda(e_{2n+3}, e_{2n+7}, \dots, e_{2m+2n-3}) \otimes \mathbb{Z}/p[x_n]/(x_n^2) & \text{if } n \text{ is even and } m \text{ is odd},\\ \Lambda(e_{2n+3}, e_{2n+7}, \dots e_{2m+2n-5}, e'_{m+n-1}) \otimes \mathbb{Z}/p[x_n]/(x_n^2) & \text{if } n \text{ and } m \text{ are even}, \end{cases}$$

and $\wp^k e_{2m(i)+1} = \binom{m(i)}{k} e_{2m(i)+2k(p-1)+1}$, where $\wp^k e_{2m(i)+1} = 0$ if indecomposable elements do not exist on the degree $2m(i)+2k(p-1)+1$.

Consider the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the fibration

$$\Omega(\mathrm{SO}(m+m)/\mathrm{SO}(n)) \hookrightarrow \mathrm{SO}(n)/\mathrm{SO}(n-1) \to \mathrm{SO}(m+n)/\mathrm{SO}(n-1).$$

We have that

$$E_2^{**} \cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p, H^*(S^{n-1}; \mathbb{Z}/p))$$

and

$$E_r^{**} \Rightarrow H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n))\,;\,\mathbb{Z}/p),$$

where $\Gamma = H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n-1)\,;\,\mathbb{Z}/p)$.

Let n be odd and m be even. Then we see that $i^*(x_{n-1}) = y_{n-1}$, where i^* is the morphism of algebras from

$$H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n-1)\,;\,\mathbb{Z}/p)$$

into

$$H^*(S^{n-1}; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_{n-1}]/(y_{n-1}^2)$$

induced from the inclusion map i. Therefore, by computing the Koszul complex, it follows that

$$E_2^{**} \cong \Gamma[s^{-1}e_{2(n-1)+3}, s^{-1}e_{2(n-1)+7}, \dots, s^{-1}e_{2m+2n-3}].$$

Hence $E_2^{**} \cong E_\infty^{**} \cong E_0^{**}$. By applying [**10**; Theorem 2.4], we obtain the desired result in the case where n is odd and m is even.

Let n and m be odd. By using the same argument as the above, we see that

$$E_0^{**} \cong \Gamma[s^{-1}e_{2(n-1)+3}, s^{-1}e_{2(n-1)+7}, \dots, s^{-1}e_{2m+2n-5}, s^{-1}e'_{m+n-1}].$$

Let S be the p-simple system of generators determined from the divided power algebra E_0^{**}. Put $2t+1 = m+n-1$. Then $2pt+1 > 2(m+n-3)+1$. Using this fact and the usual argument of the filtration degrees and the total degrees, we see that $\gamma_{p[f]}(s^{-1}e'_{m+n-1})^p \notin N^p(S)$. Furthermore, using the Steenrod operation in the Eilenberg-Moore spectral sequence, we see that $\gamma_{p[f]}(s^{-1}e'_{m+n-1}) \notin N(S)$. Hence we have our result in the case where n and m are odd.

Let n be even and m odd. We can obtain that

$$E_2^{**} \cong \Gamma[s^{-1}e_{2(n-1)+1}, s^{-1}e_{2(n-1)+5}, \dots, s^{-1}e_{2m+2n-3}] \otimes \Lambda(x_{n-1}),$$

where $\operatorname{bideg} s^{-1}e_i = (-1, i)$ and $\operatorname{bideg} x_{n-1} = (0, n-1)$.

Let $\{\overline{E}_r, \overline{d}_r\}$ denote the mod p Eilenberg-Moore spectral sequence of the path loop fibration

$$\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)) \hookrightarrow P(\mathrm{SO}(m+n)/\mathrm{SO}(n)) \to \mathrm{SO}(m+n)/\mathrm{SO}(n).$$

We see that this spectral sequence collapses at the E_2-term by applying [**6**; DHA Lemma]. Therefore $H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p)$ can be determined as a vector space. By comparing each dimension of $H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p)$ and the total complex $\bigoplus E_2^{**}$, we conclude that $E_2^{**} \cong E_\infty^{**} \cong E_0^{**}$. As usual, we can have an appropriate p-simple system of generators for $H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/p)$. By [**10**; Theorem 2.4], we get our result in the case where n is even and m is odd. In a similar manner, we can also get it in the case where n and m are even. □

In general, it is not easy to determine the algebra structure of

$$H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p)$$

from the associated bigraded algebra $\overline{E}_0^{**}$ which is obtained from the Eilenberg-Moore spectral sequence $\{\overline{E}_r, \overline{d}_r\}$ in the above proof. For example consider $\{\overline{E}_r, \overline{d}_r\}$ in the case where $p = 3$, $m = 5n-4$ and n is an even integer greater than 3. Then we see that

$$\begin{aligned}\overline{E}_0^{**} &\cong \Gamma[s^{-1}e_{2n+3}, s^{-1}e_{2n+7}, \dots, s^{-1}e_{2m+2n-5}, s^{-1}e'_{m+n-1}] \\ &\otimes \Gamma[\tau(x_n^2)] \otimes \Lambda(s^{-1}x_n)\end{aligned}$$

as an algebra, where

$$\begin{aligned}\operatorname{bideg} s^{-1}e_i &= (-1, i), \\ \operatorname{bideg} s^{-1}e'_{m+n-1} &= (-1, m+n-1) = (-1, 6n-5), \\ \operatorname{bideg} \tau(x_n^2) &= (-2, 2n).\end{aligned}$$

Since $\tau(x_n^2) = [x_n|x_n]$ in E_∞^{**}, it follows that

$$\wp_{EM}^{n-1}\tau(x_n^2) = \wp_{EM}^{n-1}[x_n|x_n] = \sum_{i+j=n-1} [\wp^i x_n|\wp^j x_n]$$

in $E_\infty^{-2,*}$. $\wp^i x_n$ is decomposable for dimensional reasons. Therefore, if $\wp^i x_n \neq 0$, then $n+2i(3-1) \geq 2n+3+2n+7$ and so $n+4i \geq 4n+10$. Hence $n+2j(3-1) < 4n+10$, because $i+j = n-1$. We can conclude that $\wp^j x_n = 0$. Finally, $[\wp^i x_n|\wp^j x_n] = 0$ for any i and j such that $i+j = n-1$. We have that $\tau(x_n^2)^3 = \wp^{n-1}\tau(x_n^2) \in \overline{F}^{-1}H^{6n-6}$. The element $\tau(x_n^2)^3$ may be equal to $s^{-1}e'_{n+m-1}$, because $s^{-1}e_{n+m-1}$ belongs to $\overline{F}^{-1}H^{6n-6}$. It is difficult to show whether $\tau(x_n^2)^3$ is equal to $s^{-1}e'_{m+n-1}$ or not even if we use the argument of the filtration and the Steenrod operation in the Eilenberg-Moore spectral sequence.

Proof of Theorem 1.14. As is known, $H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/2) \cong \Delta(x_n, x_{n+1}, \dots, x_{m+n-1})$ and

$$Sq^j x_i = \binom{i}{j} x_{i+j}, \quad \text{where } x_t = 0 \text{ if } t \geq m+n. \tag{2.16}$$

Therefore,

$$H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/2) \cong \bigotimes_{j\in J} \mathbb{Z}/2[x_j]/(x_j^{2[s_j]}),$$

where $J = L \cup L'$. Consider the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the path loop fibration on $\mathrm{SO}(m+n)/\mathrm{SO}(n)$. Then we have that

$$\begin{aligned}E_2^{**} &\cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/2, \mathbb{Z}/2) \quad (\Gamma = H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/2)) \\ &\cong \left\{ \bigotimes_{j\in J, s_j>1} (\Gamma[w_j] \otimes \Lambda(s^{-1}x_j)) \right\} \otimes \left\{ \bigotimes_{j\in J, s_j=1} \Gamma[s^{-1}x_j] \right\}\end{aligned}$$

and $E_r^{**} \Rightarrow H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/2)$, where $\operatorname{bideg} s^{-1}x_j = (-1, j)$ and $\operatorname{bideg} w_j = (-2, j\cdot 2[s_j])$. By applying **[6**; DHA Lemma], we see that this spectral sequence collapses at the E_2-term: $E_0^{**} \cong E_\infty^{**} \cong E_2^{**}$. Let S be the simple system of generators obtained from E_0^{**}. If $j \in L'$ and $s_j = 1$, then $\gamma_{2[f]}(s^{-1}x_j) \notin N(S)$. In fact, suppose that $\gamma_{2[f]}(s^{-1}x_j) \in N(S)$; we see that there exists an integer $i \in J$ such that $(j-1) = 2[f']\cdot(i\cdot 2[t]-2)$, where $f' \geq 1$ and $t \geq 1$. But the left-hand side of the equality is odd and the right-hand side is even, which is a contradiction. Next let us verify that $\gamma_{2[f]}(s^{-1}x_j)^2 \notin N^2(S)$ if $j \in L'$ and $s_j = 1$. Suppose that $\gamma_{2[f]}(s^{-1}x_j)^2 \in N^2(S)$. From **[10**; Lemma 3.1], we have the following equality:

$$Sq^{2[f]\cdot(j-1)}\gamma_{2[f]}(s^{-1}x_j) = \gamma_{2[f]}(s^{-1}x_j)^2 = \lambda\gamma_{2[f]}(s^{-1}x_i) + w$$

in $E_0^{-2[f],*}$, where $\lambda \neq 0$ and w expressed by S does not have the term $\mu\gamma_{2[f]}(s^{-1}x_i)$ $(\mu \neq 0)$. Therefore

$$Sq_{EM}^{2[f]\cdot(j-1)}[\underbrace{x_j|x_j|\cdots|x_j}_{2[f]}] = \lambda[\underbrace{x_i|x_i|\cdots|x_i}_{2[f]}] + w \quad \text{in } E_\infty^{-2[f],*}$$

and the left-hand side is equal to

$$\sum_{i(1)+\cdots+i(2[f])=2[f]\cdot(j-1)} [Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j].$$

For any term $[Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j]$, provided that there exists some integer $i(t)$ such that $i(t) \geq j$, there exists an integer $i(t')$ such that $i(t') < j-1$. Since j is even, from (2.16), we obtain that $[Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j] = 0$. Similarly, we see that the term $[Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j]$ is zero if $i(t) < j$ for any $i(t)$. Hence

$$\lambda[x_i|\cdots|x_i] + w = 0 \quad \text{in } E_\infty^{-2[f],*},$$

which is a contradiction. Thus we conclude that $\gamma_{2[f]}(s^{-1}x_j)^2 \notin N^2(S)$. Using the above fact and **[10**; Lemma 3.1], we have Theorem 1.14. (Note that $(m+n-2)/2+1 \in L$ if and only if $m+n-2 \equiv 0 \bmod 4$ and $n \leq m$.) $\square$

Proof of Theorem 1.15. Let $\{E_r, d_r\}$ denote the Eilenberg-Moore spectral sequence of the path loop fibration

$$\begin{aligned}\Omega(\mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n)) &\hookrightarrow P(\mathrm{U}(m+n)/\mathrm{U}(n)) \\ &\to \mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n).\end{aligned}$$

Note that this spectral sequence has a Hopf algebra structure. We know that

$$H^*(\mathrm{U}(m+n)/\mathrm{U}(m)\times \mathrm{U}(n);\mathbb{Z}/p)\cong \mathbb{Z}/p[c_1,\dots,c_n]/(\rho_1,\dots,\rho_n)$$

as an algebra, where $\rho_1,\dots,\rho_n$ is a regular sequence, ρ_i is decomposable for any i, $\deg c_i = 2i$, $\deg\rho_i = 2m+2i$ and $m\geq n$ (cf. [**11**]). By virtue of [**19**; Proposition 1.1], we obtain that

$$E_2^{**}\cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p,\mathbb{Z}/p)\cong \Lambda(s^{-1}c_1,\dots,s^{-1}c_n)\otimes\Gamma[\tau\rho_1,\dots,\tau\rho_n]$$

as an algebra, where $\mathrm{bideg}\, s^{-1}c_i = (-1,2i)$ and $\mathrm{bideg}\,\tau\rho_j = (-2,2m+2j)$. Since the free algebra generators with less total degree than $\mathrm{totdeg}\, s^{-1}c_n+1$ have column degree -1, by applying [**6**; DHA Lemma], it follows that those images by the differential d_r are zero for any $r\geq 2$. By applying [**10**; Theorem 2.4], we have Theorem 1.15. □

In order to prove Theorem 1.16, we will calculate a Koszul complex in the concrete.

Proof of Theorem 1.16. As is known,

$$H^*(\mathrm{SU}(9)/\mathbb{Z}/3;\mathbb{Z}/2)\cong\Lambda(e_3,e_5,\dots,e_{17}),$$

(2.17) $Sq^{2j}e_{2i-1} = \binom{i-1}{j}e_{2i+2j-1}$, where $e_{2t-1}=0$ if $t>9$,

and

(2.18)

$$H^*(E_8;\mathbb{Z}/2)$$
$$\cong \mathbb{Z}/2[x_3,x_5,x_9,x_{15}]/(x_3^{16},x_5^8,x_9^4,x_{15}^4)\otimes\Lambda(x_{17},x_{23},x_{27},x_{29}),$$
$$Sq^2x_3=x_5,\ 0\quad Sq^4x_5=x_9,\quad Sq^2x_{15}=x_{17},\quad Sq^8x_9=x_{17},$$
$$Sq^8x_{15}=x_{23},\quad Sq^4x_{23}=x_{27}\quad\text{and}\quad Sq^2x_{27}=x_{29}.$$

Let us show that

(2.19) $$i^*(x_i)=\begin{cases} e_i & \text{if } i=3,5,9,15 \text{ or } 17\\ 0 & \text{if } i=23,27 \text{ or } 29,\end{cases}$$

where $i\colon \mathrm{SU}(9)/\mathbb{Z}/3\hookrightarrow E_8$ is the inclusion map.

First, we have that $i^*(x_3)=e_3$ since $j_*\colon H_3(\mathrm{SU}(7))\to H_3(E_8)$ is an isomorphism, where $j\colon \mathrm{SU}(7)\to E_8$ is a composition of the inclusion maps $i\colon \mathrm{SU}(9)/\mathbb{Z}/3\to E_8$ and $k\colon \mathrm{SU}(7)\to\mathrm{SU}(9)/\mathbb{Z}/3$.

Using the Steenrod operation, we have that $i^*(x_5) = e_5$, $i^*(x_9) = e_9$ and $i^*(x_{17}) = e_{17}$. Let us show that $i^*(x_{15}) = e_{15}$. Since $\{e_{15}, e_3 \cdot e_5 \cdot e_7\}$ is a basis of $H^{15}(\mathrm{SU}(9)/\mathbb{Z}/3; \mathbb{Z}/2)$, we can write as follows: $i^*(x_{15}) = \lambda e_{15} + \lambda' e_3 \cdot e_5 \cdot e_7$. Therefore, applying the Steenrod operation, we have that

$$Sq^2 i^*(x_{15}) = Sq^2(\lambda e_{15} + \lambda' e_3 \cdot e_5 \cdot e_7)$$

and so

$$e_{17} = \lambda e_{17} + \lambda' e_3 \cdot e_5 \cdot e_9 .$$

We see that $\lambda = 1$ and $\lambda' = 0$. Hence we obtain that

$$i^*(x_{15}) = e_{15} . \tag{2.20}$$

From (2.17), (2.18) and (2.20), we conclude that $i^*(x_i) = 0$ if $i = 23$, 27 or 29. Thus we have (2.19).

Next let us consider the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the fibration $\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)) \hookrightarrow \mathrm{SU}(9)/\mathbb{Z}/3 \to E_8$. By using the Koszul resolution, we can obtain the explicit form of the E_2-term:

$$\mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/2, H^*(\mathrm{SU}(9)/\mathbb{Z}/3; \mathbb{Z}/2)) \cong H(\mathscr{K}, \delta),$$

where $\Gamma = H^*(E_8; \mathbb{Z}/2)$,

$$\begin{aligned} \mathscr{K} = {} & \Lambda(s^{-1}x_3, s^{-1}x_5, s^{-1}x_9, s^{-1}x_{15}) \\ & \otimes \Gamma[w_1, w_2, w_3, w_4, s^{-1}x_{17}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{29}] \\ & \otimes \Lambda(e_3, e_5, \dots, e_{17}), \end{aligned}$$

$\mathrm{bideg}\, s^{-1}x_j = (-1, j)$, $\mathrm{bideg}\, w_1 = (-2, 48)$, $\mathrm{bideg}\, w_2 = (-2, 40)$, $\mathrm{bideg}\, w_3 = (-2, 36)$, $\mathrm{bideg}\, w_4 = (-2, 60)$, $\mathrm{bideg}\, e_i = (0, i)$ and $\delta(s^{-1}x_i) = e_i$ if $i = 3, 5, 9, 15$ or 17,

$$\delta(\gamma_{2[f]}(s^{-1}x_{17})) = \gamma_{2[f]-1}(s^{-1}x_{17}) \otimes e_{17}$$

and $\delta(\alpha) = 0$ for any other algebra generator α. (The differential δ is determined from (2.19), see Figure 2.) Computing the above complex, we obtain that

$$\begin{aligned} E_2^{**} \cong {} & \Lambda(e_7, e_{11}, e_{13}) \\ & \otimes \Lambda(s^{-1}x_3 \otimes e_3, s^{-1}x_5 \otimes e_5, s^{-1}x_9 \otimes e_9, s^{-1}x_{15} \otimes e_{15}) \\ & \otimes \Gamma[w_1, w_2, w_3, w_4] \otimes \Gamma[s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{29}] \end{aligned}$$

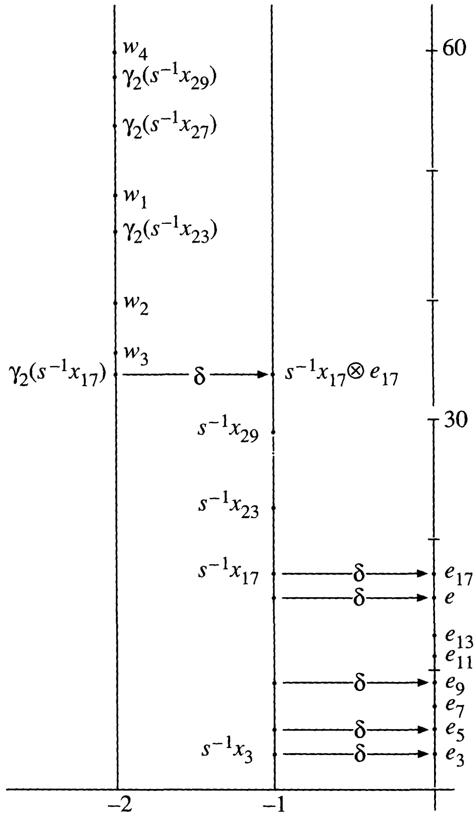

FIGURE 2

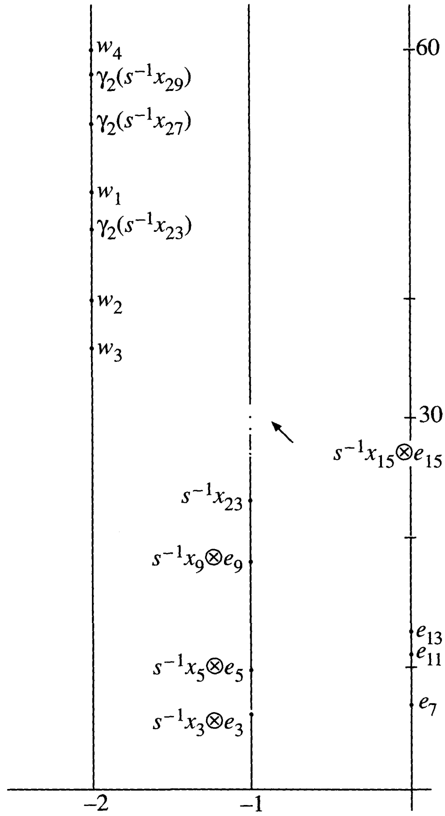

FIGURE 3

(see Figure 3). Applying the same argument as the proof of Theorem 1.15, we see that

$$d_r(\alpha) = 0 \text{ for } r \geq 2 \text{ and any algebra generator } \alpha \text{ in } E_2^{**}.$$

Hence $E_0^{**} \cong E_\infty^{**} \cong E_2^{**}$. Put

$$\begin{aligned} AG = &\{e_7, e_{11}, e_{13}, s^{-1}x_5 \otimes e_5, s^{-1}x_9 \otimes e_9, s^{-1}x_{15} \otimes e_{15}\} \\ &\cup \{\gamma_{2[f_1]}(w_1), \gamma_{2[f_2]}(w_2), \gamma_{2[f_3]}(w_3), \gamma_{2[f_4]}(w_4)\}_{f_i \geq 0} \\ &\cup \{\gamma_{2[g_1]}(s^{-1}x_{23}), \gamma_{2[g_2]}(s^{-1}x_{27}), \gamma_{2[g_3]}(s^{-1}x_{29})\}_{g_j \geq 0}. \end{aligned}$$

The usual argument of the filtration of $H^*(\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)); \mathbb{Z}/2)$ allows us to conclude that $e_i^2 = 0$ and $(s^{-1}x_i \otimes e_i)^2 = 0$. Furthermore, $h(\alpha) = 2$ for any $\alpha \in AG$ because there is no pair of non-negative integers (f, f') which satisfies

$$22 \cdot 2^f = 26 \cdot 2^{f'}, 28 \cdot 2^{f'}, 46 \cdot 2^{f'}, 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$26 \cdot 2^f = 28 \cdot 2^{f'}, 46 \cdot 2^{f'}, 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$28 \cdot 2^f = 46 \cdot 2^{f'}, 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$46 \cdot 2^f = 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$38 \cdot 2^f = 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$34 \cdot 2^f = 58 \cdot 2^{f'}$$

(see [**10**; Theorem 2.4]).

We have Theorem 1.16 by [**10**; Theorem 2.4].

References

[1] S. Araki, *Differential Hopf algebras and the cohomology* mod 3 *of the exceptional Lie groups* E_7 *and* E_8, Ann. Math., **73** (1961), 43–65.

[2] P. F. Baum and W. Browder, *The cohomology of quotients of classical groups*, Topology, **3** (1965), 305–336.

[3] A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compact connexes*, Amer. J. Math., **76** (1954), 273–342.

[4] R. Bott, *The space of loops on a Lie group*, Michigan Math. J., **5** (1958), 35–61.

[5] R. M. Kane, *On the loop spaces without* p *torsion*, Pacific J. Math., **60** (1975), 189-201.

[6] ——, *The Homology of Hopf Spaces*, North-Holland Math. Library, **40** (1988).

[7] A. Kono, *Hopf algebra structure of simple Lie groups*, J. Math. Kyoto. Univ., **17** (1977), 259–298.

[8] A. Kono and K. Kozima, *The* mod 2 *homology of the space of loops on the exceptional Lie group*, Proc. Royal Soc. Edinburgh, **112** A (1989), 187–202.

[9] D. Kraines, *The kernel of the loop suspension map*, Illinois J. Math., **21** (1977), 91–108.

[10] K. Kuribayashi, *The extension problem and the* mod 2 *cohomology of the space of loops on* $\mathrm{Spin}(N)$, Proc. Royal Soc. Edinburgh, **121A** (1992), 91–99.

[11] ——, *On the* mod p *cohomology of spaces of free loops on the Grassmann and Stiefel manifolds*, J. Math. Soc. Japan, **43**, 2, (1991), 331–346.

[12] J. McCleary, *User's guide to spectral sequences*, Publish or Perish Inc., (1985).

[13] J. M. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math., **81** (1965), 211–236.

[14] M. Mimura and H. Toda, *Cohomology operations and the homotopy of compact Lie groups* - I, Topology, **9** (1969), 317–336.

[15] D. L. Rector, *Steenrod operations in the Eilenberg-Moore spectral sequence*, Comment. Math. Helv., **45** (1970), 540–552.

[16] M. Rothenberg and N. E. Steenrod, *The cohomology of classifying spaces of H-spaces*, Bull. Amer. Math. Soc., **71** (1965), 872–875.

[17] L. Smith, *Homological algebra and the Eilenberg-Moore spectral sequence*, Trans. Amer. Math. Soc., **129** (1967), 58–93.

[18] ——, *The cohomology of stable two stage Postnikov systems*, Illinois J. Math., **11** (1968), 310–329.

[19] ——, *Cohomology of* $\Omega(G/U)$, Proc. Amer. Math. Soc., **19** (1968), 399–404.

[20] ——, *On the Künneth theorem* I, Math. Z., **166** (1970), 94–140.

[21] ——, *On the characteristic zero cohomology of the free loop space*, Amer. J. Math., **103** (1981), 887–910.

[22] E. Thomas, *Steenrod squares and H-spaces*, Ann. Math., **77** (1963), 306–317.

[23] T. Watanabe, *The homology of the loop space of the exceptional group F_4*, Osaka J. Math., **15** (1978), 463–474.

Received August 22, 1991 and in revised form November 20, 1991.

GUNMA COLLEGE OF TECHNOLOGY
GUNMA 371, JAPAN

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

A NOTE ON MURASUGI SUMS

ABIGAIL THOMPSON

We give two examples to show that the genus of knots is neither sub- nor super-additive under the Murasugi sum operation.

A number of "addition" operations can be defined on pairs of knots in S^3; the connected sum is the most obvious of these, but there are several other more complicated possibilities. A general question one can ask is: which properties of knots behave "nicely" under these operations? It has long been known that the genus of a knot is additive under connect sum. Schubert [Sc] showed that bridge number is additive minus one under connect sum.

Outstanding questions are how crossing number, unknotting number and tunnel number behave under connect sum. Only the most obvious inequalities are currently available, and they are quite weak—for example, the crossing number is obviously sub-additive, as is the unknotting number, and it is easy to show that the tunnel number of the connect sum of K_1 and K_2 is less than or equal to the sum of their tunnel numbers plus one.

A more complicated operation on pairs of knots is the band-connect sum. This operation is not well defined, since it depends on how the band is chosen. Gabai and Scharlemann simultaneously established the superadditivity of genus under band-connect sum [G1], [S].

Yet another operation combining knots is the Murasugi sum of two knots (see [G2] for a definition); this depends on a choice of Seifert surfaces for the knots as well as a choice of disks along which to do the sum. Gabai [G2] nevertheless has shown that under reasonable conditions many geometric properties of the Seifert surfaces are retained under the Murasugi sum. In particular, he has shown that the Murasugi sum of K_1 and K_2 along minimal genus Seifert surfaces R_1 and R_2 yields a minimal genus Seifert surface R for the resulting knot K, so genus is additive under Murasugi sum provided the addition is done along minimal genus surfaces. Taking the Murasugi sum of two knots can thus be considered a "natural" operation on pairs consisting of knots together with minimal genus Seifert surfaces. However, the

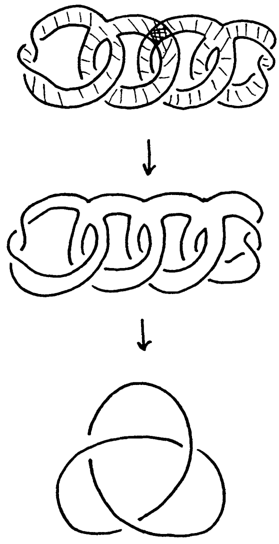

FIGURE 1

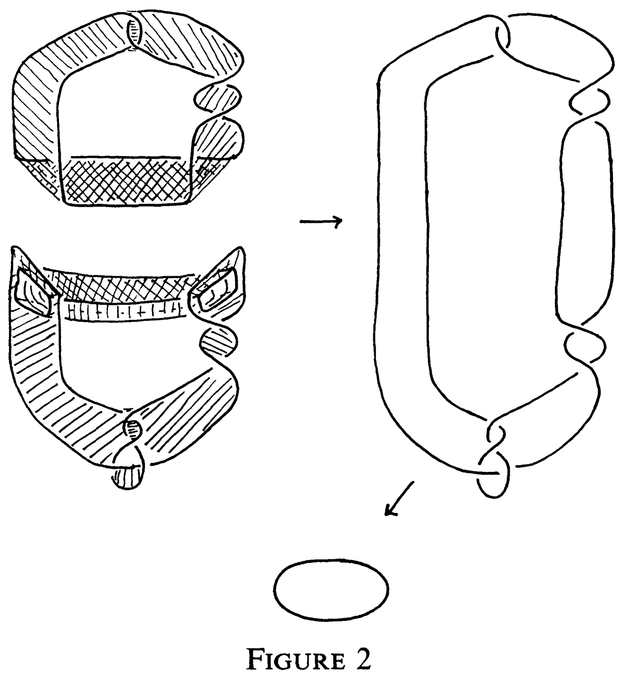

FIGURE 2

operation of constructing a Murasugi sum is not confined to minimal genus or even incompressible Seifert surfaces; we give two examples to illustrate that the genus does not behave in a predictable way in this larger category. The first [Figure 1] is an example of two trivial knots, each bounding a (compressible) genus one surface, summed along a square to yield a trefoil. The second example [Figure 2] is two figure eight knots, one bounding a genus one surface and the other bounding a (compressible) genus two surface, summed along a square to yield the trivial knot.

REFERENCES

[G1] D. Gabai, *Genus is superadditive under band connected sum*, Topology, **26**, No. 2 (1987), 209–210.

[G2] ——, *The Murasugi sum is a natural geometric operation*, Contemp. Math., vol. 20, Amer. Math. Soc., Providence, RI, 1983, pp. 131–143.

[S] M. Scharlemann, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom., **29** (1989), 557–614.

[Sc] H. Schubert, *Uber eine numerische Knoteninvariante*, Math. Z., **61** (1954), 245–288.

Received April 4, 1992. Partially supported by an NSF grant and a grant from the Alfred P. Sloan Foundation.

UNIVERSITY OF CALIFORNIA
DAVIS, CA 95616

PACIFIC JOURNAL OF MATHEMATICS

Volume 163 No. 2 April 1994

0030-8730(1994)163:2;1-I