Pacific Journal of Mathematics

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Volume 163 No. 2 April 1994

PACIFIC JOURNAL OF MATHEMATICS
Vol. 163, No. 2, 1994

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Let X be a simply connected space whose $\bmod p$ cohomology is isomorphic to that of a compact, simply connected, simple Lie group as an algebra over the Steenrod algebra. We determine the algebra structure of the $\bmod p$ cohomology of ΩX algebraically. Moreover we give a method to determine the algebra structure of the $\bmod p$ cohomology of the space of loops on a homogeneous space.

0. Introduction. Let G be a compact simply connected Lie group and ΩX the space of loops on a space X. In **[4]**, R. Bott has given a method to obtain generators of the Pontryagin ring $H_*(\Omega G)$ and has determined its Hopf algebra structure explicitly for $G = \mathrm{SU}(m)$, $\mathrm{Spin}(m)$ and G_2. By applying this method, T. Watanabe **[23]** has determined the Hopf algebra structure of $H_*(\Omega F_4)$. A. Kono and K. Kozima **[8]** have determined the Hopf algebra structure over the Steenrod algebra $\mathscr{A}(2)$ of $H_*(\Omega G; \mathbb{Z}/2)$ for $G = F_4, E_6, E_7$ and E_8, without using Bott's method. In order to determine the algebra structure, they have made use of the Eilenberg-Moore spectral sequence **[16]** which converges to $H^*(G; \mathbb{Z}/2)$ and whose E_2-term is isomorphic to $\mathrm{Ext}^{**}_{H_*(\Omega G;\mathbb{Z}/2)}(\mathbb{Z}/2, \mathbb{Z}/2)$. Moreover a homotopy fiber of $\Omega x_4 \colon \Omega BG \to \Omega K(\mathbb{Z}, 4)$ has been used to examine the coalgebra structure, where $x_4 \colon BG \to K(\mathbb{Z}, 4)$ is a map representing the generator of $H^4(BG)$. The consideration of the dual of those results (**[4]**, **[8]**, **[23]**) enables us to determine the Hopf algebra structure of the $\bmod p$ cohomology of ΩG for the Lie groups G. On the other hand, we can decide the coalgebra structure of $H^*(\Omega G; \mathbb{Z}/p)$ algebraically from the algebra $H^*(G; \mathbb{Z}/p)$ over the Steenrod algebra $\mathscr{A}(p)$. The following result is due to R. M. Kane **[5]**.

THEOREM 0.1. *Suppose that X is a simply connected H-space and* (0.1): *there exists a compact, simply connected, simple Lie group G such that $H^*(X; \mathbb{Z}/p) \cong H^*(G; \mathbb{Z}/p)$ as an algebra over the* $\bmod p$ *Steenrod algebra $\mathscr{A}(p)$.* (*We do not require the existence of any map between X and G which induces the isomorphism.*)
Then $H^*(\Omega X; \mathbb{Z}/p) \cong H^*(\Omega G; \mathbb{Z}/p)$ as a coalgebra.

This result motivates the conjecture that $H^*(\Omega X; \mathbb{Z}/p)$ is isomorphic, as an algebra, to $H^*(\Omega G; \mathbb{Z}/p)$ under the condition in Theorem 0.1. In this paper, we will show

THEOREM 0.2. *If X is a simply connected space and satisfies* (0.1), *then $H^*(\Omega X; \mathbb{Z}/p) \cong H^*(\Omega G; \mathbb{Z}/p)$ as an algebra.*

(*Note X is merely a simply connected space. We do not assume that it space is an H-space.*)

Theorem 0.2 is obtained as a consequence of algebraic calculation of the algebras $H^*(\Omega G; \mathbb{Z}/p)$. In particular, when $H_*(G)$ is p-torsion free, the algebra structure of $H^*(\Omega G; \mathbb{Z}/p)$ is determined by virtue of Proposition 1.6, which asserts that algebraic calculation of $H^*(\Omega X; \mathbb{Z}/p)$ is possible when $H^*(X; \mathbb{Z}/p)$ is an exterior algebra. In order to calculate the algebra $H^*(\Omega G; \mathbb{Z}/p)$, we make use of the Steenrod operations in the Eilenberg-Moore spectral sequence ([**15**], [**20**]) and [**10**, Theorem 2.3], which is an answer to extension problems in spectral sequences.

In the latter half of this paper, we examine the algebra structure of the cohomology rings of spaces of loops on homogeneous spaces. In [**19**], L. Smith has shown the following.

THEOREM ([**19**; Theorem P2]). *Let G be a compact simply connected Lie group, U a closed connected subgroup of G and $i: U \hookrightarrow G$ the inclusion map. Consider $H^*(U; \mathbb{Z}/p)$ as an $H^*(G; \mathbb{Z}/p)$ module via the map $i^*: H^*(G; \mathbb{Z}/p) \to H^*(U; \mathbb{Z}/p)$. Then if $H^*(G; \mathbb{Z}/p)$ is an exterior algebra on odd dimensional generators, there is a filtration $\{F^{-n}H^*(\Omega(G/U); \mathbb{Z}/p); n \geq 0\}$ such that $E_0^{**}(H^*(\Omega(G/U); \mathbb{Z}/p)) \cong \mathrm{Tor}^{**}_{H^*(G;\mathbb{Z}/p)}(\mathbb{Z}/p, H^*(U; \mathbb{Z}/p))$ as a Hopf algebra.*

From this theorem and [**10**; Theorem 2.4], we will obtain a proposition (Proposition 1.10) on the algebra structure of $H^*(\Omega(G/U); \mathbb{Z}/p)$. By applying our proposition, the mod p cohomology rings of

$$\Omega(\mathrm{SU}(m+n)/\mathrm{SU}(n)), \qquad \Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(n)),$$
$$\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(m) \times \mathrm{Sp}(n))$$

can be computed. But if G is not simply connected or $H^*(G; \mathbb{Z}/p)$ is not an exterior algebra, it is not easy to calculate the cohomology ring of $\Omega(G/U)$ in general. In order to determine the algebra structure of

$$H^*(\Omega(U(m+n)/U(m) \times U(n)); \mathbb{Z}/p),$$
$$H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p), \quad H^*(\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)); \mathbb{Z}/2),$$

we cannot apply Proposition 1.10 because $U(m+n)$ and $\mathrm{SO}(m+n)$ are not simply connected and $H^*(E_8;\mathbb{Z}/2)$ is not an exterior algebra. In the concrete, we will attempt to compute the mod p cohomology rings of

$$\Omega(\mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n)), \qquad \Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n))$$

and the mod 2 cohomology ring of

$$\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)).$$

This paper is organized as follows. In §1, we state our results. In §2, we prove them by using results of **[1]**, **[2]**, **[3]**, **[7]**, **[14]** and **[22]**.

1. Results. In this paper, we may denote p^f by $p[f]$ for any prime number p. $\mathbb{K}_p$ means a field of characteristic p. In this section, for algebras A and B, $A\cong B$ means that A is isomorphic to B as an algebra.

Let G be an exceptional Lie group. When $H^*(G)$ has p-torsion, the algebra structure of the mod p cohomology of the space of loops on the exceptional Lie group G is determined by considering the Eilenberg-Moore spectral sequence converging to $H^*(\Omega G;\mathbb{Z}/p)$.

THEOREM 1.1.

(1) $$H^*(\Omega G_2;\mathbb{Z}/2)\cong\mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^4)\otimes\Gamma[w_{10},y_8],$$
$$\deg s^{-1}x_3=2,\quad \deg y_8=8,\quad \deg w_{10}=10.$$

(2) $$H^*(\Omega F_4;\mathbb{Z}/2)\cong\mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^4)\otimes\Gamma[w_{10},y_8,s^{-1}x_{15},s^{-1}x_{23}],$$
$$\deg s^{-1}x_i=i-1,\quad \deg y_8=8,\quad \deg w_{10}=10.$$

(3) $$H^*(\Omega E_6;\mathbb{Z}/2)\cong\mathbb{Z}/2[s^{-1}x_8]/(s^{-1}x_8^{16})\otimes\{\otimes_{f\geq 1}\mathbb{Z}/2[e_f]/(e_f^8)\}\otimes\Gamma[w_{10},s^{-1}x_{15},s^{-1}x_{23}]$$
$$\deg s^{-1}x_i=i-1,\quad \deg w_{10}=10,\quad \deg e_f=2^{f+2}.$$

(4) $$H^*(\Omega E_7;\mathbb{Z}/2)\cong\mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^{16})\otimes\Gamma[w_{10},w_{18},w_{34},y_{32},s^{-1}x_{15},s^{-1}x_{23},s^{-1}x_{27}],$$
$$\deg s^{-1}x_i=i-1,\quad \deg w_i=i,\quad \deg y_{32}=32.$$

(5) $H^*(\Omega E_8; \mathbb{Z}/2)$

$$\cong \mathbb{Z}/2[s^{-1}x_3]/(s^{-1}x_3^{16}) \otimes \mathbb{Z}/2[s^{-1}x_{15}]/(s^{-1}x_{15}^4)$$
$$\otimes \Gamma[w_{46}, w_{38}, w_{34}, w_{58}, y_{32}, y_{56}, s^{-1}x_{23}, s^{-1}x_{27}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg w_i = i, \quad \deg y_i = i.$$

THEOREM 1.2.

(1) $$H^*(\Omega F_4; \mathbb{Z}/3) \cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^9) \otimes \Gamma[w_{22}, y_{18}, s^{-1}x_{11}, s^{-1}x_{15}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg y_{18} = 18, \quad \deg w_{22} = 22.$$

(2) $$H^*(\Omega E_6; \mathbb{Z}/3) \cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^9) \otimes \Gamma[w_{22}, y_{18}, s^{-1}x_9, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{17}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg y_{18} = 18, \quad \deg w_{22} = 22.$$

(3) $$H^*(\Omega E_7; \mathbb{Z}/3) \cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^{27}) \otimes \{\otimes_{f\geq 1}\mathbb{Z}/3[e_f]/(e_f^9)\} \otimes \Gamma[w_{22}, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{27}, s^{-1}x_{35}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg w_{22} = 22, \quad \deg e_f = 6\cdot 3^f.$$

(4) $H^*(\Omega E_3; \mathbb{Z}/3)$

$$\cong \mathbb{Z}/3[s^{-1}x_3]/(s^{-1}x_3^{27})$$
$$\otimes \Gamma[w_{22}, w_{58}, y_{54}, s^{-1}x_{15}, s^{-1}x_{27}, s^{-1}x_{35}, s^{-1}x_{39}, s^{-1}x_{47}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg y_{54} = 54, \quad \deg w_i = i.$$

THEOREM 1.3.

$H^*(\Omega E_8; \mathbb{Z}/5)$

$$\cong \mathbb{Z}/5[s^{-1}x_3]/(s^{-1}x_3^{25})$$
$$\otimes \Gamma[w_{58}, y_{50}, s^{-1}x_{15}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{35}, s^{-1}x_{39}, s^{-1}x_{47}],$$
$$\deg s^{-1}x_i = i-1, \quad \deg y_{50} = 50, \quad \deg w_{58} = 58.$$

Before we state the algebra structure of the mod p cohomology of the space of loops on G whose integral cohomology has no p-torsion, let us define some notation.

NOTATION 1.4. Let k be a non-negative integer, p a prime number and $\wp^i$, Sq^i the Steenrod operations. Put $P(k, m) = \wp^{p^{k-1}\cdot m}\cdots\wp^m$ where $k > 0$, $\wp^i = Sq^{2i}$ if $p = 2$, and $P(0, m) = \mathrm{id}$.

The following lemma will be needed to study the Steenrod operations in the Eilenberg-Moore spectral sequence.

LEMMA 1.5. *Let H^* be a Hopf algebra over $\mathscr{A}(p)$. Suppose that H^* is isomorphic, as an algebra, to an exterior algebra on odd dimensional generators. Then we can choose generators x_i which satisfy the following properties.*

(1.1) $H^* \cong \Lambda(x_1, \dots, x_S)$, *where* $\deg x_i = 2m(i)+1$.

$P(k, m(i))x_i = \varepsilon x_j$ *for any* $k \geq 0$ *and* i, *where* $\varepsilon = 0$ *or* 1, $\deg x_j = 2m(i)p^k + 1$ *and* $x_j = 0$ *if* $(QH^*)^{2m(i)p^k+1} = 0$.

Also, for any i *and* j $(i \neq j)$, *if* $P(k, m(i))x_i = P(k', m(i))x_j$, *then* $P(k, m(i))x_i = P(k', m(j))x_j = 0$.

In Proposition 1.6, we treat a space X which satisfies the following:

(A) X is a simply connected space and

$$H^*(X; \mathbb{K}_p) \cong \Lambda(x_{2\cdot m(1)+1}, \dots, x_{2\cdot m(s)+1}),$$

where $\deg x_{2m(i)+1} = 2m(i)+1$ and $m(1) \leq \cdots \leq m(s)$.

(B) When $\mathbb{K}_p = \mathbb{Z}/p$, $H^*(X; \mathbb{Z}/p)$ has a Hopf algebra structure over $\mathscr{A}(p)$. Moreover if we choose generators $x_{2m(i)+1}$ satisfying (1.1), then one of the conditions (1.2) or (1.3) is satisfied for any $i \in J$, where $J = \{i | x_{2\cdot m(i)+1} \neq P(k, m(j))x_{2\cdot m(j)+1}$ for any $k > 0$ and $j\}$.

(1.2): $m(j)\cdot p[f] \neq m(i)$ for any $j \in J$ and $f \geq 1$.

(1.3): If there exist $j \in J$ and $f \geq 1$ such that $m(j)\cdot p[f] = m(i)$, then $f \leq k(j)$, where $k(j) = \min\{k | P(k, m(j))x_{2\cdot m(j)=1} = 0\}$. If $m(i)\cdot p[k(i)+f] = m(j)\cdot p[t]$ for some $j \in J$, $t < k(j)$ and $f \geq 1$, then $k(i) \geq k(j)$.

Let $\{F^n\}_{n\leq 0}$ be the decreasing filtration of $\Gamma = H^*(\Omega X; \mathbb{Z}/p)$ which is obtained from the Eilenberg-Moore spectral sequence converging to Γ. Roughly speaking, the condition (1.2) or (1.3) is sufficient for deciding whether, for any algebra generator x of Γ belonging to F^n, x^p and the algebra generators of Γ belonging to F^{n+1} are independent.

PROPOSITION 1.6. (1) *If* $p = 0$ *and* X *satisfies the condition* (A), *then*

$$H^*(\Omega X; \mathbb{K}_0) \cong \mathbb{K}_0[s^{-1}x_{2\cdot m(1)+1}, \dots, s^{-1}x_{2\cdot m(s)+1}],$$

where $\deg s^{-1}x_{2\cdot m(i)+1} = 2\cdot m(i)$.

(2) *Suppose that* $\mathbb{K}_p$ *is a perfect field whose characteristic is non-zero,* X *satisfies the condition* (A), *and that* $m(1)\cdot p > m(s)$. *Then*

$$H^*(\Omega X; \mathbb{K}_p) \cong \Gamma[s^{-1}x_{2\cdot m(1)+1}, \dots, s^{-1}x_{2\cdot m(s)+1}],$$

where $\deg s^{-1}x_{2\cdot m(i)+1} = 2\cdot m(i)$.

(3) *If* X *satisfies the conditions* (A) *and* (B), *then*

$$H^*(\Omega X; \mathbb{Z}/p) \cong \bigotimes_{i\in J}\left\{\bigotimes_{f\geq 0}\mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2\cdot m(i)+1})]/(\gamma_{p[f]}(s^{-1}x_{2\cdot m(i)+1})^{p[k(i)]})\right\},$$

where $\deg\gamma_{p[f]}(s^{-1}x_{2\cdot m(i)+1}) = 2\cdot m(i)\cdot p[f]$, *and* $\gamma_1(s^{-1}x_{2\cdot m(i)+1}) = s^{-1}x_{2\cdot m(i)+1}$. *Throughout Proposition* 1.6, $s^{-1}x_t$ *transgresses to* x_t.

By making use of Proposition 1.6, we can determine the algebra structure of the mod p cohomology of ΩG, where G is a compact, simply connected, simple Lie group whose integral cohomology has no p-torsion.

In Proposition 1.6, if X is a simply connected Lie group G whose type is $(2n(1)+1, \dots, 2n(t)+1)$, then $S^{2n(1)+1}\times\cdots\times S^{2n(t)+1}$ is mod 0 equivalent to G. Therefore, Proposition 1.6 (1) holds clearly in this case. Since $S^3\times S^5\times\cdots\times S^{2n-1}\simeq_p \mathrm{SU}(n)$ (mod p-equivalence) if $p\geq n$, and $S^3\times S^7\times\cdots\times S^{4n-1}\simeq_p \mathrm{Sp}(n)$ if $p\geq 2n$, Proposition 1.6 (2) holds clearly in the cases where $X=\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$.

REMARK. In the assumption of Theorem 0.2, if the condition "G is simple" is omitted, then we cannot deduce the assertion of Theorem 0.2 by applying Proposition 1.6. In fact, the condition (1.3) does not hold in general for cohomology of semi-simple Lie groups. For example, let us consider the mod 3 cohomology ring

$$H^*(\mathrm{SU}(2)\times \mathrm{Spin}(20); \mathbb{Z}/3) \cong \Lambda(x_3)\otimes\Lambda(e_3, e_7, e_{11}, \dots, e_{23}, e_{27}, e_{31}, e_{35})\otimes\Lambda(y_{19}).$$

If we take notice of the elements x_3 and y_{19}, then it follows that condition (1.3) is not satisfied because $m(j)=1$ and $m(i)=9$, that is $1\cdot 3^2 = 9$ and $f=2>1=k(j)$. This means that we cannot determine, by using our method, the mod 3 cohomology ring of the space of loops on a simply connected space X whose mod 3 cohomology is isomorphic to $H^*(\mathrm{SU}(2)\times\mathrm{Spin}(20); \mathbb{Z}/3)$.

Applying Proposition 1.6 (3), we have

THEOREM 1.7.

(1) $H^*(\Omega G_2; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{11}] & \text{if } p=3 \text{ or } p>5, \\ \bigotimes_{f\geq 0}\mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{25}) & \text{if } p=5. \end{cases}$$

(2) $H^*(\Omega F_4; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{23}] & \text{if } p > 11, \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} \\ \qquad \otimes\Gamma[s^{-1}x_{11}(5), s^{-1}x_{15}(7), s^{-1}x_{23}(11)] & \text{if } p = 5, 7 \text{ or } 11. \end{cases}$$

(3) $H^*(\Omega E_6; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_9, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{17}, s^{-1}x_{23}] & \text{if } p > 11, \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} \\ \otimes\Gamma[s^{-1}x_9, s^{-1}x_{11}(5), s^{-1}x_{15}(7), s^{-1}x_{17}, s^{-1}x_{23}(11)] & \text{if } p = 5, 7 \text{ or } 11. \end{cases}$$

(4)

$H^*(\Omega E_7; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{11}, s^{-1}x_{15}, s^{-1}x_{19}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{35}] & \text{if } p > 17, \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} \\ \otimes\Gamma[s^{-1}x_{11}(5), s^{-1}x_{15}(7), s^{-1}x_{19}, s^{-1}x_{23}(11), s^{-1}x_{27}(13), \\ \qquad s^{-1}x_{35}(17)] & \text{if } p = 5, 7, 11, 13 \text{ or } 17. \end{cases}$$

(5) $H^*(\Omega E_8; \mathbb{Z}/p)$

$$\cong \begin{cases} \Gamma[s^{-1}x_3, s^{-1}x_{15}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{35}, s^{-1}x_{39}, \\ \qquad s^{-1}x_{47}, s^{-1}x_{59}] & \text{if } p > 29, \\ \{\bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_3)]/(\gamma_{p[f]}(s^{-1}x_3)^{p[2]})\} \\ \otimes\Gamma[s^{-1}x_{15}(7), s^{-1}x_{23}(11), s^{-1}x_{27}(13), \\ \qquad s^{-1}x_{35}(17), s^{-1}x_{39}(19), s^{-1}x_{47}(23), s^{-1}x_{59}(29)] \\ & \text{if } p = 5, 7, 11, 13, 17, 19, 23 \text{ and } 29. \end{cases}$$

Throughout Theorem 1.7, $\deg s^{-1}x_i = i - 1$, $\deg s^{-1}x_i(q) = i - 1$, *and* $s^{-1}x_i(q)$ *is removed from the divided polynomial algebra if* $p = q$. *Moreover* $s^{-1}x_i$ $(s^{-1}x_i(q))$ *transgresses to* x_i, *which is a suitable free algebra generator of* $H^*(G; \mathbb{Z}/p)$.

Before we state results about the cohomology rings of spaces of loops on classical groups, let us define the following

NOTATION 1.8. Let T be a set consisting of some natural numbers. Put $M(T, p) = \{n \in T | n \neq mp^f$ for any $m \in T$ and $f \geq 1\}$ and $t(m, k) = \min\{t | 2mp^t + 1 > k\}$ for $m \in M(T, p)$.

THEOREM 1.9.

(1)

$$H^*(\Omega\,\mathrm{Spin}(2n+1); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,4n-1)]}) \right\}$$

where $T = \{1, 3, \dots, 2n-1\}$ *and* $p \neq 2$.

(2)

$$H^*(\Omega\,\mathrm{Spin}(2n); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,4n-5)]}) \right\} \otimes \Gamma[s^{-1}e'_{2n-1}]$$

where $T = \{1, 3, \dots, 2n-3\}$ *and* $p \neq 2$.

(3)

$$H^*(\Omega\,\mathrm{SU}(n); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,2n-1)]}) \right\}$$

where $T = \{1, 2, \dots, n-1\}$.

(4)

$$H^*(\Omega\,\mathrm{Sp}(n); \mathbb{Z}/p) \cong \bigotimes_{m \in M(T,p)} \left\{ \bigotimes_{f \geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}e_{2m+1})]/(\gamma_{p[f]}(s^{-1}e_{2m+1})^{p[t(m,4n-1)]}) \right\}$$

where $T = \{1, 3, \dots, 2n-1\}$ *and* $p \neq 2$.

(5) $$H^*(\Omega\,\mathrm{Sp}(n); \mathbb{Z}/2) \cong \Gamma[s^{-1}x_3, s^{-1}x_7, \dots, s^{-1}x_{4n-1}].$$

Throughout Theorem 1.9, the free algebra generator $s^{-1}e_i$ *(resp.* $s^{-1}e'_i$, $s^{-1}x_i$ *and* $s^{-1}x'_i$*) transgresses to an appropriate free algebra generator*

e_i (*resp.* e'_i, x_i *and* x'_i) *of* $H^*(G; \mathbb{Z}/p)$ (*see the proof of Theorem* 1.9 *in* §2).

Let G be a simply connected Lie group whose $\bmod p$ cohomology is an exterior algebra on odd dimensional generators, U a closed connected subgroup of G, and $i: U \hookrightarrow G$ the inclusion map. By [**13**; 7.20 Theorem (Samelson-Leray)], we see that the sub-Hopf algebra $H^*(G; \mathbb{Z}/p)\backslash\backslash i^*$ (= sub-ker i^*; see [**18**; Notation, p. 312]) of $H^*(G; \mathbb{Z}/p)$ is an exterior algebra on odd dimensional generators. Moreover, from the method of construction of $H^*(G; \mathbb{Z}/p)\backslash\backslash i^*$ (see [**18**; Proposition 1.4]), we see that $H^*(G; \mathbb{Z}/p)\backslash\backslash i^*$ is a sub-Hopf algebra of $H^*(G; \mathbb{Z}/p)$ over $\mathscr{A}(p)$. Under the above conditions and notations, the following proposition holds.

PROPOSITION 1.10. *Suppose that the condition* (1.2) *or* (1.3) *is satisfied in the algebra*

$$H^*(G; \mathbb{Z}/p)\backslash\backslash i^* \cong \Lambda(x_{2m(1)+1}, \dots, x_{2m(s)+1}),$$

where $x_{2m(i)+1}$ *are algebra generators satisfying* (1.1), *and that*

$$Q(H^*(U; \mathbb{Z}/p)//i^*)^{2m(i)\cdot p[k(i)+f]} = 0$$

for any $i \in J$ *and* $f \geq 0$. *Then*

$$\begin{aligned} &H^*(\Omega(G/U); \mathbb{Z}/p) \\ &\quad\cong \bigotimes_{i\in J}\left\{\bigotimes_{f\geq 0}\mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2m(i)+1})]/(\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^{p[k(i)]})\right\} \\ &\quad\quad \otimes H^*(U; \mathbb{Z}/p)//i^* \end{aligned}$$

as an algebra.

Applying Proposition 1.10, we have the following:

THEOREM 1.11.

(1)

$$\begin{aligned} &H^*(\Omega(\mathrm{SU}(m+n)/\mathrm{SU}(n)); \mathbb{Z}/p) \\ &\quad\cong \bigotimes_{s\in M(T,p)}\left\{\bigotimes_{f\geq 0}\mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-1)]})\right\} \end{aligned}$$

where $T = \{n, n+1, \dots, m+n-1\}$.

(2)

$$H^*(\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(n));\mathbb{Z}/p) \cong \bigotimes_{s\in M(T,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,4m+4n-1)]}) \right\}$$

where $T=\{2n+1, 2n+3, \dots, 2m+2n-1\}$ *and* $p\neq 2$.

(3)
$$H^*(\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(n));\mathbb{Z}/2) \cong \Gamma[s^{-1}x_{4n+3}, s^{-1}x_{4n+7}, \dots, s^{-1}x_{4m+4n-1}].$$

THEOREM 1.12.

$$H^*(\Omega(\mathrm{Sp}(m+n)/\mathrm{Sp}(m)\times\mathrm{Sp}(n));\mathbb{Z}/p) \cong \Gamma[s^{-1}x_{4m+3}, s^{-1}x_{4m+7}, \dots, s^{-1}x_{4m+4n-1}] \otimes \Lambda(x'_3, x'_7, \dots, x'_{4n-1})$$

where $\deg s^{-1}x_j = j-1$, $\deg x'_i = i$ *and* $m\geq n$.

The following theorems are obtained by computing in the concrete.

THEOREM 1.13.

$$H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n));\mathbb{Z}/p) \cong \begin{cases} \displaystyle\bigotimes_{s\in M(T_1,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-3)]}) \right\} & \text{if } n \text{ is odd, } m \text{ is even,} \\ \displaystyle\bigotimes_{s\in M(T_2,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-5)]}) \right\} \otimes\Gamma[s^{-1}e'_{m+n-1}] & \text{if } n \text{ and } m \text{ are odd,} \\ \displaystyle\bigotimes_{s\in M(T_3,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-3)]}) \right\} \otimes\Lambda(x_{n-1}) & \text{if } n \text{ is even, } m \text{ is odd,} \\ \displaystyle\bigotimes_{s\in M(T_4,p)} \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/p[\gamma_{p[f]}(s^{-1}x_{2s+1})]/(\gamma_{p[f]}(s^{-1}x_{2s+1})^{p[t(s,2m+2n-5)]}) \right\} \otimes\Gamma[s^{-1}e'_{m+n-1}]\otimes\Lambda(x_{n-1}) & \text{if } n \text{ and } m \text{ are even,} \end{cases}$$

where $p\neq 2$, $T_1=\{n, n+2, \dots, m+n-2\}$, $T_2=\{n, n+2, \dots, m+n-3\}$, $T_3=\{n-1, n+1, \dots, m+n-2\}$ *and* $T_4=\{n-1, n+1, \dots, m+n-3\}$.

THEOREM 1.14. *When* $n \geq 2$,

$$
\begin{aligned}
&H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n))\,;\mathbb{Z}/2) \\
&\quad \cong \left\{ \bigotimes_{j\in L,\, s_j>1} \Gamma[w_j] \right\} \\
&\quad \otimes \Lambda \otimes \left\{ \bigotimes_{m(j)\in M(T,2)} \mathbb{Z}/2[s^{-1}x_j]/(s^{-1}x_j^{2^{t(m(j),\,m+n-1)}}) \right\} \\
&\quad \otimes \left\{ \bigotimes_{j\in L'} \Gamma[w_j] \otimes \Lambda(s^{-1}x_j) \right\}
\end{aligned}
$$

where

$$L = \{j \mid j = 2m(j)+1,\, n \leq j \leq m+n-1\}, \qquad T = \{m(j) \mid j \in L\},$$

$$L' = \{j \mid j = 2m(j),\, n \leq j < \min(2n,\, m+n)\},$$

$$
\Lambda = \begin{cases}
\bigotimes_{j\in L,\, s_j=1} \Gamma[y_{4m(j)}] \quad \textit{if } m+n-2 \not\equiv 0 \bmod 4 \textit{ or } n > m, \\
\bigotimes_{j\in L,\, s_j=1} \Gamma[y_{4m(j)}] \\
\otimes \left\{ \bigotimes_{f\geq 0} \mathbb{Z}/2[\gamma_{2[f]}(y_{m+n-2})]/(\gamma_{2[f]}(y_{m+n-2})^4) \right\} \\
m(j) \neq (m+n-2)/4,\, (m+n-2)/2 \\
\qquad\qquad \textit{if } m+n-2 \equiv \bmod 4 \textit{ and } n \leq m,
\end{cases}
$$

$j\cdot 2[s_j-1] < m+n \leq j\cdot 2[s_j]$, $\deg w_j = j\cdot 2[s_j]-2$, $\deg s^{-1}x_i = i-1$, $\deg y_t = t$.

THEOREM 1.15.

$$
\begin{aligned}
&H^*(\Omega(\mathrm{U}(m+n)/\mathrm{U}(m)\times \mathrm{U}(n))\,;\mathbb{Z}/p) \\
&\quad \cong \Gamma[\tau\rho_1,\, \tau\rho_2,\, \ldots,\, \tau\rho_n] \otimes \Lambda(s^{-1}c_1,\, s^{-1}c_2,\, \ldots,\, s^{-1}c_n),
\end{aligned}
$$

where $\deg \tau\rho_i = 2m+2i-2$, $\deg s^{-1}c_i = 2i-1$, $m \geq n$.

THEOREM 1.16.

$$
\begin{aligned}
&H^*(\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3))\,;\mathbb{Z}/2) \\
&\quad \cong \Lambda(e'_7,\, e'_{11},\, e'_{13},\, u_5,\, u_9,\, u_{17},\, u_{29}) \\
&\quad\quad \otimes \Gamma[w_{38},\, w_{34},\, w_{46},\, w_{58},\, v_{22},\, v_{26},\, v_{28}]
\end{aligned}
$$

where $\deg e'_i = i$, $\deg u_j = j$, $\deg w_l = l$, $\deg v_m = m$.

Furthermore, $j^*(e_i) = e'_i$ *if* $i = 7, 11$ *or* 13, *and* $j^*(e_i) = 0$ *if* $i = 3, 5, 9, 15$, *or* 17, *where* j *is the inclusion map in the fibration*

$$\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)) \overset{j}{\hookrightarrow} \mathrm{SU}(9)/\mathbb{Z}/3 \to E_8$$

and

$$e_i \in H^*(\mathrm{SU}(9)/\mathbb{Z}/3; \mathbb{Z}/2) \cong \Lambda(e_3, e_5, \dots, e_{17}).$$

2. Proofs. In this section, we will prove all the results stated in §1.

Proof of Theorems 1.1, 1.2 *and* 1.3. Let (G, p) be one of pairs of the exceptional Lie group and the prime number in Theorem 1.1, 1.2 or 1.3. For an appropriate algebra generator x_i of $H^*(G; \mathbb{Z}/p)$, choose a continuous map f to the Eilenberg-MacLane space $K(\mathbb{Z}/p, \deg x_i)$ from G representing the generator x_i. We can compare the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ converging to $H^*(\Omega G; \mathbb{Z}/p)$ with the spectral sequence converging to $H^*(K(\mathbb{Z}/p, \deg x_i - 1); \mathbb{Z}/p)$ by using the morphism of spectral sequences which is induced by the map $f: G \to K(\mathbb{Z}/p, \deg x_i)$. By applying [**18**; Lemma 3.9], all differentials d_r are determined. This enables us to obtain the explicit form of E_∞^{**}. We have Theorems 1.1, 1.2 and 1.3 by virtue of [**10**; Theorem 2.4]. (Cf. the proof of Lemma 2.2.)

In order to prove Lemma 1.5, we will prepare a lemma.

Notation. Put $U = \{u | u \not\equiv 0 \bmod p\}$. For any $u \in U$, let $i(u)$ be the least integer i which satisfies $(QH^*)^{2up^i+1} \neq 0$.

Lemma 2.1. *For any* $u \in U$, *put* $m = up^{i(u)}$. *Under the assumptions of Lemma* 1.5, *for any* l, *we can choose a basis* $\{x_1, \dots, x_v\}$ *for* $\bigotimes_{0\le t\le l+1}(QH^*)^{2mp^t+1}$ *so as to satisfy the following conditions.*

(i) $x_1, \dots, x_v$ *are primitive.*

(ii) *If* $\deg P(k, m(i))x_i \le 2mp^{l+1} + 1$ $(\deg x_i = 2m(i) + 1)$, *then* $P(k, m(i))x_i = \varepsilon x_j$, *where* $\varepsilon = 1$ *or* 0, $\deg x_j = 2m(i)p^k + 1$ *and* $x_j = 0$ *if* $(QH^*)^{2m(i)p^k+1} = 0$.

(iii) *For any* i *and* j $(i \neq j)$, *if*

$$\deg P(k, m(i))x_i = \deg P(k', m(j))x_j \le 2mp^{l+1} + 1$$

and

$$P(k, m(i))x_i = P(k', m(j))x_j,$$

then

$$P(k, m(i))x_i = P(k', m(j))x_j = 0.$$

Proof. All basis elements x_j can be replaced by primitive elements modulo decomposables, by the Samelson-Leray theorem and associativity of homology. Let us prove this lemma by induction on dimensions. Suppose that Lemma 2.1 holds up to an integer l, that is, we can choose a basis

$$M = \{P(t_i, mp^{l(i)})x_i\}_{i\in J, 0\le t_i\le s(i)}$$

for $\bigoplus_{0\le t\le l+1}(QH^*)^{2mp^t+1}$ so that x_i is primitive, where $\deg x_i = 2mp^{l(i)}+1$ and $s(i)$ is the lesser of $l+1-l(i)$ and the integer t satisfying

$$P(t+1, mp^{l(i)})x_i = 0 \quad \text{and} \quad P(t, mp^{l(i)})x_i \neq 0.$$

We can see that basis elements of $(QH^*)^{2mp^{l+1}+1}$ can be uniquely expressed as $P(l+1-l(j), mp^{l(j)})x_j$. Let S be a subset

$$\{\wp^{mp^{l+1}}\cdot P(l+1-l(j), mp^{l(j)})x_j\}$$

of $(QH^*)^{2mp^{l+2}+1}$ which is obtained from the basis

$$\{P(l+1-l(j), mp^{l(j)})x_j\}.$$

Choose a maximal subset S' consisting of linearly independent elements of S. The subset S' is written as

$$\{\wp^{mp^{l+1}}\cdot P(l+1-l(j_i), mp^{l(j_i)})x_{j_i}\}_{1\le i\le N}.$$

If there exists an integer $j\in J-\{j_1, \dots, j_N\}$ such that

$$\wp^{mp^{l+1}}\cdot P(l+1-l(j), mp^{l(j)})x_j \neq 0,$$

then, from the maximality of S', we have that

$$\begin{aligned}\wp^{mp^{l+1}}&\cdot P(l+1-l(j), mp^{l(j)})x_j \\ &= \sum_{1\le i\le N}(-\lambda_i)\wp^{mp^{l+1}}\cdot P(l+1-l(j_i)\cdot mp^{l(j_i)})x_{j_i},\end{aligned}$$

where the coefficients λ_i are not all zero. Choose an element x_{j_t} of maximal degree from the elements x_{j_i} $(1\le i\le N)$ such that $\lambda_i\neq 0$. Put $y_{j_t} = x_{j_t} + \sum_{0\le i\le N, i\neq t}\lambda'_i P(l(j_t)-l(j_i), mp^{l(j_i)})x_{j_i}$, where $x_{j_0} = x_j$, $\lambda_0 = 1$ and $\lambda'_i = \lambda_i/\lambda_t$. By replacing x_{j_t} with y_{j_t} and $P(k, mp^{l(j_t)})x_{j_t}$ with $P(k, mp^{l(j_t)})y_{j_t}$ for all $k\le l+1-l(j_t)$, we see that $\wp^{mp^{l+1}}\cdot P(l+1-l(j_t), mp^{l(j_t)})y_{j_t} = 0$. The subset of H^* obtained from M by this replacement is a basis for H^* and satisfies the conditions (i), (ii) and (iii) up to the integer l.

If the argument started from the unique expression of the base of $(QH^*)^{2mp^{l+1}+1}$ continues infinitely, then we obtain infinitely many bases $P(l+1-l(j), mp^{l(j)})x_j$ of $(QH^*)^{2mp^{l+1}+1}$ such that $\wp^{mp^{l+1}} \cdot P(l+1-l(j), mp^{l(j)})x_j = 0$, which is a contradiction. Finally, by repeating the argument, we can obtain a basis $\{P(l+1-l(j), mp^{l(j)})x_j\}$ for $(QH^*)^{2mp^{l+1}+1}$ such that all non-zero elements

$$\wp^{mp^{l+1}} \cdot P(l+1-l(j), mp^{l(j)})x_j$$

are linearly independent. From such elements, we can obtain a basis for $\bigoplus_{0\le t\le l+2}(QH^*)^{2mp^t+1}$ which satisfies the conditions (i), (ii) and (iii). (Note that all basis elements x_j for QH^* are primitive.) Similarly, we can choose a basis for $\bigoplus_{0\le t\le 1}(QH^*)^{2mp^t+1}$ so as to satisfy (i), (ii) and (iii). This completes the proof of Lemma 2.1. □

Proof of Lemma 1.5. The vector spaces

$$P(k, up^{i(u)+r}) \cdot (QH^*)^{2up^{\iota(u)+r}+1} \quad \text{and} \quad \bigoplus_{0\le t}(QH^*)^{2u'p^{\iota(u')+t}+1}$$

do not intersect for any k, r and u, $u' \in U$ $(u \neq u')$. Therefore Lemma 1.5 follows from Lemma 2.1. □

Proof of Proposition 1.6 (1) *and* (2). Let $\{E_r, d_r\}$ be the Eilenberg-Moore spectral sequence (with $\mathbb{K}_p$-coefficients) of the path-loop fibration $\Omega X \hookrightarrow PX \to X$. Put $\Gamma = H^*(X; \mathbb{K}_p)$.

(1) In the case where $p = 0$, since $\Gamma \cong \Lambda(x_{2m(1)+1}, \dots, x_{2m(s)+1})$, we see that $E_2 \cong \mathrm{Tor}_\Gamma^{**}(\mathbb{K}_0, \mathbb{K}_0) \cong \mathbb{K}_0[s^{-1}x_{2m(1)+1}, \dots, s^{-1}x_{2m(s)+1}]$. Since the total degree of each algebra generator in E_2^{**} is even, this spectral sequence collapses at the E_2-term. Hence, by [**12**; Example 11 (page 25)], we have (1).

(2) By the same argument as in the proof of (1), we can conclude that $E_0^{**} \cong E_\infty^{**} \cong E_2^{**} \cong \Gamma[s^{-1}x_{2m(1)+1}, \dots, s^{-1}x_{2m(s)+1}]$. Therefore, a subset $S = \{\gamma_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f\ge 0, 1\le i\le s}$ of $H^*(\Omega X; \mathbb{K}_p)$ is a p-simple system of generators. In order to apply [**10**; Theorem 2.4], we must verify that

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \notin N(S) \tag{2.1}$$

and

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^P \notin N^p(S) \quad \text{for any } i \ (1 \le i \le s) \tag{2.2}$$

(see [**10**; Notation 2.2]). If there exists some integer i such that $\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \in N(S)$, then we have an equation

$$\sum_j \lambda_j \alpha_j^p + w = \gamma_{p[f]}(s^{-1}x_{2m(i)+1}), \tag{2.3}$$

where $\alpha_j \in S$ and w represented by S does not have a term

$$\lambda\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \qquad (\lambda \neq 0).$$

Comparing the degree of the elements in the equation, we see that $p^{f'+1} \cdot 2m(j) = p^f \cdot \deg s^{-1}x_{2m(i)+1} = p^f \cdot 2m(i)$ when the filtration degree of α_i is $p[f']$. Suppose that $m(j) < m(i)$. Then $f'+1 > f$ and so $p < p^{f'+1-f} = \frac{m(i)}{m(j)} \leq \frac{m(s)}{m(1)}$. But this contradicts the assumption $\frac{m(s)}{m(1)} < p$. For a similar reason, the case $m(j) > m(i)$ does not occur. Hence we have that $m(j) = m(i)$. Thus each α_j in the equation (2.3) is written as $\gamma_{p[f-1]}(s^{-1}x_{2m(t_j)+1})$, where $m(t_j) = m(i)$. The element α_j^p is in a smaller filter than the filter including $\gamma_{p[f]}(s^{-1}x_{2m(i)+1})$. From the equation (2.3), we have that

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) - w = \sum_j \lambda_j \alpha_j^p. \tag{2.4}$$

Let l be the least of the filtration degrees of the terms in the left-hand side of (2.4). Consider the equation (2.4) in $E_0^{l,*}$. The right-hand side of (2.4) is zero and the left-hand side is non-zero. Finally, we obtain (2.1). In a similar manner, we have (2.2). From the above argument, we see that $h(\gamma_{p[f]}(s^{-1}x_{2m(i)+1})) = p$ for any i (see [**10**; Theorem 2.4]). Hence we have (2) by applying [**10**; Theorem 2.4]. □

In order to prove Proposition 1.6 (3) by using [**10**; Theorem 2.4], we must obtain a good p-simple system of generators for $H^*(\Omega X; \mathbb{Z}/p)$. First, applying the same argument as in the proof of (2), we can conclude that $E_0^{**} \cong E_\infty^{**} \cong E_2^{**} \cong \Gamma[s^{-1}x_{2m(1)+1}, \dots, s^{-1}x_{2m(s)+1}]$, where $\{E_r, d_r\}$ is the Eilenberg-Moore spectral sequence (with $\mathbb{Z}/p$-coefficients) of the path loop fibration $\Omega X \hookrightarrow PX \to X$. Therefore, we can choose a subset $S = \{\gamma_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f\geq 0,\, 1\leq i\leq s}$ of $H^*(\Omega X; \mathbb{Z}/p)$ as a p-simple system of generators for $H^*(\Omega X; \mathbb{Z}/p)$. The following lemma guarantees that we can choose a good p-simple system of generators.

LEMMA 2.2. *A p-simple system of generators*

$$\widetilde{S} = \{\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f\geq 0,\, 1\leq i\leq s}$$

for $H^*(\Omega X; \mathbb{Z}/p)$ *which satisfies the following conditions* (2.5), (2.6) *and* (2.7) *can be organized from the system* S.

(2.5) $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^r+1})^p = \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^{r+1}+1})$ *for any* $i \in J$ *and* $0 \le r \le k(i) - 2$. (*About the integer* $k(i)$ *and the set* J *of integers, see the remarks following Lemma* 1.5.)

(2.6) $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \notin N(\widetilde{S})$.

(2.7) $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^{k(i)-1}+1})^p \notin N^p(\widetilde{S})$ *for any* $i \in J$.

Proof of Proposition 1.6 (3). Let AG be a subset

$$\{\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})\}_{i\in J, f\ge 0}$$

of $\widetilde{S}$. By Lemma 2.2, we see that the conditions of [**10**; Theorem 2.4] are satisfied and that $h(\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})) = p[k(i)]$. Thus we have (3) by virtue of [**10**; Theorem 2.4]. □

Lemma 2.2 can be proved by virtue of the following lemma.

LEMMA 2.3. *In the module* $F^{-p[f]}H^*(\Omega X; \mathbb{Z}/p)$, *if* $k(i) = 1$, *then* $\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^p = w_0$ *and if* $k(i) > 1$, *then* $\gamma_{p[f]}(s^{-1}x_{2m(i)p^t+1})^p = \gamma_{p[f]}(s^{-1}x_{2m(i)p^{t+1}+1}) + w_t$ *for any* $0 \le t \le k(i) - 2$, *where* $w_n \in F^{-p[f]+1}H^*(\Omega X; \mathbb{Z}/p)$. (*See Figure* 1.)

Proof. By [**15**], we know that the module E_r^{**} is an $\mathscr{A}(p)$-module and that the isomorphisms $E_0^{**} \cong E_\infty^{**}$ and $E_\infty^{**} \cong E_2^{**}$ are morphisms of $\mathscr{A}(p)$-modules, where $\mathscr{A}(p)$ is the Steenrod algebra. Let us consider $\wp_{EM}^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1})$ in E_∞^{**} for any $i \in J$. By identifying the $\mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p, \mathbb{Z}/p)$ which is obtained from the Koszul resolution and that which is obtained from the bar resolution, we can regard $\gamma_{p[f]}(s^{-1}x_{2m(i)+1})$ as

$$\underbrace{[x_{2m(i)+1}|x_{2m(i)+1}|\cdots|x_{2m(i)+1}]}_{p[f]}.$$

(See [**19**; Proposition 1.1] and [**11**; Proposition 1.2].) Therefore

$$\begin{aligned}\wp_{EM}^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) &= \wp_{EM}^{m(i)\cdot p[f]}[x_{2m(i)+1}|\cdots|x_{2m(i)+1}] \\ &= [\wp^{m(i)}x_{2m(i)+1}|\cdots|\wp^{m(i)}x_{2m(i)+1}] \\ &\quad + \sum[\wp^{l(1)}x_{2m(i)+1}|\cdots|\wp^{l(n(f))}x_{2m(i)+1}], \\ &\qquad l(1) + \cdots + l(n(f)) = m(i)\cdot p[f], \\ &\qquad (l(1), \dots, l(n(f))) \ne (m(i), \dots, m(i)).\end{aligned}$$

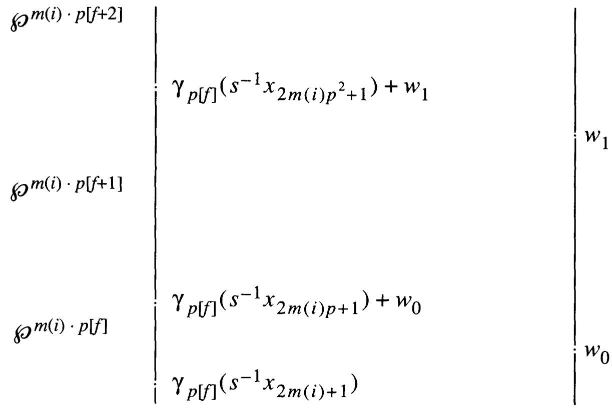

FIGURE 1

In the above last expression, the second summations are zero from the instability axiom of the Steenrod operation. From (1.1),

$$\wp^{m(i)}x_{2m(i)+1} = \begin{cases} x_{2m(i)p+1} & \text{if } k(i) > 1 \\ 0 & \text{if } k(i) = 1 \end{cases} \quad \text{in } H^*(X; \mathbb{Z}/p).$$

Note that there is an integer j such that $m(i)p = m(j)$ if $k(i) > 1$. Hence we obtain that

$$\wp_{EM}^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) = \begin{cases} \gamma_{p[f]}(s^{-1}x_{2m(i)p+1}) & \text{if } k(i) > 1, \\ 0 & \text{if } k(i) = 1, \end{cases}$$

in $E_\infty^{-p[f],*}$.

Therefore, we see that

$$\begin{aligned} \gamma_{p[f]}(s^{-1}x_{2m(i)+1})^p &= \wp^{m(i)\cdot p[f]}\gamma_{p[f]}(s^{-1}x_{2m(i)+1}) \\ &= \begin{cases} \gamma_{p[f]}(s^{-1}x_{2m(i)p+1}) & \text{if } k(i) > 1, \\ 0 & \text{if } k(i) = 1, \end{cases} \end{aligned}$$

in $E_0^{-p[f],*}$, where $\wp^i$ is the ordinary Steenrod operation. This fact allows us to conclude that

$$\gamma_{p[f]}(s^{-1}x_{2m(i)+1})^p = \begin{cases} \gamma_{p[f]}(s^{-1}x_{2m(i)p+1}) + w_0 & \text{if } k(i) > 1 \\ w_0 & \text{if } k(i) = 1, \end{cases}$$

in $F^{-p[f]}H^*(\Omega X; \mathbb{Z}/p)$, where $w_0 \in F^{-p[f]+1}H^*(\Omega X; \mathbb{Z}/p)$. Using the same argument as above, it follows that the latter half of Lemma 2.3 holds.

Proof of Lemma 2.2. Put

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^r+1}) = \gamma_{p[f]}(s^{-1}x_{2m(i)p^r+1}) + w_0^{p[r-1]} + w_1^{p[r-2]} + \cdots + w_{r-1}$$

for $1 \le r \le k(i)-1$, and put $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) = \gamma_{p[f]}(s^{-1}x_{2m(i)+1})$. From Lemma 2.3, it follows that (2.5) holds.

Let $\widetilde{S}$ be the subset of $H^*(\Omega X; \mathbb{Z}/p)$ which is obtained from S by replacing $\gamma_{p[f]}(s^{-1}x_{2m(i)p^r+1})$ with $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p^r+1})$ in S for any $i \in J$. Let us prove (2.6). If $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \in N(\widetilde{S})$ for some $i \in J$, then we have following:

$$(2.8) \qquad \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) = \sum_j \mu_j \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})^p + w$$

in $H^*(\Omega X; \mathbb{Z}/p)$, where $\mu_j \neq 0$ and w represented by $\widetilde{S}$ does not have a term $\lambda \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})$ $(\lambda \neq 0)$.

First let us consider the case where i satisfies (1.2). Choose an integer j in the right-hand side of the equality (2.8) such that $j \in J$. By comparing the degrees of the elements in the equality (2.8), we have that $2m(i) \cdot n(f) = p \cdot 2m(j) \cdot p[f(j)]$. From (1.2), we can conclude that $m(i) = m(j)$. Hence

$$(2.9) \qquad \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) \in F^{-p[f-1]}H^*(\Omega X; \mathbb{Z}p).$$

Choose an integer j so that $j \notin J$. Then there exist some integers $t \in J$ and $n \in \mathbb{N}$ such that $x_{2m(j)+1} = P(n, m(t))x_{2m(t)+1}$. Since $\gamma_{p[f(j)]}(s^{-1}x_{2m(j)+1}) = \gamma_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n]}$, from (2.8), we see that $2m(t) \cdot [f(j)] \cdot p[n+1] = 2m(i) \cdot p[f]$. From the condition (1.2), we have that $m(i) = m(t)$ and $f > f-n-1 = f(j)$. Thus we can conclude that

$$\begin{aligned}(2.10) \quad \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) &= P(n, m(t))\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1}) \\ &= P(n, m(t))\tilde{\gamma}_{p[f-n-1]}(s^{-1}x_{2m(t)+1}) \\ &\in F^{-p[f-n-1]}H^*(\Omega X; \mathbb{Z}/p).\end{aligned}$$

From (2.9) and (2.10), we see that the equality (2.8) causes a contradiction to the module structure of E_0^{**}. Thus we have (2.6).

Next let us consider the case that i satisfies the condition (1.3). Assume that there exists an element $\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})$ which satisfies $f(j) > f$ in (2.8). Applying the same argument as above, we see that there exist integers $t \in J$ and $n \in \mathbb{N}$ such that $\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) = \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n]}$. If $n+1 < k(t)$, then

$$\text{fil-deg}\, \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n+1]} = -p[f(j)].$$

Therefore, by using the usual argument of the filtration, we see that (2.8) causes a contradiction. Hence $n+1 \ge k(t)$. From the argument of the total degree in (2.8), we obtain that $p[f]\cdot 2m(i) = p[f(j)]\cdot p[n+1]\cdot 2m(t)$ and so $p[f]\cdot m(i) = p[f(j)+n+1]\cdot m(t)$. But this equality contradicts the condition(1.3) because $f(j) > f$ and $n+1 \geq k(t)$. Hence we conclude that $f(j) \leq f$ for any j in (2.8). Suppose that $f(j) = f$ for some j. From the condition (1.3) and the fact that $m(i) = p[n+1]\cdot m(t)$, where i and $t \in J$, it follows that

$$\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n+1]} \in F^{-p[f]+1}H^*(\Omega X;\mathbb{Z}/p)$$

if $n+1 = k(t)$ and that

$$\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t)+1})^{p[n+1]} = \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(t)p^{n+1}+1})$$

if $n+1 < k(t)$. From (2.8), we have an equality:

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) = \sum \lambda_u \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(u)+1}) + w \quad \text{in } E_0^{-p[f],*},$$

where $\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \neq \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(u)+1})$ and $\lambda_u \neq 0$. But this equality contradicts the fact that $\widetilde{S}$ is a p-simple system of generators for E_0^{**}. Finally, $f(j) < f$ for any j, which is a contradiction. We have (2.6).

Let us verify (2.7). If there exists an integer i such that

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p[k(i)-1]+1})^p \in N^p(\widetilde{S}),$$

then we have the following:

$$(2.11)\quad \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p[k(i)-1]+1})^p = \sum_j \lambda_j \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}) + w'$$

in $H^*(\Omega X;\mathbb{Z}/p)$, where w' expressed by $\widetilde{S}$ does not include terms $\lambda\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})$ $(\lambda \neq 0)$.

Suppose that there exists an integer j in (2.11) such that $j \in J$. By applying the same argument as the proof of (2.6), we see that the equality (2.11) causes a contradiction. Hence it follows that $j \notin J$ for any j in (2.11). For any j, there exist integers $t_j \in J$ and n_j such that

$$\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t_j)+1})^{p[n_j]} = \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1}).$$

From (2.11), we have the following equality:

$$(2.12)\quad \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})^{p[k(i)]} = \sum_j \lambda_j \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t_j)+1})^{p[n_j]} + w'.$$

We can suppose that the element

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)p[k(i)-1]+1})^p \quad (= \tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1})^{p[k(i)]})$$

has the least degree of elements $\tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(j)+1})^{p[k(j)]}$ which belong to $N^p(\widetilde{S})$. Hence

$$\tilde{\gamma}_{p[f]}(s^{-1}x_{2m(i)+1}) \neq 0 \quad \text{and} \quad \tilde{\gamma}_{p[f(j)]}(s^{-1}x_{2m(t_j)+1}) \neq 0$$

in $\bigoplus_{u<p[f+k(i)]\cdot 2m(i)}(QH^*(\Omega X; \mathbb{Z}/p))^u$ (cf. the proof of [**10**; Proposition 2.5]). When the condition (1.2) is satisfied, it follows that the equality (2.3) induces a contradiction (compare the degrees of the elements in the left-hand side of (2.12) with those of the right-hand side). If the condition (1.3) is satisfied, then $k(i) \geq k(t_j)$ for any j. Therefore, by regarding $H^*(\Omega X; \mathbb{Z}/p)$ as an algebra which is a tensor product of monogenic algebras, we see that the equality (2.12) induces an equality contradicting the algebra structure of $H^*(\Omega X; \mathbb{Z}/p)$ (cf. the proof of [**10**; Proposition 2.5] and [**13**; 7.11 Theorem (Borel)]). Finally, we have (2.7). □

Proof of Theorem 1.7. By using the result in [**14**] concerning the Steenrod operation in $H^*(G; \mathbb{Z}/p)$ and Proposition 1.6 (3), we can have this theorem. □

Proof of Theorem 1.9. (1) As is known,

$$H^*(\mathrm{Spin}(2n+1); \ \mathbb{Z}/p) \cong \Lambda(e_3, e_7, \dots, e_{4n-1})$$

and

$$\wp^k e_{2m(i)+1} = \binom{m(i)}{k} e_{2m(i)+2k(p-1)+1} \tag{2.13}$$

if there exists the algebra generator $e_{2m(i)+2k(p-1)+1}$, and $\wp^k e_{2m(i)+1} = 0$ if indecomposable elements do not exist on the degree $2m(i) + 2k(p-1)+1$. Therefore the set $\{m(i) | i \in J\}$ is equal to $M(T, p)$ and the number $k(i)$ is equal to $t(m, n)$. By virtue of Proposition 1.6 (3), we have (1).

(2) Since $\mathrm{Spin}(2n-1) \times S^{2n-1} \simeq_p \mathrm{Spin}(2n)$, it follows that

$$\Omega\,\mathrm{Spin}(2n-1) \times \Omega S^{2n-1} \simeq_p \Omega\,\mathrm{Spin}(2n).$$

Hence we obtain (2) from (1). (In this case, since the condition (1.3) is satisfied, (2) can be proved by applying Proposition 1.6 (3) without using (1).)

(3) and (4). If $p \neq 2$, then (2.13) holds in

$$H^*(\mathrm{SU}(n); \ \mathbb{Z}/p) \cong \Lambda(e_3, e_5, \dots, e_{2n-1})$$

and

$$H^*(\mathrm{Sp}(n);\ \mathbb{Z}/p) \cong \Lambda(e_3, e_7, \dots, e_{4n-1}).$$

If $p = 2$, then $Sq^{2j}e_{2i-1} = \binom{i-1}{j}e_{2i+2j-1}$ in $H^*(\mathrm{SU}(n); \mathbb{Z}/2)$, where $e_{2t-1} = 0$ if $t > n$. By applying Proposition 1.6 (3), we can obtain (3) and (4).

(5) By considering the degrees of the subalgebra generators of $H^*(\Omega X; \mathbb{Z}/2)$, we see that $P(k, m)x_{2m+1} = 0$ for any m and $k > 0$. Therefore, $J = \{1, 2, \dots, n\}$, $m(i) = 2i-1$ and $k(i) = 1$ in Proposition 1.6 (3). We have (5) by Proposition 1.6 (3). □

The method used to prove Theorems 1.1, 1.2, 1.3, 1.7, 1.9 and [**10**; Theorem 2.4] is indeed algebraic, that is, properties of G as Lie groups are not used. Therefore we can have Theorem 0.2.

Proposition 1.10 can be deduced from the results of [**19**].

Proof of Proposition 1.10. By virtue of [**19**; Theorem P2], it follows that

$$\begin{aligned}(2.14)\quad E_0^{**}(H^*(\Omega(G/U); \mathbb{Z}/p)) &\cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p, H^*(U; \mathbb{Z}/p)) \\ &\cong H^*(U; \mathbb{Z}/p)//i^* \otimes \mathrm{Tor}_{\Gamma\backslash\backslash i^*}^{**}(\mathbb{Z}/p, \mathbb{Z}/p),\end{aligned}$$

as Hopf algebras (see [**19**; Proposition 1.5]), where $\Gamma = H^*(G; \mathbb{Z}/p)$,

$$\begin{aligned}&[H^*(U; \mathbb{Z}/p)//i^* \otimes \mathrm{Tor}_{\Gamma\backslash\backslash i^*}^{**}(\mathbb{Z}/p, \mathbb{Z}/p)]^{s,t} \\ &\quad = \bigoplus_{m+n=t} [(H^*(U; \mathbb{Z}/p)//i^*)^m \otimes \mathrm{Tor}_{\Gamma\backslash\backslash i^*}^{s,n}(\mathbb{Z}/p, \mathbb{Z}/p)].\end{aligned}$$

Moreover, from the proof of [**19**; Theorem P2], we see that the filtration $\{F^{-n}H^*(\Omega(G/U); \mathbb{Z}/p)\}$ is given from the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of a fibration $\Omega(G/U) \hookrightarrow U \xrightarrow{i} G$, and that the isomorphism (2.14) is as follows:

$$E_0^{**}(H^*(\Omega(G/U); \mathbb{Z}/p)) \cong E_\infty^{**} \cong E_2 \cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p, H^*(U; \mathbb{Z}/p)).$$

Therefore, we can conclude that

(2.15) the isomorphism (2.14) is a morphism of $\mathscr{A}(p)$-modules.

Since $H^*(U; \mathbb{Z}/p)//i^*$ is a Hopf algebra, by the Hopf-Borel theorem ([**13**; 7.11 Theorem]), it follows that $H^*(U; \mathbb{Z}/p)//i^*$ is isomorphic to

$$\Lambda(y_1, \dots, y_t) \otimes \mathbb{Z}/p[u_1, \dots, u_m]/(u_1^{p[f(1)]}, \dots, u_m^{p[f(m)]})$$

as an algebra, where y_i and u_i are appropriate algebra generators.

Since $\Gamma\backslash\backslash i^* = H^*(G; \mathbb{Z}/p)\backslash\backslash i^* \cong \Lambda(x_{2m(1)+1}, \dots, x_{2m(s)+1})$, we obtain that

$$\mathrm{Tor}_{\Gamma\backslash\backslash i^*}^{**}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \Gamma[s^{-1}x_{2m(1)+1}, \dots, s^{-1}x_{2m(s)+1}].$$

Let us express the element in E_0^{**} and its representative element with the same notation. Let S be a subset

$$\{y_i\}_{1\leq i\leq t} \cup \{u_j\}_{1\leq j\leq m} \cup \{\gamma_{p[f]}(s^{-1}x_{2m(i)+1})\}_{f\geq 0,\, 1\leq i\leq s}$$

of $H^*(\Omega(G/U); \mathbb{Z}/p)$, where $\deg \gamma_{p[f]}(s^{-1}x_{2m(i)+1}) = p[f] \cdot 2m(i)$. Then S is a p-simple system of generators for $H^*(\Omega(G/U); \mathbb{Z}/p)$. From (2.15), by using the same argument as the proof of Proposition 1.6 (3), we can have this proposition. □

Proof of Theorem 1.11. Let i: $\mathrm{SU}(n) \to \mathrm{SU}(m+n)$ be the inclusion map. We know that $i^*(x_{2i-1}) = x_{2i-1}$ if $1 \leq i \leq n$ and that $i^*(x_{2i-1}) = 0$ if $n < i \leq m+n$, where i^* is the morphism of algebras from $H^*(\mathrm{SU}(m+n); \mathbb{Z}/p) \cong \Lambda(x_3, x_5, \dots, x_{2m+2n-1})$ into $H^*(\mathrm{SU}(n); \mathbb{Z}/p) \cong \Lambda(x_3, x_5, \dots, x_{2n-1})$, and the x_i are appropriate generators of each algebra. Hence we can conclude that

$$H^*(\mathrm{SU}(n); \mathbb{Z}/p)//i^* = 0$$

and that

$$H^*(\mathrm{SU}(m+n); \mathbb{Z}/p)\backslash\backslash i^* = \Lambda(x_{2n+1}, x_{2n+3}, \dots, x_{2m+2n-1}).$$

By applying Proposition 1.10, we can obtain Theorem 1.11 (1). Similarly, we have Theorem 1.11 (2) and (3). □

Proof of Theorem 1.12. Let i: $\mathrm{Sp}(m) \times \mathrm{Sp}(n) \to \mathrm{Sp}(m+n)$ be the inclusion map and Bi: $B\,\mathrm{Sp}(m) \times B\,\mathrm{Sp}(n) \to B\,\mathrm{Sp}(m+n)$ the map which is induced from i. We know that $Bi^*(q_i) = \sum_{j+k=i} q'_j \cdot q''_k$, where Bi^* is the morphism of algebras from

$$H^*(B\,\mathrm{Sp}(m+n); \mathbb{Z}/p) \cong \mathbb{Z}/p[q_1, q_2, \dots, q_{m+n}]$$

into

$$H^*(B\,\mathrm{Sp}(m) \times B\,\mathrm{Sp}(n); \mathbb{Z}/p) \cong \mathbb{Z}/p[q'_1, q'_2, \dots, q'_m, q''_1, \dots, q''_n],$$

q_i, q'_i, and q''_i are appropriate algebra generators of each algebra, and $\deg q_i = \deg q'_i = \deg q''_i = 4i$. Therefore, we see that $i^*(x_{4i-1}) = x'_{4i-1} + x''_{4i-1}$ if $1 \leq i \leq n$, $i^*(x_{4i-1}) = x'_{4i-1}$ if $n+1 \leq i \leq m$, and $i^*(x_{4i-1}) = 0$ if $m+1 \leq i$, where i^* is the morphism of algebras from i^*: $H^*(\mathrm{Sp}(m_n); \mathbb{Z}/p) \cong \Lambda(x_3, x_7, \dots, x_{4m+4n-1})$ into

$$\begin{aligned} &H^*(\mathrm{Sp}(m) \times \mathrm{Sp}(n); \mathbb{Z}/p) \\ &\quad \cong \Lambda(x'_3, x'_7, \dots, x'_{4m-1}, x''_3, x''_7, \dots, x''_{4n-1}), \end{aligned}$$

and the algebra generators x_i, x'_i and x''_i transgress to q_i q'_i and q''_i respectively. Hence we can conclude that

$$H^*(\mathrm{Sp}(m) \times \mathrm{Sp}(n); \mathbb{Z}/p)//i^* \quad \text{and} \quad H^*(\mathrm{Sp}(m+n); \mathbb{Z}/p)\backslash\backslash i^*$$

are isomorphic to

$$\Lambda(x'_3, x'_7, \dots, x'_{4n-1}) \quad \text{and} \quad \Lambda(x_{4m+3}, x_{4m+7}, \dots, x_{4m+4n-1})$$

respectively. We have Theorem 1.12 by virtue of Proposition 1.10. □

Proof of Theorem 1.13. Let p be an odd prime. As is known,

$$H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/p) \cong \begin{cases} \Lambda(e_{2n+1}, e_{2n+5}, \dots, e_{2m+2n-3}) & \text{if } n \text{ is odd and } m \text{ is even}, \\ \Lambda(e_{2n+1}, e_{2n+5}, \dots, e_{2m+2n-5}, e'_{m+n-1}) & \text{if } n \text{ and } m \text{ are odd}, \\ \Lambda(e_{2n+3}, e_{2n+7}, \dots, e_{2m+2n-3}) \otimes \mathbb{Z}/p[x_n]/(x_n^2) & \text{if } n \text{ is even and } m \text{ is odd}, \\ \Lambda(e_{2n+3}, e_{2n+7}, \dots e_{2m+2n-5}, e'_{m+n-1}) \otimes \mathbb{Z}/p[x_n]/(x_n^2) & \text{if } n \text{ and } m \text{ are even}, \end{cases}$$

and $\wp^k e_{2m(i)+1} = \binom{m(i)}{k} e_{2m(i)+2k(p-1)+1}$, where $\wp^k e_{2m(i)+1} = 0$ if indecomposable elements do not exist on the degree $2m(i)+2k(p-1)+1$.

Consider the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the fibration

$$\Omega(\mathrm{SO}(m+m)/\mathrm{SO}(n)) \hookrightarrow \mathrm{SO}(n)/\mathrm{SO}(n-1) \to \mathrm{SO}(m+n)/\mathrm{SO}(n-1).$$

We have that

$$E_2^{**} \cong \mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p, H^*(S^{n-1}; \mathbb{Z}/p))$$

and

$$E_r^{**} \Rightarrow H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p),$$

where $\Gamma = H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n-1); \mathbb{Z}/p)$.

Let n be odd and m be even. Then we see that $i^*(x_{n-1}) = y_{n-1}$, where i^* is the morphism of algebras from

$$H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n-1); \mathbb{Z}/p)$$

into

$$H^*(S^{n-1}; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_{n-1}]/(y_{n-1}^2)$$

induced from the inclusion map i. Therefore, by computing the Koszul complex, it follows that

$$E_2^{**} \cong \Gamma[s^{-1}e_{2(n-1)+3}, s^{-1}e_{2(n-1)+7}, \dots, s^{-1}e_{2m+2n-3}].$$

Hence $E_2^{**} \cong E_\infty^{**} \cong E_0^{**}$. By applying [**10**; Theorem 2.4], we obtain the desired result in the case where n is odd and m is even.

Let n and m be odd. By using the same argument as the above, we see that

$$E_0^{**} \cong \Gamma[s^{-1}e_{2(n-1)+3}, s^{-1}e_{2(n-1)+7}, \dots, s^{-1}e_{2m+2n-5}, s^{-1}e'_{m+n-1}].$$

Let S be the p-simple system of generators determined from the divided power algebra E_0^{**}. Put $2t+1 = m+n-1$. Then $2pt+1 > 2(m+n-3)+1$. Using this fact and the usual argument of the filtration degrees and the total degrees, we see that $\gamma_{p[f]}(s^{-1}e'_{m+n-1})^p \notin N^p(S)$. Furthermore, using the Steenrod operation in the Eilenberg-Moore spectral sequence, we see that $\gamma_{p[f]}(s^{-1}e'_{m+n-1}) \notin N(S)$. Hence we have our result in the case where n and m are odd.

Let n be even and m odd. We can obtain that

$$E_2^{**} \cong \Gamma[s^{-1}e_{2(n-1)+1}, s^{-1}e_{2(n-1)+5}, \dots, s^{-1}e_{2m+2n-3}] \otimes \Lambda(x_{n-1}),$$

where $\operatorname{bideg} s^{-1}e_i = (-1, i)$ and $\operatorname{bideg} x_{n-1} = (0, n-1)$.

Let $\{\overline{E}_r, \overline{d}_r\}$ denote the mod p Eilenberg-Moore spectral sequence of the path loop fibration

$$\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)) \hookrightarrow P(\mathrm{SO}(m+n)/\mathrm{SO}(n)) \to \mathrm{SO}(m+n)/\mathrm{SO}(n).$$

We see that this spectral sequence collapses at the E_2-term by applying [**6**; DHA Lemma]. Therefore $H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p)$ can be determined as a vector space. By comparing each dimension of $H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p)$ and the total complex $\bigoplus E_2^{**}$, we conclude that $E_2^{**} \cong E_\infty^{**} \cong E_0^{**}$. As usual, we can have an appropriate p-simple system of generators for $H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/p)$. By [**10**; Theorem 2.4], we get our result in the case where n is even and m is odd. In a similar manner, we can also get it in the case where n and m are even. □

In general, it is not easy to determine the algebra structure of

$$H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/p)$$

from the associated bigraded algebra $\overline{E}_0^{**}$ which is obtained from the Eilenberg-Moore spectral sequence $\{\overline{E}_r, \overline{d}_r\}$ in the above proof. For example consider $\{\overline{E}_r, \overline{d}_r\}$ in the case where $p = 3$, $m = 5n-4$ and n is an even integer greater than 3. Then we see that

$$\overline{E}_0^{**} \cong \Gamma[s^{-1}e_{2n+3}, s^{-1}e_{2n+7}, \dots, s^{-1}e_{2m+2n-5}, s^{-1}e'_{m+n-1}] \otimes \Gamma[\tau(x_n^2)] \otimes \Lambda(s^{-1}x_n)$$

as an algebra, where

$$\operatorname{bideg} s^{-1}e_i = (-1, i),$$
$$\operatorname{bideg} s^{-1}e'_{m+n-1} = (-1, m+n-1) = (-1, 6n-5),$$
$$\operatorname{bideg} \tau(x_n^2) = (-2, 2n).$$

Since $\tau(x_n^2) = [x_n|x_n]$ in E_∞^{**}, it follows that

$$\wp_{EM}^{n-1}\tau(x_n^2) = \wp_{EM}^{n-1}[x_n|x_n] = \sum_{i+j=n-1} [\wp^i x_n|\wp^j x_n]$$

in $E_\infty^{-2,*}$. $\wp^i x_n$ is decomposable for dimensional reasons. Therefore, if $\wp^i x_n \neq 0$, then $n+2i(3-1) \geq 2n+3+2n+7$ and so $n+4i \geq 4n+10$. Hence $n+2j(3-1) < 4n+10$, because $i+j = n-1$. We can conclude that $\wp^j x_n = 0$. Finally, $[\wp^i x_n|\wp^j x_n] = 0$ for any i and j such that $i+j = n-1$. We have that $\tau(x_n^2)^3 = \wp^{n-1}\tau(x_n^2) \in \overline{F}^{-1}H^{6n-6}$. The element $\tau(x_n^2)^3$ may be equal to $s^{-1}e'_{n+m-1}$, because $s^{-1}e_{n+m-1}$ belongs to $\overline{F}^{-1}H^{6n-6}$. It is difficult to show whether $\tau(x_n^2)^3$ is equal to $s^{-1}e'_{m+n-1}$ or not even if we use the argument of the filtration and the Steenrod operation in the Eilenberg-Moore spectral sequence.

Proof of Theorem 1.14. As is known, $H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/2) \cong \Delta(x_n, x_{n+1}, \dots, x_{m+n-1})$ and

$$Sq^j x_i = \binom{i}{j} x_{i+j}, \quad \text{where } x_t = 0 \text{ if } t \geq m+n. \tag{2.16}$$

Therefore,

$$H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/2) \cong \bigotimes_{j\in J} \mathbb{Z}/2[x_j]/(x_j^{2[s_j]}),$$

where $J = L \cup L'$. Consider the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the path loop fibration on $\mathrm{SO}(m+n)/\mathrm{SO}(n)$. Then we have that

$$E_2^{**} \cong \operatorname{Tor}_\Gamma^{**}(\mathbb{Z}/2, \mathbb{Z}/2) \quad (\Gamma = H^*(\mathrm{SO}(m+n)/\mathrm{SO}(n); \mathbb{Z}/2))$$
$$\cong \left\{ \bigotimes_{j\in J, s_j>1} (\Gamma[w_j] \otimes \Lambda(s^{-1}x_j)) \right\} \otimes \left\{ \bigotimes_{j\in J, s_j=1} \Gamma[s^{-1}x_j] \right\}$$

and $E_r^{**} \Rightarrow H^*(\Omega(\mathrm{SO}(m+n)/\mathrm{SO}(n)); \mathbb{Z}/2)$, where $\mathrm{bideg}\, s^{-1}x_j = (-1, j)$ and $\mathrm{bideg}\, w_j = (-2, j\cdot 2[s_j])$. By applying **[6**; DHA Lemma], we see that this spectral sequence collapses at the E_2-term: $E_0^{**} \cong E_\infty^{**} \cong E_2^{**}$. Let S be the simple system of generators obtained from E_0^{**}. If $j \in L'$ and $s_j = 1$, then $\gamma_{2[f]}(s^{-1}x_j) \notin N(S)$. In fact, suppose that $\gamma_{2[f]}(s^{-1}x_j) \in N(S)$; we see that there exists an integer $i \in J$ such that $(j-1) = 2[f']\cdot(i\cdot 2[t]-2)$, where $f' \geq 1$ and $t \geq 1$. But the left-hand side of the equality is odd and the right-hand side is even, which is a contradiction. Next let us verify that $\gamma_{2[f]}(s^{-1}x_j)^2 \notin N^2(S)$ if $j \in L'$ and $s_j = 1$. Suppose that $\gamma_{2[f]}(s^{-1}x_j)^2 \in N^2(S)$. From **[10**; Lemma 3.1], we have the following equality:

$$Sq^{2[f]\cdot(j-1)}\gamma_{2[f]}(s^{-1}x_j) = \gamma_{2[f]}(s^{-1}x_j)^2 = \lambda\gamma_{2[f]}(s^{-1}x_i) + w$$

in $E_0^{-2[f],*}$, where $\lambda \neq 0$ and w expressed by S does not have the term $\mu\gamma_{2[f]}(s^{-1}x_i)$ $(\mu \neq 0)$. Therefore

$$Sq_{EM}^{2[f]\cdot(j-1)}[\underbrace{x_j|x_j|\cdots|x_j}_{2[f]}] = \lambda[\underbrace{x_i|x_i|\cdots|x_i}_{2[f]}] + w \quad \text{in } E_\infty^{-2[f],*}$$

and the left-hand side is equal to

$$\sum_{i(1)+\cdots+i(2[f])=2[f]\cdot(j-1)} [Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j].$$

For any term $[Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j]$, provided that there exists some integer $i(t)$ such that $i(t) \geq j$, there exists an integer $i(t')$ such that $i(t') < j-1$. Since j is even, from (2.16), we obtain that $[Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j] = 0$. Similarly, we see that the term $[Sq^{i(1)}x_j|\cdots|Sq^{i(2[f])}x_j]$ is zero if $i(t) < j$ for any $i(t)$. Hence

$$\lambda[x_i|\cdots|x_i] + w = 0 \quad \text{in } E_\infty^{-2[f],*},$$

which is a contradiction. Thus we conclude that $\gamma_{2[f]}(s^{-1}x_j)^2 \notin N^2(S)$. Using the above fact and **[10**; Lemma 3.1], we have Theorem 1.14. (Note that $(m+n-2)/2+1 \in L$ if and only if $m+n-2 \equiv 0 \bmod 4$ and $n \leq m$.) □

Proof of Theorem 1.15. Let $\{E_r, d_r\}$ denote the Eilenberg-Moore spectral sequence of the path loop fibration

$$\Omega(\mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n)) \hookrightarrow P(\mathrm{U}(m+n)/\mathrm{U}(n)) \\ \to \mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n).$$

Note that this spectral sequence has a Hopf algebra structure. We know that

$$H^*(\mathrm{U}(m+n)/\mathrm{U}(m)\times\mathrm{U}(n);\mathbb{Z}/p)\cong\mathbb{Z}/p[c_1,\dots,c_n]/(\rho_1,\dots,\rho_n)$$

as an algebra, where $\rho_1,\dots,\rho_n$ is a regular sequence, ρ_i is decomposable for any i, $\deg c_i = 2i$, $\deg\rho_i = 2m+2i$ and $m\ge n$ (cf. [**11**]). By virtue of [**19**; Proposition 1.1], we obtain that

$$E_2^{**}\cong\mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/p,\mathbb{Z}/p)\cong\Lambda(s^{-1}c_1,\dots,s^{-1}c_n)\otimes\Gamma[\tau\rho_1,\dots,\tau\rho_n]$$

as an algebra, where $\mathrm{bideg}\, s^{-1}c_i = (-1,2i)$ and $\mathrm{bideg}\,\tau\rho_j = (-2,2m+2j)$. Since the free algebra generators with less total degree than $\mathrm{totdeg}\, s^{-1}c_n+1$ have column degree -1, by applying [**6**; DHA Lemma], it follows that those images by the differential d_r are zero for any $r\ge 2$. By applying [**10**; Theorem 2.4], we have Theorem 1.15. □

In order to prove Theorem 1.16, we will calculate a Koszul complex in the concrete.

Proof of Theorem 1.16. As is known,

$$H^*(\mathrm{SU}(9)/\mathbb{Z}/3;\mathbb{Z}/2)\cong\Lambda(e_3,e_5,\dots,e_{17}),$$

(2.17) $Sq^{2j}e_{2i-1} = \binom{i-1}{j}e_{2i+2j-1}$, where $e_{2t-1}=0$ if $t>9$,

and

(2.18)

$$H^*(E_8;\mathbb{Z}/2)$$
$$\cong\mathbb{Z}/2[x_3,x_5,x_9,x_{15}]/(x_3^{16},x_5^8,x_9^4,x_{15}^4)\otimes\Lambda(x_{17},x_{23},x_{27},x_{29}),$$
$$Sq^2x_3=x_5,0\quad Sq^4x_5=x_9,\quad Sq^2x_{15}=x_{17},\quad Sq^8x_9=x_{17},$$
$$Sq^8x_{15}=x_{23},\quad Sq^4x_{23}=x_{27}\ \text{ and }\ Sq^2x_{27}=x_{29}.$$

Let us show that

(2.19) $$i^*(x_i)=\begin{cases} e_i & \text{if } i=3,5,9,15 \text{ or } 17\\ 0 & \text{if } i=23,27 \text{ or } 29,\end{cases}$$

where $i\colon \mathrm{SU}(9)/\mathbb{Z}/3\hookrightarrow E_8$ is the inclusion map.

First, we have that $i^*(x_3)=e_3$ since $j_*\colon H_3(\mathrm{SU}(7))\to H_3(E_8)$ is an isomorphism, where $j\colon \mathrm{SU}(7)\to E_8$ is a composition of the inclusion maps $i\colon\mathrm{SU}(9)/\mathbb{Z}/3\to E_8$ and $k\colon\mathrm{SU}(7)\to\mathrm{SU}(9)/\mathbb{Z}/3$.

Using the Steenrod operation, we have that $i^*(x_5) = e_5$, $i^*(x_9) = e_9$ and $i^*(x_{17}) = e_{17}$. Let us show that $i^*(x_{15}) = e_{15}$. Since $\{e_{15}, e_3 \cdot e_5 \cdot e_7\}$ is a basis of $H^{15}(\mathrm{SU}(9)/\mathbb{Z}/3; \mathbb{Z}/2)$, we can write as follows: $i^*(x_{15}) = \lambda e_{15} + \lambda' e_3 \cdot e_5 \cdot e_7$. Therefore, applying the Steenrod operation, we have that

$$Sq^2 i^*(x_{15}) = Sq^2(\lambda e_{15} + \lambda' e_3 \cdot e_5 \cdot e_7)$$

and so

$$e_{17} = \lambda e_{17} + \lambda' e_3 \cdot e_5 \cdot e_9 .$$

We see that $\lambda = 1$ and $\lambda' = 0$. Hence we obtain that

$$i^*(x_{15}) = e_{15} . \tag{2.20}$$

From (2.17), (2.18) and (2.20), we conclude that $i^*(x_i) = 0$ if $i = 23$, 27 or 29. Thus we have (2.19).

Next let us consider the Eilenberg-Moore spectral sequence $\{E_r, d_r\}$ of the fibration $\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)) \hookrightarrow \mathrm{SU}(9)/\mathbb{Z}/3 \to E_8$. By using the Koszul resolution, we can obtain the explicit form of the E_2-term:

$$\mathrm{Tor}_\Gamma^{**}(\mathbb{Z}/2, H^*(\mathrm{SU}(9)/\mathbb{Z}/3; \mathbb{Z}/2)) \cong H(\mathscr{K}, \delta),$$

where $\Gamma = H^*(E_8; \mathbb{Z}/2)$,

$$\begin{aligned} \mathscr{K} = {} & \Lambda(s^{-1}x_3, s^{-1}x_5, s^{-1}x_9, s^{-1}x_{15}) \\ & \otimes \Gamma[w_1, w_2, w_3, w_4, s^{-1}x_{17}, s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{29}] \\ & \otimes \Lambda(e_3, e_5, \dots, e_{17}), \end{aligned}$$

$\mathrm{bideg}\, s^{-1}x_j = (-1, j)$, $\mathrm{bideg}\, w_1 = (-2, 48)$, $\mathrm{bideg}\, w_2 = (-2, 40)$, $\mathrm{bideg}\, w_3 = (-2, 36)$, $\mathrm{bideg}\, w_4 = (-2, 60)$, $\mathrm{bideg}\, e_i = (0, i)$ and $\delta(s^{-1}x_i) = e_i$ if $i = 3, 5, 9, 15$ or 17,

$$\delta(\gamma_{2[f]}(s^{-1}x_{17})) = \gamma_{2[f]-1}(s^{-1}x_{17}) \otimes e_{17}$$

and $\delta(\alpha) = 0$ for any other algebra generator α. (The differential δ is determined from (2.19), see Figure 2.) Computing the above complex, we obtain that

$$\begin{aligned} E_2^{**} \cong {} & \Lambda(e_7, e_{11}, e_{13}) \\ & \otimes \Lambda(s^{-1}x_3 \otimes e_3, s^{-1}x_5 \otimes e_5, s^{-1}x_9 \otimes e_9, s^{-1}x_{15} \otimes e_{15}) \\ & \otimes \Gamma[w_1, w_2, w_3, w_4] \otimes \Gamma[s^{-1}x_{23}, s^{-1}x_{27}, s^{-1}x_{29}] \end{aligned}$$

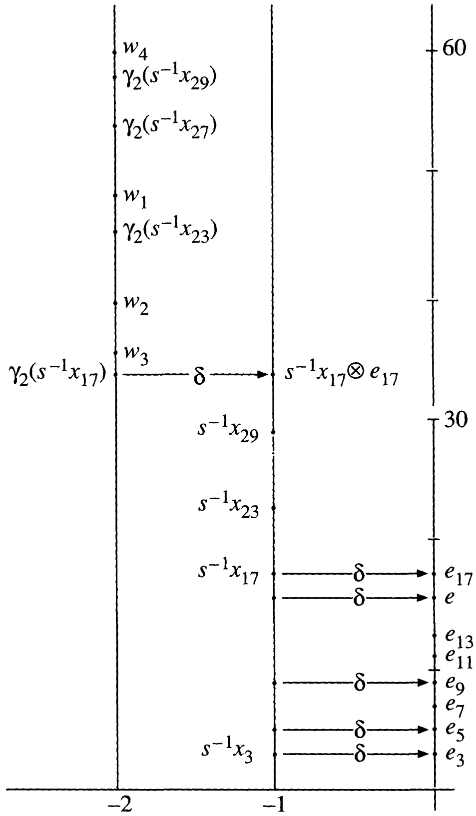

FIGURE 2

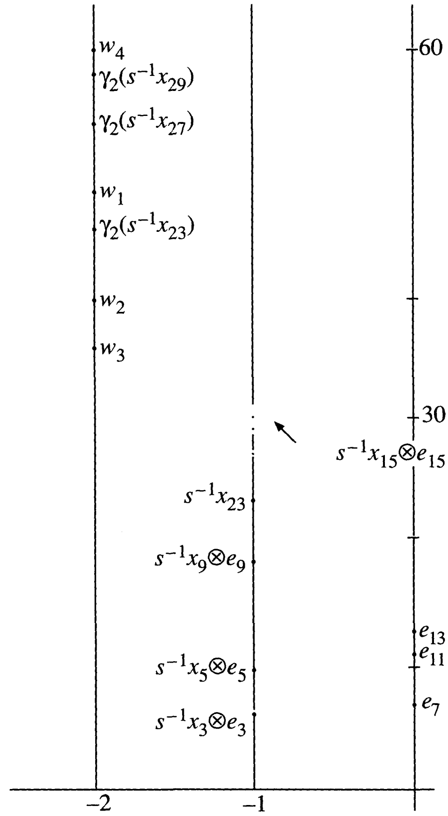

FIGURE 3

(see Figure 3). Applying the same argument as the proof of Theorem 1.15, we see that

$$d_r(\alpha) = 0 \text{ for } r \geq 2 \text{ and any algebra generator } \alpha \text{ in } E_2^{**}.$$

Hence $E_0^{**} \cong E_\infty^{**} \cong E_2^{**}$. Put

$$\begin{aligned} AG = \{&e_7, e_{11}, e_{13}, s^{-1}x_5 \otimes e_5, s^{-1}x_9 \otimes e_9, s^{-1}x_{15} \otimes e_{15}\} \\ \cup \{&\gamma_{2[f_1]}(w_1), \gamma_{2[f_2]}(w_2), \gamma_{2[f_3]}(w_3), \gamma_{2[f_4]}(w_4)\}_{f_i \geq 0} \\ \cup \{&\gamma_{2[g_1]}(s^{-1}x_{23}), \gamma_{2[g_2]}(s^{-1}x_{27}), \gamma_{2[g_3]}(s^{-1}x_{29})\}_{g_j \geq 0}. \end{aligned}$$

The usual argument of the filtration of $H^*(\Omega(E_8/(\mathrm{SU}(9)/\mathbb{Z}/3)); \mathbb{Z}/2)$ allows us to conclude that $e_i^2 = 0$ and $(s^{-1}x_i \otimes e_i)^2 = 0$. Furthermore, $h(\alpha) = 2$ for any $\alpha \in AG$ because there is no pair of non-negative integers (f, f') which satisfies

$$22 \cdot 2^f = 26 \cdot 2^{f'}, 28 \cdot 2^{f'}, 46 \cdot 2^{f'}, 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$26 \cdot 2^f = 28 \cdot 2^{f'}, 46 \cdot 2^{f'}, 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$28 \cdot 2^f = 46 \cdot 2^{f'}, 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$46 \cdot 2^f = 38 \cdot 2^{f'}, 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$38 \cdot 2^f = 34 \cdot 2^{f'}, 58 \cdot 2^{f'},$$
$$34 \cdot 2^f = 58 \cdot 2^{f'}$$

(see [**10**; Theorem 2.4]).

We have Theorem 1.16 by [**10**; Theorem 2.4].

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Received August 22, 1991 and in revised form November 20, 1991.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 163 No. 2 April 1994

0030-8730(1994)163:2;1-I