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In this note we prove that if $N \subset M \subset P$ is an inclusion of II_1 factors with finite Jones index such that $N \subset P$ has finite depth, then $N \subset M$ and $M \subset P$ have finite depth. We show this result by studying the iterated basic constructions for $M \subset P$ and $N \subset P$. In particular our proof gives detailed information about the graphs for $N \subset M$ resp. $M \subset P$. Furthermore, we give an abstract characterization of intermediate subfactors in terms of Jones projections in $N' \cap P_1$, where $N \subset P \subset P_1$ is the basic construction for $N \subset P$ and give examples showing that if $N \subset M$ and $M \subset P$ have finite depth, then $N \subset P$ does not necessarily have finite depth.

1. Introduction. The problem of classifying subfactors of the hyperfinite II_1 factor is one of the most challenging problems in operator algebras. Starting with an inclusion $N \subset M$ of hyperfinite II_1 factors with finite Jones index $[M : N] < \infty$, one constructs the associated Jones tower of factors $N \subset M \subset M_1 \subset M_2 \subset \ldots$, where M_{i+1} is the II_1 factor obtained from the *Jones basic construction* for $M_{i-1} \subset M_i$ (see **[Jo1]**). The centralizer algebras $\{M_i' \cap M_j\}_{i \leq j}$ are finite dimensional C^*-algebras sitting in the envelopping II_1 factor $M_\infty = \overline{\bigcup M_k}^w$. Furthermore, inclusions of four such algebras

$$\begin{array}{ccccccc} M_i' & \cap & M_k & \subset & M_i' & \cap & M_{k+1} \\ & \cup & & & & \cup & \\ M_{i+1}' & \cap & M_k & \subset & M_{i+1}' & \cap & M_{k+1} \end{array}$$

satisfy certain symmetry conditions: they form what is called a *commuting square* (**[Po2]**, see also **[GHJ]**). All the information contained in this double sequence of finite dimensional algebras is actually contained in the following sequence of commuting squares

$$\begin{array}{ccccccccc} M' & \cap & M_k & \subset & M' & \cap & M_{k+1} & \subset \cdots \\ & \cup & & & & \cup & & \\ M_1' & \cap & M_k & \subset & M_1' & \cap & M_{k+1} & \subset \cdots \end{array}$$

which is an invariant for the inclusion $N \subset M$, called the *standard invariant* (**[Po4]** or *paragroup* **[Oc1]**). From this sequence one can form the inclusion $\overline{\bigcup_k M' \cap M_k}^w \subset \overline{\bigcup_k M_1' \cap M_k}^w$ of hyperfinite II_1 von Neumann algebras and ask if these algebras form a model for $N \subset M$,

i.e. are (anti-)isomorphic to the inclusion $N \subset M$. Popa introduced recently a concept of *amenability* for inclusions $N \subset M$ ([**Po3**], [**Po4**]) and showed that precisely the amenable subfactors of R, the hyperfinite II_1 factor, are classified by their standard invariant. A particular, but important class of amenable subfactors of R are the *finite depth* subfactors, referring to the condition $\sup_k \dim Z(M' \cap M_k) < \infty$, where $Z(M' \cap M_k)$ denotes the center of $M' \cap M_k$. Equivalently, this condition expresses the fact that the width of the Bratteli diagram describing the inclusions $\mathbb{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \ldots$ is bounded from a certain point on. Popa showed in ([**Po2**], see also [**Oc1**]) that finite depth subfactors N of the hyperfinite II_1 factor M are classified by an initial commuting square

$$\begin{array}{ccccccc} M' & \cap & M_{k_0} & \subset & M' & \cap & M_{k_0+1} \\ & \cup & & & & \cup & \\ M_1' & \cap & M_{k_0} & \subset & M_1' & \cap & M_{k_0+1} \end{array}$$

for k_0 large enough (which can be made precise). Subfactors of index < 4 are automatically of finite depth and the associated commuting squares can be classified in terms of graphs of Coxeter-Dynkin type A, D, E and certain connections on them ([**B-N**], [**I1**], [**I2**], [**Jo1**], [**Ka**], [**Oc1**], [**Oc2**], [**SV**]). Wenzl constructed interesting series of finite depth subfactors via braid group representations, generalizing Jones' original construction of subfactors of the hyperfinite II_1 factor. It is by now well-known that Jones' discovery of certain remarkable braid group representations in the higher relative commutants of every finite index subfactor lead him to the construction of his link invariant, the Jones polynomial. Similarly, Wenzl's subfactors carry representations of the braid group in their higher relative commutants which can be used to obtain the HOMFLY and Kauffman polynomials using the same method as Jones' original construction of his link invariant ([**Jo1**], [**Jo2**], [**We1**], [**We2**]). The simplest finite depth subfactors are obtained by letting a finite group G act by properly outer automorphisms of R and considering the inclusion $R \subset R \rtimes G$. The canonical (classifying) commuting square of this inclusion contains all the information on G and its representation theory: G can be completely recovered from the inclusion. Similarly, if $H \subset G$ is a subgroup of G of finite index, then $R \rtimes H \subset R \rtimes G$ is again a finite depth inclusion and the associated canonical commuting square can be described explicitly in terms of induced representations (for details of all this and more examples coming from groups, see [**Bi2**], [**KY**]).

It is a well-known theorem in the theory of extensions of von Neumann algebras that if G is a countable discrete group of outer automorphisms on the II_1 factor N and P is a subfactor with $N \subset P \subset N \rtimes G$, then there is a subgroup $H \subset G$ such that $P = N \rtimes H$ ([**NT**], [**Su**]). This result is quite apparent for a finite group G and it is natural to ask if a similar result holds for finite depth subfactors, where the role of the group is played by the more general object, the canonical commuting square or the paragroup. In other words, given an inclusion of II_1 factors $N \subset M \subset P$ such that $N \subset P$ has finite depth, does this force the finite depth condition on $N \subset M$ and $M \subset P$? We prove that this statement is indeed true, more precisely we show the following theorem:

THEOREM. *Let $N \subset M \subset P$ be an inclusion of II_1 factors with $[P : N] < \infty$ and assume $N \subset P$ has finite depth. Then $N \subset M$ and $M \subset P$ have finite depth.*

This theorem will follow from a detailed study of the basic construction for $N \subset P$ and $M \subset P$. We are able to describe the higher relative commutants of $M \subset P$ completely in terms of the higher relative commutants of the inclusion $N \subset P$. In particular we obtain information on the graphs for $N \subset M$ and $M \subset P$ and our proof provides an algorithm for computing these graphs from the graphs for $N \subset P$.

In §2 we collect for the convenience of the reader some facts about the basic construction, fix the notation and prove some useful lemmas. We proceed then with the proof of our theorem. Furthermore, we give some examples showing that the converse of our theorem does not hold: if $N \subset M$ and $M \subset P$ have finite depth, then $N \subset M$ need not have finite depth, in fact $N \subset M$ need not even be amenable in the sense of Popa ([**Po4**]).

In §3 we give an abstract characterization of intermediate subfactors M of a given irreducible inclusion $N \subset P$ in terms of Jones projections in $N' \cap P_1$, where $N \subset P \subset P_1$ is the basic construction for $N \subset P$. This allows us to recognize intermediate subfactors by looking at the projections in $N' \cap P_1$ and reconstruct the subfactor from these projections.[1]

2. The proof of the theorem. Let $N \subset M$ be an inclusion of II_1 factors with finite Jones index $[M : N]$. We denote by $L^2(M, \mathrm{tr}_M)$

[1] Y. Kawahigashi informed us that A. Ocneanu has a characterization of intermediate subfactors using his Fourier transform.

the closure of M in the Hilbert norm $\|x\|_2 = \mathrm{tr}_M(x^*x)^{1/2}$ induced by the unique trace tr_M on M. Let $e_N^M : L^2(M, \mathrm{tr}_M) \to L^2(N, \mathrm{tr}_N)$ be the orthogonal projection and let $J_M : L^2(M, \mathrm{tr}_M) \to L^2(M, \mathrm{tr}_M)$ be the canonical conjugation defined by $J_M(x) = x^*$, $x \in M$ viewed as a vector in $L^2(M, \mathrm{tr}_M)$. The algebra $M_1 = vN(M, e_N^M) = \langle M, e_N^M \rangle$, i.e. the von Neumann algebra generated by M and e_N^M in $B(L^2(M, \mathrm{tr}_M))$ is called the *basic construction* for $N \subset M$ ([**Jo1**]). We recall ([**Jo1**]):

(1) $e_N^M x e_N^M = E_N^M(x) e_N^M$, where $E_N^M : M \to N$ is the unique trace preserving conditional expectation from M onto N.
(2) $N = \{e_N^M\}' \cap M$.
(3) $e_N^M \langle M, e_N^M \rangle e_N^M = N e_N^M \simeq N$.
(4) $\langle M, e_N^M \rangle = J_M N' J_M$.
(5) $[J_M, e_N^M] = 0$.
(6) There is a unique trace tr_{M_1} on M_1 such that $\mathrm{tr}_{M_1}(x e_N^M) = [M : N]^{-1} \mathrm{tr}_M(x)$ for all $x \in M$.
(7) $M = \overline{\mathrm{span}\, N e_N^M N}^{w}$ ([**PiPo1**]).

It is easy to see that $N \subset M$ has finite depth iff $M \subset M_1$ has finite depth iff $\sup_k \dim Z(M' \cap M_k) < \infty$ iff $\sup_k \dim Z(M' \cap M_{2k}) < \infty$ iff $\sup_k \dim Z(M' \cap M_{2k+1}) < \infty$ iff $\sup_k \dim Z(N' \cap M_{2k}) < \infty$ iff $\sup_k \dim Z(N' \cap M_{2k+1}) < \infty$ (see for instance [**Po2**] for a proof). We will use at various instances the following simple abstract characterization of the basic construction ([**PiPo2**]): Let $N \subset M$ be an inclusion of II_1 factors with $[M : N] < \infty$ and let P be a II_1 factor containing M and a projection p such that $[p, N] = 0$ and $E_M^P(p) = [P : M]^{-1} 1_M = [M : N]^{-1} 1_M$, then P is the basic construction for $N \subset M$, i.e. there is an isomorphism from M_1 onto P leaving M pointwise fixed and carrying e_N^M to p.

It will be useful for the proof of our theorem to study the basic construction for certain reduced algebras. This is done in the next few lemmas.

LEMMA 2.1. *Let $N \subset M$ be II_1 factors, $[M : N] < \infty$. Let $p \in N$, $q \in N' \cap M$ be projections and consider the inclusion $qpNpq \subset pqMpq$ with trace $\mathrm{tr}_{pqMpq}(qpxpq) = \mathrm{tr}_M(p)^{-1} \mathrm{tr}_N(q)^{-1} \mathrm{tr}_M(pqxpq)$, $x \in M$. Then the unique trace preserving conditional expectation $E_{pqNpq}^{pqMpq} : pqMpq \to pqNpq$ is given by*

$$E_{pqNpq}^{pqMpq}(pqxpq) = \mathrm{tr}_M(q)^{-1} p E_N^M(qxq) pq, \qquad x \in M.$$

Proof. The proof is a straightforward computation and left to the reader. □

Recall that $N \subset M$ is called *extremal* iff $\mathrm{tr}_M|_{N'\cap M} = \mathrm{tr}_{N'}|_{N'\cap M}$ ([**PiPo1**]), which happens for instance whenever $N \subset M$ has finite depth.

LEMMA 2.2. *Let $N \subset M$ be an extremal inclusion of II_1 factors, $[M : N] < \infty$ and let $N \subset M \subset M_1 = \langle M, e_N^M\rangle$ be the basic construction. Suppose we have projections $p \in N' \cap M$ and $q \in M' \cap M_1$ with $\mathrm{tr}_M(p) = \mathrm{tr}_{M_1}(q)$ such that $\mathrm{tr}_{M_1}(pqe_N^M) = [M : N]^{-1}\,\mathrm{tr}_M(p)$. Then*

$$Npq \subset (pMp)q \subset pqM_1pq \tag{1}$$

is the basic construction for the pair $(Npq \subset (pMp)q) \simeq (Np \subset pMp)$.

Proof. Note that

$$[qpMpq : Npq] = [pqM_1pq : pqMpq] = \mathrm{tr}_M(p)^2[M : N]$$

since $N \subset M$ is extremal ([**PiPo1**]).

Consider $e := \mathrm{tr}_M(p)^{-1}pqe_N^Mpq \in pqM_1pq$, then

$$\begin{aligned}\mathrm{tr}_{pqM_1pq}(e) &= \mathrm{tr}_M(p)^{-2}\,\mathrm{tr}_{M_1}(e) = \mathrm{tr}_M(p)^{-3}\,\mathrm{tr}_{M_1}(pqe_N^M)\\ &= \mathrm{tr}_M(p)^{-2}[M : N]^{-1} = [pqM_1pq : pqMpq]^{-1}.\end{aligned}$$

Let m be the unique element in M satisfying $me_N^M = qe_N^M$, i.e. $m = [M : N]E_M^{M_1}(qe_N^M)$. Then $m \in N' \cap M$. We have $e^2 = \mathrm{tr}_M(p)^{-2}E_N^M(pm)pqe_N^Mpq = e$, since by hypothesis

$$\begin{aligned}[M : N]^{-1}\,\mathrm{tr}_M(p) &= \mathrm{tr}_{M_1}(pqe_N^M) = \mathrm{tr}_{M_1}(pme_N^M)\\ &= \mathrm{tr}_{M_1}(E_N^M(pm)e_N^M) = E_N^M(pm)[M : N]^{-1},\end{aligned}$$

which implies $E_N^M(pm) = \mathrm{tr}_M(p)$. In order to see that (1) is indeed the basic construction we use the above-mentioned characterization of the basic construction ([**PiPo1**]). Since e clearly commutes with Npq, we only need to check that

$$E_{pqMpq}^{pqM_1pq}(e) = \mathrm{tr}_M(p)^{-1}[M : N]^{-1}pq. \tag{2}$$

By Lemma 2.1 we know that

$$\begin{aligned}E_{pqMpq}^{pqM_1pq}(e) &= \mathrm{tr}_M(p)^{-2}pE_M^{M_1}(qe_N^Mq)pq\\ &= \mathrm{tr}_M(p)^{-2}pm[M : N]^{-1}m^*pq\\ &= \mathrm{tr}_M(p)^{-2}[M : N]^{-1}pmm^*pq.\end{aligned} \tag{3}$$

Computing traces on both sides of (3) gives

$$\mathrm{tr}_M(p)^2 = \mathrm{tr}_{M_1}(pmm^*pq) = \|pm\|_2^2 \,\mathrm{tr}_{M_1}(q)$$

and hence

$$\|pm - p\|_2^2 = \|pm\|_2^2 - \mathrm{tr}_M(p) = 0\,,$$

i.e. $pm = p$. Note that this implies in particular that $pqe_N^M = pe_N^M$ (the condition $pm = p$ is actually equivalent to the condition on the traces in the statement of the lemma). Thus

$$E_{pqMpq}^{pqM_1pq}(e) = \mathrm{tr}_M(p)^{-1}[M : N]^{-1}pq\,,$$

which completes the proof. □

REMARK 2.3. Note that if $N \subset M$ is extremal, given a projection p as in the lemma, we can always find a projection $q \in M' \cap M_1$ such that $\mathrm{tr}_{M_1}(pqe_N^M) = \mathrm{tr}_M(p)[M : N]^{-1}$, $\mathrm{tr}_{M_1}(q) = \mathrm{tr}_M(p)$. Namely, let $q := J_MpJ_M \in J_M(N' \cap M)J_M = M' \cap M_1$, where J_M denotes as usual the canonical conjugation on $L^2(M, \mathrm{tr}_M)$. We have then clearly $\mathrm{tr}_{M_1}(q) = \mathrm{tr}_M(p)$ (extremality) and $\mathrm{tr}_{M_1}(pJ_MpJ_Me_N^M) = [M : N]^{-1}\,\mathrm{tr}_M(p)$ since $pJ_MpJ_Me_N^M = pe_N^M$.

The proof of the following lemma is trivial.

LEMMA 2.4. *Let $N \subset M$ be II_1 factors, $[M : N] < \infty$, $p \in N$ a projection and $N \subset M \subset M_1 \subset \ldots$ the basic construction. Then*

$$pNp \subset pMp \subset pM_1p \subset \ldots$$

is that basic construction for $pNp \subset pMp$.

We describe now the construction which will be used to prove the theorem. Let

$$M \subset P \subset Q_1 \subset Q_2 \subset \ldots \tag{4}$$

be the Jones tower of factors obtained by iterating the basic construction for $M \subset P$. Similarly, let

$$N \subset P \subset P_1 \subset P_2 \subset \ldots \tag{5}$$

be the tower for $N \subset P$. Note that $N \subset M \subset P \subset Q_1 \subset P_1 \subset B(L^2(P, \mathrm{tr}_P))$. Let $\tilde{Q}_2 := \langle P_1, e_{Q_1}^{P_1}\rangle$ be the basic construction for $Q_1 \subset P_1$, then $P \subset Q_1 \subset P_1 \subset \tilde{Q}_2 \subset P_2 \subset B(L^2(P_1, \mathrm{tr}_{P_1}))$. Continuing this construction we obtain

$$N \subset M \subset P \subset Q_1 \subset P_1 \subset \tilde{Q}_2 \subset P_2 \subset \tilde{Q}_3 \subset P_3 \subset \ldots\,, \tag{6}$$

where $\tilde{Q}_{i-1} \subset P_i \subset \tilde{Q}_i = \langle P_{i-1}, e_{\tilde{Q}_{i-1}}^{P_{i-1}} \rangle$ is the basic construction. If we set $\alpha := [M : N]$, $\beta := [P : M]$, then $\alpha\beta = [P : N]$ and the indices of the various inclusions are indicated as follows

$$N \subset^{\alpha} M \subset^{\beta} P \subset^{\beta} Q_1 \subset^{\alpha} P_1 \subset^{\alpha} \tilde{Q}_2 \subset^{\beta} P_2 \subset^{\beta} \tilde{Q}_3 \subset \dots, \tag{7}$$

and

$$\begin{aligned} \dots \subset^{\alpha} \tilde{Q}_{2k-2} \subset^{\beta} P_{2k-2} \subset^{\beta} \tilde{Q}_{2k-1} \subset^{\alpha} P_{2k-1} \subset^{\alpha} \tilde{Q}_{2k} \\ \subset^{\beta} P_{2k} \subset^{\beta} \tilde{Q}_{2k+1} \subset^{\alpha} P_{2k+1} \subset^{\alpha} \dots. \end{aligned} \tag{8}$$

We will denote the Jones' projections in the following way: for instance $e_{\tilde{Q}_{2k-1}}^{P_{2k-1}}$ denotes the Jones projection which implements the conditional expectation from P_{2k-1} onto $\tilde{Q}_{2k-1}$.

The computation in the next lemma will be used in the theorem.

LEMMA 2.5. *With the notation as above we have*

(1) $E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = \alpha^{-1} e_{\tilde{Q}_{2k}}^{P_{2k}}$ *and hence* $e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k-1}}^{P_{2k}} e_{P_{2k}}^{P_{2k+1}} = \alpha^{-1} e_{\tilde{Q}_{2k}}^{P_{2k}} e_{P_{2k}}^{P_{2k+1}}$, $k \geq 1$.

(2) $E_{\tilde{Q}_{2k}}^{P_{2k}}(e_{P_{2k-2}}^{P_{2k-1}}) = \beta^{-1} e_{\tilde{Q}_{2k-1}}^{P_{2k-1}}$ *and hence* $e_{\tilde{Q}_{2k}}^{P_{2k}} e_{P_{2k-2}}^{P_{2k-1}} e_{P_{2k-1}}^{P_{2k}} = \beta^{-1} e_{\tilde{Q}_{2k-1}}^{P_{2k-1}} e_{P_{2k-1}}^{P_{2k}}$, $k \geq 1$, $(\tilde{Q}_1 := Q_1)$.

Proof. The proof of (1) and (2) are identical, so we prove only (1). Since $e_{\tilde{Q}_{2k}}^{P_{2k}} \in \tilde{Q}_{2k+1}$, we have that

$$e_{\tilde{Q}_{2k}}^{P_{2k}} E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{\tilde{Q}_{2k}}^{P_{2k}} e_{P_{2k-1}}^{P_{2k}}) = E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}).$$

By **[PiPo1]** we know that there is a unique element $m \in P_{2k}$ such that $E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) e_{\tilde{Q}_{2k}}^{P_{2k}} = m e_{\tilde{Q}_{2k}}^{P_{2k}}$. Applying $E_{P_{2k}}^{\tilde{Q}_{2k+1}}$ to both sides of the equation gives

$$m = \mathrm{tr}_{P_{2k+1}}(e_{\tilde{Q}_{2k}}^{P_{2k}})^{-1} E_{P_{2k}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) = \beta[P : N]^{-1} 1 = \alpha^{-1} 1.$$

Since

$$\begin{aligned} e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k-1}}^{P_{2k}} e_{P_{2k}}^{P_{2k+1}} &= e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k-1}}^{P_{2k}} e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} e_{P_{2k}}^{P_{2k+1}} \\ &= E_{\tilde{Q}_{2k+1}}^{P_{2k+1}}(e_{P_{2k-1}}^{P_{2k}}) e_{P_{2k}}^{P_{2k+1}}, \end{aligned}$$

the second part also follows. □

We restate now the theorem and give then the proof.

THEOREM 2.6. *Let $N \subset M \subset P$ be an inclusion of II_1 factors with $[P : N] < \infty$ and assume $N \subset P$ has finite depth. Then $N \subset M$ and $M \subset P$ have finite depth.*

Proof. (a) We show first that $M \subset P$ has finite depth. We actually prove the following statement by induction:

$$(9) \qquad (P \subset Q_2 \subset Q_4 \subset \cdots \subset Q_{2k}) \simeq (Pf_{k-1} \subset f_{k-1}P_2f_{k-1} \subset \cdots \subset f_{k-1}P_{2k}f_{k-1}),$$

where $f_k := e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} \cdots e_{\tilde{Q}_{2k+1}}^{P_{2k+1}}$ (note that f_k is a projection since all the $e_{\tilde{Q}_r}^{P_r}$'s commute). First we show

$$(10) \qquad (P \subset Q_1 \subset Q_2) \simeq (P \simeq Pe_{Q_1}^{P_1} \subset e_{Q_1}^{P_1}\tilde{Q}_2 e_{Q_1}^{P_1} \subset e_{Q_1}^{P_1}P_2e_{Q_1}^{P_1}).$$

Since $\tilde{Q}_2 = \langle P_1, e_{Q_1}^{P_1}\rangle$ we have $e_{Q_1}^{P_1}\tilde{Q}_2e_{Q_1}^{P_1} = Q_1e_{Q_1}^{P_1}$ and therefore $(P \subset Q_1) \simeq (Pe_{Q_1}^{P_1} \subset Q_1e_{Q_1}^{P_1})$ and hence

$$(11) \qquad \begin{aligned}(P \subset Q_1 \subset Q_2) &\simeq (Pe_{Q_1}^{P_1} \subset Q_1e_{Q_1}^{P_1} \subset \langle Q_1e_{Q_1}^{P_1}, Pe_{Q_1}^{P_1}\rangle) \\ &\simeq (Pe_{Q_1}^{P_1} \subset e_{Q_1}^{P_1}\tilde{Q}_2e_{Q_1}^{P_1} \subset e_{Q_1}^{P_1}P_2e_{Q_1}^{P_1}),\end{aligned}$$

where the last isomorphism is checked by using again the abstract characterization of the basic construction ([**PiPo2**]): Set $e := e_{Q_1}^{P_1}e_P^{P_1}e_{Q_1}^{P_1}$ and note that actually $e_{Q_1}^{P_1}e_P^{P_1} = e_P^{P_1}$. Thus e is a projection in $e_{Q_1}^{P_1}P_2e_{Q_1}^{P_1}$ of the right trace, namely β. Clearly $[e, Pe_{Q_1}^{P_1}] = 0$ and it remains to show that

$$(12) \qquad E_{e_{Q_1}^{P_1}\tilde{Q}_2e_{Q_1}^{P_1}}^{e_{Q_1}^{P_1}P_2e_{Q_1}^{P_1}}(e) = \beta^{-1}e_{Q_1}^{P_1}.$$

To simplify the notation we set $A := e_{Q_1}^{P_1}P_2e_{Q_1}^{P_1}$ and we recall that $\tilde{Q}_2 = \overline{\operatorname{span} P_1e_{Q_1}^{P_1}P_1}^{w}$. We need to check that

$$\operatorname{tr}_A(e_{Q_1}^{P_1}xe_{Q_1}^{P_1}e) = \beta^{-1}\operatorname{tr}_A(e_{Q_1}^{P_1}xe_{Q_1}^{P_1}),$$

for all $x \in \tilde{Q}_2$. Let $y \in Q_1$ with $e_{Q_1}^{P_1}xe_{Q_1}^{P_1} = ye_{Q_1}^{P_1}$, then

$$\begin{aligned}\operatorname{tr}_A(e_{Q_1}^{P_1}xe_{Q_1}^{P_1}e_P^{P_1}e_{Q_1}^{P_1}) &= [P_1 : Q_1]\operatorname{tr}_{P_2}(ye_{Q_1}^{P_1}e_P^{P_1}) = \alpha\operatorname{tr}_{P_2}(ye_P^{P_1}) \\ &= [P_1 : P]^{-1}\alpha\operatorname{tr}_{P_2}(y) = (\beta)^{-1}\alpha\operatorname{tr}_{P_2}(ye_{Q_1}^{P_1}) \\ &= \beta^{-1}\operatorname{tr}_A(e_{Q_1}^{P_1}xe_{Q_1}^{P_1}).\end{aligned}$$

This proves (12). In particular we have $(P \subset Q_2) \simeq (Pe_{Q_1}^{P_1} \subset e_{Q_1}^{P_1} P_2 e_{Q_1}^{P_1})$ (note that this implies already that $P \subset Q_2$ and hence $M \subset P$ have finite depth since $P \subset P_2$ does (assuming we know that reduced subfactors of finite depth subfactors have finite depth). Since we want to get an explicit description of the higher relative commutants, we want to prove more, namely (9)).

For clarity of exposition let us also do the next step of the induction. Since $P \subset P_2 \subset P_4$ is the basic construction, $e_{Q_1}^{P_1} \in P' \cap P_2$, $e_{\tilde{Q}_3}^{P_3} \in P_2' \cap P_4$, $\operatorname{tr}_{P_2}(e_{Q_1}^{P_1}) = \operatorname{tr}_{P_4}(e_{\tilde{Q}_3}^{P_3}) = \alpha^{-1}$, we only need to check

$$\operatorname{tr}_{P_4}(e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} e_P^{P_2}) = [P_2 : P]^{-1} \operatorname{tr}_{P_2}(e_{Q_1}^{P_1}) = [P : N]^{-2}\alpha^{-1} \tag{13}$$

in order to be able to apply Lemma 2.2. By [**PiPo2**] we know that $e_P^{P_2} = [P : N] e_{P_1}^{P_2} e_P^{P_1} e_{P_2}^{P_3} e_{P_1}^{P_2}$ and hence

$$\begin{aligned}
\operatorname{tr}_{P_4}(e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} e_P^{P_2}) &= [P : N] \operatorname{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_P^{P_1} e_{P_2}^{P_3} e_{P_1}^{P_2} e_{\tilde{Q}_3}^{P_3}) \\
&= [P : N]\alpha^{-1} \operatorname{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_P^{P_1} e_{P_2}^{P_3} e_{\tilde{Q}_2}^{P_2}) \qquad \text{(Lemma 2.5)} \\
&= \alpha^{-1} \operatorname{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_P^{P_1} e_{\tilde{Q}_2}^{P_2}) \\
&= \alpha^{-1}\beta^{-1} \operatorname{tr}_{P_4}(e_{Q_1}^{P_1} e_{P_1}^{P_2} e_{Q_1}^{P_1}) \qquad \text{(Lemma 2.5)} \\
&= [P : N]^{-2} \operatorname{tr}_{P_4}(e_{Q_1}^{P_1}) = [P : N]^{-2}\alpha^{-1}.
\end{aligned}$$

Thus we can apply Lemma 2.2 to $(P \subset Q_2) \simeq (Pe_{Q_1}^{P_1} \subset e_{Q_1}^{P_1} P_2 e_{Q_1}^{P_1}) \simeq (Pe_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} \subset e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3} P_2 e_{Q_1}^{P_1} e_{\tilde{Q}_3}^{P_3})$ and obtain

$$(P \subset Q_2 \subset Q_4) \simeq (Pf_1 \subset f_1 P_2 f_1 \subset f_1 P_4 f_1).$$

This shows (9) for $k = 1, 2$. Now suppose (9) holds for k, and we will show it for $k + 1$. To this end it is enough to show that

$$(Q_{2k-2} \subset Q_{2k} \subset Q_{2k+2}) \simeq (f_k P_{2k-2} f_k \subset f_k P_{2k} f_k \subset f_k P_{2k+2} f_k). \tag{14}$$

Note that $f_{k-2} \in P_{2k-2}$, thus by Lemma 2.4 (and [**PiPo2**]) we know that $f_{k-2}P_{2k-2}f_{k-2} \subset f_{k-2}P_{2k}f_{k-2} \subset f_{k-2}P_{2k+2}f_{k-2}$ is the basic construction. We want to apply Lemma 2.2 with $N \leftrightarrow f_{k-2}P_{2k-2}f_{k-2}$, $M \leftrightarrow f_{k-2}P_{2k}f_{k-2}$, $M_1 \leftrightarrow f_{k-2}P_{2k+2}f_{k-2}$, $p \leftrightarrow f_{k-1}$ and $q \leftrightarrow e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} f_{k-2}$. Since $f_{k-2}P_{2k-2}f_{k-2} \subset f_{k-2}P_{2k}f_{k-2}$ is clearly extremal (because $P_{2k-2} \subset P_{2k}$ has finite depth) and $\operatorname{tr}_{f_{k-2}P_{2k+2}f_{k-2}}(e_{\tilde{Q}_{2k+1}}^{P_{2k+1}} f_{k-2}) =$

$\mathrm{tr}_{f_{k-2}P_{2k}f_{k-2}}(f_{k-1}) = \alpha^{-1}$, we only need to check the trace condition in Lemma 2.2, i.e.

$$\mathrm{tr}_{f_{k-2}P_{2k+2}f_{k-2}}(f_k e^{f_{k-2}P_{2k}f_{k-2}}_{f_{k-2}P_{2k-2}f_{k-2}}) = \alpha^{-1}[P:N]^{-2}. \tag{15}$$

But using [**PiPo2**] and Lemma 2.5 we compute

$$\begin{aligned}
\mathrm{tr}_{f_{k-2}P_{2k+2}f_{k-2}}(f_k e^{f_{k-2}P_{2k}f_{k-2}}_{f_{k-2}P_{2k-2}f_{k-2}}) &= \alpha^{k-1}\,\mathrm{tr}_{P_{2k+2}}(f_k e^{P_{2k}}_{P_{2k-2}}) \\
&= \alpha^{k-1}[P:N]\,\mathrm{tr}_{P_{2k+2}}(f_k e^{P_{2k}}_{P_{2k-1}} e^{P_{2k-1}}_{P_{2k-2}} e^{P_{2k+1}}_{P_{2k}} e^{P_{2k}}_{P_{2k-1}}) \\
&= \alpha^{k-1}[P:N]\alpha^{-1}\,\mathrm{tr}_{P_{2k+2}}(f_{k-1} e^{P_{2k}}_{P_{2k-1}} e^{P_{2k-1}}_{P_{2k-2}} e^{P_{2k+1}}_{P_{2k}} e^{P_{2k}}_{Q_{2k}}) \\
&= \alpha^{k-2}\,\mathrm{tr}_{P_{2k+2}}(f_{k-1} e^{P_{2k}}_{P_{2k-1}} e^{P_{2k-1}}_{P_{2k-2}} e^{P_{2k}}_{Q_{2k}}) \\
&= \alpha^{k-1}\alpha^{-1}\beta^{-1}\,\mathrm{tr}_{P_{2k+2}}(f_{k-1} e^{P_{2k}}_{P_{2k-1}}) \\
&= [P:N]^{-2}\alpha^{k-1}\,\mathrm{tr}_{P_{2k}}(f_{k-1}) = [P:N]^{-2}\alpha^{-1}.
\end{aligned}$$

Applying Lemma 2.2 gives (14) and completes the induction, i.e. (9) holds. Therefore we proved that

$$P' \cap Q_{2r} \simeq f_{k-1}(P' \cap P_{2r})f_{k-1}, \qquad 1 \le r \le k, \tag{16}$$

which implies that $P \subset Q_2$ and hence $M \subset P$ have finite depth since $P \subset P_2$ does by assumption ($P \subset P_2$ has finite depth iff $N \subset P$ has finite depth, [**Po2**]).

(b) The fact that $N \subset M$ has finite depth follows now from a simple duality argument. We can choose P_{-1}, M_{-1} such that $P_{-1} \subset M_{-1} \subset N \subset M \subset P$ and $P_{-1} \subset N \subset P$ and $M_{-1} \subset N \subset M$ are basic constructions. $P_{-1} \subset N$ has finite depth since $N \subset P$ does by hypothesis and hence $M_{-1} \subset N$ has finite depth by what we just proved. But $M_{-1} \subset N$ has finite depth iff $N \subset M$ has finite depth, which completes the proof of the theorem. □

The main motivation for giving a detailed proof of the theorem is the fact that we want to obtain information on the principal graphs (see [**GHJ**] for terminology) for $N \subset M$ and $M \subset P$ in terms of the principal graphs for $N \subset P$. Some information can indeed be obtained by looking at the Bratteli diagrams of the inclusions of higher relative commutants associated to $N \subset P$. We summarize in the next corollary what can be read off the above proof.

COROLLARY 2.7. *Let $N \subset M \subset P$ be as in the theorem. Then the Bratteli diagram of $N \subset M$ (from 2 to 2 steps) is obtained as a*

subdiagram of the Bratteli diagram for $N \subset P$ *(from* 2 *to* 2 *steps). Similarly, the Bratteli diagram for* $P \subset Q_1$ *(from* 2 *to* 2 *steps), which is the "dual" Bratteli diagram for* $M \subset P$*, is obtained as a subdiagram of the Bratteli diagram for* $P \subset P_1$ *(from* 2 *to* 2 *steps), which is the "dual" Bratteli diagram for* $N \subset P$ *(from* 2 *to* 2 *steps). Furthermore, the method in the above proof gives an explicit algorithm to compute these Bratteli diagrams (see* (9), (16)).

Note that the corollary generalizes what happens in the situation $N \subset N \rtimes H \subset N \rtimes G$, where $H \subset G$ are finite groups and H is a subgroup of finite index of G (see for instance **[KY]**). Even for these subfactors it is impossible to find a general and more explicit relation between the principal graphs of the "big" inclusion and the ones of the two "smaller" inclusions.

Let us also remark that since finite depth subfactors are classified by their canonical commuting squares or paragroups (**[Po1]**, **[Oc1]**), our theorem can be viewed as defining a *quotient* of the canonical paragroup associated to $N \subset P$ by the one associated to $N \subset M$: the result is again a paragroup, namely the one associated to $M \subset P$. As pointed out previously by Ocneanu, the quotient G/H of two groups (viewed as paragroups) $H \subset G$ with $[G : H] < \infty$ is always a paragroup. We intend to explore these ideas further in a future paper.

We mention that Popa has shown independently the analogous statement of the theorem with "finite depth subfactor" replaced by "amenable subfactor", which does not imply our theorem.

Finally we give some examples of finite depth subfactors $N \subset M$, $M \subset P$ such that $N \subset P$ is not of finite depth and/or amenable in the sense of Popa (**[Po3]**, **[Po4]**). Let N be the hyperfinite II_1 factor and consider $N \subset N \rtimes_\alpha \mathbb{Z}_2 \subset (N \rtimes_\alpha \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2 =: P$, with α and β outer actions of $\mathbb{Z}_2$ on N such that period $(\alpha\beta) = \infty$. Then $N \subset P$ has standard graph D_∞, i.e. is not of finite depth (**[Po3]**). Haagerup showed in **[Ha]** that if there are subfactors $N \subset M \subset P$ of the hyperfinite II_1 factor where $N \subset M$ and $M \subset P$ have index 2 resp. $4\cos^2 \pi/5$ (hence are of finite depth), then $N \subset P$ cannot be amenable. Another such example was mentioned to us by V.F.R. Jones: take $\mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ with generators α and β. Let α and β act on the hyperfinite II_1 factor by properly outer automorphisms such that the action of $\mathrm{PSL}(2, \mathbb{Z})$ is ergodic on central sequences (**[Jo3]**) and consider the inclusion $N^\alpha \subset N \subset N \rtimes_\beta \mathbb{Z}_3$ of index $2 \cdot 3 = 6$, which cannot be amenable since all the central sequences for P contained in the subfactor are trivial (**[Bi1]**).

Given an inclusion of II_1 factors $N \subset P$, $[P : N] < \infty$, we would like to determine all intermediate subfactors of $N \subset P$ from looking only at this given inclusion. This can indeed be done and we settle this problem in the following section.

3. Abstract characterzation of intermediate subfactors. Consider II_1 factors $N \subset P$, $[P : N] < \infty$, not necessarily of finite depth and let $N \subset P \subset P_1$ be the basic construction. If there is an intermediate subfactor $N \subset M \subset P$, then the Jones' projection $e_M^P \in N' \cap P_1$ can be abstractly characterized among the projections in $N' \cap P_1$. Let us first collect the properties of e_M^P.

PROPOSITION 3.1. *Let $N \subset M \subset P$ be II_1 factors with $[P : N] < \infty$. Let $N_1 \subset N \subset P \subset P_1$, $M_{-1} \subset N \subset M$ and $M \subset P \subset Q_1$ be basic constructions with $N \subset M \subset P \subset Q_1 \subset P_1$ and $N_1 \subset M_{-1} \subset N \subset M \subset P$. Then*

(1) $e_M^P \in N' \cap P_1$.
(2) $e_M^P e_N^P = e_N^P$.
(3) $E_P^{P_1}(e_M^P) = [P : M]^{-1} 1_P$.
(4) $e_M^P P e_M^P = M e_M^P \subset P e_M^P$.
(5) $e_M^P e_{N_1}^N e_N^P = E_M^P(e_{N_1}^N) e_N^P$ *and* $E_M^P(e_{N_1}^N) = [P : M]^{-1} e_{M_{-1}}^N$.

Proof. (1)–(4) follow from properties of Jones projections, (5) is proved using the method of Lemma 2.5. □

It will turn out that (1)–(5) characterize the Jones projections coming from intermediate subfactors, but that actually not all these properties are needed to give this characterization.

Consider the basic construction $N \subset P \subset P_1$, choose a subfactor $N_1 \subset N$ such that $N_1 \subset N \subset P$ is the basic construction and and define the set

$\mathrm{IS}(N, P) := \{q \in N' \cap P_1$ projection such that
$(1) q e_N^P = e_N^P$.
$(2) E_P^{P_1}(q) \in \mathbb{C}$.
(3) Let m be the unique element in P satisfying
$q e_{N_1}^N e_N^P = m e_N^P$, i.e. $m = E_P^{P_1}(q e_{N_1}^N e_N^P)[P : N]$.
Assume that m is a scalar multiple of a projection.$\}$

Note that (2) is equivalent to requiring that $E_P^{P_1}(q) = \mathrm{tr}_{P_1}(q) 1_P$ and that (3) does not depend on the choice of the subfactor $N_1 \subset N$

(two different choices are conjugate by a unitary in N). Furthermore, it is clear that we always have 1, $e_N^P \in \mathrm{IS}(N, P)$ and that every Jones projection e_M^P coming from an intermediate subfactor M lies in $\mathrm{IS}(N, P)$. Conversely, we prove that any projection in $\mathrm{IS}(N, P)$ is a Jones projection coming from an intermediate subalgebra $N \subset M \subset P$.

THEOREM 3.2. *Let $N \subset P$ be II_1 factors with $[P : N] < \infty$. Then every projection $q \in \mathrm{IS}(N, P)$ implements a conditional expectation from P onto the intermediate subalgebra $M := \{q\}' \cap P$. If M is a factor, then its index $[P : M]$ in P is equal to $\mathrm{tr}_{P_1}(q)^{-1}$. In particular, $q = 1$ corresponds to the subfactor P and $q = e_N^P$ to N.*

Proof. Let $q \in \mathrm{IS}(N, P)$ and $m = E_P^{P_1}(qe_{N_1}^N e_N^P)[P : N] = \lambda p$ for some $\lambda \in \mathbb{C}$ and a projection $p \in P$. We show first that q implements a conditional expectation from P onto $M := \{q\}' \cap P$, a finite von Neumann algebra. Since $qPq = \overline{\mathrm{span}\, qNe_{N_1}^N Nq} = \overline{\mathrm{span}\, Nqe_{N_1}^N qN}$, we need to determine $qe_{N_1}^N q$. But

$$qe_{N_1}^N q = [P : N]qe_{N_1}^N e_N^P e_{N_1}^N q = [P : N]\lambda^2 pe_N^P p.$$

We first compute λ:

$$\mathrm{tr}_P(p) = \lambda^{-1}\,\mathrm{tr}_{P_1}(qe_{N_1}^N e_N^P)[P : N] = \lambda^{-1}[P : N]^{-1}.$$

Since $me_{N_1}^N = [P : N]E_P^{P_1}(qe_{N_1}^N e_N^P e_{N_1}^N) = \mathrm{tr}_{P_1}(q)e_{N_1}^N$, we get that $pe_{N_1}^N = e_{N_1}^N p$ is a projection. But $pe_{N_1}^N = \lambda^{-1}\,\mathrm{tr}_{P_1}(q)e_{N_1}^N$, thus $\lambda = \mathrm{tr}_{P_1}(q)$.

We show now that $pe_N^P p = \mathrm{tr}_P(p)pq$. It is easy to see that $E_N^P(m) = [P : N]^{-1}1_N$, hence $E_N^P(p) = \mathrm{tr}_{P_1}(q)^{-1}[P : N]^{-1}1_N = \mathrm{tr}_P(p)$. Thus

$$\begin{aligned}\|pe_N^P p - \mathrm{tr}_P(p)pq\|_2^2 &= \mathrm{tr}_{P_1}(e_N^P pe_N^P p) - 2\,\mathrm{tr}_P(p)\,\mathrm{tr}_{P_1}(qpe_N^P p) + \mathrm{tr}_P(p)^3\,\mathrm{tr}_{P_1}(q)\\ &= \mathrm{tr}_P(p)^2[P : N]^{-1} - 2\,\mathrm{tr}_P(p)\,\mathrm{tr}_{P_1}(e_N^P p) + \mathrm{tr}_P(p)^2[P : N]^{-1} = 0.\end{aligned}$$

This implies that $qe_{N_1}^N q = [P : N]\,\mathrm{tr}_{P_1}(q)^2\,\mathrm{tr}_P(p)pq = \mathrm{tr}_{P_1}(q)pq$. We have therefore $qPq = \overline{(\mathrm{span}\, NpN)}q$, in particular $qPq \subset Pq$. This allows us to define explicitly the desired conditional expectation. Let $x \in P$, $y \in P$ with $qxq = yq$, then $y = E(x) := \mathrm{tr}_{P_1}(q)^{-1}E_P^{P_1}(qxq)$. Suppose $x = x^* \in P$, then $yq = qxq = (qxq)^* = (yq)^* = qy$, which shows that $E(x) \in M$ for all $x \in P$. If $x \in M$, then $E(x) = \mathrm{tr}(q)^{-1}E_P^{P_1}(xq) = \mathrm{tr}(q)^{-1}xE_P^{P_1}(q) = x$. Furthermore, if we let tr_M

be the trace on M induced from the trace on P, then $\mathrm{tr}_M(E(x)) = \mathrm{tr}(q)^{-1}\,\mathrm{tr}_{P_1}(qxq) = \mathrm{tr}(q)^{-1}\,\mathrm{tr}(q)\,\mathrm{tr}_P(x) = \mathrm{tr}_P(x)$, i.e. E is indeed the unique trace preserving conditional expectation from P onto M with $qxq = E(x)q$, for all $x \in P$. Hence $qPq = Mq = \overline{(\mathrm{span}\, NpN)}q$, which implies $M = \overline{\mathrm{span}\, NpN}$. Note that factoriality of M does not follow automatically. Using the Pimsner-Popa estimate ([**PiPo1**]) it is now easy to see that $\lambda(M, N)^{-1} = \mathrm{tr}_P(p)^{-1}$ ($\lambda(M, N)$ denotes the generalized index for non-factors ([**PiPo1**])) and hence $\lambda(P, M)^{-1} = [P : N]\mathrm{tr}(p) = \mathrm{tr}_{P_1}(q)^{-1}$, which says $[P : M] = \mathrm{tr}_{P_1}(q)^{-1}$ if M is a factor. □

The following corollary gives the desired abstract characterization of Jones projections coming from intermediate subfactors of an irreducible inclusion $N \subset P$.

COROLLARY 3.3. *Let $N \subset P$ be II_1 factors with $[P : N] < \infty$ and suppose $N' \cap P = \mathbb{C}$. Then $\mathrm{IS}(N, P)$ is precisely the set of Jones projections coming from intermediate subfactors $N \subset M \subset P$ and gives therefore a complete description of the intermediate subfactors of $N \subset P$.*

Proof. Apply Proposition 3.1 and Theorem 3.3. □

REMARK 3.4. (1) Property (3) used in the definition of the set $\mathrm{IS}(N, P)$ can be replaced by the following condition: (3)′ $qPq \subset Pq$. One can then show that (1), (2), (3)′ are equivalent to (1), (2) and (3), thus giving an alternative definition of the set $\mathrm{IS}(N, P)$.

(2) Conditions (1)–(3) in the definition of $\mathrm{IS}(N, P)$ do not insure factoriality of the intermediate subalgebra M obtained from $q \in \mathrm{IS}(N, P)$ in general. Of course, if $N \subset P$ is irreducible, then all intermediate subalgebras are factors. Condition (2) will imply factoriality in many cases: if M is not a factor, then $E_P^{P_1}(e_M^P) = E_M^{P_1}(e_M^P)$ is a central element, i.e. of the form $E_P^{P_1}(e_M^P) = \sum_{i=1}^r \alpha_i p_i$, where $Z(M) = \bigoplus_{i=1}^r \mathbb{C}p_i$, $\sum_{i=1}^r p_i = 1_P = 1_M$. Then $E_P^{P_1}(e_M^P)$ will be a scalar iff $\alpha_i \equiv$ const., $1 \leq i \leq r$. Whether this happens or not will depend on the traces of the minimal central projections p_i. For example, if $N \subset M \subset P$ is an intermediate subfactor, then $M \vee (M' \cap P)$ is an intermediate subalgebra, which will not be a factor in general, however the Jones projection corresponding to it may a priori be in $\mathrm{IS}(N, P)$. Conversely, if M is an intermediate subalgebra (not necessarily of factor) of $N \subset P$, then it is not clear whether (3) holds in general or not.

(3) Since we are interested mainly in irreducible subfactors, the corollary gives the desired description of intermediate subfactors in terms of information just coming from $N \subset P$. Furthermore, the Jones projections coming from all intermediate *factors* are contained in the set $\mathrm{IS}(N, P)$, which will be enough information in many concrete examples.

Subfactors $N \subset P$ with intermediate subfactors as in the corollary are of course easily obtained from group actions, i.e. $N := R \subset P := R \rtimes G$, G a finite (for instance non-simple) group acting properly outer on the hyperfinite II_1 factor R. We define

DEFINITION 3.5. Let $N \subset P$ be II_1 factors, $[P : N] < \infty$, then the inclusion $N \subset P$ is called *maximal* if there is no subfactor M of P such that $N \subset M \subset P$ other than N and P themselves. Equivalently, $N \subset P$, $N' \cap P = \mathbb{C}$, is maximal iff $\mathrm{IS}(N, P) = \{1, e_N^P\}$.

Note that clearly $R \subset R \rtimes G$ is maximal iff the group G has only the trivial group as a subgroup. Since an inclusion $N \subset P$ can only be non-maximal if the index is a product of two indices, we see that all inclusions of index < 4 and those with index $\in (4, 8\cos^2 \frac{\pi}{5})$ are certainly maximal. We gave above examples of non-maximal inclusions at index 4, index $8\cos^2 \frac{\pi}{5}$ and index 6.

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